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Rainbow copies of $C_4$ in edge-colored hypercubes

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Abstract

For positive integers $k$ and $d$ such that $4 \leq k < d$ and $k \neq 5$, we determine the maximum number of rainbow colored copies of $C_4$ in a $k$-edge-coloring of the $d$-dimensional hypercube $Q_d$. Interestingly, the $k$-edge-colorings of $Q_d$ yielding the maximum number of rainbow copies of $C_4$ also have the property that every copy of $C_4$ which is not rainbow is monochromatic.

1 Introduction

For a graph $G$, an edge-coloring $\varphi : E(G) \to \{1, 2, \ldots \}$ of $G$ is rainbow if no two edges receive the same color. Throughout this note, we will denote the $d$-dimensional hypercube by $Q_d$. A convenient way to consider $Q_d$ is as a graph with vertices corresponding to binary sequences of length $d$ and edges as pairs of vertices with corresponding binary sequences of Hamming distance 1.

Various problems concerning edge-colorings of hypercubes have been studied, see e.g. [1, 2, 3, 4]. In particular, Faudree, Gyárfás, Lesniak and Schelp [5] proved that there is a $d$-edge-coloring of $Q_d$ such that every $C_4$ is rainbow for $d = 4$ or $d > 5$.

Our main result determines the maximum number of rainbow copies of $C_4$ in a $k$-edge-coloring of $Q_d$ for any positive integers $k$ and $d$ such that $4 \leq k < d$ and $k \neq 5$. Note that when $k = d$, by [5], there is an edge-coloring of $Q_d$ using $d$ colors where every $C_4$ is rainbow.

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Theorem 1. Fix integers $k$ and $d$ such that $4 \leq k < d$ and $k \neq 5$ and write $d = ka + b$ such that $a$ is a non-negative integer and $b \in \{0, 1, 2, \ldots, k - 1\}$. Then the maximum number of rainbow copies of $C_4$ in a $k$-edge-coloring of $Q_d$ is

$$2^{d-2} \left[ \binom{d}{2} - k \binom{a}{2} - ba \right].$$

Interestingly, the $k$-edge-colorings of $Q_d$ that yield the maximum number of rainbow copies of $C_4$ have the additional property that every non-rainbow $C_4$ is monochromatic.

2 Proof of Theorem [1]

Proof. First we prove the upper bound. Assume that $Q_d$ is $k$-edge-colored such that the number of rainbow copies of $C_4$ is maximized. At each vertex $v$ there are $\binom{d}{2}$ incident copies of $C_4$. For a set of $t$ edges of the same color incident to $v$, none of the $\binom{t}{2}$ pairs form a rainbow copy of $C_4$. If there are $t_i$ edges of color $i \in [k]$ incident with $v$, then there are at most

$$\binom{d}{2} - \sum_{i \in [k]} \binom{t_i}{2} \leq \binom{d}{2} - (k - b) \binom{a}{2} - b \binom{a + 1}{2} = \binom{d}{2} - k \binom{a}{2} - ba \quad (1)$$

rainbow copies of $C_4$ at $v$. Summing up (1) for each of the $2^d$ vertices of $Q_d$ counts each $C_4$ four times, which gives the desired upper bound.

Now we prove the lower bound. For each binary sequence coding a vertex of $Q_d$, we partition the first $(k - b)a$ binary digits into $(k - b)$ blocks, each of length $a$, and the last $b(a + 1)$ binary digits into $b$ blocks, each of length $a + 1$. This yields $k$ blocks of consecutive binary digits each of length $a$ or $a + 1$. Computing the sum of the terms in each block modulo 2 yields a binary sequence of length $k$. Thus we have associated a binary sequence of length $k$ with each vertex of $Q_d$. This gives a map, $h$, of the vertices of $Q_d$ to the vertices of $Q_k$. Recall that the edges of $Q_d$ are pairs of vertices such that their corresponding binary sequences of length $d$ have Hamming distance 1. If $u, v \in V(Q_d)$ have Hamming distance 1, then $h(u)$ and $h(v)$ also have Hamming distance 1 since they differ exactly in one block. Therefore, we can also consider $h$ as a map from $E(Q_d)$ to $E(Q_k)$. By [5], there is an edge-coloring, say $\varphi$, of the edges of $Q_k$ with $k$ colors such that every $C_4$ is rainbow. Now let us color the edges of $Q_d$ with the color of their image under $h$ in $Q_k$ i.e. the color of an edge $e$ in $Q_d$ is $\varphi(h(e))$.

