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MATHEMATICAL PRINCIPLES OF DATA INVERSION

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ABSTRACT

Inverse elastic-wave scattering problems are shown to be ill-posed in general. Standard simplifying assumptions—the first Born approximation, the physical optics approximation, and the stopping power approximation of tomography—are shown to reduce to the same ill-posed mathematical problem, that of inverting a Fourier transform. Practical implications of this fact are illustrated with examples.

INTRODUCTION

This paper points out general features of the mathematical problems to which one is led in recovering information about materials' defects, from scattered elastic waves. These problems typically present two complications: non-linearity and ill-posedness. The non-linearity appears because the same defects which serve as sources of scattered elastic waves also affect the waves' propagation. The ill-posedness reflects the fact that significant but sharply localized flaws may cause only insignificant scattering. These two complicating features, which are common to many inverse scattering problems, must be borne in mind when making rational inferences from scattered elastic waves.

Standard simplifying assumptions bring one to linear approximate versions of inverse scattering problems. The ill-posed character of the original problem remains. Data inversion for each of three cases considered here—the first Born approximation, the physical optics approximation, and the stopping-power approximation of tomography—all reduce to the same mathematical problem, that of evaluating the inverse Fourier transform of a function determined from experimental data. Examples are given of an approach for treating this standard regularizable ill-posed problem, and its treatment in general is discussed.

A GENERAL INVERSION PROBLEM

Many cases are covered by the following general problem:

Elastic waves are excited in a piece $R$ of a linear elastic material by an input $f(t)$ to a transducer of known characteristics, located on a part of the surface $\partial R$ of $R$. Responses $g_i(f,t)$ are observed, possibly by a second transducer, located near points $s$ of a subset $A$ of $\partial R$. (Observations are limited to $\partial R$ because, if the interior of $R$ were accessible for observations, there would perhaps be no need for any nondestructive evaluation.) The responses $g_i$ might be the three displacement components at points of parts of $\partial R$ not constrained in the test or the $g_i$ might be traction components at parts of $\partial R$ whose displacements are constrained by the test. The piece $R$ includes a region $B$, in which elastic properties may differ from standards. The flawed region $B$ might be a crack, a void, or an inclusion; it might be a region of anomalous elastic constants. The nonstandard elastic properties of the material in $B$ are associated with nonstandard values of some set $Q$ of characteristics of the performance of the piece $R$ in service; $Q$ might include fatigue life or yield stress.

It being impossible or inconvenient to observe $Q$ directly in a nondestructive way, the objective of the test is to estimate the actual value of $Q$ for a given sample part $R$, from knowledge of the input $f(t)$ and a set of observed responses $g_i(f,t)$, for $g$ in $A$ and $t$ in some interval. It is important to note that the solution of the general inversion problem is a set of estimated values of performance factors, and not necessarily the shape of the defective region $B$.

In the next section, we'll discuss the ideas of ill-posed and well-posed problems. After that, we'll show by an example that inverse elastic wave scattering problems are in general ill-posed.

WELL-POSED AND ILL-POSED PROBLEMS

A well-posed problem, as defined by Hadamard about 1904, is a problem that has a unique solution, which depends continuously on the problem's data. For example, consider the problem of finding the deflection $Y(x)$ of a string fixed at both ends and subjected to a distributed load $W(x)$ reduces, for an appropriate system of units, to finding $Y(x)$ such that

$$Y''(x) = -W(x), \quad Y(0) = 0 = Y(1).$$

The solution of this problem is

$$Y(x) = \int_0^1 K(x,t)W(t)dt,$$

where

$$K(x,t) = \begin{cases} t(1-x) & 0 \leq t \leq x \\ x(1-t) & x \leq t \leq 1 \end{cases}.$$
Clearly the problem defined by (1) and (2) has a unique solution \( Y(x) \), for any continuous \( W(x) \). Moreover, if \( W(x) \) changes by an amount \( \delta W \), then \( Y \) changes by \( \delta Y \), with

\[
\delta Y(x) = \int_0^1 K(x,t) \delta W(t) dt
\]

(5)
since

\[
|K(x,t)| \leq \frac{1}{4}
\]

(6)
for all \( x \) and \( t \) of interest.

