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**Abstract**

This paper studies fertility choices and fertility policies when children's earning abilities are random and parents are altruistic. We characterize equilibrium allocations arising in endowment economies with either complete or incomplete markets. Both models can replicate a number of empirical regularities, such as inequality, social mobility and fertility decreasing with ability, but the incomplete markets model provides a number of more plausible predictions. We find that fertility policies are generally welfare detrimental in our models even when fertility is inefficiently high.
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# 1 Introduction

At least since Malthus it has been recognized that high fertility rates are associated to poverty. Such association motivates many family planning programs around the world that seek to reduce fertility as a way to alleviate poverty.[1] Understanding the link between poverty and fertility, and more generally, between inequality and fertility is an important part of the theory of distribution.

This paper studies fertility choices and fertility policies in economies where the underlying force driving inequality, poverty, social mobility and fertility differentials is uninsurable idiosyncratic risk. Our model integrates two leading streams of the literature on inequality and fertility. One the one hand, inequality and social mobility is driven by idiosyncratic shocks, as in Huggett (1993) or Aiyagari (1994). On the other hand, fertility is purely motivated by altruistic reasons, as in Barro and Becker (1989) and Becker and Barro (1988). Individuals in our models are fully rational, altruistic toward their descendants, and heterogeneous in their abilities. Earning abilities are randomly determined at birth and potentially correlated with parental abilities. Insurance markets are available but parents cannot leave negative bequests to their children. Altruistic models with idiosyncratic risk are central to modern macroeconomics, particularly when studying issues of inequality and redistribution, but with the important exception of Alvarez (1999), these models assume exogenous fertility. This paper is the first to provide a characterization of the endowment version of a Bewley economy extended to include endogenous fertility.[2]

The first main contribution of the paper is to characterize complete markets allocations in the presence of idiosyncratic shocks and endogenous fertility. Surprisingly, while perfect risk sharing and flat consumption are optimal when fertility is exogenous, they are not optimal when fertility is endogenous. Instead consumption and fertility remain stochastic in the complete markets economy. The reason is that when a life is costly to create, optimal consumption is an utilization rate and the higher the net cost of creating a new life the higher its utilization rate, as shown by Barro and Becker (1989). Since the net cost of a child is not constant in our environment, as it is tied

[1] For example, a United Nations Population Fund pamphlet asserts that "effective family planning programmes targeted to meet the needs of poor populations can reduce the fertility gap between rich and poor people, and make a powerful contribution to poverty reduction and the achievement of the Millennium Development Goals." Reported by UNPF at http://www.unfpa.org/rh/planning/mediakit/docs/sheet4.pdf. Retrieved April 2 2014.

[2] See Ljungqvist and Sargent (2012, chapter 18) for a presentation of Bewley models.

to both the parent's and the child's abilities, then neither flat consumption nor flat fertility is optimal. We also show that the complete markets model predicts a negative relationship between fertility and parental ability, inequality and social mobility. However, we also document some key counterfactual implications. For example, the model predicts that all children of the same parent have the same consumption, that the actual ability of an individual does not affects his/her own consumption, or that maximum fertility is often optimal for plausible calibrations of the income process.

We then proceed to characterize the incomplete markets economy, one that arises when the non-negative bequest constraint binds. The model can replicate a number of stylized facts: inequality, persistence as well as social mobility, and fertility decreasing with ability. The equilibrium is characterized by a Markov branching process satisfying the *Conditional Stochastic Monotonicity* property. This means that if a kid from a poor family and a kid from a rich family both fall into one of the poorest classes, it is more likely that the poor kid will be poorer than the rich kid. But the model also requires a significant degree of social mobility in order to rationalize why the poor have more children. Absent social mobility, fertility rates would be equal among the poor and the rich. Furthermore, significant social mobility is a distinguishing feature of the data.

A negative relationship between fertility and ability is not simple to obtain within dynastic altruistic models, as discussed by Jones, Schoonbroodt and Tertilt (2011). Our explanation for the negative relationship is novel and the second main contribution of the paper. It arises from the interplay of two opposites forces. On the one hand, higher ability individuals face a larger opportunity cost of having children as children take valuable parental time. On the other hand, higher ability individuals enjoy a larger benefit of having children when abilities are intergenerationally persistent because the utility of a child is positively related to parental ability. We find that the effect of ability on the marginal cost dominates its effect on the marginal benefit if the intergenerational persistence of abilities is not perfect. Regression to the mean in abilities means that low ability parents expect their children to be, on average, of higher ability while high ability parents expect their children to be of lower ability. This explains why fertility decreases with ability, and income, in a fully rational, homothetic and altruistic environment.

Golosov, Jones and Tertilt (2007) have shown that allocations are efficient in frictionless Barro-Becker models. Of particular interest is whether fertility is inefficiently high or inefficiently low when markets are incomplete. We show analytically that steady state fertility can be inefficiently high when children are a net financial cost to parents and markets are incomplete. In that case, binding bequest constrains prevent early generations from extracting resources from later generations making future generations effectively richer. Since fertility is a normal good, the fertility of early generations falls while fertility of later generations, and in particular steady state fertility, increases. Providing conditions under which fertility is inefficiently high in incomplete markets economies is the third main contribution of the paper. Our paper complements Schoonbroodt and Tertilt (2014) who have shown that fertility is inefficiently low for early generations.

Finally, we study the consequences and optimality of fertility policies, such as family planning programs aiming at reducing fertilities rates. We show that policies restricting the fertility of the poor result in a sequence of income distributions that dominates the original distribution in all periods in the first order stochastic sense. In particular, average income and consumption increase in all periods. This result arises from two forces. First, average ability of (born) individuals increases because the poor has proportionally more low ability children as a result of the assumed conditional stochastic monotonicity property. Second, consumption and income of the poor strictly increases because they spend less time and resources raising children. These results seemingly provide the theoretical support to family planning programs seeking to reduce the fertility of the poor (Chu and Koo 1990).

In spite of these positive implications, the fourth main result of the paper is to show that fertility restrictions of any type, not only for the poor, unequivocally reduce individual and social welfare in our model, even when fertility is inefficiently high. As we show, a policy that restricts fertility reduces the set of feasible choices and invariably reduces welfare of all individuals in all generations, even those whose fertility is not directly affected. This is because altruistic parents care not only about their own consumption and fertility but also care about the consumption and fertility of all their descendants. Regardless of current ability there is a positive probability that a descendant of the dynasty will fall into the group directly affected in finite time. Furthermore, the welfare of those individuals who are not born under the new policy also falls, or at least does not increase. Social welfare falls because the welfare of all individuals, born and unborn, either falls or remain the same. This is the case, for example, if social welfare is defined as classical (Bentham) utilitarianism, a weighted sum of the welfare of all present and future individuals. The result also holds for versions of classical utilitarianism that are consistent with the Barro-Becker concept of diminishing altruism. An interpretation of our results is that the positive effect on welfare of fertility restrictions, namely higher average consumption, is dominated by the negative effect of a smaller dynasty size.

If social welfare is defined as average (Mills) utilitarianism rather than classical utilitarianism, social welfare could increase even if the welfare of all individuals falls if population falls even more. In this case, the net effect of fertility restrictions on social welfare depends on the relative strength of two opposite forces. On the one hand, distributions of abilities and income improve for all periods. On the other hand, welfare of all individuals fall. We present two analytical cases in which the later force dominates and hence social welfare defined as average welfare falls not only in present value but also for all periods. Our quantitative exercises also show that these results hold more generally.

The negative impact of fertility policies on individuals' welfare applies not only to policies aiming at reducing fertilities, but also to policies compelling individuals to increase their fertilities. An example of such policy is the seemingly official Chinese policy of stigmatizing unwed women older than 27 as "leftover women".[3]

We also study the effects on steady state social welfare of taxes or subsidies seeking to increase or reduce the cost of raising children using a calibrated version of the model. The government is required to run a balanced budget. The results suggest steady state average social welfare could be increased but only by a very small tax, say one less than 2% the time cost of raising every child. Out of this range, neither tax nor subsidy would improve social welfare. The consequence on individual welfare varies across ability types and is determined by the magnitude of the tax/subsidy.

In addition to the papers already mentioned, our paper is related to Alvarez (1999). He studies an economy with idiosyncratic shocks, incomplete markets and endogenous fertility choices by altruistic parents but does not study the high fertility of the poor nor the consequences of fertility policies. Our incomplete markets economy is a version of his model, one with non-negative bequest constraints. In equilibrium no individual leaves positive bequests. This is a stronger degree of market incompleteness than that in Alvarez. Similar degree of market incompleteness is exploited by Krusell et al. (2011) to obtain closed form solution for asset prices in a Huggett (1993) model. Similarly, we are able to derive various closed form solutions and provide analytical proofs of the welfare effects of various fertility policies. Our paper is also related to Hosseini et al. (2013) who study a related

[3] Fincher, Leta Hong (12 October 2012). "OP-ED CONTRIBUTOR; China's 'Leftover' Women". The New York Times. Retrieved 29 March 2014.

problem using an optimal contracting approach, and to Sommer (2013) who studies fertility in the presence of idiosyncratic shocks but parents are not dynastically altruistic as in Barro and Becker.

There is a related literature that studies fertility policies in general equilibrium. A recent example is Liao (2013) who studies the *One Child Policy* using a calibrated deterministic dynastic altruism model with two types of individuals, skilled and unskilled, in the spirit of Doepke (2004). Although Liao's model can generate fertility differentials, Doepke (2004) documents that this channel alone is relatively weak. Part of the issue is that the model only generates upward mobility in equilibrium. Our model, in contrast, generates significant upward and downward mobility that can lead to significant fertility differentials. The mechanisms are different and therefore complementary. We are also able to derive sharp analytical results. For example, we prove that fertility reducing policies, like the one child policy, decrease every individual's welfare for sure while Liao's calibrated result suggests it's true for almost all generations but not all.[4]

The rest of the paper is organized as follows. Section 2 analyzes a simple deterministic model of fertility. We show that when children are a net financial cost to parents and markets are incomplete then steady state fertility is inefficiently high. Section 3 introduces idiosyncratic shocks into the model. We characterize the optimal fertility as well as its relationship with earning abilities. Section 4 studies the effect of fertility restriction policies on individual and social welfare. Numerical simulations, policy experiments and robustness checks are performed in section 5. Section 6 concludes. Proofs are in the Appendix.

## 2 A deterministic model

It is convenient to consider first a deterministic version of the model. We use the model to show analytically under what conditions fertility may be inefficiently high when markets are incomplete. A similar result is obtained for the full model but using numerical simulations. Schoonbroodt and Tertilt (2014) have shown that, under certain assumptions, incomplete markets models can result in inefficiently low fertility. In those cases, policies directed to promote higher fertility may be welfare enhancing. We show that, under

[4] Another related paper is Moav (2005). In his model, individuals exhibit warm-glow altruism, and comparative advantage in the education of children explains differential fertility rates among rich and poor individuals as well as the persistence of poverty. He does not study fertility policies and social mobility does not occur in equilibrium.

different assumptions, incomplete markets models can result in inefficiently high, not low, fertility. If that is the case then policies limiting fertility could, in principle, be welfare improving. As we later show, this is not the case. Restricting fertility in incomplete markets models, even if fertility is inefficiently high, is generally welfare detrimental.

The key assumption explaining why fertility could be inefficiently low or inefficiently high is whether children are a net financial cost or a net financial gain to parents. When children are a net financial gain, as in Schoonbroodt and Tertilt (2014) or Cordoba and Ripoll (2014), then market incompleteness generates inefficiently low fertility. But when children are a net cost, as in Barro and Becker (1989) and Becker and Barro (1988), then market incompleteness generates inefficiently high *steady state* fertility. Short term fertility, on the other hand, in particular the fertility of the first generation, is inefficiently low under market incompleteness regardless of whether children are a net financial cost or benefit.

The deterministic models of this section do not generate a negative relationship between fertility and income. Therefore, they do not help to rationalize the negative fertility-income relationship nor the high fertility of the poor. The model in the next section, with idiosyncratic shocks, can generate a negative relationship between fertility and income as well as inefficiently high fertility rates.

## 2.1 Complete markets

Individuals live for two periods, one as a child and one as an adult. Children do not consume. The lifetime utility of an adult individual, or just an individual, at time $t$ is of the Barro-Becker type

$$u(c_t) + \Phi(n_t) U_{t+1}, \; t = 0, 1, 2, \cdots ,$$

where $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $\sigma \in (0,1)$, is the utility from consumption, $n$ is the number of children and $\Phi(n) = \beta n^{1-\varepsilon}$, satisfying $\epsilon \in (0, \sigma)$, $\Phi(\bar{n}) < 1$, and $n \leq \bar{n}$. $\Phi(n)$ is the weight that parents attach to the welfare of their $n$ children and $U_{t+1}$ is the welfare of an individual at $t+1$. The upper limit on $\epsilon$ is needed for an interior solution of fertility, as discussed in Barro and Becker (1988). The upper limit on $\Phi(\overline{n})$ is required to guarantee bounded utility.

Individuals are endowed with one unit of time that is used either to work or to raise children. An adult's labor supply is $l_t = 1 - \lambda n_t$, where $\lambda$ is the time cost of raising a child. In additional to labor income, individuals receive transfers, or bequest, from their parents in the amount $b_t$. Let $w$ be the wage rate per unit of labor supply and $r$ be the interest rate at time. An individual's budget constraint is given by

$$w(1-\lambda n_t) + (1+r) b_t \geq c_t + n_t b_{t+1}.$$

The maximum number of children, $\overline{n}$, needs to satisfy two restrictions. First, $\overline{n} \leq 1/\lambda$ is required so that parents labor supply, $1-\lambda n_t$, is nonnegative. Second, $\Phi(\overline{n}) < 1$ implies $\overline{n} < \beta^{\frac{1}{\varepsilon-1}}$. These restrictions are satisfied if $\lambda^{1-\varepsilon} > \beta$. A final assumption is required so that maximum fertility is not always optimal: $\beta^{\frac{1}{\varepsilon}} (1+r)^{\frac{1}{\varepsilon}} < \overline{n}$.

Barro and Becker (1989) characterized optimal consumption and fertility allocations, given prices and $b_0$, of the sequence version of the problem above under the assumption $(1+r)\lambda w > w$. This assumption requires children to be a net financial burden to parents. The following proposition characterizes optimal allocations.

**Proposition 1** *If $(1+r)\lambda w > w$ then the optimal solutions for $t \geq 1$ are given by*

$$c_t^C = \frac{1-\sigma}{\sigma-\varepsilon}[(1+r)\lambda w - w]$$

*and*

$$n_t^C = \beta^{\frac{1}{\varepsilon}} (1+r)^{\frac{1}{\varepsilon}}.$$

*If $(1+r)\lambda w < w$ the optimal solutions for $t \geq 1$ satisfy $n_t = \overline{n}$ and $\frac{c_{t+1}}{c_t} = \left(\beta \overline{n}^{-\varepsilon}(1+r)\right)^{1/\sigma}$.*

The first part of the Proposition was proven by Barro and Becker (1989) while the second part is proven in the Appendix. According to the proposition, if $(1+r)\lambda w > w$ then there is no transitional dynamics after the first period, consumption is proportional to the net cost of raising a child, and fertility is proportional to the interest rate but independent of wage income. However, if $w > (1+r)\lambda w$ then maximum fertility is optimal and the economy is always in transition, except if parameters are such that $\beta \overline{n}^{-\varepsilon}(1+r) = 1$. Maximum fertility is optimal because children earn enough income to compensate their parents for the costs of raising them and such compensation scheme is possible when markets are complete.

