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# Accept-Reject Decisions and Probabilistic Failure Prediction

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## **Abstract**

The purpose of this paper is to discuss recent progress in the development of an NDE decision formalism applied to the case of brittle fracture in ceramics. The on-line input into the formalism is a set of non-destructive (ND) measurements and the on-line output is the probability of failure conditioned on the above measurements. The final accept-reject decision depends only upon the comparison of the above output with a threshold related to the concerns of the user. The formalism involves stochastic physical models of the ND measurement process, the failure process (assuming a given stress environment), and the a priori statistics of defects. The present formalism goes beyond that reported at the previous meeting in several respects: (a) a greater variety of possible defect types are included, (b) a correspondingly greater variety of competing failure processes are considered, and (c) a more diverse set of ND measurements are incorporated. An important general modification of the total formalism has been introduced: namely, the earlier formalism involving a single most significant defect concept has been replaced by a more realistic formalism in which all possible combinations of defects are taken into account. The model of a priori statistics of defects has been accordingly modified with the removal of the extreme value feature. We will present false-rejection false-acceptance probability curves for various sets of synthetic test data and various values of model parameters.

## **Keywords**

Nondestructive Evaluation

## **Disciplines**

Materials Science and Engineering

# ACCEPT-REJECT DECISIONS AND PROBABILISTIC FAILURE PREDICTION\*

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## ABSTRACT

The purpose of this paper is to discuss recent progress in the development of an NDE decision formalism applied to the case of brittle fracture in ceramics. The on-line input into the formalism is a set of non-destructive (ND) measurements and the on-line output is the probability of failure conditioned on the above measurements. The final accept-reject decision depends only upon the comparison of the above output with a threshold related to the concerns of the user. The formalism involves stochastic physical models of the ND measurement process, the failure process (assuming a given stress environment), and the a priori statistics of defects. The present formalism goes beyond that reported at the previous meeting in several respects: (a) a greater variety of possible defect types are included, (b) a correspondingly greater variety of competing failure processes are considered, and (c) a more diverse set of ND measurements are incorporated. An important general modification of the total formalism has been introduced: namely, the earlier formalism involving a single most significant defect concept has been replaced by a more realistic formalism in which all possible combinations of defects are taken into account. The model of a priori statistics of defects has been accordingly modified with the removal of the extreme value feature. We will present false-rejection false-acceptance probability curves for various sets of synthetic test data and various values of model parameters.

## INTRODUCTION

The purpose of this talk is to discuss recent progress in the development of the NDE decision formalism (i.e., the determination of the probability of failure conditioned on ND measurements and the associated optimization of accept/reject criteria). Here the application is limited to brittle fracture in ceramics, although the general methodology with suitable modifications is applicable to any mode of failure in any material.

We first discuss the general aspects of probabilistic failure prediction, partly for the purpose of defining notation. We then present a review of past work based upon the dominant-defect approximation. This work involves low-frequency acoustical ND measurements and two alternative failure processes due to voids and subcritical inclusions, respectively. The last part of the talk deals with a more realistic formalism in which explicit consideration is given to all combinations of defects that can occur.

## GENERALITIES ON PROBABILISTIC FAILURE PREDICTION

It has been shown<sup>1</sup> that, in the case of single dominant defect, the conditional probability of performance  $c$  ( $c = 1 \rightarrow$  survival and  $c = 0 \rightarrow$  failure), is given by

$$P(c|y) = \frac{1}{P(y)} \int dx P(y|x) P(c|x) P(x) \quad (1)$$

$$P(y) = \int dx P(y|x) P(x) \quad (2)$$

where  $y$  is an  $n$ -dimensional vector representing the results of all nondestructive measurements and

$x$  is an  $m$ -dimensional state vector representing the characteristics of the single significant defect. The state vector is assumed to have the property

$$P(y, c|x) = P(y|x) P(c|x) \quad (3)$$

at least to a sufficient degree of approximation.

Now, in the case of many defects where the single significant defect assumption is not adequate, the above formalism has to be extended. This will be done in a later section.

With any given loss function, we always obtain an accept-reject decision rule of the game form, namely

$$\begin{aligned} P(o|y) > \lambda &\Rightarrow \text{reject} \\ P(o|y) < \lambda &\Rightarrow \text{accept} \end{aligned} \quad (4)$$

where the value of the threshold depends on the a priori failure probability and on the various cost components of the loss function. This formalism is depicted in Fig. 1.

It is perhaps desirable to review also the concept of operating characteristic curve, i.e., the plot of the false-reject probability vs the false-accept probability, since this curve will be used frequently as a measure of the effectiveness of the NDE formalism in several cases. The false-accept probability (i.e., the fraction of the failing population that was accepted) is given by

$$e_0 = \int dy H[\lambda - P(o|y)] P(y|0) \quad (5)$$

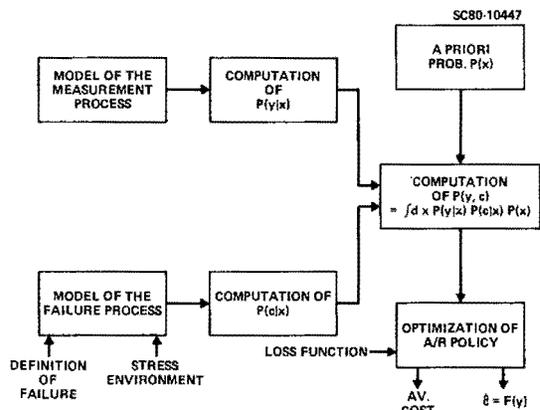


Fig. 1 NDE formalism.

where  $H(\cdot)$  is the Heaviside (or unit-step) function defined by

$$H(u) = \begin{cases} 0 & u < 0 \\ 1 & u > 0 \end{cases} \quad (6)$$

where  $u$  is a dummy variable. On the other hand, the false-reject (i.e., the fraction of the surviving population that was rejected) is given by

$$e_1 = \int dy H[P(o|y) - \lambda] P(y|1) \quad (7)$$

In (5) and (7) the integrations span the entire  $n$ -dimensional measurement vector space. In the case of a single scalar measurement variable the nature of the calculations of  $e_0$  and  $e_1$  are illustrated in Fig. 2. Most of the probabilistic quantities of interest to the user of an NDE system can be derived from  $e_0$ ,  $e_1$ , and  $P(0)$ .

