2013

Note on Nordhaus-Gaddum Problems for Colin de Verdière type Parameters

Wayne Barrett  
*Brigham Young University*

Shaun M. Fallat  
*University of Regina*

H. Tracy Hall  
*Brigham Young University*

Leslie Hogben  
*Iowa State University*, hogben@iastate.edu

Follow this and additional works at: [http://lib.dr.iastate.edu/math_pubs](http://lib.dr.iastate.edu/math_pubs)

Part of the [Algebra Commons](http://lib.dr.iastate.edu/math_pubs), and the [Discrete Mathematics and Combinatorics Commons](http://lib.dr.iastate.edu/math_pubs)

The complete bibliographic information for this item can be found at [http://lib.dr.iastate.edu/math_pubs/62](http://lib.dr.iastate.edu/math_pubs/62). For information on how to cite this item, please visit [http://lib.dr.iastate.edu/howtocite.html](http://lib.dr.iastate.edu/howtocite.html).

---

This Article is brought to you for free and open access by the Mathematics at Iowa State University Digital Repository. It has been accepted for inclusion in Mathematics Publications by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
Note on Nordhaus-Gaddum problems for Colin de Verdière type parameters

Wayne Barrett
Department of Mathematics
Brigham Young University
Provo, UT, USA
wayne@math.byu.edu

Shaun M. Fallat†
Department of Mathematics and Statistics
University of Regina,
Regina, SK, Canada
sfallat@math.uregina.ca

H. Tracy Hall
Department of Mathematics
Brigham Young University
Provo, UT, USA
H.Tracy@gmail.com

Leslie Hogben
Department of Mathematics
Iowa State University
Ames, IA, USA
American Institute of Mathematics
Palo Alto, CA, USA
LHogben@iastate.edu, hogben@aimath.org

Submitted: Jul 19, 2012; Accepted: Sep 24, 2013; Published: Oct 7, 2013
Mathematics Subject Classifications: 05C50, 05C40, 05C83, 15A03, 15B57

Abstract

We establish the bounds \( \frac{4}{3} \leq b_\nu \leq b_\xi \leq \sqrt{2} \), where \( b_\nu \) and \( b_\xi \) are the Nordhaus-Gaddum sum upper bound multipliers, i.e., \( \nu(G) + \nu(\overline{G}) \leq b_\nu |G| \) and \( \xi(G) + \xi(\overline{G}) \leq b_\xi |G| \) for all graphs \( G \), and \( \nu \) and \( \xi \) are Colin de Verdière type graph parameters. The Nordhaus-Gaddum sum lower bound for \( \nu \) and \( \xi \) is conjectured to be \( |G| - 2 \), and if these parameters are replaced by the maximum nullity \( M(G) \), this bound is called the Graph Complement Conjecture in the study of minimum rank/maximum nullity problems.

Keywords: Nordhaus-Gaddum; Colin de Verdière type parameter; Graph Complement Conjecture; maximum nullity; minimum rank; graph complement

*This research began at University of Regina. The authors thank U. Regina and NSERC for their support.
†Research supported in part by an NSERC research grant.
1 Introduction

Nordhaus-Gaddum problems have been studied for many different graph parameters, including chromatic number, independence number, domination number, Hadwiger number, etc. (see, for example, [6] and the references therein). In this note we discuss the Nordhaus-Gaddum sum upper bounds for the Colin de Verdière type parameters $\nu$, $\xi$, and $\mu$. The Graph Complement Conjecture in the study of minimum rank/maximum nullity problems is a conjectured lower bound for related Nordhaus-Gaddum problems.

All graphs in this paper are simple, undirected, and finite. The complement of a graph $G = (V,E)$ is the graph $\overline{G} = (V,\overline{E})$, where $\overline{E}$ consists of all possible edges between vertices in $V$ that are not in $E$. Let $G$ be a graph with vertices $\{1,\ldots,n\}$ and let $S_n$ denote the set of symmetric $n \times n$ real matrices. For $A = [a_{ij}] \in S_n$, the graph of $A$, denoted by $G(A)$, is the graph with vertices $\{1,\ldots,n\}$ and edges $\{ij: a_{ij} \neq 0 \text{ and } i \neq j\}$. The set of symmetric matrices associated with $G$ is $S(G) = \{A \in S_n : G(A) = G\}$. The maximum nullity of $G$ is $M(G) = \max\{\text{null } A : A \in S(G)\}$, and the maximum positive semidefinite nullity of $G$ is $M_+(G) = \max\{\text{null } A : A \in S(G) \text{ and } A \text{ is positive semidefinite}\}$.

