On Languages with Very High Information Content

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The main result of this paper is the following: Any language in ESPACE that is bounded truth-table reducible in polynomial time to a set with very high space-bounded Kolmogorov complexity must be bounded truth-table reducible in polynomial time to a sparse set.

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On Languages with Very High Information Content

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Abstract

The main result of this paper is the following: Any language in $\text{ESPACE}$ that is bounded truth-table reducible in polynomial time to a set with very high space-bounded Kolmogorov complexity must be bounded truth-table reducible in polynomial time to a sparse set.

Introduction

The subject of this paper is the class of languages that are bounded reducible in polynomial time to languages with essentially maximal information content. Specifically, it is shown that any language recognizable in exponential space (denoted $\text{ESPACE}$) that is bounded truth-table reducible in polynomial time to a set with very high space-bounded Kolmogorov complexity must be bounded truth-table reducible in polynomial time to a sparse set. Using a recent result of Ogiwara and Watanabe, this implies that no language with very high space-bounded Kolmogorov complexity can be $\leq_{\text{NT}}^P$-hard for $\text{NP}$, unless $P=\text{NP}$; similar results are given for other classes.

Languages with very high space-bounded Kolmogorov complexity are very complex in the sense that each such language has essentially maximal information content. Such languages are frequently encountered since it is known that almost every language has Kolmogorov complexity of this magnitude. The results presented here provide evidence that the information in languages with very high space-bounded Kolmogorov complexity is encoded in such a way that very little of it is computationally useful when bounded truth-table reducibilities computable in polynomial time are used to retrieve that information.

From the proof of the main result, other conclusions can be drawn. For example, Watanabe has shown that the class of languages recognizable deterministically in exponential time (denoted $\text{E}$) is not included in the class of languages that are bounded truth-table reducible in polynomial time to sparse languages; we can conclude that no language with very high space-bounded Kolmogorov complexity can be $\leq^P_{\text{NT}}$-hard for $\text{E}$.

Given a machine $M$, a function $t : \mathcal{N} \rightarrow \mathcal{N}$, a language $L \subseteq \{0,1\}^*$, and a natural number $n$, the $t$-space-bounded Kolmogorov complexity of $L_\leq n$ relative to $M$ is defined as

$$K_{\text{S}_M}(L_\leq n) = \min \{|\pi| : M(\pi,n) = \chi_{L_\leq n} \text{ in } t(2^n) \text{ space} \}.$$

Thus, $K_{\text{S}_M}(L_\leq n)$ is the length of the shortest string $\pi$ such that $M$ on input $(\pi,n)$ outputs the $(2^n + 1)$-bit characteristic string of $L_\leq n$ and halts while using no more than $t(2^n)$ workspace. This quantity is often interpreted as the “amount of information” that is contained in $L_\leq n$ and is “accessible” by means of computations that use at most $t(2^n)$ workspace.

Well-known simulation techniques show that there exists a universal machine $U$ that is optimal in the sense that for each machine $M$ there is a constant $c$ such that for all $t, I$, and $n$, we have

$$K_{\text{S}_{U,t}}(L_\leq n) \leq K_{\text{S}_M}(L_\leq n) + c.$$

Hence, we fix an optimal machine $U$ and omit it from the notation. (See [2] or [4] for additional discussion of Kolmogorov complexity.)

The main result involves languages with essentially maximal information content, that is, languages $B$ such that for every polynomial $q$, $K_{\text{S}_q}(B_\leq n) > 2^{n+1} - 2n$ a.e. (Notice that for every language $A$, $K_{\text{S}_q}(A_\leq n) < 2^{n+1} + c$ for some absolute constant $c$; this justifies the phrase “essentially maximal.”)
Main Theorem

Let $A \in \textsf{ESPACE}$ and let $k > 0$ be an integer. Suppose that there is a language $B$ such that $A \leq^p_{k-\text{tt}} B$ and, for every polynomial $q$, $K^q(B_{\leq n}) > 2^{q(n)} + 2n$ a.e. Then there exists a sparse set $S$ such that $A \leq^p_{k-\text{tt}} S$.

The proof of the Main Theorem is contained in the Appendix. We proceed to discuss its applications.

The class $\textsf{HIGH}$ is the collection of all languages $B$ that for every polynomial $q$, satisfy $K^q(B_{\leq n}) > 2^{q(n)} + 2n$ a.e.

As noted above, if a language is in $\textsf{HIGH}$, then it has essentially maximal information content.