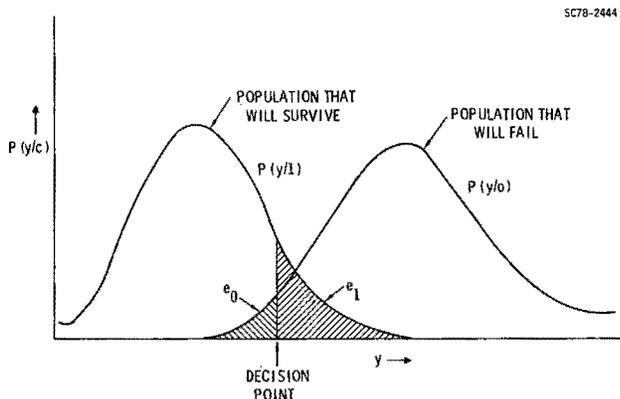


Fig. 2 Calculation of false-accept ( $e_0$ ) and false-reject ( $e_1$ ) probabilities.

Eqs. (5 and (7) give  $e_0$  and  $e_1$  as functions of the threshold  $\lambda$  and thus they provide a parametric representation of a curve in  $(e_0, e_1)$ -space. This curve is called the NDE operating characteristic. The curve is independent of the loss function (actually, it represents the locus of points representing all possible loss functions) and the a priori failure probability. Thus this curve characterizes the performance of the NDE system in a manner that is independent of external considerations. The nature of  $e_0$  and  $e_1$  is illustrated in

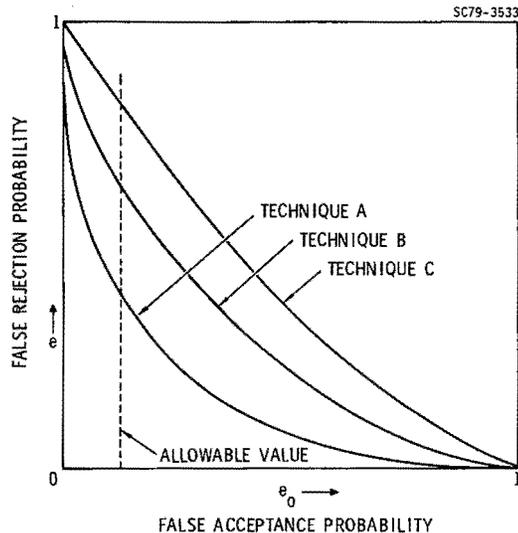


Fig. 3 The operating characteristic as a measure of NDE system effectiveness.

Fig. 3. The closer the operating characteristic curve is to the axes, the better the performance. From Fig. 3, for example, it is seen that technique A is superior to techniques B and C.

#### REVIEW OF APPLICATIONS

In this section we review previous results in the application of probabilistic failure prediction based on the single dominant defect approximation. In a later section, the many-defect case will be discussed. Here, we consider two failure mechanisms in ceramics: one involving voids with peripheral microcracks and the other involving subcritical inclusions. In the interest of brevity we have omitted as many details as possible; these are discussed in an earlier paper by Richardson, Fertig and Evans.<sup>2</sup> These examples are discussed in the following two subsections. Unfortunately, neither real failure data nor real ND measurement data were available for either example and consequently synthetic theoretical data were used throughout. Various kinds of statistical outputs were computed, but the greatest emphasis was placed upon the so-called NDE operating characteristics.

#### Failure Due to a Void with Peripheral Microcracks-

It is known that voids, which are almost always present in ceramics, are frequent sites for the initiation of crack growth, and hence lead to catastrophic failure under sufficiently large applied stresses. An adequate model of this process involves a random set of microcracks on the periphery of the void. Each crack has a certain probability of propagating to failure depending upon the stress distribution that would exist in the neighborhood of the crack if the crack were not there. We present a treatment of the perhaps over simplified case in which it is assumed that the probability of propagating to failure depends only on the stress at the void surface and that the void is approximately spherical.

Three independent models are involved in the assembly of a decision framework; the estimate of the pertinent defect dimensions from the inspection measurement  $y$  given the defect state  $x$ ; the probability of performance  $c$  at a specified ap-

plied stress level  $\sigma_{\alpha\beta}^{\infty}$ , given the defect state  $x$ ; and the a priori probability density of the state  $x$  of the dominant defect. Each of these models is examined separately and then combined to provide the optimal accept/reject decision rule and associated decision performance measures.

The relevant conditional probability density  $P(y|x)$  is implied by the stochastic measurement process

$$y = na^3 + r \quad (8)$$

where  $y$  is a possible measured value of  $A(\omega)/\omega^2$ , i.e., the scattering amplitude for longitudinal-to-longitudinal backscatter divided by the square of the frequency  $\omega$ , evaluated at a sufficiently small value of  $\omega$ . The quantity  $na^3$  is the theoretical value of the above quantity when the state  $x = a$  (the void radius) is assumed to be known. The coefficient  $n$  depends only on the known properties of the host material. The additive term  $r$  represents measurement error and is assumed to be a Gaussian random variable with zero mean and variance  $C_r$ .