In [7] and [8], Colin de Verdière introduced the parameters $\mu(G)$ and $\nu(G)$, defined to be the maximum nullity among matrices $A \in S(G)$ that satisfy the Strong Arnold Hypothesis and additional conditions. In [4] the parameter $\xi(G)$ was defined to be the maximum nullity among matrices $A \in S(G)$ that satisfy the Strong Arnold Hypothesis. A real symmetric matrix $A$ satisfies the Strong Arnold Hypothesis provided there does not exist a nonzero real symmetric matrix $X$ satisfying $AX = 0$, $A \circ X = 0$, and $I \circ X = 0$, where $\circ$ denotes the Hadamard (entry-wise) product and $I$ is the identity matrix. The Strong Arnold Hypothesis is equivalent to the requirement that certain manifolds intersect transversally (see [13]). For $\nu$, the only additional condition (besides the Strong Arnold Hypothesis) is that the matrix must be positive semidefinite. For $\mu$, the additional conditions are that the matrix must be a generalized Laplacian (i.e., have nonpositive off-diagonal entries) and have exactly one negative eigenvalue. Clearly, $\nu(G) \leq M_+(G) \leq M(G)$, $\nu(G) \leq \xi(G) \leq M(G)$, and $\mu(G) \leq \xi(G)$, and each of these inequalities can be strict (see [2]).

An important property of Colin de Verdière type parameters is minor monotonicity. The contraction of edge $e = uv$ of $G$ is obtained by identifying the vertices $u$ and $v$, deleting any loops that arise in this process, and replacing any multiple edges by a single edge. A minor of $G$ arises by performing a sequence of deletions of edges, deletions of isolated vertices, and/or contractions of edges. The notation $H \preceq G$ means that $H$ is a minor of $G$. A graph parameter $\beta$ is minor monotone if for any minor $H$ of $G$, $\beta(H) \leq \beta(G)$. In [7], [8], and [4] (respectively) it is shown that $\mu$, $\nu$, and $\xi$ are minor monotone.

For any graph $G$, the Hadwiger number $h(G)$ is the maximum order of a clique minor of $G$. It was shown in [7] and [8] that $\mu(K_n) = n - 1$ and $\nu(K_n) = n - 1$ (where $K_n$ denotes the complete graph on $n$ vertices), so by minor monotonicity $h(G) - 1 \leq \mu(G)$ and $h(G) - 1 \leq \nu(G)$.

Let $\kappa(G)$ denote the vertex connectivity of $G$, i.e., if $G$ is not complete, the smallest number $k$ such that there is a set of vertices $S$, with $|S| = k$, for which the graph obtained
by deleting the vertices in $S$ and all edges incident with a vertex in $S$, denoted by $G - S$, is disconnected (by convention, $\kappa(K_n) = n - 1$). It is proved in [15, 16] that $\kappa(G) \leq M_+(G)$ for every graph $G$. It was noted in [12] that the proof in [15] establishes $\kappa(G) \leq \nu(G)$ for all $G$. As defined in [2], the minor monotone ceiling of $\kappa$ is $[\kappa](G) = \max\{\kappa(H) : H \preceq G\}$.

It follows from the definition that $h(G) - 1 \leq [\kappa](G)$, since the $K_{h(G)}$ minor of $G$ implies $\kappa(K_{h(G)}) \leq [\kappa](G)$, and $[\kappa](G) \leq \nu(G)$, since $\kappa(G) \leq \nu(G)$ and $\nu$ is minor monotone (see [2] for more detail).

A Nordhaus-Gaddum type result is a (sharp) lower or upper bound on the sum or product of a parameter of a graph and of its complement. The Graph Complement Conjecture for $\nu$ [3] is a Nordhaus-Gaddum sum lower bound.