Almost every language in $\textsf{HIGH}$, this follows from the fact [8] that $\textsf{RAND} \subseteq \textsf{HIGH}$, where $\textsf{RAND}$ is the set of algorithmically random languages defined by Martin-Löf [7]. Martin-Löf showed that almost every language is in $\textsf{RAND}$. In fact, the inclusion of $\textsf{RAND}$ in $\textsf{HIGH}$ is proper since almost every recursive language is in $\textsf{HIGH}$ [5] while no recursively enumerable (hence, no recursive) language is in $\textsf{RAND}$. On the other hand, $\textsf{HIGH} \subseteq \textsf{ESPACE} = \emptyset$, that, is, no language recognized by a machine that uses workspace $O(2^{cn})$ for any $c > 0$ is in $\textsf{HIGH}$ [5].

The Main Theorem shows that for every integer $k > 0$, $P_{k-\text{tt}}(\textsf{HIGH}) \cap \textsf{ESPACE} \subseteq P_{k-\text{tt}}(\textsf{SPARSE})$. The following result shows that it is highly unlikely that any language in $\textsf{HIGH}$ is $\leq^p_{\text{tt}}$-hard for many of the classes studied in structural complexity theory. Recall that the reducibility $\leq^p_{\text{tt}}$ is transitive.

**Theorem 1** Let $K$ be any class chosen from \{PSPACE, NP, PP, C=P, MOD_2 P, MOD_3 P, \ldots\}. If there is a language in $\textsf{HIGH}$ that is $\leq^p_{\text{tt}}$-hard for $K$, then $K=P$.

Theorem 1 follows from the Main Theorem by results of Ogijima and Watanabe [10] and Ogijima and Lozano [9].

Theorem 1 shows that for any class $K$ chosen from PSPACE, NP, PP, C=P, MOD_2 P, MOD_3 P, \ldots, if $P \neq K$, then no language in $\textsf{HIGH}$ can be $\leq^p_{\text{tt}}$-hard for $K$. A similar consequence holds for $E = $ DTIME($2^{n}$) with no unproven hypothesis since Watanabe [11] has shown that no sparse set is $\leq^p_{\text{tt}}$-hard for $E$.

**Theorem 2** No language in $\textsf{HIGH}$ is $\leq^p_{\text{tt}}$-hard for $E$.

The Main Theorem was stated in terms of a fixed integer $k$ that bounds the number of queries. Instead, one could consider a function $k(n) = O(\log n)$ that is computable in polynomial time and replace the fixed integer $k$ with the function $k(n)$ to obtain a bound on the number of queries. Thus, $P_{k[n]-\text{tt}}(\textsf{HIGH}) \cap \textsf{ESPACE} \subseteq P_{k[n]-\text{tt}}(\textsf{SPARSE})$. In addition, the proof of the Main Theorem allows one to conclude that $P_{\log n}(\textsf{HIGH}) \cap \textsf{ESPACE} \subseteq P_{\text{tt}}(\textsf{SPARSE})$ and $P_{\log n}(\textsf{HIGH}) \cap \textsf{ESPACE} \subseteq P_{\log n}^{\text{tt}}(\textsf{SPARSE})$.

The results have been stated in terms of languages reducible to languages with extremely high space-bounded Kolmogorov complexity. No such language is in $\textsf{ESPACE}$. Such statements tell us nothing about languages to which languages in $\textsf{ESPACE}$ can be reduced. However, the proof of the Main Theorem yields a bound on the Kolmogorov complexity of languages to which languages in $\textsf{ESPACE}$ can be reduced and this bound allows us to make additional conclusions.

**Corollary 3** For every $A \in \textsf{ESPACE} - P_{\text{tt}}(\textsf{SPARSE})$, there exists a polynomial $r$ such that for all languages $B$, if $A \leq^p_{\text{tt}} B$, then $K^r(B_{\leq n}) < 2^{r(n)} + 2n$ i.o.

Corollary 3 implies that hard languages have unusually low Kolmogorov complexity. Moreover, the fact that the polynomial $r$ is fixed enables us to strengthen Theorem 1 as follows.

**Theorem 4** (a) If $P \neq \textsf{NP}$, then there is a fixed polynomial $q$ such that every $\leq^p_{\text{tt}}$-hard language $H$ for NP has space-bounded Kolmogorov complexity $K^q(H_{\leq n}) < 2^{q(n)} + 2n$ i.o.