## 2.2 Incomplete markets

Consider now an incomplete markets case. Specifically, suppose bequests are constrained to be larger than certain amount, $\underline{b}$. This is a natural restriction because parents are legally unable to leave debts to their children. The first order condition for optimal bequests can be written as a standard Euler equation but across generations:

$$n_t u'(c_t) \geq \Phi(n_t) u'(c_{t+1})(1+r) \text{ with equality if } b_{t+1} > \underline{b}. \tag{1}$$

The left hand side of this equation is the cost of endowing $n$ children with additional $db$ bequests per-child while the right hand side is the benefit. Denote $n^I$ the steady state fertility in the incomplete markets model. The following proposition states that fertility is higher in the incomplete markets model when children are a net cost to parents.

**Proposition 2** *Suppose* $(1+r)\lambda w > w$. *Then* $n^I \geq n^C$.
***Proof.*** *In steady state, equation (1) simplifies to* $1 \geq \beta(1+r)\left(n^I\right)^{-\varepsilon}$ *or*

$$n^I \geq \beta^{\frac{1}{\varepsilon}}(1+r)^{\frac{1}{\varepsilon}} = n^C.$$

■

The intuition for this key result is the following. When bequest constraints bind, early generations cannot extract as much resources from later generations. Since children are normal goods, this redistribution of resources from early to later generations reduces fertility at early times but makes future generations effectively richer and therefore increases fertility at later times. The assumption $(1+r)\lambda w > w$ is important because in that case $n^C$ is the relevant fertility when markets are complete. Otherwise, maximum fertility is optimal in the complete markets case and therefore larger or equal than the incomplete markets case.

The results above show that incomplete markets models may produce inefficiently high steady state fertility. In those instances policies seeking to reduce fertility may be welfare enhancing. The findings also put in context of the results of Schoonbroodt and Tertilt (2014). Their study is related with cases where children are a financial gain to parents and therefore market incompleteness generates too little fertility. It is also the case that even when children are a net financial cost, fertility by early generations may be inefficiently low. But over the longer term, inefficiently high fertility arises when markets are incomplete and children are a net financial cost.

Whether children are a financial benefit or cost is an open question. Cordoba and Ripoll (2014) find that children in the U.S. are likely a net financial benefit. Lee (2000) estimates intergenerational transfers for different societies and finds that lifetime transfers run from children to parents in relative rich societies, and from parents to children in relative poor societies.

The results of this section take prices as given. In that sense, they correspond to either partial equilibrium or to a small open economy. The Appendix shows that these results can also be obtained in general equilibrium. A limitation of the models so far is that they are silent about why fertility rates are larger among poorer individuals, a key motivating for many family planning policies. The next section shows that an extension of the model that allows for idiosyncratic shocks can explain this regularity.

# 3 Idiosyncratic shocks

This section introduces idiosyncratic shocks into the model of the previous section. We consider complete markets and incomplete markets arrangements. Models with idiosyncratic risk are central in modern macroeconomics, particularly when studying issues of inequality and redistribution but, with the exception of Alvarez (1999), there exists no dynastic altruistic models with endogenous fertility. Given the focus of family planning policies on poverty and inequality reduction, it is natural to study those issues within a standard model where inequality is ultimately the result of idiosyncratic risk.

The remaining of the paper derives a series of novel results for what can be called Bewley models with endogenous fertility. A key contribution is to show that a negative relationship between fertility and income arises naturally both in the complete and incomplete markets versions of the models. We also show that the complete markets model has some counterfactual predictions, and focus the rest of the discussion in the incomplete markets version.

## 3.1 Environment

Assume there is not aggregate uncertainty, goods are perishable, all information is public, and markets open every period.

### 3.1.1 Evolution of abilities

Consider an endowment economy populated by a large number of dynastic altruistic individuals who live for two periods, one as a child and one as an adult. Children do not consume. Individuals differ in their labor endowments, or earning abilities. Let $\Omega \equiv \{\omega_1, \omega_2, ..., \omega_K\}$ be the set of possible earning abilities, where $0 < \omega_1 < ... < \omega_K$. Earning abilities are drawn at the beginning of the adult life from the Markov chain

$M(\omega', \omega) = \Pr(\omega_{t+1} = \omega' | \omega_t = \omega)$ where $\omega_t$ is the ability of the parent and $\omega_{t+1}$ is the ability of the child. We use the following assumption, due to Chu and Koo (1990), to guarantee intergenerational persistence of abilities in the first order stochastic sense.

**Assumption 1** Conditional Stochastic Monotonicity (CSM):

$$\frac{\sum_{i=1}^{I} M_{i1}}{\sum_{j=1}^{J} M_{j1}} \geq \frac{\sum_{i=1}^{I} M_{i2}}{\sum_{j=1}^{J} M_{j2}} \geq ... \geq \frac{\sum_{i=1}^{I} M_{iK}}{\sum_{j=1}^{J} M_{jK}},\ 1 \leq I \leq J \leq K.$$

Assumption 1 means that if a low ability kid and a high ability kid both fall into one of the lowest ability classes, it is more likely that the kid born by a low ability parent be endowed with a lower ability than the kid born by a high ability parent. CSM implies first order stochastic dominance. To see this notice that when $J = K$, the condition becomes:

$$\sum_{i=1}^{I} M_{i1} \geq \sum_{i=1}^{I} M_{i2} \geq ... \geq \sum_{i=1}^{I} M_{iK},\ 1 \leq I \leq K.$$

Two examples of Markov chains satisfying Assumption 1 are an i.i.d. process and quasi-diagonal matrices of the form:

$$M = \begin{bmatrix} a+b & c & 0 & 0 & .. & 0 & 0 \\ a & b & c & 0 & .. & 0 & 0 \\ 0 & a & b & c & .. & 0 & 0 \\ .. & .. & .. & .. & .. & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & 0 & a & b+c \end{bmatrix}.$$

where $(a, b, c) \gg 0$, $a + b + c = 1$ and $b > 0.5$.

Assume further that $M$ has a unique invariant distribution, $\mu$, satisfying:

$$\mu(\omega_j) = \sum_{\omega_i \in \Omega} \mu(\omega_i) M(\omega_j, \omega_i) \text{ for all } \omega_j \in \Omega. \quad (2)$$

Let $\omega^t = [\omega_0, \omega_1, ..., \omega_t] \in \Omega^{t+1}$ denote a particular history of ability realizations up to time $t$. We call $\omega^t$ a family history or a family branch. With some abuse of notation, let $\omega^t = \left(\omega^{t-1}, \omega_t\right)$ for $t > 1$ and $\omega^0 = \omega_0$.

### 3.1.2 Individual resources

The technology of production is linear in ability: one unit of labor produces one unit of perishable output. The total resources available to an individual of ability $\omega_t$ at time $t$ are labor income $\omega_t (1 - \lambda n_t)$ and transfers from their parents. Let $c_t \left(\omega^t\right)$ and $n_t \left(\omega^t\right)$ denote consumption and fertility of an individual whose family history is $\omega^t$. Let $b_t \left(\omega^t\right)$ denote transfers, or bequests, received from parents.[5] Resources are used to consume and to leave transfers to children. Insurance market exists as parents can leave bequests contingent on the ability of their children. Let $q_t(\omega^t, \omega_i)$ be the time$-t$ price of a contingent asset $b_{t+1} \left(\omega^t, \omega_i\right)$ that delivers one unit of consumption at $t+1$ contingent on $\omega_{t+1} = \omega_i$ given family history $\omega^t$. The budget constraint of an individual at time $t$ with history $\omega^t$ is:

$$c_t \left(\omega^t\right) + n_t \left(\omega^t\right) \sum_{i=1}^{K} q_t(\omega^t, \omega_i) b_{t+1} \left(\omega^t, \omega_i\right) \leq \omega_t \left(1 - \lambda n_t \left(\omega^t\right)\right) + b_t \left(\omega^t\right). \quad (3)$$

### 3.1.3 Aggregate resources

Let $N_0 \left(\omega^0\right) \equiv 1$. $N_t \left(\omega^t\right) = N_{t-1} \left(\omega^{t-1}\right) n_{t-1} \left(\omega^{t-1}\right) M \left(\omega_t, \omega_{t-1}\right)$ is the population size at time $t$ of a particular branch of the family tree, the one described by $\omega^t$. Assume goods are perishable, total consumption cannot exceed the total endowment of the economy at any point in time:

$$\sum_{\omega^t \in \Omega^{t+1}} N_t \left(\omega^t\right) \left[\omega_t \left(1 - \lambda n_t \left(\omega^t\right)\right) - c_t \left(\omega^t\right)\right] \geq 0 \text{ for all } t.$$

### 3.1.4 Preferences

The lifetime utility of an individual born at time $t$ is of the Barro-Becker expected-utility type:

$$u \left(c_t \left(\omega^t\right)\right) + \Phi \left(n_t \left(\omega^t\right)\right) E_t \left[U_{t+1} \left(\omega^{t+1}\right) | \omega^t\right], \; t = 0, 1, 2, \cdots \quad (4)$$

where $u(c)$ and $\Phi(n)$ satisfy the same properties of the previous section, $E_t$ is the mathematical expectation operator conditional on the information up to time $t$ and $E_t U_{t+1}$ is the expected utility of individuals in the next generation.

[5] We denote $b_t$ bequest for short. This bequest includes all net transfers from a parent to each of his/her adult child. In particular, it includes inter vivos transfers.

### 3.1.5 Market arrangements

All markets are competitive and insurance prices are actuarially fair. Let $p_t = \sum_i q_t(\omega^t, \omega_i)$ be the price of a riskless bond. Under the assumption of no aggregate uncertainty, $p_t = p$. The riskless interest rate is then given by $1 + r = 1/p$. Furthermore, actuarially fair prices must satisfy the arbitrage condition

$$q_t(\omega^t, \omega_i) = pM(\omega_i, \omega_t). \tag{5}$$

so that $q_t(\omega^t, \omega_i) = q(\omega_t, \omega_i)$.

We study two different market arrangements. We first consider a complete markets economy in which families are only subject to natural debt limits, limits that prevent Ponzi schemes but that do not bind in equilibrium. In particular, parents can leave negative bequests to their children in a complete markets economy. We also consider an incomplete markets economy in which parents cannot leave negative bequests to their children. In particular, in this case we assume:

$$b_{t+1}\left(\omega^t, \omega_{t+1}\right) \geq 0 \text{ for all } \omega^t \in \Omega^{t+1}, \omega_{t+1} \in \Omega \text{ and all } t \geq 0. \tag{6}$$

## 3.2 Complete markets

### 3.2.1 Optimal consumption and fertility

Consider first the complete markets economy. We find convenient to write the problem recursively. Let $V_t\left(b_t, \omega^t\right)$ be the maximum lifetime utility of a time$-t$ adult with family history $\omega^t$. Then

$$V_t\left(b_t, \omega^t\right) = \max_{n_t \in [0,\overline{n}], b_{t+1}(\omega^t, \omega_i)} u\left(c_t\right) + \beta n_t^{1-\varepsilon} E_t\left[V_{t+1}\left(b_{t+1}\left(\omega^{t+1}\right), \omega^{t+1}\right) | \omega_t\right]$$

subject to

$$c_t + n_t \sum_{i=1}^{K} q(\omega^t, \omega_i) b_{t+1}(\omega^t, \omega_i) \leq \omega_t\left(1 - \lambda n_t\right) + b_t \text{ for all } t \text{ and } \left(\omega^t, \omega_i\right). \tag{7}$$

The following Proposition generalizes Proposition 1 for the case of stochastic earning abilities and children being a net financial costs to parents. Proofs are in the Appendix.

**Proposition 3** *Suppose* $(1 + r)\lambda\omega_t > E\left(\omega_{t+1}|\omega_t\right)$ *for all* $\omega_t$. *Then the optimal interior solutions for consumption and fertility are given by:*

$$c_{t+1} = c\left(\omega_t\right) = \frac{1-\sigma}{\sigma - \varepsilon}\left[\left(1 + r\right)\lambda\omega_t - E\left(\omega_{t+1}|\omega_t\right)\right] \text{ for all } t \geq 0 \tag{8}$$

*and*

$$n_t = n\left(\omega_{t-1}, \omega_t\right) = \beta^{\frac{1}{\varepsilon}}\left(1+r\right)^{\frac{1}{\varepsilon}}\left(\frac{\left(1+r\right)\lambda\omega_{t-1} - E\left(\omega_t|\omega_{t-1}\right)}{\left(1+r\right)\lambda\omega_t - E\left(\omega_{t+1}|\omega_t\right)}\right)^{\frac{\sigma}{\varepsilon}} \text{ for all } t \geq 1. \tag{9}$$

*If* $(1+r)\lambda\omega_t > E(\omega_{t+1}|\omega_t)$ *for some* $t$*, then the optimal fertility at time* $t$ *is* $\bar{n}$*.*

According to the proposition, consumption of all generations, except the first one, is proportional to the net expected financial cost of raising a child, $(1+r_t)\lambda\omega_t - E(\omega_{t+1}|\omega_t)$. Equation (8) implies that individual consumption is a random variable, one that increases with the ability of the parent and falls with the expected ability of the individual. The randomness of consumption is perhaps surprising because with exogenous fertility consumption it is not random but constant. In that case, individuals are able to insure all idiosyncratic risk, and consumption obeys the permanent income hypothesis.

But consumption in the endogenous fertility case obeys a completely different logic. In contrast to the exogenous fertility case, the existence of a next period consumer, the child, is not guaranteed nor costless. When a life is costly to create, optimal consumption becomes an utilization rate and the higher the net cost of creating a new variety, a new child, the higher the utilization rate. Furthermore, if costs and/or benefits of raising a child is random so is consumption. The randomness of consumption also implies that there exists social mobility in the endogenous-fertility complete-markets model. Equation (8) also implies that all children of the same parent has the same consumption, and that the actual ability of an individual does not affects his/her own consumption. These two predictions are particularly problematic. Evidence shows that there exists significant consumption inequality among siblings and that consumption increases with earnings (e.g., Mulligan 1997 and Gaviria 2002).

The solution for optimal fertility described by equation (9) is a restatement of the intergenerational Euler equation, $u'(c_t) = \beta n_t^{-\varepsilon}(1+r_{t+1})E_t u'(c_{t+1})$, where $\beta n_t^{-\varepsilon}$ is the average degree of altruism. Fertility, rather than consumption, now plays the key role of smoothing family welfare. If parental consumption is high while children consumption is low, then high fertility is required to smooth family utility intertemporally. More precisely, equation (9) shows that fertility depends negatively on the ability of the parent, and positively on the ability of the grandparent. The model thus provides a novel explanation for the negative fertility income relationship, an explanation that relies purely on the random nature of abilities. Conditional on grandparents' abilities, the model predicts that high ability parents have fewer children because children are more costly to those parents.

Notice that optimal fertility is the lowest for high ability parents with low ability grandparents. This is due to the fact that, with complete markets, unlucky grandparents can borrow against the income of their high ability children which reduces the wealth of those children, and therefore their fertility. As we show below, when markets are incomplete unlucky grandparents cannot borrow against their children's income and therefore the fertility of high ability parents with low ability grandparents is higher under incomplete markets.

Corner solutions are not only possible but likely in complete markets models with idiosyncratic shocks. First, if children are a net financial benefit to parents, then maximum fertility is optimal, $n_t = \overline{n}$. Second, equation (9) can easily imply $n_t > \overline{n}$ for parents whose ability is far below that of their grandparents. As we document below, using a calibrated version of the model, this feature of the complete markets model is problematic because it leads to unrealistic high fertility rates for many families.