**Conjecture 1.1 (GCC$\nu$).** For any graph $G$,

$$\nu(G) + \nu(\overline{G}) \geq |G| - 2. \quad (1)$$

It is not possible to raise the lower bound $|G| - 2$ since equality is attained for any tree that includes a $P_4$: For such a tree, it is shown in [1] that $M_+(\overline{T}) = |T| - 3$. Since $M_+(T) = 1$, $M_+(T) + M_+(\overline{T}) = |T| - 2$. It is shown in [17] that GCC$\nu$ is true for graphs with tree-width at most 3, and thus for trees. Thus GCC$\nu$ conjectures that $|G| - 2$ is a tight Nordhaus-Gaddum sum lower bound for $\nu$. This conjecture is studied in [3], where it is established for certain graphs. Various other forms of this conjecture have appeared, including: GCC$_+$, i.e., $M_+(G) + M_+(\overline{G}) \geq |G| - 2$; GCC, i.e., $M(G) + M(\overline{G}) \geq |G| - 2$, [3]; GCC, i.e., $M_+(G) + M_+(\overline{G}) \geq |G| - 2$, [3]; and GCC$\xi$, i.e., $\xi(G) + \xi(\overline{G}) \geq |G| - 2$, [9]. Of course GCC$\nu$ implies GCC$\xi$ implies GCC. The graph complement conjecture for $\mu$, i.e., $\mu(G) + \mu(\overline{G}) \geq |G| - 2$, appeared in [14].

Here we discuss values of the multiplier $b$ for a Nordhaus-Gaddum sum upper bound for the parameter $\beta$ where $\beta$ is one of $h$, $[\kappa]$, $\nu$, $\xi$, or $\mu$. We denote by $b_\beta$ the least value of $b$ making

$$\beta(G) + \beta(\overline{G}) \leq b|G|$$

true for every graph of order at least two, and call $b_\beta$ the *NG upper multiplier for $\beta$*. Stiebitz [18] has shown that

$$h(G) + h(\overline{G}) \leq \frac{6}{5}|G|$$

and there exist graphs achieving $h(G) + h(\overline{G}) = \frac{6}{5}(1 - \varepsilon)|G|$ for arbitrarily small $\varepsilon$, so $b_h = \frac{6}{5}$. We establish bounds for $b_{[\kappa]}$, $b_\nu$, and $b_\xi$. Clearly $b_h \leq b_{[\kappa]} \leq b_\nu \leq b_\xi$, and $b_h \leq b_\mu \leq b_\xi$. In Section 2 we construct a family of graphs to show that $b_{[\kappa]} \geq \frac{4}{3}$. In Section 3 we show that $b_\xi \leq \sqrt{2}$. In Section 4 we summarize our conclusions. Note that the Nordhaus-Gaddum sum upper bound for the parameters $M$ and $M_+$ is not interesting because it is trivially $2|G| - 1$:

$$M(K_n) + M(\overline{K_n}) = M_+(K_n) + M_+(\overline{K_n}) = (n - 1) + n = 2|K_n| - 1.$$
2 Lower bound for NG upper multiplier for \([\kappa]\)

In this section we construct a self-complementary graph \(H(a)\) on \(12a - 4\) vertices for \(a \geq 2\), and show that \(H(a)\) has a minor \(\hat{H}(a)\) with \(\delta(\hat{H}(a)) = \kappa(\hat{H}(a)) = 8a - 4\), where \(\delta(G)\) denotes the minimum degree of a vertex of \(G\). It is shown that this example implies that \(b_{[\kappa]} \geq 4/3\).