Clearly, each vertex in $Q_d$ is incident to $a$ edges of each of $k - b$ colors and it is also incident to $a + 1$ edges of each of the remaining $b$ colors. To complete the proof, we need to check that each pair of edges of different color incident to the same vertex is contained in a rainbow $C_4$. Among the four vertices in any $C_4$ the maximum Hamming distance is 2. Thus all differences among the length $d$ binary sequences of the four vertices of the $C_4$ occur in at most 2 blocks. If all the differences occur in the same block, then the four edges of the $C_4$ are mapped to the same edge in $Q_k$, and thus, the $C_4$ is monochromatic. If the differences
occur in 2 distinct blocks, then the four edges of the $C_4$ are mapped to a $C_4$ in $Q_k$ and thus receive different colors in the coloring of $Q_d$.

### 3 Remarks

Theorem 1 omits the case $k = 5$. This is because there is no 5-edge-coloring of $Q_5$ where every copy of $C_4$ is rainbow, which was proved in [5]. Using a computer, we showed that the maximum number of rainbow copies of $C_4$ in a 5-edge-coloring of $Q_5$ is 73 (there are 80 copies of $C_4$ in $Q_5$). Of course, our blow-up method can be applied on a 5-edge-coloring of $Q_5$ with 73 rainbow copies of $C_4$. However, the resulting bound does not match the upper bound. Moreover, it is even worse than a bound for 4-edge-coloring for large $d$. Our attempt to apply the flag algebra framework on 5-edge-colored hypercubes gave an upper bound that matched the trivial upper bound. We suspect that the trivial upper bound might be the correct order of magnitude for $d \to \infty$. More precisely, if $q_5(d)$ is the maximum number of rainbow copies of $C_4$ in a 5-edge-coloring of $Q_d$, then

$$\lim_{d \to \infty} \frac{q_5(d)}{(d^2)^{2d-2}} = \frac{4}{5}.$$ 

A related question is to determine the number of colors needed to edge-color a graph so that at least some fixed number of colors appear in each copy of a specified subgraph. For graphs $G$ and $H$ and integer $q \leq |E(H)|$, denote by $f(G, H, q)$ the minimum number of colors required to edge-color $G$ such that the edge set of every copy of $H$ in $G$ receive at least $q$ colors. Using this notation, it was shown in [5] that $f(Q_d, C_4, |E(C_4)|) = f(Q_d, C_4, 4) = d$, for $d = 4$ or $d > 5$. Mubayi and Stading [6] proved that if $k \equiv 0 \pmod{4}$, then there are positive constants, $c_1$ and $c_2$, depending only on $k$ such that

$$c_1 d^{k/4} < f(Q_d, C_k, k) < c_2 d^{k/4}.$$ 

They also showed that $f(Q_d, C_6, 6) = f(Q_d, Q_3, 12) = f(Q_d, Q_3, |E(Q_3)|)$, and that for every $\varepsilon > 0$, there exists $d_0$ such that for $d > d_0$

$$d \leq f(Q_d, Q_3, 12) \leq d^{1+\varepsilon}.$$ 

It would be interesting to determine the value of $f(Q_d, Q_{\ell}, |E(Q_{\ell})|)$ for $\ell \geq 3$. Combined with a generalization of our blow-up technique it may allow us to determine the maximum number of rainbow copies of $Q_{\ell}$ in a $k$-edge-coloring of $Q_d$ in general.

### References


