Propagation of elastic waves in equiaxed stainless-steel polycrystals with aligned [001] axes

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Abstract
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Keywords
elastic waves, plane waves, acoustic testing, anisotropy, attenuation, calculation methods, cubic lattices, phase velocity, polycrystals, stainless steels, ultrasonic waves, wave propagation, Aerospace Engineering

Disciplines
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Comments
Propagation of elastic waves in equiaxed stainless-steel polycrystals with aligned [001] axes

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The unified theory for elastic wave propagation in polycrystalline materials, developed by Stanke and Kino [J. Acoust. Soc. Am. 75, 665–681 (1984)], has been applied to compute the propagation constants in materials consisting of cubic crystallites with texture. The particular texture considered here is one in which one of the cube axes of each crystallite is aligned in a preferred direction with the other two being randomly oriented, leading to a material with macroscopic transverse isotropy. The calculations are done for plane waves under the assumption that the single crystal anisotropy is not large. Numerical results for the variations of attenuation and phase velocity with direction of propagation and frequency are presented for L, SH, and SV waves. © 1996 Acoustical Society of America.

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INTRODUCTION

A polycrystalline material is composed of numerous discrete grains, each having a regular, crystalline atomic structure. The elastic properties of the grains are anisotropic and their crystallographic axes are differently oriented. When an ultrasonic wave propagates through such a polycrystalline aggregate, it is attenuated by scattering at the grain boundaries, with the value of this attenuation and the related shift in the propagation velocity depending on the size, shape, and orientation distributions of the grains. If the grains are equiaxed and randomly oriented, these propagation properties are independent of direction, but such is not the case when the grains have preferred orientation. Thus the anisotropic nature of elastic wave propagation in a number of structural materials, such as the austenitic stainless-steel welds and cast piping used in nuclear power plants, stems from the details of their grain structures.

A close examination of these stainless steels reveals a variety of microstructures ranging from randomly oriented, equiaxed grains to highly oriented, columnar grains.1,2 A lack of the knowledge of the presence of anisotropies in the attenuation and velocities of elastic waves due to favored grain orientations and elongated shapes can create problems in ultrasonic nondestructive testing of stainless-steel components. Consequently an effort has been undertaken to understand the behavior of effective wave propagation constants (which incorporate the effects of microstructure on the attenuation coefficient and phase velocity) in these kinds of materials. This paper describes one step in this study.

The propagation of elastic waves in randomly oriented, equiaxed polycrystals has received considerable attention, with the most recent contributions for the cubic materials being made by Hirsekorn,3,4 Stanke and Kino,5,6 and Beltzer and Brauner.7 Previous references are cited in their work. Hirsekorn models the medium as a collection of spherical scatterers, and represents the fields in the medium as a Born series, thereby formally including multiple scattering. She then truncates the Born series after a second-order term to evaluate the frequency dependence of the attenuation and velocity. Stanke and Kino present an alternative “unified theory.” This is based on the second-order Keller approximation8 and the use of a geometric autocorrelation function to describe the grain size distribution. Stanke and Kino argue that their approach is to be preferred because (a) the unified theory more fully treats multiple scattering (which is lost in the second-order perturbation theory used in Hirsekorn’s calculations), (b) the unified theory avoids the high-frequency oscillations which are coherent artifacts of the single-sized, spherical grains assumed by Hirsekorn, and (c) the unified theory correctly captures the high-frequency “geometric regime” in which the Born approximation breaks down. They present numerical results for iron and aluminum, illustrating the behavior of attenuation and velocity in the Rayleigh, stochastic, and geometric regimes for cubic polycrystals of weak and strong anisotropy.

When stainless steel is produced during welding or solidification, a common microstructure finds the [001] crystallographic axes aligned parallel to the thermal gradients that existed during solidifications. The grains are generally randomly oriented about this direction, and can be elongated. The theoretical treatment of such cases is more limited. Hirsekorn has extended her theory to the case of preferred crystallographic orientation while retaining the assumption of spherical grain shape,9 and has performed numerical calculations for the case of stainless steel with fully aligned [001] axes.10 The general formalism of Stanke and Kino applies to the case of elongated grains with preferred orientation. However, no evaluations of that formalism have been reported for preferentially aligned grains. This paper reports such an extension of the theory of Stanke and Kino. Specifically, we consider elastic wave propagation in polycrystals of...
cubic crystallite symmetry with the aforementioned texture in which the [001] crystallographic axes of all grains are parallel to the z axis of the laboratory coordinate system. The remaining crystallographic axes [100] and [010] are randomly oriented leading to a transversely isotropic material. The second-order unified theory of Stanke and Kino,5 which is valid when the single crystal anisotropy is not large, has been extended to arrive at the present theory. To this order of approximation, some degree of multiple scattering by individual grains is accounted for. In order to simplify the associated algebra, the highly oriented, columnar grains have been replaced with highly oriented, equiaxed grains. Numerical results for the attenuation and velocity of both longitudinal and shear waves in this model material are calculated and compared with other existing theories. The material properties of this idealized medium are chosen to be the same as those of 304 stainless-steel austenitic weld metal and are listed in Table I. In a sequel, grain elongation will be introduced to allow the effects of grain shape to be independently examined.