Example 2.1. Construct the graph \(H(a) = (V, E)\) as follows (see Figure 1): The \(12a - 4\) vertices of \(H(a)\) are partitioned into four sets \(V_i\), \(i = 1, 2, 3, 4\) of \(r = 3a - 1\) vertices each. The sets \(V_1\) and \(V_2\) are the “sparse part” of \(H(a)\), with \(H(a)[V_i] = K_r, i = 1, 2\) (where \(G[W]\) denotes the subgraph of \(G\) induced by the subset \(W\) of the vertices of \(G\)). The sets \(V_3\) and \(V_4\) are the “dense part” of \(H(a)\), with \(H(a)[V_i] = K_r, i = 3, 4\). Every edge between a vertex in \(V_1\) and a vertex in \(V_3\) is in the edge set \(E\), and likewise for \(V_2\) and \(V_4\). There are no edges between \(V_1\) and \(V_4\), nor between \(V_2\) and \(V_3\). Regarding the edges between \(V_1\) and \(V_2\), number the vertices of \(V_1\) as \(u_{2i-1}, i = 1, \ldots, r\) and the vertices of \(V_2\) as \(u_{2i}, i = 1, \ldots, r\). Then vertex \(u_s \in V_1\) is adjacent to the \(a\) vertices \(u_{s+j} \in V_2\), \(j = 1, 3, \ldots, 2a - 3, 2a - 1\) (where for \(k = 6a - 1, \ldots, 8a - 4\), \(u_k\) is interpreted as \(u_{\ell}\) with \(\ell \equiv k \mod (6a - 2)\) and \(1 \leq \ell \leq 2a - 2\)). If \(u \in V_1 \cup V_2\), then \(\deg_{H(a)} u = r + a = 4a - 1\) (where \(\deg_G w\) denotes the degree of \(w\) in \(G\)). Regarding the edges between \(V_3\) and \(V_4\), number the vertices of \(V_3\) as \(v_{2i-1}, i = 1, \ldots, r\) and the vertices of \(V_4\) as \(v_{2i}, i = 1, \ldots, r\). Then vertex \(v_s \in V_3\) is adjacent to all vertices \(v_p \in V_4\) except for \(p = s + j, j = 1, 3, \ldots, 2a - 3, 2a - 1\). If \(v \in V_3 \cup V_4\), then \(\deg_{H(a)} v = (r - 1) + r + (r - a) = 8a - 4\). It is clear from the construction that \(\hat{H}(a) = H(a)\).

![Figure 1: Schematic diagram for the construction of \(H(a)\)](image-url)

Construct the minor \(\hat{H}(a)\) by contracting the edges \(u_{2i-1}u_{2i}, i = 1, \ldots, r\), and denote the set of these \(r\) vertices by \(V_{1,2}\). If \(v \in V_3 \cup V_4\), then \(\deg_{\hat{H}(a)} v = \deg_{H(a)} v = 8a - 4\). Note that each of the new vertices in \(V_{1,2}\) has degree equal to \(2((4a - 1) - 1) = 8a - 4\), so \(\hat{H}(a)\) is \((8a - 4)\)-regular. Furthermore, if \(w \in V_{1,2}\), \(w\) is adjacent to all \(2r = 6a - 2\) vertices in \(V_3 \cup V_4\) so \(\hat{H}(a)[V_{1,2}]\) is \((2a - 2)\)-regular. Since each vertex in \(V_3 \cup V_4\) is adjacent to \(r = 3a - 1\) vertices in \(V_{1,2}\), \(\hat{H}(a)[V_3 \cup V_4]\) is \((5a - 3)\)-regular.
To establish that $\kappa(\hat{H}(a)) = \delta(\hat{H}(a))$, we use the property that for certain circulants $C$, $\kappa(C) = \delta(C)$, establish a method for computing $\kappa$, and examine parts of $\hat{H}(a)$ separately. For $1 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$, the *consecutive circulant* $\text{Circ}_n(1, \ldots, t)$ is the graph on the vertices $\{0, 1, \ldots, n-1\}$ with vertex $i$ adjacent to vertices $i + j$ and $i - j$ for $j = 1, \ldots, t$ (with arithmetic mod $n$). We will use the fact that for a consecutive circulant the vertex connectivity is equal to the (common) degree; Harary [11] gave the consecutive circulant as an example of a graph having maximum vertex connectivity $\frac{2m}{n}$ among graphs having $n$ vertices and $m$ edges (when $\frac{2m}{n}$ is an integer), and this result is now well known.