### 3.2.2 General equilibrium

Given optimal fertility rules $n(\omega_{t-1}, \omega_t)$, let $N_{t+1}(\omega_t, \omega_{t+1})$ be the population at time $t+1$ with own type $\omega_{t+1}$ and parental type $\omega_t$. Assuming a law of large number holds, aggregate population evolves according to:

$$N_{t+1}(\omega_t, \omega_{t+1}) = \sum_{\omega_{t-1}} N_t(\omega_{t-1}, \omega_t)\, n(\omega_{t-1}, \omega_t)\, M(\omega_{t+1}, \omega_t).$$

Let $N_t \equiv \sum_{\omega_{t-1}} \sum_{\omega_t} N_t(\omega_{t-1}, \omega_t)$ be total population at time, $\pi_t(\omega_{t-1}, \omega_t) \equiv \frac{N_t(\omega_{t-1}, \omega_t)}{N_t}$ be the fraction of population of type $\omega_t$ and parental type $\omega_{t-1}$,and $1 + g_t \equiv \frac{N_{t+1}}{N_t}$ the gross growth rate of population. Then, the previous expression can be written as:

$$\pi_{t+1}(\omega_t, \omega_{t+1}) = \frac{1}{1+g_t} \sum_{\omega_{t-1}} \pi_t(\omega_{t-1}, \omega_t)\, n(\omega_{t-1}, \omega_t)\, M(\omega_{t+1}, \omega_t) \qquad (10)$$

Adding across $(\omega_t, \omega_{t+1})$ pairs, it can be seen that $1+g_t = \sum_{\omega_t} \sum_{\omega_{t-1}} \pi_t(\omega_{t-1}, \omega_t)\, n(\omega_{t-1}, \omega_t)$. The stationary distribution $\pi(\omega_{t-1}, \omega_t)$ is the invariant distribution that solves (10).

Given that production is perishable, the equilibrium interest rate has to equate aggregate consumption to aggregate endowments. Using the stationary distribution, the equilibrium condition can be expressed as

$$\sum_{\omega'}\sum_{\omega} c(\omega)\,\pi(\omega,\omega') = \sum_{\omega'}\sum_{\omega}\left[1-\lambda n(\omega,\omega')\right]\omega'\pi(\omega,\omega').$$

This equation pins down the steady state interest rate $r$ since both $c(\omega)$ and $n(\omega,\omega')$ are functions of the interest rate $r$.

## 3.3 Incomplete markets

### 3.3.1 Recursive formulation

Consider next the case of bequests constraints of the type (6). Assume output is perishable, aggregate consumption must equal aggregate production. Alternatively, aggregate savings are zero. Moreover, savings are equal to the total amount of bequests left by parents. Since all bequests are non-negative then aggregate savings are zero if and only if all bequests are zero. Therefore, in any equilibrium with bequest constraints the budget constraint (3) simplifies to:

$$c_t\left(\omega^t\right) \leq \omega_t\left(1-\lambda n_t\left(\omega^t\right)\right) \text{ for all } \omega^t \in \Omega^{t+1} \text{ and all } t \geq 1. \tag{11}$$

This is balanced budget constraint for every period and state. The lack of intergenerational transfers significantly simplifies the problem. To study the incomplete markets problem, we first show that the principle of optimality holds. Standard arguments cannot be used because the discount factor is endogenous. Alvarez (1999) shows that the principle of optimality holds for a dynastic version of this problem, while we show that it holds for the household version of the problem.[6]

Let $\hat{N}_0\left(\omega^{-1}\right)=1$, $\hat{N}_t\left(\omega^{t-1}\right)=\prod_{j=0}^{t-1} n_j\left(\omega^j\right)$ for $t \geq 1$. The problem can be written in sequential form, using (11) and (4) recursively, as follows:

$$V_0^*(\omega_0) = \sup_{\left\{\hat{N}_{t+1}(\omega^{t-1},\omega_t)\right\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \hat{N}_t\left(\omega^{t-1}\right)^{1-\varepsilon} u\left(\omega_t\left(1-\lambda\frac{\hat{N}_{t+1}\left(\omega^{t-1},\omega_t\right)}{\hat{N}_t\left(\omega^{t-1}\right)}\right)\right) \tag{12}$$

[6] The analogous dynastic problem is:

$$U(N,\omega) = \max_{N'\in[0,\overline{n}]} u\left(\omega-\lambda N'/N\right)N^{1-\epsilon} + \beta E\left[U\left(N',\omega'\right)|\omega\right].$$

In this problem the number of family members is a state variable, $N$, all members have the same ability, $\omega$, and make the same choices. The household problem does not impose these constraints. On problem is simpler than Alvarez's because there are no savings. But the problem is still dynamic.

subject to

$$0 \leq \hat{N}_{t+1}\left(\omega^{t-1}, \omega_t\right) \leq \hat{N}_t\left(\omega^{t-1}\right)/\lambda \text{ for all } \omega^{t-1} \in \Omega^t,\ \omega_t \in \Omega \text{ and } t \geq 0.$$

The recursive formulation of this problem is:

$$U(\omega) = \max_{n \in [0, \overline{n}]} u\left(\omega\left(1 - \lambda n\right)\right) + \beta n^{1-\varepsilon} E\left[U\left(\omega'\right)|\omega\right]. \tag{13}$$

The next proposition states that the principle of optimality holds for this problem.

**Proposition 4** *The functional equation (13) has a unique solution, $U(\omega)$. Moreover $U(\omega) = V_0^*(\omega)$ for $\omega \in \Omega$.*

#### 3.3.2 Optimal fertility

The optimality condition for fertility choices is

$$\lambda \omega u'\left(\left(1 - \lambda n^*\right)\omega\right) = \beta\left(1 - \varepsilon\right) n^{*-\varepsilon} E\left[U\left(\omega'\right)|\omega\right]. \tag{14}$$

The left hand side of (14) is the marginal cost of an additional child while the right hand side is the marginal benefit. The marginal cost is the product of the opportunity cost of raising a child, $\lambda\omega$, times the marginal utility of consumption. The marginal benefit to the parent is the discounted expected welfare of a child, $\beta E\left[U\left(\omega'\right)|\omega\right]$, times the parental weight associated to the last child, $(1-\varepsilon)\, n(\omega)^{-\varepsilon}$. Let $n^* = n(\omega)$ be the optimal fertility rule and $c^* = c(\omega) \equiv (1 - \lambda n(\omega))\,\omega$ the optimal consumption rule.

In contrast to the complete markets case, corner solutions are not optimal in the incomplete markets case under the assumed functional forms. Having no children is not optimal because the marginal benefit of a child is infinite while the marginal cost is finite. In particular, $E\left[U\left(\omega'\right)|\omega\right] > 0$ for all $\omega$ while $\lim_{n \to 0} n^{-\epsilon} = \infty$. Having the maximum number of children is also sub-optimal because the marginal cost is infinite when parental consumption is zero, while the marginal benefit is finite.

Consider now the relationship between fertility, $n^*$, and parental earning ability, $\omega$. According to (14), both marginal benefits ($MB$) and marginal costs ($MC$) are affected by abilities. $MB$ increase with $\omega$ because of the postulated intergenerational persistence of abilities: high ability parents are more likely to have high ability children. Regarding $MC$, there are two effects. On one hand, $MC$ tends to rise with $\omega$ because higher ability increases the opportunity cost of the parental time required to raise children. On the other hand, $MC$ tends to fall because the larger the ability the smaller the marginal utility of consumption. Given that $\sigma \in (0,1)$, the first effect dominates the second one so $MC$ increases with $\omega$. The need for $\sigma \in (0,1)$ suggests a tension between the theory and the empirics since estimates of the intertemporal elasticity of substitution are typically lower than 1. But the correct interpretation of $1/\sigma$ is an intergenerational elasticity of substitution (IGES), one controlling inter-personal consumption smoothing rather than intra-personal consumption smoothing (Cordoba and Ripoll 2011).

Since both $MB$ and $MC$ increase with $\omega$, it is not clear in principle whether fertility increases or decreases with ability. The following proposition provides the answer in three cases: i.i.d. abilities across generations, perfect intergenerational persistence of abilities without uncertainty and random walk (log) abilities.[7]

In order to analyze the case of perfect persistence of ability, it is instructive to write the first order condition in an alternative way. Using equation (14) to express (13) as:

$$U_t = u\left(c\left(\omega\right)\right) + \frac{1}{1-\varepsilon} n\left(\omega\right) \lambda \omega u'\left(c\left(\omega\right)\right) \tag{15}$$

Then use (15) to rewrite (14) as:

$$u'\left(c\left(\omega\right)\right) n\left(\omega\right)^{\varepsilon} = \beta E\left[ u'\left(c\left(\omega'\right)\right) \frac{\omega'}{\omega}\left(\frac{1}{\lambda} + \frac{\sigma-\varepsilon}{1-\sigma}\left(\frac{1}{\lambda} - n\left(\omega'\right)\right)\right) \middle| \omega\right] \tag{16}$$

This equation is useful because it only requires marginal utilities, rather than total utility as in equation (14), and corresponds to the Euler equation of the problem describing the optimal consumption rule. Although savings are zero in equilibrium, fertility allows individuals to smooth consumption across generations.[8]

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[7] Although a random walk does not satisfy some of the assumptions above, it helps to develop some intuition.

[8] Equation (14) can also be written in the form of a more traditional Euler equation. Let $1+r'$ be the gross return of "investing" in a child. It is given by $1+r' \equiv \frac{U(\omega')/u'(c')}{\lambda\omega}$. In this expression, $U(\omega')/u'(c')$ is the value of a new life, in terms of goods, while $\lambda\omega$ is the cost of creating a new individual. Then (14) can be written as:

$$u'(c) = \beta\left(1-\epsilon\right) n^{*-\epsilon} E\left[u'(c')\left(1+r'\right)|\omega\right]. \tag{17}$$

This is an Euler equation with a discount factor $\beta\left(1-\epsilon\right) n^{*-\epsilon}$. It suggests that optimal fertility choices are similar to saving decisions and that children are like an asset, as pointed out by Alvarez (1999). However, two important differences with the traditional Euler Equation are that the individual controls the discount factor and the gross return.

**Proposition 5** ***Persistence and the fertility-ability relationship.*** *(i) Fertility decreases with ability if abilities are i.i.d. across generations. In this case* $n(\omega)$ *satisfies the equation* $\frac{n(\omega)^{\varepsilon}}{(1-\lambda n(\omega))^{\sigma}} = A\omega^{\sigma-1}$ *where* $A$ *is a constant. Furthermore, fertility is independent of ability in one of the following two cases: (ii)* $M$ *is the identity matrix (abilities are perfectly persistent and deterministic); or (iii)* $\ln \omega_t = \ln \omega_{t-1} + \varepsilon_t$ *where* $\varepsilon_t \sim N(0, \sigma_{\varepsilon}^2)$.

According to Proposition 5, fertility decreases with ability when abilities are i.i.d. The intuition is that without intergenerational persistence, a higher ability of the parent only affects her or his marginal cost but not marginal benefit as $E[U(\omega')|\omega] = E[U(\omega')]$ for all $\omega \in \Omega$. On the other extreme, fertility is independent of ability when abilities are perfectly persistent across generations (cases ii and iii). This is because in those cases both the marginal cost and marginal benefit are proportional to $\omega^{1-\sigma}$. Given that fertility becomes independent of ability only in the extreme case of perfect persistent, it is natural to conjecture that fertility decreases with ability when persistence is less than perfect. We were able to confirm this conjecture numerically but analytical solutions were not obtained.

### 3.3.3 General equilibrium

Denote $N_t(\omega)$ the mass of population with ability $\omega$, $N_t \equiv \sum_{\omega\in\Omega} N_t(\omega)$ total population, $\pi_t(\omega) = \frac{N_t(\omega)}{N_t}$ the fraction of population with ability $\omega \in \Omega$ at time $t$, and $1 + g_t = \sum_{\omega\in\Omega} n(\omega)\pi_t(\omega)$ the gross growth rate of population. The initial population of different ability types $\{N_0(\omega_i)\}_{i=1}^{K}$ is given.

As in Section 3.2.2., it can be shown that the law of motion for $\pi_t(\omega_i)$ satisfies

$$\pi_{t+1}(\omega_j) = \frac{1}{1+g_t} \sum_{\omega_i\in\Omega} n(\omega_i)\pi_t(\omega_i) M(\omega_j, \omega_i) \text{ for all } \omega_j \in \Omega \tag{18}$$

Let $\pi^*(\omega) \equiv \lim_{t\to\infty} \pi_t(\omega)$ represents the limit distribution of abilities where $\omega \in \Omega = \{\omega_i\}_{i=1}^{N}$. Given the Assumption 1 and that $n(\omega_1) \geq ... \geq n(K) > 0$, a unique limit distribution exists (see Chu and Koo 1990).

We now provide some analytical results characterizing the distribution $\pi_t$, the limit distribution $\pi^*$ and their relationship to fertility for some special cases. Calibrated results are provided in Section 5. The following proposition provides a simple but important benchmark. The first part states that when fertility is identical across types then $\pi^* = \mu$ : the endogenous limit distribution of abilities equals the exogenous invariant distribution of abilities described by equation (2). In other words, the endogenous distribution of abilities just reflects the genetic distribution of abilities, what can be termed nature rather than nurture. The second part of the proposition shows that this results also holds when fertility rates are different across types but there is not persistence of abilities.

**Proposition 6** ***When*** $\boldsymbol{\pi}^*$ ***equals*** $\boldsymbol{\mu}$. *Suppose one of the following two conditions holds: (i)* $n(\omega) = n$ *for all* $\omega \in \Omega$*; or (ii)* $M(\omega', \omega)$ *is independent of* $\omega$ *for all* $\omega' \in \Omega$*. Then* $\pi^*(\omega) = \mu(\omega)$ *for all* $\omega \in \Omega$*. Moreover, if (ii) holds then* $\pi_t(\omega) = M(\omega, :)$ *for all* $\omega \in \Omega$ *and all* $t \geq 1$*.*

The following proposition uses propositions 5 and 6 to characterize fertility and the limiting ability distribution in cases of either no persistence or perfect persistence of abilities.

**Proposition 7** ***Persistence, fertility and ability distribution****. (i) If* $M(\omega', \omega)$ *is independent of* $\omega$ *then* $n(\omega)$ *decreases with* $\omega$ *and* $\pi_t(\omega) = \mu(\omega) = M(\omega, :)$ *for all* $\omega \in \Omega$ *and* $t \geq 1$; *(ii) if* $M$ *is the identity matrix then* $n(\omega) = n$ *for all* $\omega \in \Omega$ *and* $\pi_t(\omega) = \pi^*(\omega) = \pi_0(\omega)$ *for all* $\omega \in \Omega$ *and all* $t$; *(iii) if* $\ln \omega$ *follows a Gaussian random walk then* $n(\omega) = n$, *and given* $\omega_0$ *the variance of abilities diverges to* $\infty$*.*

In words, if children's abilities are independent of parental abilities, then fertility decreases with ability but the observed distribution of abilities is independent of fertility choices and determined by the Markov chain $M$ from the second period on. Furthermore, with certainty and perfect intergenerational persistence of abilities the observed distribution of abilities in any period is identical to the initial distribution of abilities. Finally, if (log) abilities follow a random walk then there is not limit distribution of abilities since its variance goes to infinite.

As for intermediate case of some persistence, the following Proposition is an application of Chu and Koo's (CK, 1990) Theorem 2. It states that if the fertility of the poor, meaning the group with lowest income, is higher than the fertility of the rest of the population then $\pi^*$ is different from $\mu$, and moreover, $\mu$ dominates $\pi^*$ in the first order stochastic sense.