Thus, changes in the solution are commensurate with changes in the data, and this is the essence of continuous dependence of the solution on the data. On the other hand, consider the problem of finding an unknown load \( W \), given the displacement \( Y \) which it produces. The solution of this problem is obvious from (1): to find \( W \) given \( Y \), differentiate \( Y \) twice and multiply by \((-1)\).

There is, thus, a unique solution of the present problem, at least for each twice continuously differentiable \( Y \). However, this problem is ill-posed, because there are functions \( Y_0 \) and \( Y_1 \) which differ arbitrarily little, whose second derivatives differ arbitrarily greatly. Specifically, let

\[
Y_0 = x(1-x); \quad Y_1 = x(1-x) + \varepsilon \sin nx, \quad n=1,2,...
\]

(8)
Both \( Y_0 \) and \( Y_1 \) satisfy boundary conditions (2), and

\[
|Y_1 - Y_0| = |\varepsilon \sin nx| \leq |\varepsilon|
\]

(9)
However,

\[
W_0 = -Y_0'' = 2
\]

while

\[
W_1 = -Y_1'' = 2 + n^2 \pi^2 \varepsilon \sin nx
\]

(10)
so that

\[
W_1 - W_0 = n^2 \pi^2 \varepsilon \sin nx
\]

(11)
Now, by choosing \( \varepsilon \) sufficiently small, \( Y_0 \) and \( Y_1 \) can be made to differ as little as one pleases; given any \( \varepsilon \), \( n \) can be chosen sufficiently large so that, at some points, \( W_1 \) and \( W_0 \) differ by as much as one pleases.

This ill-posedness due to lack of continuous dependence on the data greatly hampers the solution of the last problem in practice, since all experiments—even numerical ones—involve some error. As we'll see in the following section, the general elastic wave inverse scattering problem also exhibits discontinuous dependence of solution on data.

ILL-POSED NATURE OF INVERSE ELASTIC-WAVE SCATTERING

The case of a slab with varying elastic properties is one of the few inverse elastic wave scattering problems to be solved in great detail (1). The material of this section shows that this problem is ill-posed.

The geometry of the problem is shown in Fig. 2. Elastic waves are launched from surface \( S_1 \) of a slab whose elastic properties are functions only of \( x \). The objective of the experiment is to determine the variation of elastic-wave impedance with travel time. (Reference (1) shows that this is, in fact, all that one can infer from inverse scattering for this case) by analyzing elastic waves scattered back to surface \( S_1 \), and transmitted to surface \( S_2 \).

We can focus on the essential aspects of the problem by regarding the slab as very thick, and considering elastic wave reflection and transmission at interior points. If the material is "perfect", i.e., has constant elastic-wave impedance, then no elastic waves will be scattered, and an incident elastic wave train will propagate through the slab without attenuation.

Now, consider the following particular case of a slab with varying elastic wave impedance: the slab is of constant impedance, save for a segment of length \( 2L \) (Fig. 3). Let us consider for the moment time-harmonic waves, with frequency \( \omega \) and wave numbers

\[
k_i = \omega/c_i.
\]

One finds by elementary methods that a right-moving incident wave \( U_0 e^{i k_1 x} \) in region 1, causes a left-moving reflected wave \( -U_0 e^{-i k_1 x} \) in region 1, and a right-moving transmitted wave \( U_0 e^{i k_1 x} \) in region 3, with

\[
g = U_0 R(\omega); \quad h = U_0 T(\omega)
\]

(13)
If

\[
\frac{\omega L}{2 c_1} \ll \frac{\pi}{4}
\]

(14)
then

\[
R(\omega) = \frac{1}{2} \left( \frac{\omega}{c_2} \right)^2 \left( \frac{z_2^2}{z_1} \right) + 0 \left( \frac{\omega}{c_2} \right)^2
\]

(15)
and

\[
T(\omega) = 1 + 0 \left( \frac{\omega}{c_2} \right)^2
\]

(16)
Clearly, for any given \( \omega \) and any given \( z_1, z_2, c_1, c_2 \), one could make \( R(\omega) \) as close to zero as he pleased—and, simultaneously, make \( T(\omega) \) as close to 1 as he pleased—by taking \( L \) sufficiently small.
From this, and from the fact that \( R(\omega) \) and \( T(\omega) \) are uniformly bounded for all \( \omega \), it follows that if the actual incident wave is any given band-limited signal, then there exist two impedance profiles—the constant, and the constant with an inclusion like that of Fig. 3—for which both reflected and transmitted signals are as nearly the same as one wishes, while the profiles differ, locally, by as much as one wishes.