**Theorem 2.2.** [19, Theorem 4.1.5 (Harary)] For $1 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$, 
$$\kappa(\text{Circ}_n(1, \ldots, t)) = \delta(\text{Circ}_n(1, \ldots, t)) = 2t.$$  

**Theorem 2.3.** Let $G$ be a connected graph on $n$ vertices with $G \neq K_n$ and let $1 \leq t \leq n - 1 - \delta(G)$. Define 
$$f(t) = \max\{s \in \mathbb{Z}^+ : K_{t,s} \text{ is a subgraph of } \overline{G}\},$$ 
where $K_{s,t}$ denotes the complete bipartite graph on $s$ and $t$ vertices. Then 
$$\kappa(G) = \min\{n - (t + f(t)) : 1 \leq t \leq n - 1 - \delta(G)\}.$$ 

**Proof.** For every $t$ such that $1 \leq t \leq n - 1 - \delta(G)$, $\overline{G}$ contains a $K_{t,1}$ (by choosing a vertex $v$ of degree $\delta(G)$ as the partite set of 1 vertex, and $t$ of its non-neighbors as the other partite set), so $f(t)$ is defined.

Choose $t$ such that $1 \leq t \leq n - 1 - \delta(G)$. Let $U$ be a set of $t$ vertices and let $W$ be a set of $f(t)$ vertices such that $\overline{G}[U \cup W]$ contains a $K_{t,f(t)}$ subgraph. Then $G[U \cup W]$ is disconnected, so $\kappa(G) \leq |V \setminus (U \cup W)| = n - (t + f(t))$. Since this is true for every $t$ such that $1 \leq t \leq n - 1 - \delta(G)$, 
$$\kappa(G) \leq \min\{n - (t + f(t)) : 1 \leq t \leq n - 1 - \delta(G)\}.$$ 

Choose a set $S$ such that $|S| = \kappa(G)$ and $G - S$ is disconnected. Let $U$ be the set of vertices of one connected component, let $t_0 = |U|$, and let $W = V \setminus (U \cup S)$; note $|W| = n - (t_0 + \kappa(G))$. Then $\overline{G}$ contains $K_{t_0,n-(t_0+\kappa(G))}$ with bipartition $U,W$. Thus $f(t_0) \geq n - (t_0 + \kappa(G))$, so 
$$\kappa(G) \geq n - (t_0 + f(t_0)) \geq \min\{n - (t + f(t)) : 1 \leq t \leq n - 1 - \delta(G)\}. \quad \square$$

We now return to establishing the properties of one of the graphs constructed in Example 2.1.

**Observation 2.4.** Let $G$ be a graph whose vertex set $V$ can be partitioned as $V = X \cup Y$ such that each vertex in $X$ is adjacent to each vertex in $Y$. Let $G_X = G[X]$ and $G_Y = G[Y]$. Then
\begin{itemize}
\item $\delta(G) = \min\{\delta(G_X) + |Y|, \delta(G_Y) + |X|\}$,
\item $\kappa(G) = \min\{\kappa(G_X) + |Y|, \kappa(G_Y) + |X|\}$.
\end{itemize}

**Theorem 2.5.** For $\widehat{H(a)}$ as in Example 2.1,

$$\kappa(\widehat{H(a)}) = \delta(\widehat{H(a)}) = 8a - 4.$$

**Proof.** Let $X = V_{1,2}$ and $Y = V_3 \cup V_4$. Then $V(\widehat{H(a)}) = X \cup Y$ and every vertex in $X$ is adjacent to every vertex in $Y$. By Observation 2.4, if we show that $\kappa(\widehat{H(a)}[V_{1,2}]) = \delta(\widehat{H(a)}[V_{1,2}])$ and $\kappa(\widehat{H(a)}[V_3 \cup V_4]) = \delta(\widehat{H(a)}[V_3 \cup V_4])$, it follows that $\kappa(\widehat{H(a)}) = \delta(\widehat{H(a)})$.

Since $\widehat{H(a)}[V_{1,2}] = \text{Circ}_r(1, \ldots, a - 1)$, we have $\kappa(\widehat{H(a)}[V_{1,2}]) = \delta(\widehat{H(a)}[V_{1,2}])$ by Theorem 2.2.