We turn to the calculation of  $P(c|x)$ , the probability of the performance  $c$ , given the state  $x (=a)$  of the significant defect. In the present model it is assumed, as we have stated, that the only type of defect that is significant in the context of structural failure is a spherical void. As illustrated in Fig. 4 this void has randomly positioned cracks distributed at its surface. With a specified applied stress, each crack has the potential of propagating into a large crack, subsequently causing structural failure. The probability of this happening is a function of the local stress  $\sigma_{\alpha\beta}(\vec{r})$  in the neighborhood of the crack (i.e., that would exist at  $\vec{r}$  if the crack were absent). The cracks are, in this instance, considered to be much smaller than the void diameter, so that the effects of stress gradients into the host can be neglected. The modifications that pertain when this condition is not satisfied has been discussed by Evans, Biswas, and Fulrath.<sup>3</sup>

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$\sigma_{\alpha\beta}^{\infty}$

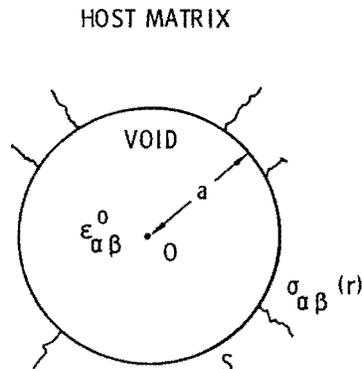


Fig. 4 Void with peripheral microcracks.

Based upon this model the probability of survival, given that the state  $x = a$  is specified, is

$$P(1|x) \equiv P(1|a) = 1 - P(0|a) \\ = \exp(-4\pi n_s a^2 \langle Q \rangle_A) \quad (9)$$

where  $n_s$  is the average surface density of cracks on the surface of the spherical void and  $Q = Q(\sigma_{\alpha\beta}(\vec{r}))$  is the probability that a crack at the position  $\vec{r}$  on the surface will propagate to failure. The symbol  $\langle Q \rangle_A$  denotes the area average of  $Q$  over the surface of the void.

Studies of defect densities in ceramics indicate that the large value extreme, of interest to fracture problems, can frequently be characterized by the cumulative distribution

$$F(a) \equiv \int_0^a P(a) da = 1 - \exp - (a/a^0)^k \quad (10)$$

where  $a^0$  is a critical radius.

Here we combine the last three equations to yield  $P(y,c)$  from which we deduce  $P(y|c)$  and the classification errors  $e_0$  and  $e_1$ .

It is desirable to introduce the dimensionless variables  $z = y/C_r^{1/2}$ ,  $\xi = a/a^0$ , and an additional dimensionless parameter  $\kappa = n(a^0)^3/C_r^{1/2}$ , which is a signal-to-noise ratio characterizing the observation of elastic waves scattered from a spherical void of radius  $a^0$ . Another useful quantity is the dimensionless failure parameter

$$\zeta = 4\pi n_s (a^0)^2 \langle Q \rangle_A \quad (11)$$

whose significance is given by  $P(1|x) = P(1|x) - \exp(-\zeta)$  when  $a = a^0$  (i.e., the void has the critical radius defined by (10)). We actually compute  $P(z|c)$  instead of  $P(y|c)$  with a scale factor introduced into the normalization.

In Fig. 5 we present plots of  $P(x|0)$  and  $P(z|1)$ , vs  $z$  for  $k = 3$ ,  $\kappa = 10$  and  $\zeta = 0.01$ . These figures show the structure of the  $c = 0$  class (i.e., the normalized population of objects that are going to fail) and  $c = 1$  class (i.e., the normalized complementary population of objects that are going to survive). Moreover, they show the nature of the overlap of the two classes.

In Fig. 6 we also given a plot of  $e_1$  vs  $e_0$  for the same parameter values. This is the so-called "operating characteristic" of the system. It is to be emphasized again that  $e_0$  is the falsely accepted fraction of objects that are actually going to fail. Conversely,  $e_1$  is the falsely rejected fraction of objects that are actually going to survive.

The above results indicate a rather poor NDE performance due, of course, to an excessive overlap of surviving and failing populations. This overlap is due almost entirely to inherent randomness in the failure process remaining even when the state  $x = a$  is known with precision. However, one must measure the width of the overlap region relative to the width of the combined populations a situation that is improved by the introduction of stress gradient effects.

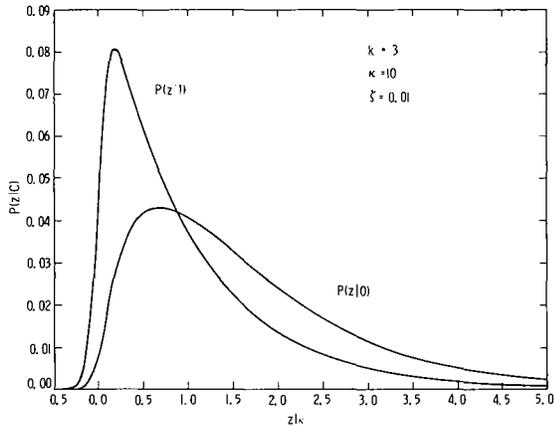


Fig. 5 Failing and surviving population.

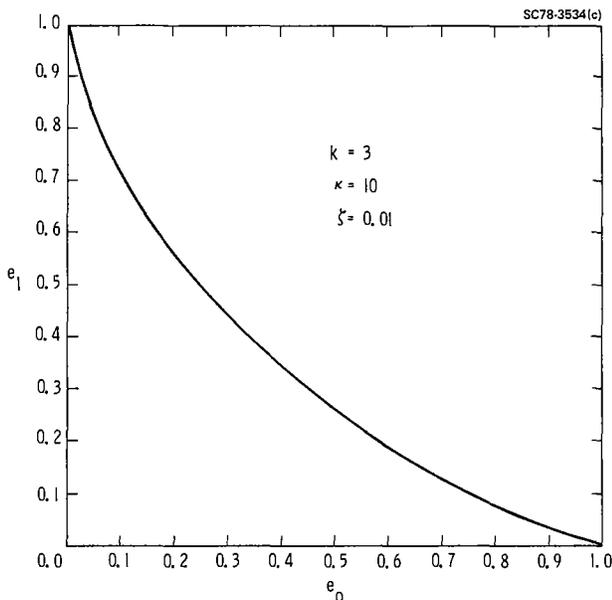


Fig. 6 NDE operating characteristic.