Thus we see that a particular inverse elastic wave scattering problem is ill-posed. This pathology is shown to be quite general, in Reference (2).

In the following sections we will see that three widely used approximate treatments of inverse elastic-wave scattering problems all lead to the same ill-posed mathematical problem.

**THE FIRST BORN APPROXIMATION**

Reference (2) gives the details of a demonstration that the Fourier transform \( U_0(x;\omega) \) of the displacement field of the elastic wave scattered by a defect in a finite region \( B \) of an otherwise unbounded homogeneous isotropic elastic material satisfies the integral equation

\[
U_0(x;\omega) = -\frac{i}{4\pi \rho_0 \omega^2} \int_B \left[ \int_B g_{k}(x-\gamma) \delta \epsilon_{\gamma mn p} \epsilon_{\gamma mn p} U_0^s(\gamma) \right] d\gamma + G_1(x;\omega)
\]

where

\[
G_1(x;\omega) = \frac{1}{4\pi \rho_0 \omega^2} \int_B \left[ \int_B g_{k}(x-\gamma) \delta \epsilon_{\gamma mn p} \epsilon_{\gamma mn p} U_0^s(\gamma) \right] d\gamma + \omega^2 \rho_0 U_0^s(x;\omega)
\]

In (17) and (18), \( \delta \epsilon_{\gamma mn p} \) is the variation of the stiffness tensor from the values \( \lambda_0 \delta \epsilon_{\gamma mn p} \) found in the unbounded medium, and \( \delta \rho(x) \) is the variation in density from the value \( \rho_0 \) found in the unbounded medium. The \( U_0^s(x;\omega) \) are the Fourier transforms of the displacement field incident wave, and \( g_{k}(x;\omega) \) is the free-space elastic Green's tensor.

This formulation deals explicitly with the scattered field, which we will find convenient.

The first Born approximation (3) here means the solution of the approximation to (17) obtained by assuming that \( \delta \epsilon_{\gamma mn p} \) and \( \delta \rho \) are "small", and that the scattered fields and their derivatives are also "small", so that the integrals in (17), which involve products of "small" terms, are negligible in comparison with \( G_1(x;\omega) \). This approximation gives \( U_0^s(x;\omega) \) immediately in terms of the \( U_0^s(x;\omega) \).

If the incident wave \( u(x,t) \) is a plane dilatation pulse, i.e., if

\[
u = 9\phi \left( t - \frac{\epsilon \cdot x + \delta}{a_0} \right)
\]

with

\[
\phi(t) = 0, t \neq 0, T
\]

and

\[
\phi(t) \in C^4, \forall t
\]

then \( U_0^s(x;\omega) \) is given (2) by

\[
U_0^s(x;\omega) = \frac{-g(\omega)}{4\pi \rho_0 a_0^2} \int_B \left[ \int_B g_{k}(x-\gamma) \left( \frac{\delta \epsilon_{\gamma mn p} \epsilon_{\gamma mn p}}{a_0^2} \right) \right] d\gamma
\]

where

\[
g(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^T \phi(u)e^{i\omega u} du
\]

and

\[
\phi(x) \equiv x \cdot e + \delta
\]

In the far field, i.e., when

\[
\frac{|x|}{|\gamma|} \ll 1, \quad k \frac{|x|}{|\gamma|} \ll \frac{\pi}{k}, \quad \gamma \in \partial B
\]

the free-space Green's tensor \( g_{k}(x;\omega) \) is given approximately by

\[
g_{k}(x;\omega) \approx k^2 \frac{\epsilon}{a_0} \left[ \frac{\epsilon}{x} - \frac{\delta \epsilon \cdot e}{x} \right]
\]

In the back-scatter direction, where \( x = -x \) and \( x = -e \), it can be shown (2) that

\[
U_0^s(x;\omega) = \frac{-g(\omega)}{4\pi \rho_0 a_0^2} \int_B \left[ \int_B g_{k}(x-\gamma) \left( \frac{\delta \epsilon_{\gamma mn p} \epsilon_{\gamma mn p}}{a_0^2} \right) \right] d\gamma
\]