### I. THEORY

#### A. Formal approach

Following Stanke and Kino,5 we note that the displacement field associated with the propagation of an elastic wave through a polycrystalline material is governed by the stochastic wave equation

\[ [C_{ijkl}^{\xi}(\mathbf{r})u_{ij}^{\xi}(\mathbf{r})]_{,j} + \rho \omega^2 u_{ij}^{\xi}(\mathbf{r}) = 0, \]  

where \( \xi \) represents a particular medium of an ensemble of possible media \( \Xi \), \( u_{ij}^{\xi}(\mathbf{r}) \) is the actual field in the medium \( \xi \), and the notation \( j \) denotes differentiation. They define the perturbation of the local elastic tensor \( C_{ijkl}^{\xi}(\mathbf{r}) \) from the isotropic homogeneous elastic tensor as

\[ \epsilon \Delta_{ijkl}^{\xi}(\mathbf{r}) = C_{ijkl}^{\xi}(\mathbf{r}) - C_{ijkl}^{0}, \]

where \( C_{ijkl}^{0} \) is the Voigt12 averaged elastic tensor. In this formalism, the quantity \( \epsilon \Delta_{ijkl} \) is a perturbation parameter, which is small with respect to \( C_{ijkl}^{0} \) when the single crystal anisotropy is not large. Our goal is to find the expected propagation constant \( k = \omega v_p - i \alpha \), where \( v_p \) is the phase velocity and \( \alpha \) is the attenuation, from the expected properties of the media. Application of the second-order Keller approximation,8 which accounts for some degree of multiple scattering since the scattered waves are generated by the mean plane wave which is itself effected by the scattering, to Eqs. (1) and (2) yields the equation for the expected displacement field,

\[ 0 = C_{ijkl}^{0}(u_{ik}^{\xi}(\mathbf{r}),j)_{,j} + \rho \omega^2 u_{ik}^{\xi}(\mathbf{r}) + \epsilon \left( \Delta_{ijkl}^{\xi}(\mathbf{r}) \int G_{km,ij}(\mathbf{r} - \mathbf{r}') \right) 
\times \langle u_{ik}^{\xi}(\mathbf{r}) \rangle_{,j} - \epsilon^2 \left( \Delta_{ijkl}^{\xi}(\mathbf{r}) \int G_{km,ij}(\mathbf{r} - \mathbf{r}') [\langle \epsilon \Delta_{mnpq}^{\xi}(\mathbf{r}') \rangle_{,n'} \cdot d\mathbf{v}]_{,j} \right) 
\times \langle u_{ik}^{\xi}(\mathbf{r}) \rangle_{,n,q} \rangle_{,j} \cdot d\mathbf{v}. \]  

(3)

The angled brackets in Eq. (3) represent the ensemble averages. The Green’s function \( G_{km} \) used above has been taken from the work of Lifshits and Parkhamovski.13

In order to simplify the averages, we will assume that the media are statistically homogeneous, i.e., the ensemble averages are independent of position. The only property of the media which affects the one-point average \( \langle \Delta_{ijkl}^{\xi}(\mathbf{r}) \rangle \) is the rotation of the crystallographic axes of the grain containing \( \mathbf{r} \) with respect to the laboratory coordinate system. In addition, following Stanke and Kino,5 we also assume that the Euler angles of the grains in the ensemble are statistically independent. Then for plane waves propagating in the \( \mathbf{k} \) direction,

\[ \langle u_{ij}(\mathbf{r}) \rangle = a \hat{u}_i e^{-ik \cdot \mathbf{r}}, \]