(b) There is a fixed polynomial $q$ such that every $\leq^p_{\text{tt}}$-hard language $H$ for $E$ has space-bounded Kolmogorov complexity $K^q(H_{\leq n}) < 2^{q(n)} + 2n$ i.o.

**Appendix**

**Proof of the Main Theorem.** Assume that the result is false, that is, assume that $A \in \textsf{ESPACE} - P_{k-\text{tt}}(\textsf{SPARSE})$. Let $f$ and $g$ be functions such that $A \leq^p_{k-\text{tt}} B$ via $(f,g)$. We will show that there is a polynomial $r$ such that

$$K^r(B_{\leq n}) < 2^{r(n)} + 2n \text{ i.o.},$$  

(1)

thus contradicting the hypothesis.

Define $f$ as follows: for each $x \in \{0,1\}^*$ and $i, 1 \leq i \leq k$, $f_i(x) = f_i(x)10^{l_i}$. Note that for each $x \in \{0,1\}^*$ and $i, 1 \leq i \leq k$,

$$|f_i(x)| > |x|.  \quad (2)$$
For each $C \subseteq \{0,1\}^*$, define $\tilde{C}$ by
$$
\tilde{C} = \{ \tilde{f}_k(x) \mid x \in \{0,1\}^*, 1 \leq i \leq k, f_i(x) \in C \}.
$$
Notice that $A \leq_{k-\text{tt}}^p B$ via $(f, g)$ implies that $A \leq_{k-\text{tt}} B$ via $(\tilde{f}, \tilde{g})$.

Now let us briefly sketch the argument. For each $n \in N$, we define a finite tree $T_n$ whose leaves at each level include partial specifications of all languages $C$ such that $A \leq_{k-\text{tt}} B$ via $(\tilde{f}, \tilde{g})$. In particular, $\tilde{B}$ will satisfy some leaf of $T_n$. For a randomly selected language $C \subseteq \{0,1\}^*$, let $\varepsilon_1$ be the event that $\tilde{C}$ satisfies some vertex at depth $\geq 1$ in $T_n$. The trees $T_n$ will be constructed so that the probability $Pr(\varepsilon_1)$ decays exponentially as $\ell$ increases. It will follow by the algorithmic construction of $T_n$ that $KS(C_{\leq n})$ is small for all $C$ such that $\tilde{C}$ satisfies some vertex that is deep in $T_n$, thus guaranteeing that $KS(B_{\leq n})$ is small for infinitely many $n$. The quantitative details will be such that (1) holds.

Some terminology will be convenient. For $n \in N$, an $n$-assignment is a partial function $\alpha$ from $\{0,1\}^{\leq n}$ into $\{0,1\}$. (Thus, an $n$-assignment is also an $m$-assignment for all $m \leq n$.) A language $C \subseteq \{0,1\}^*$ satisfies an $n$-assignment $\alpha$ if $\alpha(x) = [x \in C]$ for all $x \in \text{dom} \ \alpha$. Two $n$-assignments $\alpha$ and $\beta$ are consistent with one another if there is a language $C \subseteq \{0,1\}^*$ such that $\tilde{C}$ satisfies both $\alpha$ and $\beta$. For an $n$-assignment $\alpha$ and $x \in \{0,1\}^*$, let $\alpha(x)$ denote the string $\alpha(\tilde{f}_1(x)) \ldots \alpha(\tilde{f}_n(x)) \in \{0,1\}^k$. (Note that $\alpha(\tilde{f}(x))$ is defined if and only if $\tilde{f}_i(x) \in \text{dom} \ \alpha$.) A string $x \in \{0,1\}^*$ forbids an $n$-assignment $\alpha$ if $\alpha(\tilde{f}(x))$ is defined and $g(x) \alpha(\tilde{f}(x)) \neq [x \in A]$. (This string that implies that $A \leq_{k-\text{tt}} B$ via $(\tilde{f}, \tilde{g})$ and $x$ forbids $\alpha$, then $\tilde{C}$ does not satisfy $\alpha$.)

An $n$-assignment to $\tilde{f}(x)$ is an $n$-assignment $\alpha$ such that $\text{dom} \ \alpha = \{\tilde{f}_k(x), \ldots, \tilde{f}_n(x)\} \subseteq \{0,1\}^{\leq n}$. Finally, fix an “end-marker” $\$” and, for each $n \in N$ and $n$-assignment $\alpha$, let $x(\alpha, n) = \$” if no such $x$ exists.