**Proposition 8** *Suppose* $M$ *satisfies Assumption 1 and* $n(\omega_1) > n(\omega_i) = n$ *for all* $i > 1$*. Then* $\sum_{i=1}^{I} \pi^*(\omega_i) > \sum_{i=1}^{I} \mu(\omega_i)$ *for all* $1 \leq I \leq K$*.*

**Proof.** See Chu and Koo (1990, pp.1136). ■

# 4 Welfare analysis of fertility policies

We now consider simple types of fertility policies that directly set constraints on fertility choices. In particular, let $\underline{n}(\omega) \geq 0$ and $\overline{n}(\omega) \leq \overline{n}$ be the lower and upper bounds on fertility set by the policy. Bounds could potentially depend on individual abilities. The motivation for the upper bound is the one child policy. Although fertility policies are significantly more complex than simple bounds, the bounds capture a key essence of the intent of these policies which is to limit or, more recently, promote fertility.

## 4.1 Fertility policies and income

A key aspect of fertility policies such as family planning policies is their ability to increase the income of the poor and also, the average income of the economy. Since income equals $\omega\left(1-\lambda n\left(\omega\right)\right)$ then restricting fertility in fact increases individual income because it increases the effective labor supply. Furthermore, average earning abilities and average income are defined as:

$$E_t = \sum_{\omega\in\Omega} \omega\pi_t\left(\omega\right);\; I_t = \sum_{\omega\in\Omega} \omega\left(1-\lambda n\left(\omega\right)\right)\pi_t\left(\omega\right).$$

These expressions shows that, in addition to increasing individual income, fertility policies also have the potential to increase average income if those policies also result in better distributions of abilities, $\pi_t\left(\omega\right)$, in the first order stochastic sense. In fact, a corollary of Proposition 6 (i) and Proposition 8, is that reducing the fertility of the poor to the same level as that of others results in a limit distribution that dominates the original distribution. More generally, CK show that if fertility decreases with income, $M$ satisfies Assumption 1, and the initial distribution of incomes is at its steady state level, $\pi_0^*\left(\omega_i\right)$, then a reduction in the fertility of the poor results in a sequence of income distributions that first order stochastically dominate $\pi_0^*\left(\omega_i\right)$, that is, $\sum_{i=1}^{I}\pi_t\left(\omega_i\right) < \sum_{i=1}^{I}\pi_0^*\left(\omega_i\right)$ for all $1 \leq I \leq K$ and $t > 0$.

The positive effects of family planning policies on the income of the poor, average income and average ability are often used to provide support for these policies. We next show that, in spite of its positive effects on income, fertility policies are welfare detrimental in our incomplete markets economies.

### 4.2 Fertility policies and individual welfare

The indirect utility $U^r(\omega)$ of the problem with fertility constraints solves the Bellman equation:

$$U^r(\omega) = \max_{n \in [\underline{n}(\omega), \overline{n}(\omega)]} u((1 - \lambda n)\omega) + \beta n^{1-\varepsilon} E\left[U^r(\omega') | \omega\right]. \tag{19}$$

Let $n^r(\omega)$ denotes the corresponding optimal fertility rule. The following proposition states that binding fertility restrictions in at least one state reduce the indirect utility, or welfare, of all individuals even those whose fertility is not directly affected. The proposition also states that fertility restrictions of any type (weakly) reduce the fertility of all individuals except perhaps those whose fertility rates are at or below the lower bound.

**Proposition 9** *$U^r(\omega) \leq U(\omega)$ for all $\omega$ and it holds with strict inequality for all $\omega$ if $n(\hat{\omega}) > \overline{n}(\hat{\omega})$ or $n(\hat{\omega}) < \underline{n}(\hat{\omega})$ for at least one $\hat{\omega} \in \Omega$. Furthermore $n^r(\omega) = \underline{n}(\omega)$ if $n(\omega) \leq \underline{n}(\omega)$ and $n^r(\omega) \leq n(\omega)$ otherwise. In particular, $n^r(\omega) < n(\omega)$ for all $\omega$ if $n(\omega) > \underline{n}(\omega)$ for all $\omega$ and $n(\hat{\omega}) > \overline{n}(\hat{\omega})$ for at least one $\hat{\omega} \in \Omega$.*

Proposition 9 implies that policies such as the *One Child Policy*, which limits fertility of all individuals, or policies that compel individuals to increase their fertility such as the "leftover women" stigma in China, are detrimental to all individuals' welfare in our incomplete markets model.

Although fertility is possibly inefficiently high for some ability types and there might be potential room for policies to improve individual welfare, the proposition shows that policies restricting fertility choices do not help because it restricts individual's choices without providing any compensation. Furthermore, fertility restrictions that only affect a particular group result in lower welfare for all individuals because, regardless of current ability, there is a positive probability that a descendant of the dynasty will fall into the group directly affected in finite time. Given that welfare of every individual falls with fertility restrictions, the marginal benefits of having children also falls while the marginal cost remains the same. Thus fertility must fall for all types except perhaps for those who are constrained by the policy to increase their fertility.

### 4.3 Fertility policies and social welfare

Given that fertility policies reduce the welfare of all individuals, as shown in Proposition 9, it is natural to infer that social welfare should also fall. The answer, however, depends on how social welfare is defined and whether the policy reduces or increases population. In this section we focus on fertility policies that impose upper limits on fertility rates such as limiting the fertility of the poor or the *One Child Policy.* Other fertility policies like coercing the rich to have more children and fertility related taxes/subsidies involve more competing factors and are postponed to the quantitative exercise.

### 4.3.1 Total Utilitarian Social Welfare

Classical (Bentham) utilitarianism defines social welfare as the total discounted welfare of all (born) individuals:[9]

$$W = \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U(\omega) N_t(\omega) . \tag{20}$$

In this formulation $\beta_p(t) \geq 0$ is the weight the social planner assigns to generation $t$, $U(\omega)$ is the utility of an individual with ability $\omega$, and $N_t(\omega)$ is the size of population of generation $t$ endowed with ability $\omega$. Since individuals are altruistic toward their descendants, $\beta_p(t) > 0$ means that the planner gives additional weight to generation $t$ on top of what is implied by parental altruism. A particular case in which the planner weights only the original generation and therefore adopts its altruistic weights is the one with $\beta_p(0) = 1$ and $\beta_p(t) = 0$ for $t > 0$ :

$$W_0 = \sum_{\omega \in \Omega} U(\omega) N_0(\omega) \tag{21}$$

According to Proposition 9, upper limits affecting the fertility of at least one ability group reduce fertility of all ability groups. Therefore, upper limits on fertility unequivocally reduce population of all ability groups at all times after time 0. Given that both population and individual welfare fall for all ability types, we are able to show that fertility limits unequivocally decrease social welfare if social welfare is of the classical or Benthamite utilitarian form.

**Proposition 10** *Imposing upper limits on fertility choices reduces social welfare as defined by (20).*

[9] The results in this section are similar if the welfare of the unborn is explicitly considered as long as the unborn enjoy lower utility than the born.

An identical result is obtained if the planner exhibits positive but diminishing returns to population, say if $N_t(\omega)$ in expression (20) is replaced by $N_t(\omega)^{1-\varepsilon_p}$ where $\varepsilon_p \in (0,1)$. This formulation seems a natural extension of the Barro-Becker preferences for a planner. The result does not necessarily hold if we define social welfare as the average, or Mills, utilitarian form as discussed next.

### 4.3.2 Average Utilitarian Social Welfare

Define average, or Mills, utilitarian social welfare as

$$\overline{W} = \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U(\omega) \pi_t(\omega) \tag{22}$$

This definition of welfare is analogous to (20) but uses population shares, $\pi_t(\omega)$, rather than population, $N_t(\omega)$ as weights. A particular case is one where the planner cares only about steady state welfare so that $\beta_p(t) = 0$ for all $t$ and $\lim_{t\to\infty} \beta_p(t) = 1$, the steady state welfare function is given as

$$\overline{W}^* = \sum_{\omega \in \Omega} U(\omega) \pi^*(\omega). \tag{23}$$

This definition of social welfare is the one used by CK to argue in favor of family planning policies. It is also a commonly used criteria in social choice theory because it avoids Parfit's repugnant conclusion. Notice that average social welfare could increase even if the welfare of all individuals fall when fertility limits are enacted. The net effect depends on the relative strength of two potentially opposite forces: on the one hand individual welfare falls but on the other hand the distribution of abilities, $\pi$, may improve. CK assume that $U(\omega)$ is invariant to the policy in place and therefore the only consequence of the policy is to change $\pi$. In that case, Proposition 8 states that limiting the fertility of the poor improves the distribution of abilities, in a stochastic dominance sense, and therefore increases average welfare, given $U$. The following corollary summarizes CK main result.

**Corollary 11** *Suppose social welfare is defined by (22) where $U(\omega)$ is a non-decreasing function of ability. Furthermore, suppose $M$ satisfies CSM and fertilities are exogenously given and satisfy $n(\omega_1) > n(\omega_i) = n$ for all $1 < i \leq N$. Then (i) reducing the fertility of the lowest ability group increases social welfare; (ii) fertility policies that do not change the distribution of abilities do not change social welfare.*

This result by CK assumes, however, that $U$ is invariant to the policy in place which is not true in our endogenous fertility model. As shown above, welfare of all individuals falls with the policy. Such reduction in individual welfare may be strong enough to offset the positive effects of the policy. The following proposition shows that in fact this is possible.

**Proposition 12** *Suppose $M(\omega', \omega)$ is independent of $\omega$ for all $\omega' \in \Omega$. Then upper limits on fertility choices reduce social welfare as defined by (22).*

Proposition 12 relies on the earlier finding in Proposition 6 that, when abilities are i.i.d., the distribution of abilities among the population is independent of fertility choices and thus fully determined by the Markov matrix $M$. Therefore, in the i.i.d. case the effect of any fertility policy on social welfare, as defined by (22), is only determined by its effect on individual welfare, $U$.

The following proposition for a deterministic case of perfect persistence of abilities states that average utilitarian welfare unequivocally falls with "uniform" fertility restrictions such as the one child policy.

**Proposition 13** *Suppose $M$ is the identity matrix. Then a uniform fertility restriction $\bar{n}(\omega) = \bar{n}$ reduces social welfare as defined by (22).*

Proposition 13 provides another example in which fertility restriction do not affect $\pi$. Since in the deterministic case all ability groups have the same fertility choices and the fertility restriction affects all ability groups equally, it follows that $\pi_t = \pi_0$ for all $t$ so that the effect of the policy on social welfare is only determined by the effect on individual welfare $U$.

Proposition 12 and 13 formally provide analytical examples in which social welfare falls when fertility is restricted even under the notion of average social welfare. We now turn to quantitative exercise to investigate more general cases in which abilities are correlated but less than perfect persistent across generations. The calibrated results suggest that fertility restricting policies decrease social welfare in general, even when social welfare is defined as average welfare.

# 5 Quantitative exploration

We now explore some of the quantitative implications of the model. While the computation of the incomplete markets model is relatively simple, the computation of the complete markets model is not. The reason is that corner solutions easily arise for standard calibrations of the income process. In particular, equation (9) would easily result in fertility larger than the maximum if $\omega_{t-1}$ and $\omega_t$ are sufficiently different.

To avoid corner solutions we first consider an example that sets a relatively narrow range for the set of abilities. Although the example produces only limited inequality, it allows us to illustrate that fertility in the incomplete markets model may be inefficiently high relative to the complete markets version. The rest of the section uses a more realistic calibration of the income process but provides only results for the incomplete markets model.

### 5.1 Example

Suppose abilities follow a two-state Markov process characterized by $[\omega_H, \omega_L] = [0.85, 1.18]$ and $\Pr(\omega_H|\omega_H) = \Pr(\omega_L|\omega_L) = 0.9$, and $\Pr(\omega_H|\omega_L) = \Pr(\omega_L|\omega_H) = 0.1$. Assume the following parameters values: $\sigma = 0.5$, $\beta = 0.2$, $\varepsilon = 0.3$, and $\lambda = 0.3$. These values are similar to the ones calibrated in the next section. The risk free interest rate that clears the asset market in the complete markets model is 3.44. Table 1 reports steady state fertilities for the complete and incomplete markets cases according to formulas (9) and (14). Notice that fertility in the first environment is a function of both grandparents and parental abilities while in the second environment is only a function of parental ability.

**Table 1: Fertility in complete and incomplete markets models**

| $n_t = n(\omega_{t-1}, \omega_t)$ | $n(\omega_H, \omega_H)$ | $n(\omega_H, \omega_L)$ | $n(\omega_L, \omega_H)$ | $n(\omega_L, \omega_L)$ |
|---|---|---|---|---|
| Complete Markets | 0.67 | 1.60 | 0.28 | 0.67 |
| Incomplete Markets | 0.58 | 0.69 | 0.58 | 0.69 |

Table 1 illustrates the result that, in presence of idiosyncratic shocks, steady state fertility can be inefficiently high when markets are incomplete. For instance, the fertility rate of high ability parents with low ability grandparents under complete markets is 0.28 while the fertility of the same individual under incomplete markets is 0.58. As discussed above, when markets are complete unlucky grandparents can borrow against the income of their high ability children which reduces the wealth of those children, and therefore their fertility. However, when markets are incomplete unlucky grandparents cannot borrow against their children's income and therefore the fertility of high ability parents with low ability grandparents is higher under incomplete markets. Thus, the result that policies restricting fertility rates are welfare detrimental is not due to fertility being always inefficiently low in incomplete markets models. Even if fertility is inefficiently high, those policies still reduce welfare in our models.

### 5.2 Benchmark calibration

In this section, we use Brazilian data to calibrate the model and evaluate the welfare effects of fertility restriction policies. The use of Brazilian data takes advantage of data assembled by Lam (1986) on the intergenerational income process, and it corresponds to a relevant case of a developing economy with high fertility rates. We are also interested in evaluating the potential effects of the one child policy in China, but given data limitations, Brazil in 1986 perhaps provides a reasonable approximation for China at the time of the one child policy.

The following parameters are needed: the Markov chain of abilities $M$, ability vector $\vec{\omega}$, curvature of the utility function $\sigma$, discount factor $\beta$, parent's degree of altruism $\varepsilon$, and time cost of raising every child $\lambda$. For social welfare calculations we also need a social planner's weight on every generation $\beta_p(t)$.

Data on different income groups, fertility of each group, and the Markov chain are taken from Lam (1986) who provides estimates for Brazil. Average incomes of each of five income groups are $\overrightarrow{I} = [553, 968, 1640, 2945, 10991]$. They describe income classes of Brazilian male household heads aged from 40 to 45 in 1976. Average fertility of each income group are $\overrightarrow{n} = [6.189, 5.647, 5.065, 4.441, 3.449]/2$. We divide fertility by two to obtain fertility per-adult. Using income and fertility data, we calculate earning abilities of different groups as $\omega_i = \frac{I(\omega_i)}{1-\lambda n(\omega_i)}$ and normalize the lowest ability to be 1. The Markov chain provided by Lam is:

$$M = \begin{bmatrix} 0.50 & 0.25 & 0.15 & 0.10 & 0.05 \\ 0.25 & 0.40 & 0.20 & 0.20 & 0.10 \\ 0.15 & 0.20 & 0.35 & 0.20 & 0.20 \\ 0.05 & 0.10 & 0.20 & 0.35 & 0.25 \\ 0.05 & 0.05 & 0.10 & 0.15 & 0.40 \end{bmatrix}$$

This chain does not satisfy the CSM property required by Assumption 1, but it implies certain level of earning persistency across generations since its diagonal elements dominate their corresponding off-diagonal elements. We also considered a Markov chain provided by Chu and Koo (1990) that satisfies CSM, and obtain similar results.