Eq. (3) simplifies to the following equation for the expected propagation constant \( k \) for arbitrary crystallite symmetry with macroscopic texture:

\[ 0 = (C_{ijkl}^{0} + \epsilon \langle \Delta_{ijkl} \rangle \hat{u}_k \hat{k} \hat{l} - \rho \omega^2 k^2 \hat{u}_i + \epsilon^2 \langle \Delta_{ijkl}^{\xi} \Delta_{mnpq}^{\xi} \rangle \int G_{km}(\mathbf{s}) \)
\times \langle W(\mathbf{s}) \epsilon \Delta_{ijkl}^{\xi} \rangle_{,n} \hat{u}_k \hat{k} \hat{l} d\mathbf{v}. \]  

(5)

Here \( W(\mathbf{s}) \), which represents the geometric autocorrelation function, appears during the process of evaluating the two-point average in Eq. (3). Physically, this term represents the probability that two points separated by \( \mathbf{s} \) are in the same grain. Equation (5) is identical to Eq. (93) of Stanke and Kino.3 As they note, “This is an equation for the expected propagation constant accurate to \( \epsilon^2 \), and applicable for arbitrary symmetry in media with macroscopic texture and preferred grain elongation.” Insight into the significance of Eq. (5) can be gained by casting it into a form similar to that of Christoffel’s equation, used to describe elastic wave propagation in an anisotropic continuum. Accordingly, the three relations dictated by taking \( i = 1,2,3 \) can be written in the compact form

\[ \left[ \Gamma_{ik} - \rho \omega^2 k^2 \delta_{ik} \right] \hat{u}_k = 0, \]  

(6)

where

\[ \begin{align*}
\Gamma_{ik} &= C_{ijkl}^{0} \langle u_{ik}^{\xi}(\mathbf{r}),j \rangle_{,j} \\
&+ \rho \omega^2 u_{ik}^{\xi}(\mathbf{r}) + \epsilon \left( \Delta_{ijkl}^{\xi}(\mathbf{r}) \int G_{km,ij}(\mathbf{r} - \mathbf{r}') \right) 
\times \langle u_{ik}^{\xi}(\mathbf{r}) \rangle_{,j} - \epsilon^2 \left( \Delta_{ijkl}^{\xi}(\mathbf{r}) \int G_{km,ij}(\mathbf{r} - \mathbf{r}') [\langle \epsilon \Delta_{mnpq}^{\xi}(\mathbf{r}') \rangle_{,n'} \cdot d\mathbf{v}]_{,j} \right) 
\times \langle u_{ik}^{\xi}(\mathbf{r}) \rangle_{,n,q} \rangle_{,j} \cdot d\mathbf{v},
\end{align*} \]
The general method of solution is well known. Equation (6) admits solutions for \( \hat{a} \) only if the determinant of the matrix in brackets on the left-hand side vanishes. In the absence of scattering, this occur for three distinct real values of \( \omega^2/k^2 \). In the presence of scattering, requiring the determinant to vanish defines a transcendental equation which may support many roots. The correct root was selected by seeking the real part of the root closest to the root in the absence of scattering and requiring that the imaginary part lead to a non-negative attenuation. The velocity and attenuation of one quasilongitudinal and two quasishear modes of propagation were determined by selecting, from each pair the member with \( \alpha>0 \). The wave polarizations are given by the corresponding eigenvectors \( \hat{u}_k \). The three terms on the right side of Eq. (7) can be easily interpreted. The first term, \( C_{ijkl}^0 \), describes wave propagation in the isotropic material that would exist if the grains were randomly oriented, based on the Voigt approximation. The effects of preferred grain orientation, also within the Voigt approximation, are introduced to first order by the second term, \( \varepsilon(\Delta_{ijkl}) \). At this level of approximation, the material is viewed as a continuum, with no attenuation or dispersion in velocity. The third term introduces the effects of grain size and shape [through \( W(s) \)] and frequency (through \( e^{ik \cdot \hat{s}} \)). This may be thought of as introducing the effects of multiple scattering through the two point correlation functions. Preferred orientation also influences the third term through the factor \( e^{2\pi} \{ \langle \Delta_{ijkl}^2 \rangle - \langle \Delta_{ijkl} \rangle^2 \} \). In the long wavelength limit, this term can be expected to be negative, since the Voigt approximation is known to provide an upper bound on the velocities.