For each $n \in N$, define a tree $T_n$ as follows. Each vertex of each $T_n$ is an $n$-assignment. The tree $T_0$ consists of a single vertex, the empty 0-assignment $\epsilon_0$. (That is, $\text{dom} \ \epsilon_0 = \emptyset$. The tree $T_{n+1}$ is constructed from $T_n$ by attaching subtrees to zero or more of the leaves of $T_n$. The vertices of $T_n$ are the old vertices of $T_{n+1}$. The new vertices of $T_{n+1}$ are introduced recursively as follows. Let $\alpha$ be a leaf of $T_n$, or a new vertex of $T_{n+1}$. If $x(\alpha, n+1) = \$, then $\alpha$ is a leaf of $T_{n+1}$. Otherwise, the immediate successors of $\alpha$ in $T_{n+1}$, when they exist, are those $(n+1)$-assignments $\beta$ such that $\beta$ is consistent with $\alpha$, dom $\beta = (\text{dom} \ \alpha) \cup \{\tilde{f}_1(x(\alpha, n+1)), \ldots, \tilde{f}_n(x(\alpha, n+1))\}$, and $x(\alpha, n+1)$ does not forbid $\beta$. It is clear that, for each vertex $\alpha$ of $T_n$, $\text{dom} \ \alpha \subseteq W_n \cap \tilde{W}_n$, where $W_n = \{0,1\}^{\leq n}$.

Note that each vertex of each $T_n$ has at most $2^k - 1$ immediate successors. Also, along any path from the root of $T_n$ to a leaf of $T_n$, the domain of each vertex is a proper subset of the domain of its immediate successors, so the depth of $T_n$ is at most $\| \{0,1\}^{\leq n} \| = 2^{n+1} - 1$. Thus, each $T_n$ is a finite tree.

The key property of the trees $T_n$ is best understood in probabilistic terms. Fix $n \in N$ and consider the random experiment in which a set $S \subseteq \{0,1\}^{\leq n}$ is chosen probabilistically according to the uniform distribution on all such sets. For each $n$-assignment $\alpha$, let $F_\alpha$ be the event that $S$ satisfies $\alpha$, and let $F_{\alpha+}$ be the event that $S$ satisfies some immediate successor of $\alpha$ in $T_n$. (If $\alpha$ is not an interior vertex of $T_n$, then $F_{\alpha+} = \emptyset$. We emphasize that it is $S$, not $\tilde{S}$, which is chosen according to the uniform distribution.) The key property of our construction is that
$$
Pr(F_{\alpha+} \mid F_\alpha) \leq 1 - 2^{-k}
$$
for each interior vertex $\alpha$ of $T_n$. To see this, assume that $\alpha$ is an interior vertex of $T_n$ and let $x = x(\alpha, m)$, where $m$ is the least integer such that $x(\alpha, m) \neq \$”.

(That note $m \leq n$.) By the construction of $T_n$, there is an $n$-assignment $\beta$ such that $\beta$ is consistent with $\alpha$, dom $\beta = (\text{dom} \ \alpha) \cup \{\tilde{f}_1(x), \ldots, \tilde{f}_n(x)\}$, and $\beta$ is not an immediate successor of $\alpha$ in $T_n$ (because $x$ forbids $\beta$). Since $\beta$ is consistent with $\alpha$, there exists a set $S_n \subseteq \{0,1\}^{\leq n}$ such that $S_n$ satisfies both $\alpha$ and $\beta$.

Note that dom $\alpha$ and dom $\beta$ are subsets of $W_n \cap \tilde{W}_n$, where $W_n = \{0,1\}^{\leq n}$. Define $h : W_n \cap \tilde{W}_n \to W_{n-1}$ by $h(u10^k) = u$ for all $u10^k \in W_n \cap \tilde{W}_n$. Note that $F_\alpha = \{S \mid S \cap h(\text{dom} \ \alpha) = S_0 \cap h(\text{dom} \ \alpha)\}$ and $F_\beta = \{S \mid S \cap h(\text{dom} \ \beta) = S_0 \cap h(\text{dom} \ \beta)\}$. Let $G = \{S \mid S \cap (h(\text{dom} \ \beta) - h(\text{dom} \ \alpha))\}$.