Failure By to Subcritical Inclusion Fracture -The geometrical nature of the model of the defect and its observation by elastic wave scattering is depicted in Fig 7. We assume a semi-infinite specimen with known host material. With the Cartesian coordinate system partially shown, the boundary of the specimen is parallel to the xy-plane and the outward pointing normal lies in the positive z-direction. We assume that the defect is an ellipsoidal inclusion (although the subsequent analysis is limited for the sake of brevity to the oblate spheroidal case) with a known included material. We explicitly show a pulse-echo (i.e., backscatter) measurement with the incident wave pointed in the negative z-direction. However, additional transducer configurations will be considered later.

If the inclusion boundary is assumed to be an oblate spheroid then the state vector  $x$  need only be the 4-dimensional representation of the geometry since the included material is assumed known. We will use of the 4-dimensional state vector given by

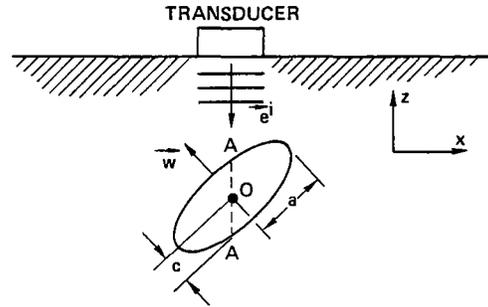


Fig. 7 Subcritical inclusion geometry.

$$x = \begin{pmatrix} a \\ c \\ \theta \\ \gamma_z \end{pmatrix} \quad (12)$$

where  $\theta$  is the azimuthal angle (in the xy plane) of the symmetry axis defined by the unit vector  $\vec{w}$  and where  $\gamma_z$  is the direction cosine of  $\vec{w}$  relative to the z-axis. The vector  $\vec{w}$  can be expressed in terms of  $\theta$  and  $\gamma_z$  as follows

$$\vec{w} = (1 - \gamma_z^2)^{1/2} (\vec{e}_x \cos \theta + \vec{e}_y \sin \theta) + \gamma_z \vec{e}_z \quad (13)$$

where  $\vec{e}_x$ ,  $\vec{e}_y$  and  $\vec{e}_z$  are the unit vectors in the Cartesian coordinate directions. As shown in Fig. 7  $a = b$  is the common length of the two major semi-axes and  $c$  is the length of the minor semi-axis.

The measurements are assumed to consist of an arbitrary number of low-frequency longitudinal-to-longitudinal backscatter processes. These are collectively represented by a standard stochastic model of the generic

$$y = f(x) + r \quad (14)$$

where  $y$ ,  $f(x)$  and  $r$  are n-dimensional vectors (but considerable attention will be devoted to the case  $n = 1$ ). In the exact theoretical measurement  $f(x)$  the  $i$ th component is given by

$$f_i(x) = A_{2,L \rightarrow L}(\vec{e}_i^s, -\vec{e}_i^i; x) \quad (15)$$

where it is assumed that  $\vec{e}_i^s = -\vec{e}_i^i = \vec{e}_i$ .

The conditional probability density  $P(y|x)$  can be expressed in the form

$$P(y|x) = G(y - f(x), C_r) \quad (16)$$

where  $G(\cdot, \cdot)$  is the n-dimensional Gaussian probability density given by

$$G(u, C) = (2\pi)^{-n/2} (\text{Det } C)^{-1/2} \exp(-\frac{1}{2} u^T C^{-1} u) \quad (17)$$

It is assumed that an uniaxial stress  $\sigma_\infty$  is applied in the x-direction. We make the rather crude assumption that at a certain value of  $\sigma_\infty$  this crack forms, as represented by the dashed

line AA in Fig. 7, a plane intersecting the geometrical center of the spheroid and having an orientation perpendicular to the axis of the applied stress, i.e., the x axis. At a sufficiently higher value of the applied stress the crack will propagate from the lower toughness inclusion (i.e., Si) into the higher toughness host material (i.e., Si<sub>2</sub>N<sub>4</sub>). We assume that the condition for this event can be adequately represented by an empirically recalibrated version of simple fracture mechanics with a Gaussian random additive variable representing inherent variability in the fracture process.

In explicit mathematical terms we assume that the performance variable c is given by

$$c = H(\sigma_f - \sigma_\infty) \quad (18)$$

where H(.) is the Heaviside unit step function,  $\sigma_\infty$  is the applied stress, and  $\sigma_f$  is the failure stress. The latter quantity is a random variable given by the random process

$$\sigma_f = \alpha + \beta \sigma_p + s \quad (19)$$

where  $\sigma_p$  is the failure stress predicted according to simple fracture mechanics,  $\alpha$  and  $\beta$  are empirical recalibration constants, and s is a Gaussian random variable with zero mean and variance  $C_s$ . The application of simple fracture mechanics (i.e., the computation of yield stress under the assumption that the ellipsoidal crack is surrounded solely by the host material) gives

$$\sigma_p = \frac{K_{IC}}{Z(\epsilon'/a')\sqrt{\pi c'}} \quad (20)$$

where  $K_{IC}$  is the fracture toughness, a' and c' are the major and minor semi-axis lengths of the fully developed internal crack, and Z(.) is a function defined by

$$Z(u) = \int_0^{\pi/2} d\psi (1 - (1 - u^2) \sin^2 \psi)^{1/2} \quad (21)$$

As stated earlier, we assume that the fully developed internal crack is represented by the cross-section formed by a plane, perpendicular to the x axis, passing through the center of the spheroid. A straight forward geometrical analysis yields the result

$$a' = a \quad (22)$$

$$c' = (a^{-2} + (c^{-2} - a^{-2})w_\parallel^2)^{-1/2} \quad (23)$$

where  $w_\parallel$  is the length of the projection of  $\vec{w}$  (the unit vector defining the axis of symmetry of the spheroid) onto the crack plane. We obtain

$$w_\parallel^2 = 1 - (1 - \gamma_z^2) \cos^2 \theta \quad (24)$$

Eqs. (16) - (24) thus give  $\sigma_p$  as a function of the state vector x defined by (12).