B. Particular case for calculations

We have carried out calculations for polycrystals of cubic symmetry where the grains are assumed to be equiaxed. This assumption enables us to replace \( W(s) \) by \( W(s) \) which greatly simplifies the evaluation of Green’s integral in Eq. (5). Following Stanke and Kino, the geometric autocorrelation function is assumed to have the form

\[
W(s) = e^{-2s/\bar{d}},
\]

where \( \bar{d} \) is the mean grain diameter. As pointed out by Stanke, this choice for \( W(s) \) is a good model for many real materials. As mentioned before, the particular texture considered in this work has the [001] crystallographic axes of all grains parallel to the \( z \) axis of the laboratory coordinate system while the [100] and [010] axes are randomly oriented. This simplifies the averaging procedure. Thus if \( \phi \) is the rotation of the [001] axis from the \( x \) axis in the laboratory system

\[
\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi.
\]

The assumption that the grains have cubic symmetry can be stated mathematically as

\[
C_{ijkl}(r) = c_{12}(\delta_{ij} \delta_{kl}) + c_{44}(\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl}) + A \alpha^j a^j \alpha^i a^i,
\]

where \( c_{11}, c_{22}, c_{44} \) are the single crystal elastic constants in abbreviated notation, \( A \) is the single crystal anisotropy \((A = c_{11} - c_{12} - 2c_{44})\), and \( \alpha^j \) is the cosine of the angle between the \( j \)th laboratory axis and the \( p \)th crystallographic axis of the grain containing \( r \). In order to perform the integration in Eq. (5), we have used the following mean-value theorem:

\[
\int e^{ik \cdot \hat{s}} ds = \int_0^\infty \int_\Sigma e^{ik \cdot \hat{s}} d\sigma ds = \int_0^\infty \frac{4\pi s}{k} \sin(ks) ds.
\]

Here \( \Sigma \) represents the surface of a sphere of radius \( s \) with \( d\sigma \) being a differential area on \( \Sigma \). Following the general procedure to obtain the complex propagation constants and polarizations as described at the end of Sec. I A, we were able to develop algebraic equations for the expected propagation constant for acoustic waves propagating along arbitrary directions in the \( y-z \) plane. In order to do this, it was found convenient to rotate the laboratory coordinate system \((x, y, z)\) by an angle \( \theta \) about the \( x \) axis resulting in a primed \((x', y', z')\) coordinate system and choose the \( z' \) axis as the propagation direction. The forms of the two-point averages are then

\[
D_{ijklmn}^{\alpha \beta \delta \epsilon \gamma \delta \epsilon} = e^{2\pi} \{ \langle \Delta_{3333} \rangle \langle \Delta_{mn33} \rangle - \langle \Delta_{3333} \rangle \langle \Delta_{mn33} \rangle \}.\]

We have calculated the following one- and two-point averages. The Appendix includes an example of the calculation \( D_{11111111} \):

\[
e^{(L)} = \frac{(\cos^4 + 3\sin^4) / (4 - 3/5)}{A},
\]

\[
e^{(S)} = \frac{(\sin^2 / (4 - 1/5))}{A},
\]

\[
D_{1111} = \frac{A^2 \sin^4 \theta}{32}, \quad D_{1122} = \frac{A^2 \cos^2 \theta \sin^4 \theta}{32},
\]

\[
D_{1133} = \frac{A^2 \sin^6 \theta}{32}, \quad D_{1212} = \frac{A^2 \cos^2 \theta \sin^4 \theta}{32},
\]

\[
D_{1313} = \frac{A^2 \sin^6 \theta}{32}, \quad D_{2222} = \frac{A^2 \cos^4 \theta \sin^4 \theta}{32},
\]

\[
D_{3333} = \frac{A^2 \cos^2 \theta \sin^6 \theta}{32}, \quad D_{2233} = \frac{A^2 \cos^2 \theta \sin^6 \theta}{32},
\]

\[
D_{2323} = \frac{A^2 \cos^2 \theta \sin^6 \theta}{32}, \quad D_{3311} = \frac{A^2 \sin^2 \theta}{32},
\]

\[
D_{1122} = \frac{A^2 \cos^2 \theta \sin^2 \theta}{32}, \quad D_{1212} = \frac{A^2 \cos^2 \theta \sin^2 \theta}{32},
\]

\[
D_{3311} = \frac{A^2 \sin^4 \theta}{32}, \quad D_{3311} = \frac{A^2 \sin^4 \theta}{32},
\]
Finally carrying out the remaining algebraic operations in Eq. (5), the expected propagation constants for $L$ waves propagating in an arbitrary direction in the $y$-$z$ plane is given by the algebraic equation

$$[1 + \frac{A (\cos^4 \theta + 3 \sin^4 \theta/4) - 3A/5}{C_{3333}}]m^L = \chi^L + \frac{\chi^2 A^2}{\chi_0^2 (C_{3333})^2} \Psi^L(\chi_L, \chi_{0L}, \chi_{0O}).$$