Then $G$ and $F_\alpha$ are independent events whose intersection is $F_\beta$, so that $Pr(F_\beta \mid F_\alpha) = Pr(G)$. Since $\| h(\text{dom} \ \beta) - h(\text{dom} \ \alpha) \| \leq \| h(\text{dom} \ \beta) \| - \| \text{dom} \ \beta - \text{dom} \ \alpha \| \leq k$, it follows that $Pr(F_{\alpha+} \mid F_\alpha) \leq 1 - Pr(F_\beta \mid F_\alpha) = 1 - Pr(G) = 1 - 2^{1-\|h(\text{dom} \ \beta) - h(\text{dom} \ \alpha)\|} \leq 1 - 2^{-k}$, confirming (3).
property 3 yields

\[ Pr(\varepsilon_{t+1}) = \sum_{a \in T_n(\ell)} Pr(F_{a+}) \]

\[ = \sum_{a \in T_n(\ell)} Pr(F_{a+} \cap F_a) \]

\[ = \sum_{a \in T_n(\ell)} Pr(F_{a+} \mid F_a) Pr(F_a) \]

\[ \leq (1 - 2^{-k}) \sum_{a \in T_n(\ell)} Pr(F_a) \]

\[ \leq (1 - 2^{-k}) Pr(\varepsilon_\ell). \]

It follows inductively that for all \( \ell \in N \),

\[ Pr(\varepsilon_\ell) \leq (1 - 2^{-k})^{\ell}. \quad (4) \]

For each \( n \in N \), define a path \( \beta_n, \beta_n, \ldots, \beta_{n,m} \) from the root of \( T_n \) to a leaf of \( T_n \) by the following recursion. First, \( \beta_n, = \alpha_q \). For the recursion step, for each interior vertex \( a \) of \( T_n \), let \( x(\alpha) = x(\alpha,m) \), where \( m \) is the least integer such that \( x(\alpha,m) \neq \emptyset \). Assume that a path \( \beta_n, \ldots, \beta_{n,j} \) has been defined so that \( \beta_{n,j} \) is an interior node of \( T_n \) and \( \beta_{n,j} \cap (x(\alpha)) = [x_{j-1}(x) \in B] \), \( \ldots, [x_{0}(x) \in B] \) for each \( x = x(\beta_{n,j}) \) with \( 0 \leq i < j \).

(Not that this hypothesis is satisfied vacuously when \( j = 0 \).) Then \( \beta_{n,j+1} \) is the unique m-assignment such that \( \beta_{n,j+1} \) is consistent with \( \beta_{n,j} \), \( \text{dom} \beta_{n,j+1} = \text{dom} \beta_{n,j} \cup \{ \hat{f}_1(x), \ldots, \hat{f}_k(x) \} \), and \( \beta_{n,j+1} \hat{f}(x) = \beta_{n,j} \hat{f}(x) \) when \( x = x(\beta_{n,j}) \). Recall that \( (\hat{f},g) \) is a \( L_{n-1}^{t_1} \)-reduction of \( A \) to \( B \), so this implies that \( g(x)(\beta_{n,j+1} \hat{f}(x)) = [x \in A] \). It follows that \( x \) does not forbid \( \beta_{n,j+1} \), so that \( \beta_{n,j+1} \) is indeed an immediate successor of \( \beta_{n,j} \) in \( T_n \).

For each \( n \in N \), \( \beta_n = \beta_{n,j} \) for some \( j \) such that \( \beta_{n,j} \) satisfies \( \beta_{n,j} \).

For each \( n \in N \), let \( S_0, S_1, \ldots, S_{n,\ell} \) be the lexicographic enumeration of all sets \( S \subseteq \{ 0,1 \}^n \) such that \( S \) satisfies some vertex \( \alpha \) of \( T_n \) whose depth in \( T_n \) is at least \( d \), where

\[ d = \frac{3}{k - \log(2^k - 1)}. \quad (5) \]

Note that this is precisely an enumeration of \( \varepsilon_{d,n} \), so that from (5) we have

\[ \log l(n) = 2^{n+1} - 1 - \log Pr(\varepsilon_{d,n}) \]

\[ \leq 2^{n+1} - 1 + d \cdot n \cdot \log(1 - 2^{-k}) \quad (6) \]

By hypothesis \( A \in \text{SPACE}_n \), and \( f \) and \( g \) were assumed to be computable in polynomial time. Thus, the trees \( T_n \) can be constructed and traversed in space polynomial in their depth. Thus, there is a machine \( M \) such that, if \( 1 \leq i \leq l(n) \) and \( i \) is written in binary, then \( M((i,n) \) outputs the \( (2^{n+1} - 1) \)-bit characteristic string of \( S_{n,i} \) using space polynomial in \( 2^n \). Since \( \beta \) satisfies each \( \beta_n \), it follows by (3.6) that there is a polynomial \( r' \) such that \( K_{S_n}(B_{\leq n}) \leq 1 + \log l(n) \leq 2^{n+1} - 3n \) for all \( n \in D \), where \( D = \{ n \in N \mid J(n) \geq an \} \). Then, by the optimality of the universal machine, there exist a polynomial \( r \) and a constant \( c \) such that

\[ K_{S_n}(B_{\leq n}) \leq 2^{n+1} - 3n + c \quad (7) \]

for all \( n \in D \).