Initial population is normalized to 1 and the initial distribution of abilities, $\pi_0$, is approximated by the stationary distribution implied by $M$ and $\overrightarrow{n}$.

Our discount factor $\beta$ and altruistic parameter, $\varepsilon$, are calibrated to match altruism function in Manuelli and Seshadri (2009) (MS henceforth). Their altruistic function in a life cycle model takes the form $e^{-\rho B}e^{-\alpha_0+\alpha_1 \ln n}$ where $\alpha_0, \alpha_1$, and $\rho$ are set to be 0.24, 0.65, 0.04, respectively. The child-bearing age $B$ is 25. So the proper mappings from our parameters to theirs are $\beta = e^{-\rho B}e^{-\alpha_0}$ and $1 - \varepsilon = \alpha_1$, which solve our $\beta$ and $\varepsilon$ as 0.29 and 0.35, respectively.

Other two key parameters are the curvature of the utility function $\sigma$ and the time cost of raising every child $\lambda$. We originally set $\sigma$ to be 0.62, the one used by MS and calibrate $\lambda$ to match the mean of fertility data because time costs are not provided by MS. But the simulated dispersion of fertility at steady state is way below that of the Brazilian fertility data. To better fit the data, we calibrate $\sigma$ and $\lambda$ to jointly match the mean and standard deviation of the fertility data. The calibrated values are $\sigma = 0.526$ and $\lambda = 0.243$. $\sigma$ is below but not far away from the one used in MS. Under this values, the mean and standard deviation of steady state fertility in the model are 2.653 and 0.433 respectively, which are close to the targets 2.648 and 0.416 in the data.

Our parameter of time cost of raising a child approximately prescribes a maximum number of 8 children per couple, or that each parent spends around 12% of their time on every child. For the social planner's weights we assume $\beta_p(t) = \delta^t$ with $\delta = 0.1$. Remember that $\delta = 0.0$ means that the planner values future generations just as much as the original generation does. The set of parameters chosen for the benchmark exercises are summarized in Table 2.

**Table 2: Parameters**

| Parameters | Concept | Values |
|---|---|---|
| $\beta$ | individual discount factor | 0.29 |
| $\varepsilon$ | altruistic parameter | 0.35 |
| $\sigma$ | elasticity of substitution | 0.526 |
| $\lambda$ | per child time cost | 0.243 |
| $\delta$ | weight of social planner | 0.1 |

### 5.3 Results

The simulated model replicates a negative relationship between fertility and ability similar to the Brazilian data[10]. As shown in Figure 1, fertility per household falls from around 6 to 2.8 as earning abilities increase from 1 to 8.5. This negative relationship arises from the interplay of two opposite forces. On the one hand, individuals with higher abilities have a larger opportunity cost of raising children. On the other hand, they enjoy a larger benefit of having children when abilities are intergenerationally persistent. The effect of ability on the marginal cost dominates as long as the intergenerational persistence of ability is less than perfect and the IGES is larger than 1.

We now use the model to perform policy experiments. First, we study policies that directly restrict or encourage fertility. Second, we consider the effects of taxes and/or subsidies on family size.

### 5.4 Policy experiments

#### 5.4.1 Restricting fertility

Consider first the effects of policies limiting the fertility rate to be no more than $\widehat{n}$ children, where $\widehat{n} < \bar{n}$. We have proved that these policies reduce total utilitarian social welfare in Proposition 10 and reduce average social welfare in certain cases, in Propositions 12 and 13. We now consider average social welfare in an empirically plausible case.

The first panel of Figure 2 shows the effect of limiting fertility on steady state average ability, $\overline{\omega}$, and average income, $\overline{y}$. These two variables increase as the upper limit on fertility decreases. As predicted by CK, tighter fertility limits, which affect lower income groups more severely, increase average income and ability. These results seemingly provide support to family planning programs. However, they ignore the negative welfare consequences of limiting family size. The second panel of Figure 2 shows that steady state average social welfare, $\overline{W}^*$, average social welfare of all generations, $\overline{W}$, total social welfare of the initial generation, $W_0$, and total social welfare of all generations, $W$, consistently increase as the upper limit on fertility is relaxed.

Knowing that policies limiting fertility reduce social welfare, we also simulate policies that impose lower limits on fertility rates. An example of this policy is recent efforts by the Chinese government to induce educated unwed

[10] By construction, our calibration targets the mean and dispersion of fertilities but not the sign of the relationship between fertility and income.

women older than 27 to marry by stigmatizing them as "leftover women". The objective of these efforts seems to be twofold. To reduce pressures due to sex imbalances brought about by the One Child Policy, and to improve the quality of the workforce. These type of policies disproportionately affect the rich, or high ability individuals, because their unconstrained fertility is typically lower. The first panel of figure 3 shows that this policy improves average ability since high ability individuals have proportionally more high ability children. On the other hand, the policies reduce average income because individuals, especially those with high abilities, spend more time raising children and this effect dominates the effect of an improved ability distribution.

The second panel of figure 3 illustrates the impact of setting lower limits on four social welfare measures. In general all four welfare measures exhibit a decreasing trend as the lower bound increases although there are certain ranges in which total welfare increase slightly. For example, total social welfare has a tiny increase by 0.08 as the lower bound increases from 5 to 6. Further increases in the lower bound, say above 6, results in all welfare measures eventually plunging.

These results confirm the main message of the paper: in general fertility restrictions, on the poor or other groups, do not have strong theoretical support for improving social welfare.

#### 5.4.2 Taxes and subsidies on fertility

We next study the effect on social welfare of taxes and/or subsidies on fertility while preserving a government balanced budget. Specifically, consider a subsidy, $s$, that helps parents offset the costs of raising children as is the spirit of recent policies in Europe. We also allow $s$ to be negative in order to consider policies deterring parents from having children. The subsidy (tax) is financed by a lump sum tax (transfer) $T(s)$ imposed on every individual. Under this policy, individual's budget constraint becomes

$$c + (\lambda - s)\,\omega n \leq \omega - T(s)\,.$$

The governments' budget constraint is

$$T(s) = s \sum_{\omega} \omega n(\omega)\, \pi^*(\omega)\,.$$

Figure 4 shows the effects of this policy on fertility and individual welfare. The horizontal axis of every panel is the subsidy, $s$, ranging from -10% to

10%. The range of subsidies is such that they do not fully compensate for the cost of raising children, $s < \lambda$. The first panel shows that fertility rates decrease as taxes increase, say as $s$ drops from 0 to -0.1, while the effect is ambiguous for subsidies.

The ambiguous impact of a subsidy on fertility is because of the income effects of the lump sum transfer. In presence of taxes or subsidies the marginal cost of an additional child becomes $(\lambda - s)\,\omega^{1-\sigma}\left(1 - (\lambda - s)\,n - T\,(s)\,\omega^{-1}\right)^{-\sigma}$ while the marginal benefit is $\beta\,(1-\varepsilon)\,n^{-\varepsilon}E\left[U\left(\omega'\right)|\omega\right]$. Marginal benefit also depends on $s$ since $U\,(\omega)$ does although it is not clear from the expression how it moves with $s$. For marginal cost, when $s > 0$, an increase in $s$ decreases the marginal cost if $T\,(s)$ is unchanged. But the lump sum tax $T\,(s)$ imposed on everyone increases with $s$, which causes marginal cost to go up. The effect of $T\,(s)$ on marginal cost is small for high ability types, so for them the effect of $s$ on $MC$ tends to dominate that of $T\,(s)$ while the inverse tends to be true for low ability individuals. This explains what happens in the first panel of Figure 4 where fertility increases for the two highest ability types and decreases for other types as $s$ increases. This is because $MC$ of the two highest ability types decrease much faster than all other types' $MC$ as subsidy increases or possibly other types' $MC$ increase with the subsidy. Likewise, when $s < 0$, $s$ becomes more negative as taxes increase, $MC$ of high ability types increase more than that of low ability types. The numerical results show that everyone's fertility falls as taxes increase. Panel 2 shows a mixed policy consequences on individual welfare. A high subsidy benefits high ability individuals while harms other types, especially low ability individuals.

Figure 5 shows the effects on social welfare, defined as steady state average social welfare. Social welfare could be improved only when there is a very low tax, e.g. up to 2% of the time cost of raising every child. Otherwise, it is neither improved by taxes nor by subsidies.

### 5.4.3 Robustness checks

We now report the results of various robustness checks. For this purpose we change one parameter at a time while keeping all the other parameters at their benchmark values and study the policy effects of reducing fertility on various welfare measures. We find that the qualitative results obtained above are mostly robust for reasonable parameters although there exist parameters for which average steady state welfare, $\overline{W}^*$, improves with fertility restrictions. The set of parameters studied is further restricted by the need to have finite utility and concavity.

We find that the results are robust to setting $\sigma$ anywhere in the range 0.45 to $1^{11}$. When $\sigma \in [0.35, 0.45]$ , fertility restrictions could moderately improves steady state average welfare $\overline{W}^*$ but only when the limit is at a very high level, as illustrated in the first panel of figure 6 for $\sigma = 0.4$. The intuition for the increase in $\overline{W}^*$ is the following. First, a small $\sigma$ implies a high IGES and therefore a small gain of smoothing consumption through fertility choices. Thus fertility restrictions when $\sigma$ is small are less harmful to individual welfare in which case people's incentive to smooth consumption through fertility is relatively weak. Second, $\overline{W}^*$ could be slightly increased by a tighter fertility restriction only in the area near the unconstrained fertility choice of the low ability group. In that case, only fertility choices of low ability individuals are directly affected. Due to their high fertility rate and low welfare, tighter restrictions could significantly improve distribution without hurting other types. So the effect on distribution dominates that on individual welfare. In summary, when $\sigma$ is low fertility restrictions starting from a high level may increase average steady state welfare because they have a minor impact on individual welfare but relatively large effect on distribution.

The results are robust to setting $\beta$ in the range $[0.27, 0.4]^{12}$. Similar with $\sigma$, as $\beta$ is below 0.27, $\overline{W}^*$ may increase when the limit on fertility is large enough as illustrated in the second panel of figure 6 for the case $\beta = 0.2$. A low $\beta$ means that parents care little about future generations. As a response, they would have fewer kids, more consumption and lower marginal utility of consumption. In this case, fertility restrictions have a minor effect on individual welfare and, as a result, the change of the ability distribution is the dominant effect determining social welfare. However, this low degree of altruism also implies that the model predicts counterfactually low fertility rates. In particular, the simulated unrestricted fertility range is between 1.14 to 3.12 per household which is below the minimum of Brazilian's fertility data, 3.449. A similar result is obtained when $\varepsilon$ is particularly large, as illustrated in the third panel for $\varepsilon = 0.52$.

We also performed robustness checks for the cost of raising children $\lambda$ over the feasible range $(0.15, 0.32)$. The lower bound is required by the concavity of utility function while the upper bound is required to guarantee the labor supply to be nonnegative. The result is robust to all $\lambda \in (0.19, 0.32)$. When $\lambda$ is lower than 0.19, fertility of everyone is too high and population

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[11] To guarantee the concavity of the utility function and nonnegative utility, $1 > \sigma > \varepsilon$ is needed.

[12] Boundedness of utility requires $\beta < \lambda^{1-\varepsilon}$. Given $\lambda = 0.243$ and $\varepsilon = 0.35$, the upper bound for $\beta$ is 0.4 .

becomes infinite in the long run and total social welfare is not well defined.

## 6 Conclusion

This paper studies optimal fertility choices and fertility policies when children's earnings are random and parents are altruistic. We characterize equilibrium allocations in endowment economies with complete and incomplete markets. In the complete markets case, consumption and fertility are not deterministic as is the case when fertility is exogenous. This novel result is a natural consequence of a key insight provided by Barro and Becker (1989): when fertility is endogenous consumption is proportional to the net cost of raising a child. We generalize this result to the case of idiosyncratic shocks.

Our analysis shows that the incomplete markets models can replicate various features of the evidence on fertility and income distribution. For example, fertility decreases with ability and social mobility occurs in equilibrium. The negative relationship between ability and fertility arises in this framework due to the combination of four factors: incomplete markets, time cost of raising children, less than perfect persistence of abilities and an intergenerational elasticity of substitution larger than 1.

We further show that incomplete markets could lead to inefficiently high fertility rates when children are a net financial burden to parents. However, this feature of the equilibrium allocation does not imply that restricting fertility is welfare improving. On the contrary, we find that fertility restrictions are detrimental to every individual's welfare, even to those whose fertility choices are not directly affected, and therefore detrimental to social welfare if welfare is defined as classical (Bentham) utilitarianism. If social welfare is defined as average (Mills) utilitarianism, then fertility restrictions may improve social welfare but only if the distribution of abilities improves strongly. We also perform policy experiments using calibrated version of the model. These experiments suggest that, in general, fertility policies such as taxes or subsidies that affect the cost of raising children do not increase social welfare.

Our models abstracts from a number of aspects that are potentially important for fertility decisions such as bequests and human capital accumulation. Liao (2013) provides a model with human capital and find similar results using a calibrated deterministic model. We are extending our results to production economies in ongoing work (Cordoba et al. 2014).

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# Appendix

### Inefficiently high fertility in a closed economy

Section 2 shows that fertility could be inefficiently high in a small-open incomplete-markets economy. We now show that a similar could be obtained in a closed economy. Suppose the production function is $f(k, l)$ where $k$ is capital and $l$ is labor. In a closed economy, interest rates and wages are given by

$$r_t = f_1 (b_t, 1 - \lambda n_t) - \delta \text{ and } w_t = f_2 (b_t, 1 - \lambda n_t) .$$

where $\delta$ is the rate of depreciation. Substituting out prices, the first order condition with respect to fertility becomes

$$u' (c_t) [f_2 (b_t, l_t) \lambda + b_{t+1}] \leq \Phi' (n_t) U_{t+1} (b_{t+1}) \tag{24}$$

Furthermore, the corresponding Euler equation is

$$u' (c_t) n_t \geq \Phi (n_t) u' (c_{t+1}) (f_1 (b_{t+1}, l_{t+1}) + 1 - \delta)$$

If bequest constraints do not bind, then steady state fertility is determined by

$$1 = \beta (f_1 (b, 1 - \lambda n) + 1 - \delta) n^{-\varepsilon} \tag{25}$$

If bequest constraint binds, then interior solution of fertility is determined by (24). The following proposition provides a case in which steady state fertility is larger when bequest constraints binds.

**Proposition A.1.** Assume the economy is closed and $f(k, l) = Ak + Bl$ where $A$, $B > 0$. If the marginal productivity of capital $A$ is large enough such that $\frac{\beta(1-\varepsilon)}{1-\sigma} (A + 1 - \delta) \lambda^{\varepsilon} > 1 + \frac{\sigma-\varepsilon}{1-\sigma} \beta \lambda^{\varepsilon-1}$, then steady state fertility when the constraint $b \geq \underline{b}$ binds is higher than the unconstrained fertility level.