The two terms in large brackets appearing in the right side of (27) have straightforward physical interpretations. The first term is polarized transversely, since

\[
(\delta \epsilon \cdot e) e_1 e_1 \equiv 0
\]

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This term represents a train of scattered shear waves. The second, polarized longitudinally, represents a train of scattered longitudinal waves. Time-gating will make it possible to separate these two trains, when the parameter $T$ of (21) is sufficiently small and $x$ is sufficiently large. Analysis of first arrival times of scattered dilatation waves, as $e$ varies over all directions, makes possible the estimation of the location of a point in $B$, so that $x$ may be regarded as known, as the source/observation point moves over $\mathbb{R}$. Thus it is possible, in principle, to recover

$$D(\omega) = \frac{1}{(2\pi)^2} \int_B \left[ \frac{1}{a_0} \delta \chi_{\text{amp}}(\chi) e^{\frac{i}{2} k \cdot \chi} + \delta \rho(\chi) \right] e^{i k \cdot \chi} e^{i m \cdot \rho} \rho \, d\chi,$$

and

$$S(\omega) = \langle 6_{e_k e} \rangle \int \delta \chi_{\text{amp}}(\chi) e^{\frac{i}{2} k \cdot \chi} e^{i m \cdot \rho} \rho \, d\chi.$$

by observing back-scattered dilatation waves in the far field.

$D(\omega)$ and $S(\omega)$ are equivalent to spatial Fourier transforms of quantities related to the stiffness and density perturbations. For example,

$$D(\omega) = \frac{1}{(2\pi)^2} \int_B \left[ \frac{1}{2 a_0} (2\pi)^2 \delta \chi_{\text{amp}}(\chi) e^{\frac{i}{2} k \cdot \chi} + \delta \rho(\chi) \right] e^{i k \cdot \chi} e^{i m \cdot \rho} \rho \, d\chi.$$

Defining

$$\rho = -2k e,$$

we see that

$$D(\omega) = \frac{1}{(2\pi)^2} \int_B \left[ \frac{1}{2 a_0} (2\pi)^2 \delta \chi_{\text{amp}}(\chi) e^{\frac{i}{2} k \cdot \chi} + \delta \rho(\chi) \right] e^{i k \cdot \chi} e^{i m \cdot \rho} \rho \, d\chi.$$

By repeated applications of Gauss' divergence theorem, under the assumption that $\delta \chi_{\text{amp}}$ and have fourth derivatives which are piecewise smooth, we find

$$D(\omega) = \frac{1}{(2\pi)^2} \int_B \left[ \frac{1}{2 a_0} (2\pi)^2 \delta \chi_{\text{amp}}(\chi) e^{\frac{i}{2} k \cdot \chi} + \delta \rho(\chi) \right] e^{i k \cdot \chi} e^{i m \cdot \rho} \rho \, d\chi.$$

Thus observations of scattered dilatation waves from a plane dilatation pulse, over all incidence directions and all frequencies would, under the assumptions of this section, yield the Fourier transform of

$$g(x) = \frac{1}{2} \delta \chi_{\text{amp}} + \delta \rho(\chi) dx.$$

In principle, $g(x)$ could be recovered from those observations. Reconstruction of a function from partial, corrupted values of its Fourier transform is an ill-posed problem (2). This ill-posedness is not, of course, unexpected: the non-linearity of the inverse scattering problem was eliminated by the assumptions of the first Born approximation, but the ill-posedness might have been expected to remain—and it does.

THE PHYSICAL OPTICS APPROXIMATION

By the "physical optics approximation" we mean the approximation obtained for the elastic waves scattered from a void or from a rigid inclusion, when, at each point of the surface of the scattering body, the traction and displacement components of the scattered wave are taken to have the values they would take on if the incident wave were scattered from a plane boundary, coincident with the tangent plane to the surface at the point in question. Reference (2) provides a proof that the dilatation wave component of the far-field back scattered elastic wave due to a plane dilatation wave incident on a void, in the physical optics approximation, is given by

$$\phi(\rho) = \frac{1}{4\pi} \int_B \langle n \cdot \rho \rangle(\rho) e^{i m \cdot \rho} d\rho.$$