We turn finally to the calculation of P(c|x). First we observe that, according to the stochastic model (19), the conditional probability density of  $\sigma_f$  is given by

$$P(\sigma_f | x) = G(\sigma_f - \alpha - \beta \sigma_p(x), C_s) \quad (25)$$

where G(.,.) is the Gaussian function defined by (17) which in the present case is specialized to the case of scalar variables, i.e.,

$$G(u, C) = (2\pi C)^{-1/2} \exp(-\frac{1}{2} C^{-1} u^2) \quad (17a)$$

We then obtain

$$P(c = 0 | x) = \int_{-\infty}^{\sigma_\infty} d\sigma_f P(\sigma_f | x) \quad (26)$$

The a priori probability density P(x), is more complicated because the state vector x is now 4-dimensional in order to characterize fully the spheroidal geometry. We assume that the semi-axis lengths are statistically independent of the angular variables. Furthermore we assume that the latter are distributed with axial symmetry about the z-axis. The detailed analytical representation of these assumptions is given in Ref. (2).

In the numerical computations we have used a Monte Carlo technique in which quantities of the type  $\int dx(\cdot) P(x)$  are replaced by

$$\frac{\sum_{x \in \xi} (\cdot)}{\sum_{x \in \xi} 1} \quad (27)$$

where the samples of the state vector in the set  $\xi$  have been drawn at random in accordance with the probability density P(x).

In all computations we will uniformly use the following assumptions and parameter values given in Ref. (2).

In the following paragraphs we present numerical results for three cases to illustrate the separate effects of randomness and completeness in the measurement process and randomness in the failure process.

Case 1 - One Measurement - Random Measurement and Failure Process - In this case we consider a single ND measurement i.e., a pulse-echo, longitudinal-to-longitudinal scattering of elastic waves with the incident propagation in the negative z-direction. Here we assume randomness in both the measurement and failure processes. The particular values for the statistical parameters are given in Ref. (2).

In Fig. 8 we show the computed curves of P(y|c) representing the failing and surviving populations. It is clear that the severe overlap will yield a rather poor NDE performance as indicated by the plot of false-rejection probability  $e_1$  vs false acceptance probability  $e_0$  shown in Fig. 9.

Case 2 - One Measurement - Deterministic Measurement and Failure Processes - Here we consider again a single measurement of the same kind as in the last case. However, for the sake of understanding we eliminate the randomness from the measurement and failure processes by setting the variances  $C_m = C_s = 0$ . The resultant NDE performance (hypothetical) is given by the  $e_1$  vs  $e_0$  plots in Fig. 10. Although there is a marked improvement in the performance, i.e., the curve has moved

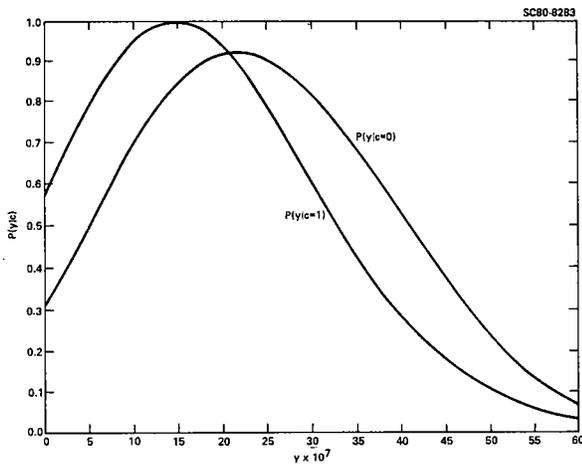


Fig. 8 Failing and surviving populations.

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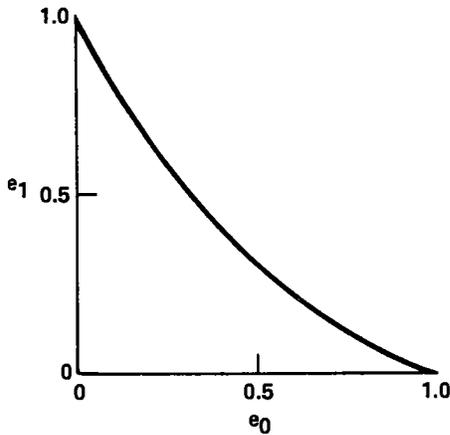


Fig. 9 NDE operating characteristic - one measurement with random measurement and failure process.