Similarly, the expected propagation-constants for $SH$ and $SV$ waves propagating in an arbitrary direction in the $y$-$z$ plane are given by the algebraic equations

$$[1 + \frac{A \sin^2 \theta/4 - A/5}{C_{1313}}]m^L = \chi^L + \frac{\chi^2 A^2}{\chi_0^2 (C_{1313})^2} \Psi^L(\chi_{SH}, \chi_{0L}, \chi_{0O}),$$

$$[1 + \frac{A \cos^2 \theta \sin^2 \theta/4 - A/5}{C_{2323}}]m^L = \chi^L + \frac{\chi^2 A^2}{\chi_0^2 (C_{2323})^2} \Psi^L(\chi_{SV}, \chi_{0L}, \chi_{0O}).$$

For brevity, the complicated expressions $\Psi^L(\chi_L, \chi_{0L}, \chi_{0O})$, $\Psi^L(\chi_{SH}, \chi_{0L}, \chi_{0O})$, and $\Psi^L(\chi_{SV}, \chi_{0L}, \chi_{0O})$ are not included in this paper. Interested parties can obtain from the authors an internal report providing the details. The definitions of the normalized frequencies, of the form $\chi = k\lambda$, may be found in Ref. 5.

II. RESULTS

To obtain the normalized attenuation coefficient $\alpha/k_{0L}$ and the normalized shift in phase velocity $(V - V_0)/V_0$ for plane waves in the considered textured medium, the algebraic equations for the expected propagation constants were solved numerically, using the Muller's method for finding complex roots, on a digital computer. Here and in what follows, $V_0$ and $k_0$ refer to phase velocity and unperturbed wave number, respectively, based on Voigt averaged elastic constants in the absence of preferred grain orientation. The subscripts “$L$” and “$S$” associated with the Voigt averaged quantities refer to $L$ and $S$ waves, respectively. In the process, the single crystal elastic constants used for stainless steel are listed in Table I and are the same as those used by Hirsekorn. In accordance with the notation employed here, the directions $x$ and $y$ are interchangeable.

A. Continuum results

At long wavelength, the elastic wave propagation properties are often predicted from the static elastic constants. In the Voigt approximation, this is equivalent to solving Eqs. (6) and (7) when the terms of order $\epsilon^2$ are omitted. Figure 1 shows the results for stainless steel. Part (a) presents the slowness surfaces, which give the reciprocal of phase velocity as a function of propagation direction. Part (b) presents the ray surfaces, which give the magnitude of the group velocity as a function of energy flow direction. Of particular interest are the cusps in the $SV$-ray surface. Other schemes for determining average elastic constants, such as the Reuss and Hill approximations, yield similar slowness and wave surfaces of slightly different numerical values. Recall that the Voigt approximation is expected to provide an upper bound to the elastic stiffness (and hence the velocities), the Reuss approximation is expected to provide a lower bound, with the Hill approximation providing an intermediate value that is often used to compare to experiment.

B. Longitudinal wave

Figure 2(a) shows the dependence of the normalized attenuation coefficient $\alpha/k_{0L}$ (attenuation per wavelength) on the normalized frequency $\chi_{0L}$ for $L$ waves propagating in the $y$ direction. Qualitatively, this curve has the same shape as the corresponding curve for the untextured material, with the attenuation in the latter being slightly higher. There appears to be a strong dependence of the attenuation coefficient on $\chi_{0L}$ in the Rayleigh region ($\chi_{0L} < 1$) while in the Rayleigh-stochastic transition region ($1 < \chi_{0L} < 10$), the attenuation is roughly proportional to square of frequency. The discrete symbols in this figure represent the corresponding results of Hirsekorn. There is a good overall agreement between the present results and those of Hirsekorn except in the transition between the Rayleigh and the stochastic region where the latter shows oscillatory behavior. These oscillations have been attributed to the fact that, in her theory, all grains were attributed to the fact that, in her theory, all grains were
$\chi_{0l}$ as a parameter. It is seen from this figure that there is no attenuation predicted at any frequency when the direction of propagation is parallel to the axis of rotation symmetry. As was noted above, the attenuation at $\theta = 90^\circ$ is slightly less than that for the corresponding randomly oriented case.