**Proof.** Equation (24) at steady state, together with the budget constraint and equilibrium prices results in

$$u' (c) (f_2 (b, l) \lambda + b) = \frac{\Phi' (n)}{1 - \Phi (n)} [f (b, 1 - \lambda n) + (1 - \delta) b - nb] \frac{u' (c)}{1 - \sigma}.$$

Using the specific production function, utility function and altruistic function specified above and collect terms, this equation becomes

$$\begin{aligned} B\lambda &= \frac{\beta(1-\varepsilon)}{1-\sigma}\left[B+(A+1-\delta)b\right]n^{-\varepsilon} - b - (b+B\lambda)\frac{\sigma-\varepsilon}{1-\sigma}\beta n^{1-\varepsilon} \\ &= \frac{\beta(1-\varepsilon)}{1-\sigma}Bn^{-\varepsilon} + b\left[\frac{\beta(1-\varepsilon)}{1-\sigma}(A+1-\delta)n^{-\varepsilon} - 1 - \frac{\sigma-\varepsilon}{1-\sigma}\beta n^{1-\varepsilon}\right] \qquad (26) \\ &\quad - B\lambda\frac{\sigma-\varepsilon}{1-\sigma}\beta n^{1-\varepsilon}. \end{aligned}$$

Denote the right hand side of this equation by $RHS(n)$. Notice that $RHS'(n) < 0$, $\lim_{n\to 0} RHS(n) = \infty$, and $\lim_{n\to\infty} RHS(n) = -\infty$. An interior solution for fertility exists if and only if $RHS\left(\frac{1}{\lambda}\right) < B\lambda$, e.g.

$$1 + \frac{\sigma-\varepsilon}{1-\sigma}\beta\lambda^{\varepsilon-1} > \frac{\beta(1-\varepsilon)}{1-\sigma}(A+1-\delta)\lambda^{\varepsilon} - B\lambda\frac{1-\beta\lambda^{\varepsilon-1}}{b}.$$

By the assumption, the term in the square bracket of (26) is positive. As a result, an exogenous increase in $b$ increases the right hand side but does not affect the left hand side, and thus leads to a bigger steady state fertility. Hence if $b$ is restricted to be higher than the unconstrained optimal choice $b^*$, e.g. $b^* < \underline{b}$, then the steady state fertility in the unconstrained case (complete markets) is smaller than that in the constrained case (incomplete markets). ■

**Proof of Proposition 1.** For the case when $(1+r)\lambda\omega > \omega$, see Barro and Becker (1989). In this proof, we focus on the case $(1+r)\lambda\omega < \omega$. First order conditions for bequests and fertility are:

$$u'(c_t)n_t = \Phi(n_t)U'_{t+1}(b_{t+1})$$

$$u'(c_t)(\lambda\omega + b_{t+1}) \leq \Phi'(n_t)U_{t+1}(b_{t+1}) \quad \text{with equality if } n_t < \bar{n}.$$

$\Phi'(0) = \infty$ excludes the possibility of zero children. Using the envelope condition $U'_t(b_t) = u'(c_t)(1+r)$, the first condition with respect to bequest becomes:

$$u'(c_t)n_t = \Phi(n_t)u'(c_{t+1})(1+r)$$

Therefore,

$$\begin{aligned} U_t(b_t) &= u(c_t) + \Phi(n_t)U_{t+1}(b_{t+1}) \\ &\geq u(c_t) + \frac{\Phi(n_t)}{\Phi'(n_t)}u'(c_t)(\lambda\omega + b_{t+1}) \\ &= u(c_t) + \frac{\Phi(n_t)}{\Phi'(n_t)n_t}u'(c_t)(\omega + (1+r)b_t - c_t) \end{aligned}$$

Forward this inequality one period and use the specific functional forms for utility and altruistic functions to obtain

$$\begin{aligned} U_{t+1}\left(b_{t+1}\right) &\geq u\left(c_{t+1}\right)+\frac{\Phi\left(n_{t+1}\right)}{\Phi'\left(n_{t+1}\right)n_{t+1}}u'\left(c_{t+1}\right)\left(\omega+\left(1+r\right)b_{t+1}-c_{t+1}\right) \\ &= u'\left(c_{t+1}\right)\frac{1}{1-\varepsilon}\left[\frac{\sigma-\varepsilon}{1-\sigma}c_{t+1}+\left(1+r\right)b_{t+1}+\omega\right] \\ &> u'\left(c_{t+1}\right)\frac{1}{1-\varepsilon}\left(1+r\right)\left(\lambda\omega+b_{t+1}\right) \\ &= u'\left(c_{t}\right)\frac{1}{\Phi'\left(n_{t}\right)}\left(\lambda\omega+b_{t+1}\right) \end{aligned}$$

for all $n_t$. The last inequality sign is due to the assumption that $(1+r)\lambda\omega < \omega$. Hence $U_{t+1}(b_{t+1})\Phi'(n_t) > u'(c_t)(\lambda\omega+b_{t+1})$ for all $n_t \in [0,\bar{n}]$ implies $n_t = \bar{n}$ is optimum. Consumption growth follows from the Euler equation. ■

**Proof of Proposition 3.** When contingent assets are available, first order condition to fertility $n_t$ is

$$\left[\lambda\omega_t+\sum_{i=1}^{K}q(\omega_t,\omega_i)b_{t+1}(\omega^t,\omega_i)\right]u'\left(c_t\right)=\beta\left(1-\varepsilon\right)n_t^{-\varepsilon}E_t\left[V_{t+1}\left(b_{t+1};\omega^{t+1}\right)|\omega_t\right] \tag{27}$$

where consumption is given by (7)

$$c_t=\omega_t\left(1-\lambda n_t\right)+b_t-n_t\sum_{i=1}^{K}q(\omega_t,\omega_i)b_{t+1}(\omega^t,\omega_i)$$

first order condition to $b_{t+1}(\omega^t,\omega_i)$ :

$$u'\left(c_t\right)q(\omega_t,\omega_i)n_t=\beta n_t^{1-\varepsilon}M\left(\omega_i,\omega_t\right)\frac{\partial V_{t+1}\left(b_{t+1};\omega^{t+1}\right)}{\partial b_{t+1}}$$

which together with envelop condition $\frac{\partial V_t\left(b_t;\omega^t\right)}{\partial b_t}=u'\left(c_t\right)$ and the actuarially fair price of $b_{t+1}(\omega^t,\omega_i)$ gives the Euler equation

$$u'\left(c_t\right)=\beta n_t^{-\varepsilon}\left(1+r\right)u'\left(c_{t+1}\right) \tag{28}$$

Notice that all children from the same family enjoy the same consumption which is independent of $\omega_{t+1}$. Substituting (27) into the objective function and use the budget constraint,

$$V_t\left(b_t;\omega^t\right)=u'\left(c_t\right)\left[\frac{c_t\left(\sigma-\varepsilon\right)}{\left(1-\sigma\right)\left(1-\varepsilon\right)}+\frac{1}{1-\varepsilon}\left(\omega_t+b_t\right)\right]$$

Forward this equation by one period, then use it and (28) to rewrite (27) as,

$$\lambda\omega_t(1+r)+(1+r)\sum_{i=1}^{K}q(\omega_t,\omega_i)b_{t+1}(\omega^t,\omega_i)=\frac{\sigma-\varepsilon}{1-\sigma}c_{t+1}+E\left[b_{t+1}(\omega^t,\omega_i)|\omega_t\right]+E\left(\omega_{t+1}|\omega_t\right)$$

After some manipulations, the consumption of every child endowed with ability $\omega'$ given parental ability $\omega$ is

$$c_{t+1}=c\left(\omega_t\right)=\frac{1-\sigma}{\sigma-\varepsilon}\left[\lambda\omega_t\left(1+r\right)-E\left(\omega_{t+1}|\omega_t\right)\right]$$

for all $t$ and $\omega_t$. Furthermore, using (28) fertility can be solved as:

$$n_t=n\left(\omega_{t-1},\omega_t\right)=\beta^{\frac{1}{\varepsilon}}\left(1+r\right)^{\frac{1}{\varepsilon}}\left(\frac{\lambda\omega_{t-1}\left(1+r\right)-E\left(\omega_t|\omega_{t-1}\right)}{\lambda\omega_t\left(1+r\right)-E\left(\omega_{t+1}|\omega_t\right)}\right)^{\frac{\sigma}{\varepsilon}}$$

When $\lambda\omega_t\left(1+r\right)<E\left(\omega_{t+1}|\omega_t\right)$, the proof follows the same logic with the second part of Proposition 1. ■

**Proof of Proposition 4.** We first show that there exists a solution $U\left(\cdot\right)$ that solves the functional equation (13). Define a set of mappings.

$$S=\left\{U:\Omega\rightarrow\mathbb{R}|\left\|U\right\|\leq M\right\}$$

where $M=\frac{u(\omega_K)}{1-\beta\lambda^{\varepsilon-1}}$, and $\|\cdot\|$ is the sup norm. $S$ is a complete metric space. Define operator $T$ as

$$TU\left(\omega\right)\equiv\max_{0\leq\ n\ \leq\frac{1}{\lambda}}u\left(\left(1-\lambda n\right)\omega\right)+\beta n^{1-\varepsilon}E\left[U\left(\omega'\right)|\omega\right] \tag{29}$$

for all $\omega\in\Omega$ and $U\in S$. Given $U\left(\cdot\right)$ and $\omega$, the right hand side of (29) has a solution that attains the maximum. First show that $T$ is a contraction. It suffices to show that $T$ satisfies two properties, monotonicity and discounting. Standard argument can show that given $U$ and $\tilde{U}\in S$ satisfying $U$ $\left(\omega\right)\leq\tilde{U}\left(\omega\right)$ for all $\omega\in\Omega$, $TU\ \left(\omega\right)\leq T\tilde{U}\left(\omega\right)$ for all $\omega\in\Omega$. The following arguments prove discounting property holds. For any given constant $b$,

$$\begin{aligned}T\left(U\left(\omega\right)+b\right)&=\max_{0\leq\ n\ \leq\frac{1}{\lambda}}u\left(\left(1-\lambda n\right)\omega\right)+\beta n^{1-\varepsilon}E\left[U\left(\omega'\right)+b|\omega\right]\\&\leq\max_{0\leq\ n\ \leq\frac{1}{\lambda}}u\left(\left(1-\lambda n\right)\omega\right)+\beta n^{1-\varepsilon}E\left[U\left(\omega'\right)|\omega\right]+\beta b\left(\frac{1}{\lambda}\right)^{1-\varepsilon}\\&=TU\left(\omega\right)+\beta\lambda^{\varepsilon-1}b\end{aligned}$$

$\beta\lambda^{\varepsilon-1} < 1$ by assumption. By Contraction Mapping Theorem, there exists a unique fixed point $U : \Omega \to \mathbb{R}$ that solves the functional equation $TU = U$. The existence of a solution $U(\cdot)$ has been proved. Next we show $U(\omega_0) = V_0^*(\omega_0)$ for all $\omega_0 \in \Omega$, that is to show $U(\omega_0)$ is the supremum of the sequential problem (12) for any given $\omega^0 = \omega_0$.

$$\begin{aligned}
U(\omega_0) &= \max_{n\in[0,\frac{1}{\lambda}]} u(\omega_0(1-\lambda n)) + \beta n^{1-\varepsilon} E_0[U(\omega_1)|\omega_0] \\
&\geqslant u(\omega_0(1-\lambda n_0(\omega_0))) + E_0 \begin{pmatrix} \beta n_0(\omega_0)^{1-\varepsilon} u\left(\omega_1\left(1-\lambda n_1\left(\omega^1\right)\right)\right) \\ +\beta^2 n_0(\omega_0)^{1-\varepsilon} E_0 n_1\left(\omega^1\right)^{1-\varepsilon} E_1[U(\omega_2)|\omega_1] \end{pmatrix} \\
&\geqslant \cdots \\
&\geqslant E_0 \sum_{t=0}^{T} \beta^t \prod_{j=0}^{t-1} n_j\left(\omega^j\right)^{1-\varepsilon} u\left(\omega_t\left(1-\lambda n_t\left(\omega^t\right)\right)\right) + \beta^{T+1} E_0 \prod_{j=0}^{T} n_j\left(\omega^j\right)^{1-\varepsilon} U(\omega_{T+1})
\end{aligned}$$

for all feasible plan $\left\{n_t\left(\omega^t\right)\right\}_{t=0}^{\infty}$. Let $\prod_{j=0}^{-1} n_j\left(\omega^j\right)^{1-\varepsilon} = 1$. The last term

$$\beta^{T+1} \prod_{j=0}^{T} n_j\left(\omega^j\right)^{1-\varepsilon} U(\omega_{T+1}) \leq \left(\beta\lambda^{\varepsilon-1}\right)^{T+1} \frac{u(\omega_K)}{1-\beta\lambda^{\varepsilon-1}}$$

The right hand side of this inequality converges to 0 as $T$ goes to infinite. Hence for all feasible plan $\left\{n_t\left(\omega^t\right)\right\}_{t=0}^{\infty}$

$$U(\omega_0) \geqslant E_0 \sum_{t=0}^{\infty} \beta^t \prod_{j=0}^{t-1} n_j\left(\omega^j\right)^{1-\varepsilon} u\left(\omega_t\left(1-\lambda n_t\left(\omega^t\right)\right)\right) \tag{30}$$

Given $\varepsilon > 0$, choosing a sequence of positive real numbers $\{\delta_t\}_{t=1}^{\infty}$ such that $\sum_{t=0}^{\infty} \left(\beta\lambda^{\varepsilon-1}\right)^t \delta_t \leq \varepsilon$. Let $n^*(\omega_t)$ be the solution that attains $U(\omega_t)$, then for all $t$

$$U(\omega_t) < u(\omega_t(1-\lambda n^*(\omega_t))) + \beta n^*(\omega_t)^{1-\varepsilon} E_t[U(\omega_{t+1})|\omega_t] + \delta_t$$

Starting from period 0, iteratively substituting the value function $U(\omega_{t+1})$ into the above inequality shows that for all $\omega_0$,

$$\begin{aligned}
U(\omega_0) < {} & E_0 \sum_{t=0}^{T} \beta^t \prod_{j=0}^{t-1} n^*(\omega_j)^{1-\varepsilon} u(\omega_t(1-\lambda n^*(\omega_t))) + \beta^{T+1} E_0 \prod_{j=0}^{T} n^*(\omega_j)^{1-\varepsilon} U(\omega_{T+1}) \\
& + E_0 \sum_{t=0}^{T} \beta^t \prod_{j=0}^{t-1} n^*(\omega_j)^{1-\varepsilon} \delta_t
\end{aligned}$$

The choice of $\{\delta_t\}$ guarantees that the last term is no more than $\varepsilon$ as $T \to \infty$. We have shown that

$$\lim_{T\to\infty} \beta^{T+1} E_0 \prod_{j=0}^{T} n^*(\omega_j)^{1-\varepsilon} U(\omega_{T+1}) = 0$$

Hence for any given $\varepsilon > 0$, there exists a feasible plan $\left\{n_t\left(\omega^t\right)\right\}_{t=0}^{\infty} = \{n^*(\omega_t)\}_{t=0}^{\infty}$ such that

$$U(\omega_0) < E_0 \sum_{t=0}^{T} \beta^t \prod_{j=0}^{t-1} n^*(\omega_j)^{1-\varepsilon} u(\omega_t(1-\lambda n^*(\omega_t))) + \varepsilon \tag{31}$$

By (30) and (31),

$$\begin{aligned} U(\omega_0) &= \sup_{\{n_t(\omega^t)\}_{t=0}^{\infty} \in \left[0, \frac{1}{\lambda}\right]} E_0 \sum_{t=0}^{\infty} \beta^t \prod_{j=0}^{t-1} n_j\left(\omega^j\right)^{1-\varepsilon} u\left(\omega_t\left(1-\lambda n_t\left(\omega^t\right)\right)\right) \\ &= \sup_{\left\{\hat{N}_{t+1}(\omega^{t-1},\omega_t)\right\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta_t^t \hat{N}\left(\omega^{t-1}\right)^{1-\varepsilon} u\left(\omega_t\left(1-\lambda \frac{\hat{N}_{t+1}\left(\omega^{t-1},\omega_t\right)}{\hat{N}_t\left(\omega^{t-1}\right)}\right)\right) \end{aligned}$$