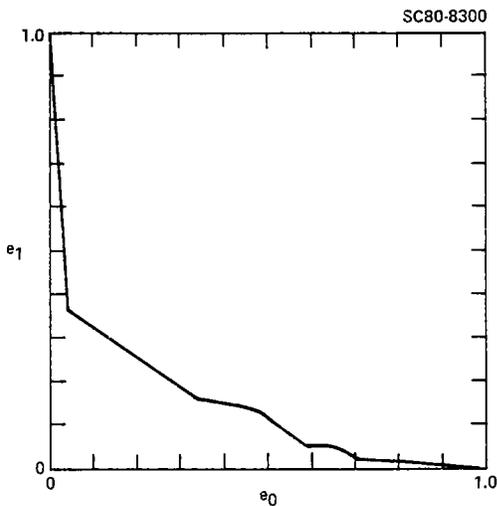


Fig. 10 NDE operating characteristic - one measurement with deterministic measurement and failure processes.

closer to the horizontal and vertical axes, the performance is of debatable merit. This is due to the serious incompleteness of the measurement set. Incidentally, the lack of smoothness of the curve is due to the relatively small fraction of Monte Carlo samples that actually affect the final answer.

Case 3 - Complete Measurement Set - Deterministic Measurement Process but a Random Failure Process

-In this case we assume a sufficient diversity of very accurate measurements that the measurement vector  $y$  implies a unique estimate of  $x$ , namely  $x(y)$ , with a negligible a posteriori variance (more precisely, a covariance matrix  $Cov(x|y)$  whose eigenvalues are sufficiently small in an appropriate sense). However, we assume the same randomness in the failure process as in Case 1. The resultant plot of  $e_1$  vs  $e_2$ , the NDE operating characteristic, is presented in Fig. 11. This highly satisfactory result demonstrates clearly that randomness in the present failure process (failure initiated in subcritical inclusions) is not a significant contributor to the degradation of NDE performance.

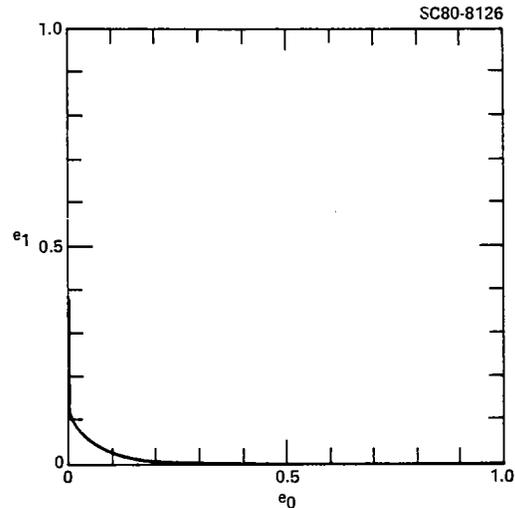


Fig. 11 NDE operating characteristic - complete measurement set.

PROBABILISTIC FAILURE PREDICTION IN MANY-DEFECT SYSTEMS

In the last section on the analysis of accept/reject decisions and the determination of unconditional and conditional probabilities of failure, we explicitly considered a single dominant defect under the assumption that, if there are others, they are insignificant contributors to failure. In the present section we consider explicitly all of the defects that may occur. To be more precise, we consider models of measurement and failure that entail a random number of defects. It is expected that the present treatment will clarify certain obscurities in the use of the dominant defect approximation. Some of these are:

- (a) The contribution to failure probability by non-dominant defects.
- (b) The contribution to failure probability by unobservable defects.

- (c) The approximate nature of the extreme value constraint in the a priori
- (d) The clarification of the distinction between detection probability and the a posteriori variance of the defect characteristics given that it has been detected.

It is expedient to consider first the case with a random number of defects of a single type. Later, the formalism will be extended to the case involving many types.

The Occupation Number Representation - It is clear that, in order to represent the state of a system containing an arbitrary, random number of defects, the single-defect formalism must be suitably extended. The simplest way to do this is to divide the single-defect state space into small cells and thus represent the state of many defects by the occupation numbers of the cells. To be more explicit, let the domain of definition  $D$  of the single-defect state space  $x$  be divided into a set of sub-domains or cells  $D_i$ ,  $i=1, \dots, p$ , where, of course,

$$\sum_{i=1}^p D_i = D \quad (28)$$

Let the centroid (or any characteristic position) of  $D_i$  be  $x_i$  and let its  $m$ -dimensional volume be denoted by  $\delta x_i$ . We will denote the number of defects in  $D_i$  by the occupation number  $n_i$ . To be useful, each  $\delta x_i$  must be so small that the probability that  $n_i > 1$  is negligible. The state of the many-defect system is now given by the set of occupation numbers

$$\{n_i\} \equiv (n_1, \dots, n_p) \quad (29)$$

If only one defect is present and has a state in  $D_i$ , then

$$\{n_i\} = (0, \dots, n_j = 1, \dots, 0) \quad (30)$$

The representation of higher numbers of defects is obvious.

It is worth noting that in the above occupation number representation of the state of a many-defect system, there is no need for an arbitrary set of labels for the defects (e. g., defect #1, defect #2, etc.). Furthermore, there is no need for a state vector of variable length as would be obtained if we represented the state of many defects by the concatenation of the states of individual defects.

Using the above occupation number representation, (1) and (2) for the one-defect case are now replaced by the following expressions for the many-defect case:

$$P(c|y) = P(y,c)/P(y) \quad (31)$$

$$P(y,c) = \sum_{\{n_i\}} P(y|\{n_i\}) P(c|\{n_i\}) P(\{n_i\}) \quad (32)$$

$$P(y) = \sum_{\{n_i\}} P(y|\{n_i\}) P(\{n_i\}) \quad (33)$$

It is also of interest to consider the unconditional probability of performance  $c$ , namely

$$P(c) = \sum_{\{n_i\}} P(c|\{n_i\}) P(\{n_i\}) \quad (34)$$

In the above expressions the symbol  $\sum_{\{n_i\}}$  denotes the summation on all modes of occupancy of the cells  $D_i$ , i.e., all combinations of the values  $n_i = 0, 1$  for  $i = 1, \dots, p$ . The interpretations of the probabilities  $P(y|\{n_i\})$ ,  $P(c|\{n_i\})$  and  $P(\{n_i\})$  are self-evident. However, here it is appropriate to emphasize that the a priori probability  $P(\{n_i\})$  does not contain any extreme value (i.e., most significant) constraint as was the case with  $P(x)$ .