If $B$ is convex, then $\delta B_1(e)$ consists of those parts of $B$ for which $n \cdot \rho > 0$. Thus for convex $B$,

$$\phi(\rho) = \frac{1}{4\pi} \int_B \langle n \cdot \rho \rangle(\rho) e^{i m \cdot \rho} d\rho.$$

We will use (34) in general, recognizing that an approximation is thereby introduced, if $B$ is not convex. Equations (32) and (33) are, apart from a constant in the definition of $\gamma$, the same as Equations (5) and (6) of Reference (3), which considered electromagnetic scattering. As in (3), straightforward manipulations show that

$$\gamma(\chi) = \frac{1}{2\pi} \int_B \langle n \cdot \rho \rangle(\rho) e^{i m \cdot \rho} d\rho.$$

where $\gamma(\chi)$, the characteristic function of $B$, is defined by

$$\gamma(\chi) = \begin{cases} 1, & \chi \in B \\ 0, & \text{otherwise} \end{cases}$$

Then, by the Fourier integral theorem,
whose normal planes fill all of three-dimensional space.

\[ y(x) = \frac{1}{(2\pi)^2} \iiint y(2\pi e^{i\xi \cdot x}) \, dv \]  
\[ \text{(36)} \]

Thus we see that in the short-wavelength limit, as in the first Born approximation, observations lead to the Fourier transform of a quantity of interest. Equation (36) is the Bojarski-Lewis identity (3), which can now be seen to apply to elastic wave scattering as well as to electromagnetic wave scattering.

**ELASTIC WAVE TOMOGRAPHY**

If elastic waves are assumed to propagate through a body without refraction, but with attenuation according to

\[ \frac{dl}{ds} = -S(r(s)), \]
\[ \text{(37)} \]

where \( s \) is arc length along the ray \( x = r(s) \) and the function \( S(x) \), the "elastic wave stopping power", characterizes flaws in the body, then it can be shown (3) that a function \( h(\mathbf{p}, e) \) which can be determined from intensity observations made on the surface of B, has the property

\[ h(\mathbf{p}, e) = \iiint e^{-i\xi \cdot x} S(x) \, dx. \]
\[ \text{(38)} \]

Here \( e \) is a unit vector in the direction of a plane elastic wave in B. The function \( h(\mathbf{p}, e) \) can be determined for any \( p \) perpendicular to \( e \). Thus, the spatial Fourier transform of \( S(x) \) can be determined, for certain spatial wave vectors \( \mathbf{p} \).

The properties of the part may allow one to find a family--or families--of propagation vectors \( e \), whose normal planes fill all of three-dimensional wave-space. This is the case, for example, if the part is any figure of revolution. In such a case, the Fourier transform of \( S(x) \) can in principle be evaluated, and \( S(x) \) reconstructed, by techniques such as the one considered in the following section.

In other cases, the Fourier transform of \( S(x) \) can be evaluated on parts of wave-number space, and this partial information may give useful insights into the nature of the flaws characterized by \( S(x) \).

**INVERSION OF FOURIER TRANSFORMS WITH NOISY DATA**

Section 5.10 of reference (2) gives a proof by example that the problem of inverting a Fourier transform is ill-posed, in the sense that there are transforms which differ arbitrarily little on all of wave-number space, but whose inverse transforms differ arbitrarily greatly at some points. Thus, while the simplifying assumptions of the first Born approximation, physical optics and tomography each eliminate the non-linear character of the general inversion problem, leaving us with the linear problem of inverting a Fourier transform, the ill-posed character of the original problem is still very much to be reckoned with. The fact that arbitrarily small deviations in transforms can lead to arbitrarily large deviations in inverse transforms, coupled with the fact that all experiments involve errors of observation, certainly warns one not to treat the inversion problem lightly. However, we'll see now that approximate inversion methods can be constructed, whose errors may be estimated. In the process we'll see how observation errors set limits on our ability to resolve details of disturbances in elastic properties.