In Fig. 3(a), we have plotted the normalized shift in phase velocity $(V_L - V_{0l})/V_{0l}$ versus the frequency parameter $\chi_{0l}$ for $L$ waves propagating in the $y$ direction. In this case we observe that, except at low frequencies, the acoustic wave is virtually nondispersive. In the quasistatic limit, the phase velocity is only approximated by the appropriate continuum theory using the Hill averaged elastic constants for a textured medium. This result is consistent with the quasistatic limit of the randomly oriented case, in which Stanke

![Graphs and images](image1.png)

**FIG. 1.** Continuum velocities of stainless steel in the Voigt approximation. (a) Slowness surfaces, (b) ray surfaces.

![Graphs and images](image2.png)

**FIG. 2.** Normalized attenuation coefficient for $L$ waves in model material. (a) Frequency dependence for propagation in $y$ direction: present, $\bigcirc$ Hirsekorn; (b) directional dependence: $\chi_{0l} = 0.1$, $1$, $2.5$, $5$, $10$. 

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and Kino observed that the unified predictions fell between the upper “Hashin bound” and lower “Shtrikman bound” lying somewhat closer to the former. As expected, there is a good agreement between the present calculations and the results of Hirsekorn except in the transition region between the Rayleigh and the stochastic regions. Figure 3(b) displays the variation of the shift in phase velocity with inclination of the propagation direction with respect to the z axis at various normalized frequencies. Predictions based on the continuum theory\textsuperscript{20} using the Voigt averaged elastic constants and the Hill averaged elastic constants, represented by symbols, are also shown in the same figure. Overall, we notice a qualitative agreement between these predictions and the present results. It is of interest to note that, by the time $\theta$ reaches about 45°, attenuation grows to a significant fraction of the maximum attenuation at $\theta=90^\circ$. This is particularly true in the Rayleigh and stochastic regions.

The frequency dependence of the normalized shift in phase velocity ($V_{SH} - V_{0\phi}$) for SH waves propagating in the y direction can be observed in Fig. 6(a). Unlike the behavior in the case of an L wave, the SH wave is highly dispersive in all frequency regimes except the geometric region. Again the phase velocity in the quasistatic limit lies somewhat above the continuum theory using the Hill averaged elastic constants. There is a good agreement between the present calculations and the results of Hirsekorn except at large frequencies. Figure 6(b) shows the variation of the shift in phase velocity with $\theta$ at various normalized frequencies. The phase velocity is higher in the direction of preferred grain orientation than at any other direction. Predictions based on the continuum theory using the Voigt averaged elastic constants and the Hill averaged elastic constants, represented by symbols, are also shown in the same figure. Again our calculations for the preferentially aligned grains is a shift in the latter transition down to $\chi_{0\phi}=6$. Consequently, the attenuation in the material with preferred orientation exceeds that of the randomly oriented material for low $\chi_{0\phi}$. The attenuation increases with grain size until entering the geometrical region and decreases thereafter. The present results compare favorably with those of Hirsekorn\textsuperscript{10} in the Rayleigh region. The agreement between these predictions become poor at higher frequencies, especially in the geometric regime. The variation of attenuation per wavelength with angle $\theta$ for various $\chi_{0\phi}$ is presented in Fig. 4(b). As expected, there is no attenuation at all frequencies when the direction of propagation is parallel to the z axis. It is of practical interest to note that, by the time $\theta$ reaches about 45°, attenuation grows to a significant fraction of the maximum attenuation at $\theta=90^\circ$. This is particularly true in the Rayleigh and stochastic regions.

C. SH wave

The frequency dependence of the normalized attenuation coefficient $ak_0$ for SH waves propagating in the y direction is shown in Fig. 4(a). Similar to L waves, this graph depicts the same qualitative behavior as the corresponding graph for the untextured material. However, some subtle changes are evident in Fig. 5, constructed for iron using the same elastic constants as Stanke and Kino.\textsuperscript{5} In the randomly oriented case, the shear wave Rayleigh region was approximately defined by $\chi_{0\phi}<0.7$. This was followed by a transition to the stochastic region (0.7$<\chi_{0\phi}$<4), the stochastic region (4$<\chi_{0\phi}$<8), and the geometric region ($\chi_{0\phi}$>8). The transition from the stochastic to the geometric region was marked by a peak in the plot of $ak_0$. One of the main differences in
we observe that, in the quasistatic limit, a continuum theory based on Voigt averaged elastic constants yields a phase velocity closer to the present calculations when \( u \) is small. On the other hand, continuum theory based on Hill averaged elastic constants appear more appropriate for \( SH \)-wave propagation perpendicular to the axis of rotation symmetry.