To prove that (1) holds, it will be sufficient to show that the set \( D \) is infinite. The remainder of the argument is devoted to that goal.

Let \( B' = \bigcup_{n \geq 0} \beta_n^{-1}(\{ 1 \}) \). Since \( \beta \) satisfies each \( \beta_n \), it is clear that \( B' \subset B \). Fix a strictly increasing polynomial \( m \) such that \( \| \hat{f}(x) \| \leq s(n) \) for all \( x \in \{ 0,1 \}^n \) and \( 1 \leq i \leq k \). Then

\[ B_{\leq n} \subseteq \beta_n^{-1}(\{ 1 \}) \quad (8) \]

for all \( n \in N \). To see that this is true, let \( y \in B_{\leq n} \).

Then \( y = \hat{f}(x) \) for some \( x \in \{ 0,1 \}^n \) and \( i, 1 \leq i \leq k \). By (2), \( |x| \leq n \), so that \( \| \hat{f}(x) \| \leq s(n) \) for all \( j, 1 \leq j \leq k \). Since \( x(\beta_{n,m}) = \emptyset \), \( x \) does not forbid any \( s(n) \)-assignment to \( \hat{f}(x) \) that is consistent with \( \beta_{n,m} \). It follows that \( y = \hat{f}(x) \) is not in \( \text{dom} \beta_{n+1} \) for any \( m \geq s(n) \). Since \( y \in B' = \bigcup_{n \geq 0} \beta_n^{-1}(\{ 1 \}) \), it follows that \( y \in \beta_n^{-1}(\{ 1 \}) \), confirming (8).

We have already noted that \( (\hat{f},g) \) is a \( L_{n-1}^{t_1} \)-reduction of \( A \) to \( B' \). In fact, \( (\hat{f},g) \) is a \( L_{n-1}^{t_1} \)-reduction of \( A \) to \( B' \). To see this, fix \( x \in \{ 0,1 \}^n \), let \( m = |x| \), and let \( m = s(n) \). Let \( \beta_n \) be the unique \( m \)-assignment such that \( \beta_n \) is consistent with \( \beta_n \), \( \text{dom} \beta_n = \{ \hat{f}_1(x), \ldots, \hat{f}_k(x) \} \), and \( \beta_n(\hat{f}(x)) = 0 \) for each \( i, 1 \leq i \leq k \), such that \( \hat{f}_i(x) \notin \text{dom} \beta_n \). Since \( x(\beta_n,m) = \emptyset \), \( x \) does not forbid \( \beta_n \).

By hypothesis \( \beta_n \) is defined, it follows by (8) that

\[ \| B_{\leq n} \| > k \cdot d \cdot s(n) \quad \text{i.o.} \quad (9) \]
On the other hand,

\[ \| \, \text{dom} \, \beta_{s(n)} \, \| \leq k \, \text{depth}(\beta_{s(n)}) = k \, J(s(n)). \quad (10) \]

By (8), (9), and (10), \( J(s(n)) > d \cdot s(n) \) for infinitely many \( n \). Since \( s \) is strictly increasing, it follows that the set \( D \) is infinite.

Since \( D \) is infinite, (1) follows from (7). This completes the proof of the Main Theorem.

Consider the reducibility specified by the pair \((f, g)\) in the proof. What properties were used? Time as such played no role. Another measure of computation such as space or time-space could have been used; for example, \( \leq_{\text{PSPACE}}^p \) could have been used instead of \( \leq_{\text{R}}^p \). The fact that the reducibility was specified by deterministic machines plays no role except for the fact that both \( f \) and \( g \) are functions; another mode of computation could have been used as long as functions are used to specify the reducibility. The hypothesis that \( A \in \text{ESPACE} \) combined with the fact that \( f \) and \( g \) could be computed in polynomial time allowed the trees \( T_n \) to be constructed and traversed in space polynomial in their depth. In each case the corresponding result would hold and the proof would be essentially the same as that of the Main Theorem.

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