In analogy with (3), the state of the many-defect system in the occupation number representation must satisfy the relation

$$P(y,c|\{n_i\}) = P(y|\{n_i\}) P(c|\{n_i\}) \quad (35)$$

Special Forms of the Various Probabilities - In this and following sections, we consider special forms of  $P(\{n_i\})$ ,  $P(c|\{n_i\})$  and  $P(y|\{n_i\})$  when certain statistical independence assumptions are introduced.

Under the assumption that the modes of occupancy of different cells are statistically independent a priori, we can write

$$P(\{n_i\}) = \prod_{i=1}^p P_i(n_i) \quad (36)$$

Since  $n_i$  takes only the two values 0 and 1, it follows that

$$P_i(n_i) = (1-n_i) (1 - P_i(1)) + n_i P_i(1) \quad (36a)$$

The a priori average value of  $n_i$  is then

$$\begin{aligned} E n_i &= \sum_{n_i} n_i [(1 - n_i) P_i(0) + n_i P_i(1)] \\ &= \sum_{n_i} n_i P_i(1) = P_i(1) \end{aligned} \quad (37)$$

Denoting the a priori average density of defects by  $\rho(x)$ , we can write

$$E n_i = P_i(1) = \rho(x_i) \delta x_i \quad (38)$$

if the variation of  $\rho(x)$  over the domain  $D_i$  is sufficiently small. We now obtain

$$P(\{n_i\}) = \prod_i [1 - n_i + (2n_i - 1) \rho(x_i) \delta x_i] \quad (39)$$

We now consider the a priori distribution of the total number of defects in an arbitrary extended region of the single-defect state space  $x$ . Let this region be denoted by  $R$  and let  $S$  be the set of integers  $i$  for which  $D_i$  is contained in  $R$ , i.e.,

$$\sum_{i \in S} D_i = R \quad (40)$$

The total number of defects in  $R$  is then

$$N_R = \sum_{i \in S} n_i \quad (41)$$

Using conventional arguments we obtain the result that  $N_R$  is Poisson-distributed, i.e.

$$P(N_R) = \exp(-\alpha) \alpha^{N_R} / (N_R)! \quad (42)$$

where

$$\alpha = EN_R = \int_R dx \rho(x) \quad (43)$$

This is a direct consequence of the assumption that the modes of occupancy of different cells are statistically independent. This is an acceptable assumption for the a priori statistics of defects but in the case of a posteriori statistics (i.e., conditioned on ND measurements) its validity may sometimes be debatable.

Another topic of interest is the relation between  $P(x)$  and  $P(\{n_i\})$  if  $P(x)$  represents the probability density of the most significant defect in a set of defects. Let the significance of a defect with state  $x$  be measured by some function  $s(x)$  which may be the volume of the defect, its probability of causing failure, or some related property. In the limit of infinitesimal cells, we obtain

$$P(x) = A \rho(x) \exp[-\int_{s(x')}^{s(x)} dx' \rho(x')] \quad (44)$$

The normalization factor  $A$  is given by

$$A = [1 - \exp(-EN)]^{-1} \quad (45)$$

a quantity that differs negligibly from unity if the average number  $EN$  of defects is large.

In the case of spherical voids with peripheral microcracks in a uniform applied stress, the single-defect state vector  $x$  reduces to a single scalar quantity, i.e., the sphere radius  $a$ . If we assume that  $a$  is also the measure of significance, i.e.,  $s(x) \rightarrow s(a) = a$ , then we obtain

$$P(a) = [1 - \exp(-EN)]^{-1} \frac{d}{da} \exp[-\phi(a)], \quad (46)$$

where

$$\phi(a) = \int_a^\infty da' \rho(a') \quad (47)$$

and

$$F(a) = [1 - \exp(-EN)]^{-1} \{ \exp[-\phi(a)] - \exp(-EN) \} \quad (48)$$

The Failure Probability - In this section we turn to the consideration of the performance probability  $P(c|\{n_i\})$  and also the unconditional probability  $P(c)$ . We will assume that the probability of failure  $P(c=0|x)$  due to a single defect with state  $x$  is given. In proceeding to the multiple defect case, we will further assume in this case that each defect behaves independently in the failure process and that survival is tantamount to no defect causing failure.

These assumptions imply that

$$P(c=1|\{n_i\}) = \prod_i P(c_i = 1|n_i) \quad (49)$$

where it is assumed that each factor  $P(c=1|n_i)$  is the probability that the cell  $D_i$  with  $n_i$  defects ( $n_i=0$  or  $1$ ) does not cause failure. We assume that  $P(c_i=1|n_i)$  is defined by the relations

$$P(c_i = 1|n_i = 0) = 1 \quad (50)$$

$$P(c_i = 1|n_i = 1) = P(c_i = 1|x_i) \quad (51)$$

where  $P(c_i = 1|x_i)$  is the probability that a single flaw with state  $x_i$  does not cause failure. Thus we can write

$$P(c_i = 1|n_i) = 1 - P(c_i = 0|x_i) n_i \quad (52)$$

and

$$P(c = 1|\{n_i\}) = \prod_i [1 - P(c_i = 0|x_i) n_i] \quad (53)$$

It is understood that  $P(c=0|x_i)$  depends implicitly upon the stress environment (random or deterministic) associated with the single-defect state  $x_i$ . It is easy to verify that (53) is consistent with the assumptions stated at the beginning of this section.