For a first approach to this task, we may note that, according to (36), (35), and (38) observations may yield values of Fourier transforms of quantities of interest at discrete points of wave-number space. Now, suppose we know that there is some cube, of edge \( 2L \), inside which the defect region \( B \) must lie. Then the quantity \( g(\lambda) \) which characterizes the imperfections, e.g.,

\[ g_0(\lambda) = \sum_{\lambda} \delta c_{\lambda m n q} \, \omega_{m n q} + v^* \delta p \]

for the first Born approximation or

\[ g(\lambda) = \gamma(\lambda) \]

for the physical optics approximation will, if sufficiently smooth, be given by the sum of a Fourier series. That is, there will be coefficients \( c_{\lambda m n} \) such that

\[ g_0(\lambda) = \sum_{\lambda} c_{\lambda m n} \, \omega_{m n} \]

The connection between the \( c_{\lambda m n} \) and \( g(\lambda) \) is

\[ c_{\lambda m n} = \frac{1}{8L^3} \iiint e^{-i\xi \cdot x} g_0(\lambda) \, dx \]
\[ \text{(40)} \]

Now, Equations (30), (35), and (38) show that values of integrals of precisely the form of (40) are directly observable, when the appropriate approximations hold. This suggests a testing program--one should observe those waves which generate the integrals of (40)--as well as a data reduction scheme.

Given experimental values \( \hat{c}_{\lambda m n} \) for some of the \( c_{\lambda m n} \), one produces

\[ \hat{g}(\lambda) = \sum_{\lambda} \hat{c}_{\lambda m n} \, \omega_{m n} \]
\[ \text{(41)} \]

as an approximate solution to the inverse scattering problem. The family of approximate solutions (41) is an example of a regularizer for the ill-posed inversion problem (17, 20). Now let us explore the connection between the approximation \( \hat{g}(\lambda) \) and the actual \( g(\lambda) \). If

\[ \hat{c}_{\lambda m n} = c_{\lambda m n} + \delta c_{\lambda m n} \]

where \( \delta c_{\lambda m n} \) includes both observation and modeling errors, then

\[ \hat{g}_N(\lambda) - g(\lambda) = \sum_{\lambda} \delta c_{\lambda m n} \, \omega_{m n} \]

\[ -2 \text{Re} \sum_{N+1} [A_{\lambda m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n}] \]

\[ -2 \text{Re} \sum_{N+1} [A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n}] \]

\[ -2 \text{Re} \sum_{N+1} [A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n} + A_{\lambda, m n}] \]

where

\[ h_{\lambda, m n} = \frac{1}{8L^3} \iiint e^{-i\xi \cdot x} \, dx \]
\[ \text{(43)} \]
In writing the second error term, we have used the fact that $c_{m-n} = c_{m-n}^*$. A study of the behavior of the two error terms,

$$E_1 \equiv \sum_{-N}^{N} \delta c_{mn} e^{i \pi (x_1 + nx_2 + nx_3)}$$

and

$$E_2 \equiv 2 \text{Re} \sum_{N+1}^{\infty} (A_{0jk} + A_{0jk}^* + A_{0jk} + A_{0jk})$$

$$+ 2 \text{Re} \sum_{N+1}^{\infty} (A_{0jk} + A_{0jk}^* + A_{0jk} + A_{0jk})$$

$$+ 2 \text{Re} \sum_{N+1}^{\infty} (A_{0jk} + A_{0jk}^* + A_{0jk} + A_{0jk})$$

is instructive. Because the series of (39) converges, $E_2$ must tend to zero as $N$ increases. Working with very large values of $N$ is not, however, the way to ensure small errors in $g_N(x)$ because $E_1$ generally increases rapidly with increasing $N$. Indeed, if we know only that $|\delta c_{mn}| < \alpha$, then we may say

$$|E_1| \leq \alpha (2N+1)^3$$

Unfortunately, the estimate is "sharp", in that there are choices of $\delta c_{mn}$ with magnitude not exceeding $\alpha$, for which the equality sign holds in (46) at some $x$. Consequently, if $N$ is taken too large, $E_1$ may be large. On the other hand, if $N$ is taken too small, $E_2$ will be large. It follows that, in fact, there is an optimum number $N^*$ of terms to take, for which $E_1 + E_2$ is a minimum. This optimum number may be estimated, if one knows enough about $g(x)$ to characterize the truncation error $E_1$. Some examples of the behavior of $g_N(x)$ for various values of $N$ are shown in Figs. 4a and 4b. These figures show how the width of the principal lobe of $g_N(x)$ decreases with increasing $N$. This width is, of course, a measure of the resolution attained by the approximation (49). The presence of the error term $E_1$ in (42) limits the value of $N$ which can be used effectively. It is in this way that the presence of errors sets resolution limits for the regularizer of (41). In general, errors limit the amount of fine-scale detail which can be recovered in solving regularizable ill-posed problems. Figures 5a and 5b show the results of applying (41) to perfect (approximately 15-place) data, and Figs. 6a and 6b show the results of applying (41) to noisy data with values of $N$ which are about equal to, and greater than the optimal value $N^*$.