Therefore

$$U(\omega_0) = V^*(\omega_0)$$

■

**Proof of Proposition 5.** (i) In this case, equation (14) can be written as $\frac{n(\omega)^{\varepsilon}}{(1-\lambda n(\omega))^{\sigma}} = A\omega^{\sigma-1}$ where $A = \frac{\beta(1-\varepsilon)}{\lambda} E[u(\omega')]$ is a constant. Using the implicit function theorem, it follows that $n'(\omega) = -\frac{(1-\sigma)/\omega}{\frac{\varepsilon}{n(\omega)} + \frac{\lambda\sigma}{1-\lambda n(\omega)}} < 0$. (ii) In deterministic case, $\omega' = \omega$. Equation (16) simplifies to:

$$n^{*\varepsilon} = \beta\left(\frac{1}{\lambda} + \frac{\sigma-\varepsilon}{1-\sigma}\left(\frac{1}{\lambda} - n^*\right)\right) \tag{32}$$

The left hand side of equation (32) is strictly increasing in $n^*$ while the right hand side is strictly decreasing in $n^*$. Obviously $n^* > 0$. An interior solution with $n^* < 1/\lambda$ exists since $\lambda^{1-\varepsilon} > \beta$. (iii) Let $n^*$ denotes the optimal fertility given $\omega$. Plug functional form of $u(\cdot)$ into equation (15)

$$U(\omega) = h(n^*)\,\omega^{1-\sigma} \tag{33}$$

where

$$h(n^*) \equiv \frac{1}{1-\sigma}(1-\lambda n^*)^{1-\sigma} + \frac{1}{1-\varepsilon}\lambda n^*(1-\lambda n^*)^{-\sigma} \tag{34}$$

We make a guess on the value function and let it take the form: $U(\omega) = A\omega^{1-\sigma}$ where $A$ is a constant, independent of $\omega$. Equating this guess with (33) results in:

$$A = h(n^*) \tag{35}$$

Thus, in order for $A$ to be independent of $\omega$, we must verify that the results $n^*$ is independent of $\omega$. Notice that,

$$E\left[U\left(\omega'\right)|\omega\right] = E\left[A\omega'^{1-\sigma}|\omega\right] = A\omega^{1-\sigma}e^{\frac{(1-\sigma)^2\sigma_\varepsilon^2}{2}}$$

The last equality holds because the assumption that $\omega'$ is lognormal distributed with $\ln\omega$ and $\sigma_\varepsilon$ as the mean and variance of $\ln\omega'$. Plug this equality into (14) to obtain:

$$\lambda(1-\lambda n^*)^{-\sigma}\omega^{1-\sigma} = A\beta(1-\varepsilon)n^{*-\varepsilon}e^{\frac{(1-\sigma)^2\sigma_\varepsilon^2}{2}}\omega^{1-\sigma}$$

$\omega$ cancels out of this equation and therefore $n^*$ is independent of $\omega$ confirming our guess. This expression together with (34) and (35) gives a rule to solve the optimal fertility $n^*$.

$$\frac{\lambda(1-\sigma)}{\beta(1-\varepsilon)n^{*-\varepsilon}}e^{-\frac{(1-\sigma)^2\sigma_\varepsilon^2}{2}} = 1-\lambda n^* + \frac{n^*}{1-\varepsilon}\lambda(1-\sigma)$$

Manipulate terms

$$\frac{\lambda(1-\sigma)n^{*\varepsilon}}{\beta(1-\varepsilon)}\left[e^{-\frac{(1-\sigma)^2\sigma_\varepsilon^2}{2}} - \beta n^{*1-\varepsilon}\right] = 1-\lambda n^*$$

The solution of $n^*$ does not depend on $\omega$ which confirms the guess on $U(\omega)$. In case (ii) and (iii), fertility is independent of ability. ■

**Proof of Poposition 6.** (i) If fertility is exogenously the same for every individual,

$$N_{t+1} = N_t\sum_{\omega_i\in\Omega} n\pi_t(\omega_i) = N_t n$$

By equation (18),

$$\pi_{t+1}(\omega_j) = \frac{nN_t}{N_{t+1}}\sum_{\omega_i\in\Omega}\pi_t(\omega_i)M(\omega_j,\omega_i) = \sum_{\omega_i\in\Omega}\pi_t(\omega_i)M(\omega_j,\omega_i)$$

Taking limit to both sides of the expression with $\pi$, we get

$$\pi^*(\omega_j) = \lim_{t\to\infty}\sum_{\omega_i\in\Omega}\pi_t(\omega_i)M(\omega_j,\omega_i) = \sum_{\omega_i\in\Omega}\pi^*(\omega_i)M(\omega_j,\omega_i)$$

Hence $\pi^*(\cdot) = \mu(\cdot)$ is the invariant distribution of $M$. (ii) $M(\cdot\,,\omega_i)$ is independent of $\omega_i$ implies $M(\omega_j,\omega_i) = M(\omega_j,:)$ for every $\omega_j \in \Omega$. By (18),

$$\begin{aligned}
\pi_{t+1}(\omega_j) &= \frac{M(\omega_j,:)}{N_{t+1}} \sum_{\omega_i\in\Omega} n(\omega_i)\,\pi_t(\omega_i)\,N_t \\
&= M(\omega_j,:) = \sum_{\omega_i\in\Omega} \pi_t(\omega_i)\,M(\omega_j,:)
\end{aligned}$$

for all $t \geq 0$, these equalities imply $\pi_{t+1}(\omega_j) = \sum\limits_{\omega_i\in\Omega} \pi_t(\omega_i)\,M(\omega_j,\omega_i)$ and hence $\pi^*(\omega_j) = \pi_{t+1}(\omega_j) = \mu(\omega_j) = M(\omega_j,:)$ for all $\omega_j$. ■

**Proof of Proposition 7.** Part (i) directly applies Proposition 5(i) and Proposition 6 (ii). For part (ii), we can apply Proposition 5 (ii), in which fertility is independent of ability when $M$ is identity. We use this result to prove the distribution of every period as well as the limit distribution is the same with the initial one.

$$\begin{aligned}
\pi_{t+1}(\omega_j) &= \frac{N_t}{N_{t+1}} \sum_{\omega_i\in\Omega} n(\omega_i)\,\pi_t(\omega_i)\,M(\omega_j,\omega_i) \\
&= \frac{N_t n}{N_{t+1}} \sum_{\omega_i\in\Omega} \pi_t(\omega_i)\,M(\omega_j,\omega_i) \\
&= \sum_{\omega_i\in\Omega} \pi_t(\omega_i)\,M(\omega_j,\omega_i) = \pi_t(\omega_j)
\end{aligned}$$

The last equality holds because $M$ is identity matrix. Therefore $\pi^*(\omega) = \pi_t(\omega) = \pi_0(\omega)$ for all $\omega$ and all $t$. Part (iii) follows Proposition 5 (iii). The conditional variance of $\ln\omega_t$ diverges to infinite because $\ln\omega_t = \ln\omega_0 + \sum\limits_{i=1}^{t} \varepsilon_i$, $E(\ln\omega_t|\omega_0) = \ln\omega_0$, $Var(\ln\omega_t|\omega_0) = t^2\sigma_\varepsilon^2$ and $\lim_{t\to\infty} Var(\ln\omega_t|\omega_0) = \infty$. ■

**Proof of Proposition 9.** Notice that

$$\begin{aligned}
U(\omega) &= \max_{n_t\in[0,1/\lambda]} u((1-\lambda n)\,\omega) + \beta n^{1-\varepsilon} E\left[U(\omega')\,|\omega\right] \\
&\geq \max_{[\underline{n}(\omega),\overline{n}(\omega)]} u((1-\lambda n)\,\omega) + \beta n^{1-\varepsilon} E\left[U(\omega')\,|\omega\right] := U^1(\omega') \\
&\geq \max_{[\underline{n}(\omega),\overline{n}(\omega)]} u((1-\lambda n)\,\omega) + \beta n^{1-\varepsilon} E\left[U^1(\omega')\,|\omega\right] := U^2(\omega') \\
&\;.. \\
&\geq \max_{[\underline{n}(\omega),\overline{n}(\omega)]} u((1-\lambda n)\,\omega) + \beta n^{1-\varepsilon} E\left[U^r(\omega')\,|\omega\right] = U^r(\omega')
\end{aligned}$$

where the first inequality is strict if a constraint is binding for any particular $\omega$, the remaining inequalities follow from the contraction mapping recursion, and the final inequality uses the contraction mapping theorem. Furthermore, a strict inequality for a particular $\omega$ translates into a strict inequality for all $\omega' s$ since $M$ is a regular Markov chain meaning that, regardless of initial ability there is positive probability that someone in the dynasty will reach a binding state in finite time. The second part of the proposition follows because fertility restrictions do not change the marginal costs of having children but it decreases the marginal benefits by reducing $U(\omega)$ for all $\omega$ (see equation (14)). Hence an upper bound of fertility makes people have fewer children than the unrestricted case as long as fertility upper bounds affect at least one of those types. ■

**Proof of Proposition 10.** Let $N_t^r(\omega)$ be the size of population with ability $\omega$ at time $t$ in presence of fertility upper limits. By Proposition 9, for all $\omega_j$

$$\begin{aligned} N_1(\omega_j) &= \sum_{\omega_i \in \Omega} n(\omega_i) N_0(\omega_i) M(\omega_j, \omega_i) \\ &\geq \sum_{\omega_i \in \Omega} n^r(\omega_i) N_0^r(\omega_i) M(\omega_j, \omega_i) = N_1^r(\omega_j) \end{aligned}$$

where initial population is not affected by policies $N_0(\omega_i) = N_0^r(\omega_i)$. Given the inequality $N_1(\omega_j) \geq N_1^r(\omega_j)$, an inductive argument shows

$$N_{t+1}(\omega_j) = \sum_{\omega_i \in \Omega} n(\omega_i) N_t(\omega_i) M(\omega_j, \omega_i) \geq \sum_{\omega_i \in \Omega} n^r(\omega_i) N_t^r(\omega_i) M(\omega_j, \omega_i) = N_{t+1}^r(\omega_j)$$

for all $\omega_j$ and all $t \geq 0$. Apply this result,

$$\begin{aligned} W^r(\beta_p) &= \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U^r(\omega) N_t^r(\omega) \\ &\leq \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U(\omega) N_t^r(\omega) \\ &\leq \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U(\omega) N_t(\omega) = W(\beta_p) \end{aligned}$$

■

**Proof of Proposition 12.** When $M(\cdot, \omega)$ is independent of $\omega$, the proof of Proposition 6 (ii) gives

$$\pi_{t+1}(\omega_j) = M(\omega_j, :) \quad \text{for all } t \geq 0$$

Initial ability distribution $\pi_0(\omega)$ is given and not affected by the policy in place. So restriction of a fertility upper bound only reduces individual utility by Proposition 9 but does not affect the ability distribution of any period. It decreases social welfare defined by (22). ■

**Proof of Proposition 13.** This Proposition relies on Proposition 7 (ii)'s results, $n(\omega) = n$ and $\pi_t(\omega) = \pi^*(\omega) = \pi_0(\omega)$ when $M$ is an identity. Similar with Proposition 12, restriction does not alter distribution, which together with Proposition 9, finishes the proof. ■

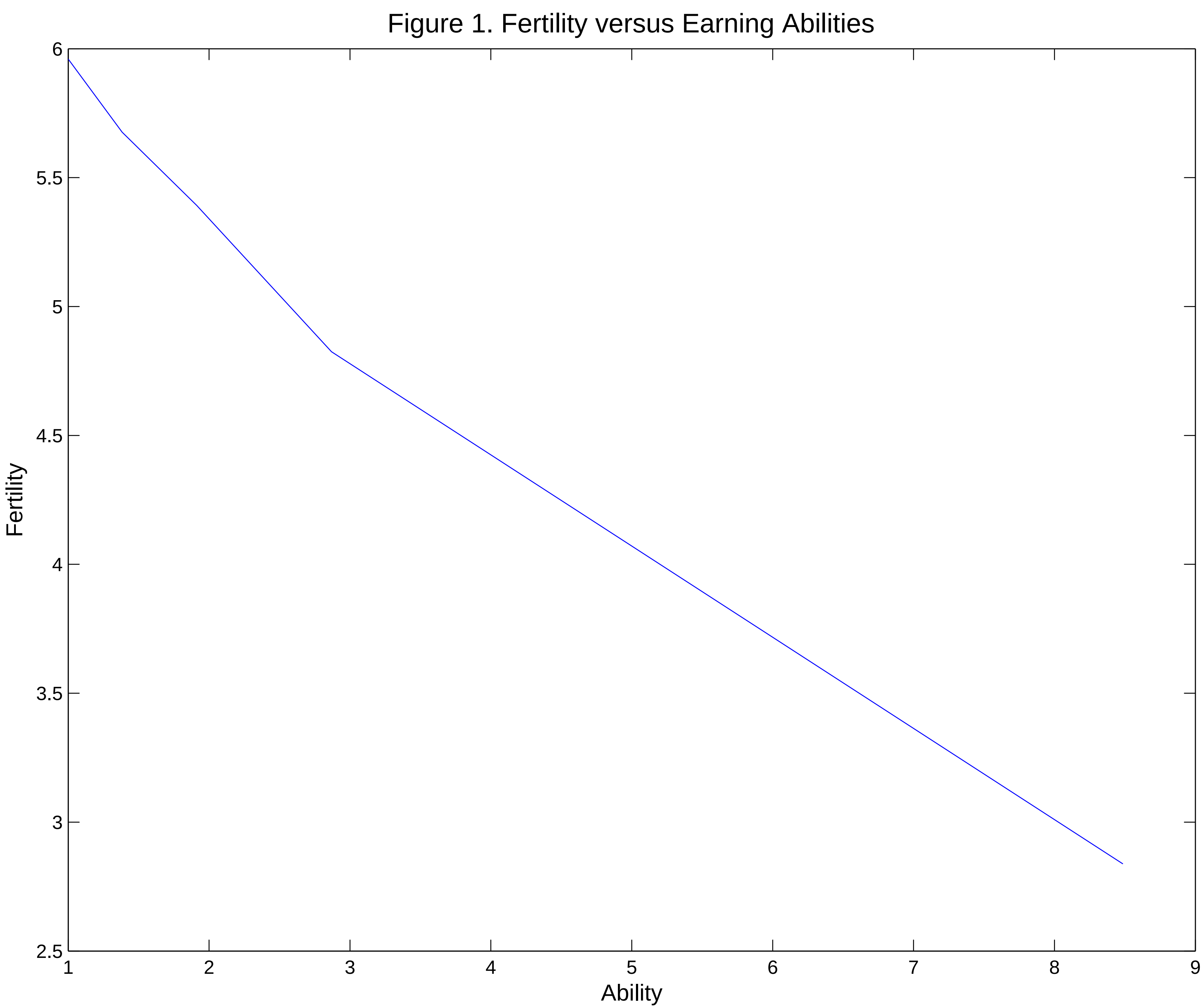

Figure 1. Fertility versus Earning Abilities
Fertility
Ability
6
5.5
5
4.5
4
3.5
3
2.5
1
2
3
4
5
6
7
8
9