The calculation of the a priori survival probability involves the consideration of

$$P(c=1) = \sum_{\{n_i\}} P(c=1|\{n_i\}) P(\{n_i\}) \quad (54)$$

from which we obtain

$$P(c = 1) = \prod_i [1 - P(c_i = 0|x_i) \rho(x_i) \delta x_i] + \exp[-\int dx \rho(x) P(c = 0|x)] \quad (55)$$

in the limit of infinitesimal cells. It is to be understood the  $P(c = 0|x)$  is the probability that a single flaw with state  $x$  causes failure.

Conditioning on ND Measurements - The problem of conditioning on nondestructive measurements in the case of many-defects systems involves some non-trivial difficulties. Considerable progress has been made in dealing with some of these difficulties using some approaches that commonly used in statistical mechanics. The full discussion of these results will take up far more space than is available here and therefore this discussion will be relegated to a future communication. Here we will describe a rather crude approximation that, besides being easy to describe, has the additional advantage of ready compatibility with the previous work on post-detection processing algorithms.

Let the total domain  $D$  of  $x$ -space contain a set or regions  $R_j$ ,  $j = 1, \dots, J$  in which zero or one defect is detected by ND measurements with an acceptable degree of assurance. After subtracting the  $R_j$ 's from  $D$ , one obtains a nonzero domain corresponding to unobservable defects (i.e., those that are too small to be detectable). Let the vector  $y_j$  be the measurement vector associated with region  $R_j$ . Also, let  $N_j$  be the number (0 or 1) of defects observed in  $R_j$ . If  $N_j = 1$  (i.e., one defect is observed in  $R_j$ ) the remaining aspects of this ND measurement are represented by the probability density  $P(x|y_j)$ ,  $x \in R_j$ , where the normalization condition holds:

$$\int_{R_j} dx P(x|y_j) = 1 \quad (56)$$

If  $N_j = 0$  (i.e., no defects are observed in  $R_j$ ) then no meaningful probability density exists. The function  $P(x|y_j)$  is closely analogous to the function  $P(x|y)$  introduced in the earlier treatment involving the dominant-defect approximation. However, there is an important difference here; namely,  $P(x|y_j)$  is the conditional probability density of the only defect in  $R_j$  which  $P(x|y)$  is the conditional probability density of the dominant defect in  $D$ .

In the case in which the conditional probability of failure is very small (or at least the probability of failure due to each observed defect is very small) we can write the conditional probability of survival of the entire many-defect system in the form

$$P(c = 1 | \{y_j\}) = \exp(-\int dx \rho(x | \{y_j\})) P(c = 0 | x) \quad (57)$$

where the conditional mean density is given by

$$\begin{aligned} \rho(x | \{y_j\}) &= \rho(x), \quad x \notin R_j, \quad j = 1, \dots, J, \\ &= P(x | y_j), \quad x \in R_j, \quad N_j = 1 \\ &= 0, \quad x \in R_j, \quad N_j = 0 \end{aligned} \quad (58)$$

It is worthy of note that conditional mean density is equal to the conditional probability density in the regions containing one (and only one) defect.

Extensions - In this subsection we discuss certain extensions that could be made in the formalism in order to circumvent the limitations of the simplifying assumptions used up until now.

The first of these assumptions involves the limitation to a single defect type. Here the collective state of an arbitrary number of such defects is given by the occupation numbers associated with cells in the single defect state space  $x$ . If all types of defects are described (aside from the type label) in the same state space  $x$ , then the extension of the previous formalism to the case of many defect types is accomplished in a very simple manner. One merely makes the substitution

$$x \rightarrow \begin{pmatrix} \gamma \\ x \end{pmatrix} \quad (59)$$

where  $\gamma$  is a discrete-valued variable labelling the defect types. In the case of integrations on  $x$ , one makes the corresponding substitution

$$\int dx (\cdot) \rightarrow \sum_{\gamma} \int dx (\cdot) \quad (60)$$

Clearly the occupation number  $n_i$  associated with cell  $D_i$  in  $x$ -space must also be subject to the substitution

$$n_i \rightarrow n_{\gamma i} \quad (61)$$

The other modifications are obvious.

However, in most cases it is not appropriate to use the same  $x$ -space for all defect types. It

is usually necessary to use a distinct version of  $x$ -space for each type. For example, in the case of a surface-opening crack it is appropriate to choose  $x$  to be a 5-dimensional vector whose components consist of two surface coordinates defining position, an angle variable giving orientation, and two lengths defining the length and depth of the crack. But in the case of an ellipsoidal inclusion (of given material) in the bulk, it is appropriate to choose  $x$  to be a 9-dimensional vector giving the position, shape, size, and orientation of the ellipsoidal boundary. There remains an important question concerning the ignorability of position coordinates. In the case of a uniform applied stress, these coordinates can be ignored in the model of the failure process. However, the ignorability in the model of the measurement process requires a careful analysis.

In any case, one must associate with each value of  $\gamma$  a corresponding vector  $x_{\gamma}$  whose dimensionality and selection of components depend on  $\gamma$ . Thus the substitutions (59) and (60) must now be replaced by

$$x \rightarrow \begin{pmatrix} \gamma \\ x_{\gamma} \end{pmatrix}, \quad (62)$$

and

$$\int dx (\cdot) \rightarrow \sum_{\gamma} \int dx_{\gamma} (\cdot) \quad (63)$$

It is useful to give specific examples of the extension of certain mathematical expressions derived in the previous sections. With the appropriate substitutions, (55) is changed to

$$P(c=1) = \exp[-\sum_{\gamma} \int dx_{\gamma} \rho(\gamma, x_{\gamma}) P(c=0 | \gamma, x_{\gamma})] \quad (64)$$

similarly (57) is changed to

$$P(c=1 | y) = \exp[-\sum_{\gamma} \int dx_{\gamma} \rho(\gamma, x_{\gamma} | y) P(c=0 | \gamma, x_{\gamma})] \quad (65)$$

The remaining modifications are straightforward.

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