While the "best possible" estimate of the function $g(x)$ obtained from a given regularizer applied to experiments with given error processes may be discouraging, it will often be the case, as pointed out above that this function is not itself the solution to the inversion problem under consideration, but only an intermediate step. It may be, that an error analysis of the actual solution presents a brighter picture. Reference (2) shows that, generally speaking, low-order integral moments of $g(x)$ may be estimated with much more confidence than can $g(x)$ itself. If one knows in advance the form of $g(x)$, if, for example one knows a priori that $g(x)$ is a sphere with a given center so that only one parameter need be estimated to determine $g(x)$, the picture presented by the error analysis may be brighter still. This is borne out by the work of Richardson (7), and suggests that use of long-wavelength limit data to evaluate moments of quantities associated with defects, as discussed by Kohn and Rice (8), may be a good approach to practice.
Values of $G_n(x) \times |v|$ for $N = 3$ (Fig. 4a), and $N = 5$ (Fig. 4b). The same arbitrary scale is used for the abscissas of both figures, but the ordinate scale varies from figure to figure.

Partial sums of the Fourier series of a function of three variables as its argument varies along a straight line. The function is the characteristic function of a sphere, whose center is off-set from the center of the cube within which the series is determined. Values of the function itself are shown on the dashed curve of Fig. 5a for reference. Figure 5a corresponds to $N = 3$ in Equation (41), while Fig. 5b corresponds to $N = 5$. The same arbitrary scale is used for both figures.
Figure 6.

The same partial sums, respectively, as in Fig. 5, with noisy values of the \( c_{ijk} \). The same arbitrary scale is used for both figures.
SUMMARY DISCUSSION
(D. A. Lee)

Jack Cohen (Denver Applied Analytics): I have seen the pictures of Kino and pictures of Quate, and we realize there must be something wrong here because they're certainly getting images. I think what's wrong is that the Fourier transform in particular and the (inaudible) to the fact well-conditioned, not ill-conditioned. The particular Fourier transform (inaudible) a unitary matrix. What more could you want?

David Lee: For most of the inverse problems we are talking about, I can exhibit data which differ by as little as you please, which correspond exactly to solutions which differ by as much as you please.

Jack Cohen: After you discretize unwisely. If you discretize wisely, you have unitary measurements.

David Lee: I'm not taking any approximate solutions at all. I'm simply talking about a property of the problems themselves. I claim I can exhibit data which differ by as little as you please, which correspond exactly to arbitrarily greatly differing solutions. There is no approximate solution involved. Now, I certainly do think you can find wise approximate solution procedures which minimize these effects, but the effects are there. My claim is that one should be aware of them.

Jack Cohen: I really disagree (inaudible).

John Richardson (Rockwell Science Center): I would like to make a comment which may clarify the previous discussion, namely, that indeed if you have a set of spatial frequencies from your actual scattering data that are completely congruent to the set of spatial frequencies you're trying to determine in your image, then there is no ill-posedness at all.

David Lee: That's true.

John Richardson: Now, there are two kinds of ill-posedness that may enter, and one is when you have a set of spatial frequencies available in the scattering data which are (inaudible) the set of spatial frequencies of using to represent their function to be determined, and that's where the particular difficulties come into discussion. But there is also the super resolution type of problem where you're trying to determine something which is of higher resolution than your data justified, and there is another kind of proposedness which can only be handled by a priori information.

David Lee: I agree with that. In fact, the thing I'm advocating is when you're doing business with inverse scattering problems, it is very important that you are aware of the ill-posed character of the problem and that you do just what I understand you have been saying: you think about the errors that you have in your data and you ask yourself what frequencies will allow you to infer rationally: if in fact they allow you to infer, with acceptable error, those frequencies which you need to do the reconstruction you want to do. Then everything is fine. The time you get into trouble is just as you said, when you try to infer things about frequencies which the data won't allow you to do.

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