D. \( SV \) wave

The frequency dependence of the normalized attenuation coefficient \( a/k_0 \) for \( SV \) waves propagating at 45° with respect to the \( y \)-axis in the \( y-z \) plane is shown in Fig. 7(a). Similar to \( SH \) waves, this graph depicts the same qualitative behavior as the corresponding graph for the untextured material. Again, the peak, marking the transition from the stochastic to the geometric regime, is shifted downward, in this case to \( x_0 = 5 \). The attenuation per wavelength first increases rapidly with \( x_0 \) in the Rayleigh and the Rayleigh-stochastic transition regions, and again decrease in the geometric regime. The agreement of the present calculation with that of Hirsekorn is rather poor at all but the lowest frequency examined. The variation of attenuation per wavelength with angle \( \theta \) for various \( x_0 \) is presented in Fig. 7(b). As expected, there is no attenuation at any frequency when the direction of propagation is either parallel to the \( z \)-axis or perpendicular to it. For small values of \( x_0 \), the normalized attenuation coefficient is symmetric about the 45° direction relative to the \( z \)-axis. At higher frequencies, this symmetry is lost and the maximum attenuation per wave length is seen to peak at increasingly higher angles measured relative to the direction of rotation symmetry.

The frequency dependence of the normalized shift in phase velocity \( (V_{SV} - V_{0s})/V_{0s} \) for \( SV \) waves propagating at 45° with respect to the \( z \)-axis measured in the \( y-z \) plane is plotted in Fig. 8(a). Unlike the behavior in the case of an \( SH \) wave, the \( SV \) wave is virtually nondispersive at all frequencies. There is a good agreement between the present calculations and the results of Hirsekorn at all frequencies. Figure 8(b) shows the variation of the shift in phase velocity with \( \theta \) at various normalized frequencies. The phase velocity is lower at 45° with respect to the preferred grain orientation direction than at any other direction. Predictions based on the continuum theory using the Voigt averaged elastic constants and the Hill averaged elastic constants, represented by sym-

![Graph](image1)

![Graph](image2)
Bols, are also shown in the same figure. For this case, the continuum theory based on the Voigt averaged elastic constants yields phase velocity consistently closer to the present calculations than that given by the Hill averaged elastic constants.

III. DISCUSSION

There are several aspects of the results which deserve further discussion. The first concerns the relationship of the quasistatic limit of the predicted velocities to the predictions.
of the anisotropic continuum theories of Voigt and Hill. For all three wave types, the predictions of the unified theory were identical to that of the Voigt theory when \( \theta = 0^\circ \) (propagation along direction of preferential alignment). This is a consequence of the fact that the wave speed controlling component of the elastic stiffness tensor is the same in each grain for propagation in this direction. Mathematically, this implies that \( \Delta_{ijkl}(r) \) is a constant, and hence \( (\Delta_{ijkl}\Delta_{mnpq}) - (\Delta_{ijkl})\langle \Delta_{mnpq} \rangle \) vanishes. Symmetry conditions require that the SV wave phase velocity at \( \theta = 90^\circ \) equals that at \( \theta = 0^\circ \). Hence \( V_{SV}(90^\circ) \) also is identically equal to the predictions based on the Voigt average. At other angles, the velocities generally move toward the predictions of the Hill average. This behavior can be seen at \( \theta = 90^\circ \) for both the L and SH waves and, to a lesser extent, at \( \theta = 45^\circ \) for the SV waves. It will also be noted that, for many values of \( \theta \) in the SV case, the predictions of the unified theory fall above the anisotropic Voigt predictions. The origin of this effect has not yet been fully investigated. We suggest, however, that it is a result of our assumption (not necessarily a part of the unified theory) that the polarizations were purely longitudinal or transverse. The fact that at \( \theta = 45^\circ \) for SV waves, for which this was an excellent assumption, the unified theory does make a prediction between the Voigt and Hill models, is consistent with the suggested explanation.

In general, we have seen the same types of frequency

FIG. 8. Normalized variation of SV-wave phase velocity in model material. (a) Frequency dependence for propagation at 45° with respect to z direction: ——— present, \( \odot \) Hirsekorn; Hill limit off scale at \( -0.378 \). (b) Directional dependence: \( \chi_{0s} = \quad 0.1, \quad -0.5, \quad -1, \quad -2.5, \quad -5, \quad \odot \) continuum (Hill), \( \square \) continuum (Voigt).

FIG. 9. Directional dependence of normalized attenuation coefficient for SV waves in model material: \( \chi_{0s} = \quad 1.0, \quad -2.5, \quad -4, \quad -5, \quad -10, \quad \Box \ 20 \).