Figure 2. Policy Effects of Reducing Fertility

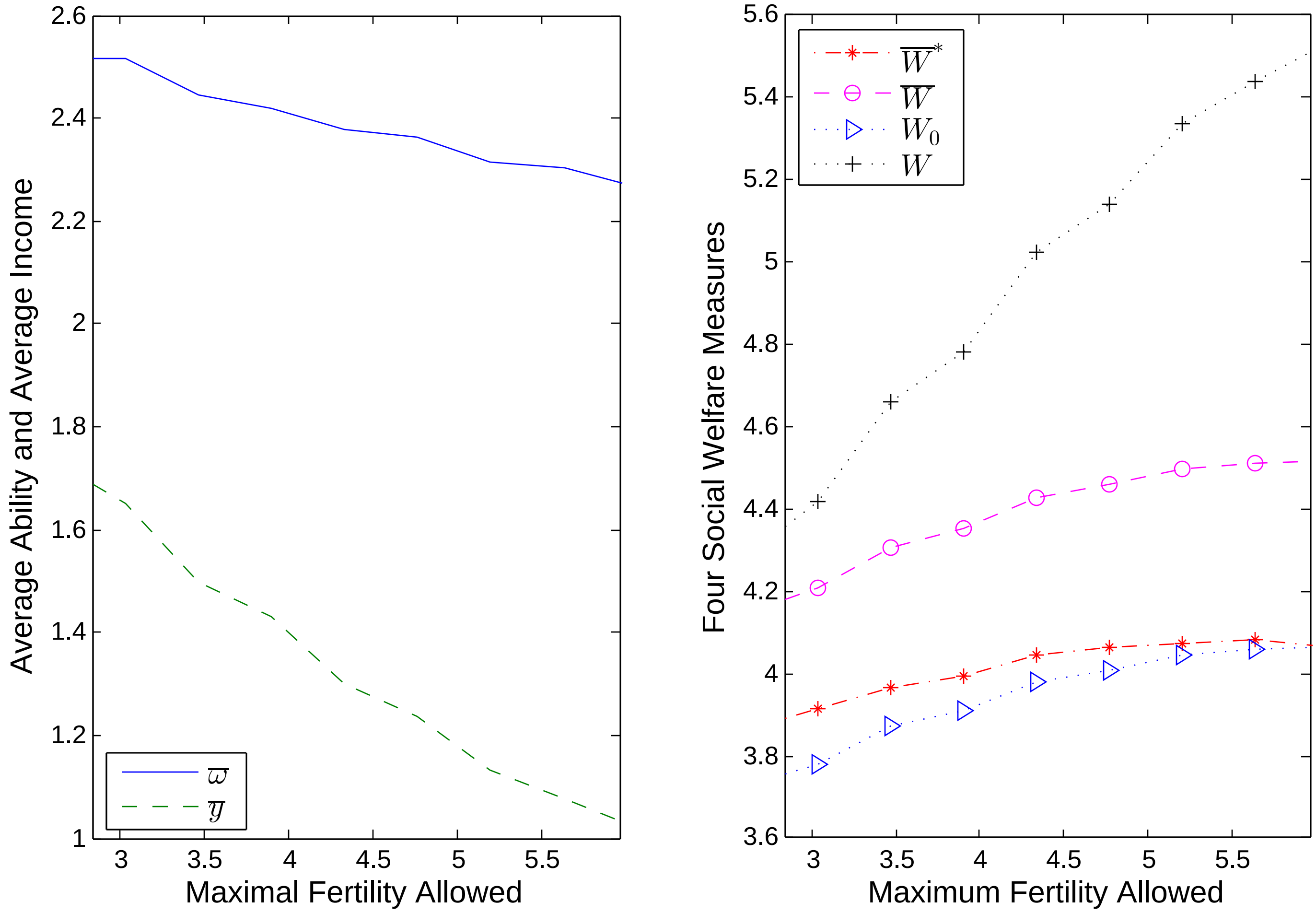

Figure 3. Policy Effects of Raising Fertility

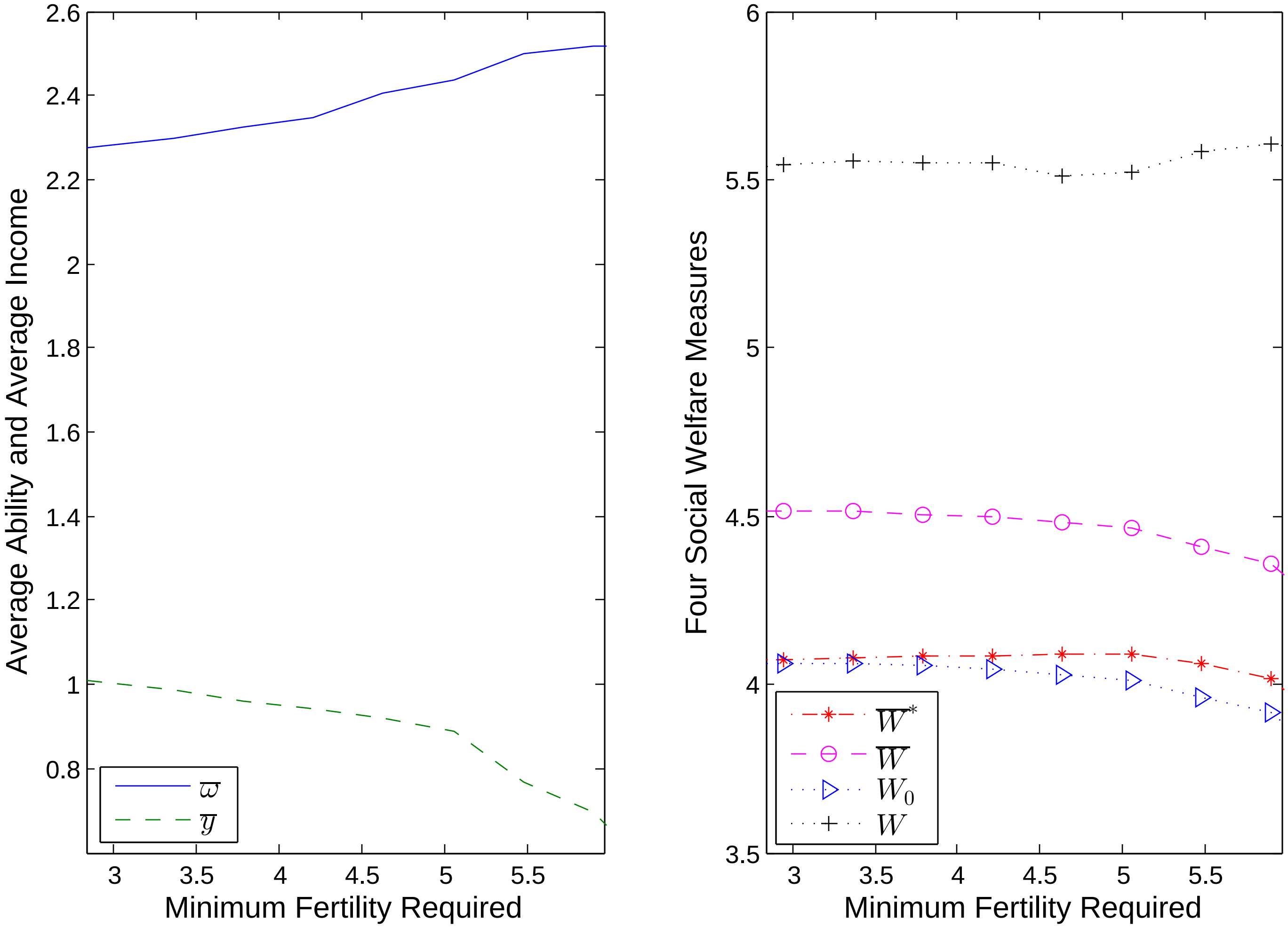

Figure 4. Policy Effects of Taxes and Subsidies

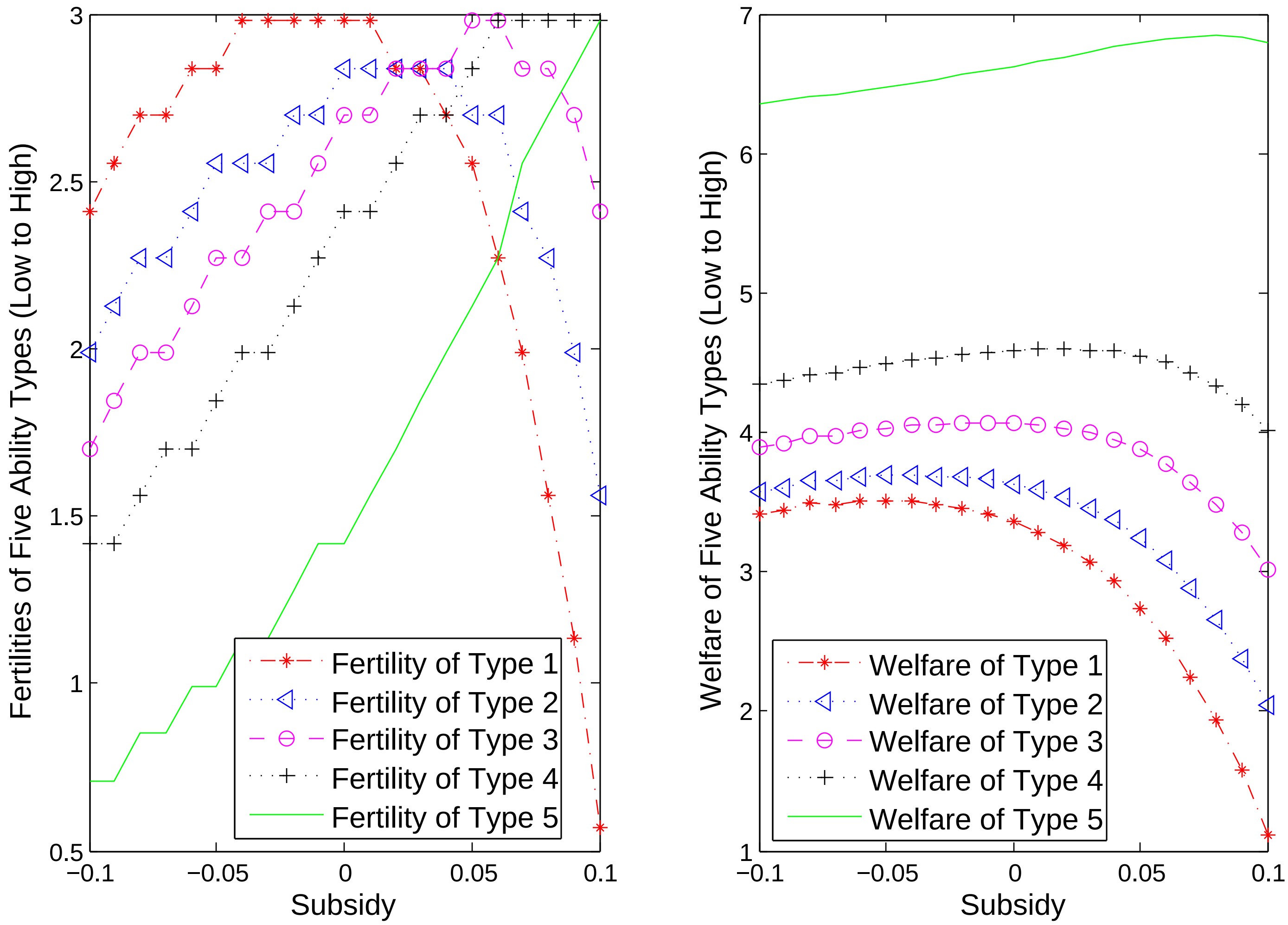

Figure 5. Effects of Taxes and and Subsidies on Steady State Average Social Welfare

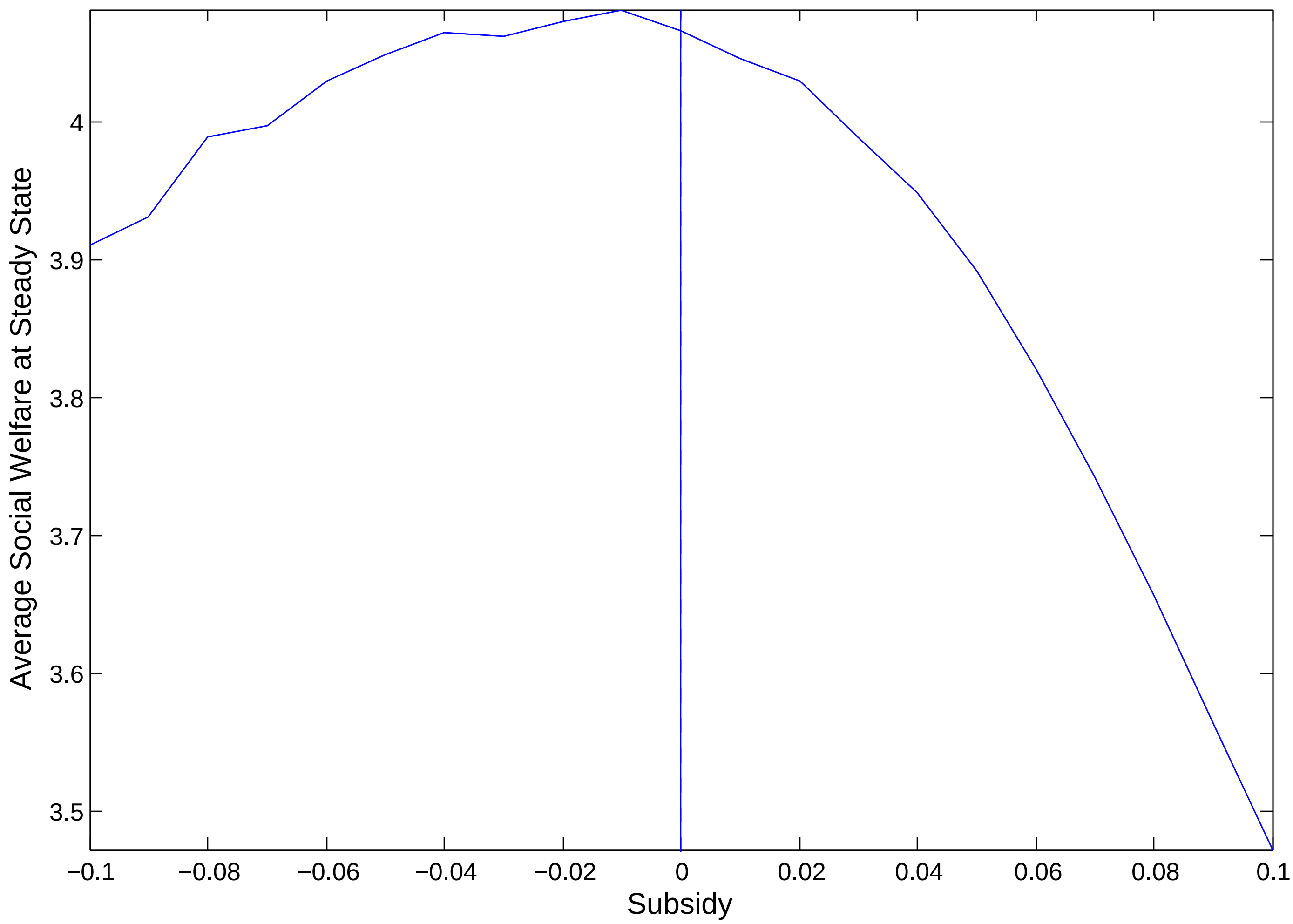

Figure 6. Robustness Checks