FIG. 10. Frequency dependence of normalized attenuation coefficient for SV waves propagating in the y-z plane in model material: ——— \( \theta = 30^\circ \), \( \square \) \( \theta = 45^\circ \), \( \square \) \( \theta = 52.5^\circ \), \( \square \) \( \theta = 60^\circ \), \( \square \) \( \theta = 67.5^\circ \), \( \square \) \( \theta = 75^\circ \), \( \odot \) \( \theta = 82.5^\circ \).
dependent behavior reported by Stanke and Kino for the randomly oriented case, with the attenuation clearly exhibiting the Rayleigh, stochastic and geometric regimes. The primary effect of preferred orientation has been to change numerical details, such as the particular values of frequency at which the various transitions occur.

There is, however, one strikingly different behavior predicted by our calculations. Examination of Fig. 7 shows that, at $\chi_{0x}=5$, the symmetry of the plots of attenuation per wavelength versus $\theta$ begins to break down. In attempts to identify the origin of this effect, several additional calculations were made. Figure 9 presents the angular variation of the $SV$-wave attenuation per wavelength for $\chi_{0x}$ ranging from 1 to 20. Here, the breakdown of symmetry observed at $\chi_{0x}=5$ is seen to be the beginning of a two-peaked structure, which has become quite pronounced at $\chi_{0x}=20$.

At first glance, this behavior might seem unlikely. However, as shown in Fig. 10, for each case examined, the frequency dependence exhibits the classical Rayleigh-stochastic-geometric regimes. Further detail for the particular case of $\theta=69.5^\circ$ is presented in Fig. 11. Here the attenuation is seen to peak near $\chi_{0x}=18$. By comparing Figs. 9–11, it appears that the peaks in the angular variation of the attenuation are associated with a shift in the stochastic-geometric region transition to higher frequency as $\theta$ moves from $45^\circ$ toward $90^\circ$ (or from $45^\circ$ toward $0^\circ$). For example, consider the case of $\chi_{0x}=20$. At $\theta=45^\circ$, one is well into the geometric regime. However, as $\theta$ increases, the shift of the transition to higher $\chi_{0x}$ causes the attenuation to increase until one reaches the angle of approximately $70^\circ$ at which the peak occurs at $\chi_{0x}=20$. Further increases in angle allow the point of operation to move into the stochastic and the Rayleigh regions with the attenuation falling rapidly. It is interesting to note that the propagation directions at which the peak attenuation occurs appear to asymptotically approach the direction of the cusps on the ray surface (inflection points on the slowness surface). We have not yet identified a simple physical explanation for these predictions.

IV. CONCLUDING REMARKS

We have extended the unified theory of Stanke and Kino\(^9\) to determine the propagation constants in a textured polycrystalline material where the crystallites have cubic symmetry. The particular texture considered here assumes that one of the cube axes of each crystallite is aligned in a preferred direction with the other two being randomly oriented, leading to a transversely isotropic medium. Since our formulation is based on the second-order Keller approximation,\(^8\) it accounts for some degree of multiple scattering and is valid for all frequencies. Our numerical results for $L$ and $SH$ waves are in good quantitative agreement with other existing theories. On the other hand, present calculations for $SV$ waves yield attenuation coefficients smaller than those predicted by the work of Hirsekorn.\(^10\) To simplify the problem at hand, we have assumed that the grains are equiaxed. We have also developed a similar formulation for the case of the elongated grains\(^21\) which will be presented in detail elsewhere.

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Combining Eqs. (2), (9), and (11) leads to the following expression for $D_{1111}^L$:

\[
D_{1111}^L = \frac{1}{2\pi} \int \left[ C_{3311}(\theta, \phi) - c_0^{3311} \right]^2 d\phi \\
- \left( \frac{1}{2\pi} \int \left[ C_{3311}(\theta, \phi) - C_0^{3311} \right] d\phi \right)^2,
\]

where $C_{3311}(\theta, \phi)$ is the indicated component of the elastic stiffness in the primed coordinate system $(x', y', z')$ obtained by rotating the laboratory coordinate system by angle $\theta$ about the $x$ axis and $\phi$ is the rotation of the $[001]$ axis from the $x$ axis in the laboratory system (see Fig. A1).

![Diagram](attachment:image.png)

**FIG. A1.** Orientation of the laboratory coordinate system $(x,y,z)$ with respect to axis of rotational symmetry $[001]$.  