Recall that in $\widehat{H(a)}[V_3 \cup V_4] = H(a)[V_3 \cup V_4]$, the vertices of $V_3$ are numbered as $v_{2i-1}, i = 1, \ldots, r$ and the vertices of $V_4$ as $v_{2i}, i = 1, \ldots, r$, and vertex $v_s \in V_3$ is adjacent to all vertices $v_p \in V_4$ except for $p = s+j, j = 1, 3, \ldots, 2a-3, 2a-1$ (where for $p > 6a-2$, $v_p$ means $v_{p-(6a-2)}$). Thus the vertex $v_s$ in $H(a)[V_3 \cup V_4]$ is adjacent to precisely the vertices $v_p, p = s+j, j = 1, 3, \ldots, 2a-3, 2a-1$. This is a bipartite regular graph with a great deal of symmetry, so in determining the shared neighborhood of two vertices, no generality is lost by considering the vertices 1 and $1 + 2d$ (with $d \leq \lfloor \frac{3a-1}{2} \rfloor$). The size of the shared neighborhood is $\max(a - d, 0)$. For $1 \leq t \leq a = 2r - 1 - (5a - 3) = |H(a)[V_3 \cup V_4]| - 1 - \delta(\widehat{H(a)}[V_3 \cup V_4])$, the maximum neighborhood intersection of a set of $t$ vertices happens when those vertices are consecutive in the same bipartition set, and it follows that in this case $f(t) = a + 1 - t$. So for all $t \in \{1, \ldots, a\}$, $|H(a)[V_3 \cup V_4]| - (t + f(t)) = 5a - 3$. Thus $\kappa(\widehat{H(a)}[V_3 \cup V_4]) = 5a - 3 = \delta(\widehat{H(a)}[V_3 \cup V_4])$. \hfill $\square$

**Corollary 2.6.** For the graph $H(a)$ in Example 2.1,

$$[\kappa](H(a)) \geq 8a - 4$$

and

$$[\kappa](H(a)) + [\kappa](\widehat{H(a)}) \geq \frac{4}{3}(1 - \frac{1}{6a - 2})|H(a)|.$$

Thus

$$b_{[\kappa]} \geq \frac{4}{3}.$$

**Proof.** By Theorem 2.5, $[\kappa](H(a)) \geq \kappa(\widehat{H(a)}) = 8a - 4$. Since $H(a)$ is self-complementary, $[\kappa](\widehat{H(a)}) \geq 8a - 4$ also, and thus $[\kappa](H(a)) + [\kappa](\widehat{H(a)}) \geq 16a - 8$. The second statement can then be established by arithmetic. Since $b_{[\kappa]} \geq \frac{1}{|H(a)|}([\kappa](H(a)) + [\kappa](\widehat{H(a)}))$, by taking the limit as $a \to \infty$ we see that $b_{[\kappa]} \geq \frac{4}{3}$. \hfill $\square$
3 Upper bound for NG upper multiplier for $\xi$

In this section we show that the NG upper multiplier $b_\xi$ is at most $\sqrt{2}$.

**Theorem 3.1.** [10] Let $G = (V_G, E_G)$ be a connected graph. Then

$$|E_G| + a \geq \frac{\xi(G)(\xi(G) + 1)}{2}$$

where $a = 1$ if $G$ is bipartite and every optimal matrix for $\xi(G)$ has zero diagonal, and $a = 0$ otherwise.

Since $\xi(G)$ is the maximum of $\xi(G_i)$ taken over the connected components $G_i$ of $G$, the hypothesis that $G$ is connected is unnecessary.

**Corollary 3.2.** Let $G = (V_G, E_G)$ be a graph. Then

$$|E_G| + 1 \geq \frac{\xi(G)(\xi(G) + 1)}{2}.$$  \hspace{1cm} (3)

**Corollary 3.3.** Let $G = (V_G, E_G)$ be a graph with at least one edge. Then

$$\xi(G) \leq \sqrt{2|E_G|}.$$  \hspace{1cm} (4)

**Proof.** Algebraic manipulation of (3) gives $\xi(G) \leq \frac{1}{2}(-1 + \sqrt{8|E_G| + 9})$. Further manipulation shows that the inequality $\frac{1}{2}(-1 + \sqrt{8|E_G| + 9}) \leq \sqrt{2|E_G|}$ is equivalent to $2 \leq |E_G|$, so (4) is true if $G$ has at least two edges. If $G$ has exactly one edge then $G$ has components $K_2$ and possibly some $K_1$'s, and thus $\xi(G) = 1 < \sqrt{2}$. \hfill $\Box$

Nordhaus-Gaddum bounds usually take one of two forms: additive or multiplicative. The form of inequality (4) suggests a third category of Nordhaus-Gaddum bound: Pythagorean.

**Corollary 3.4.** Let $G = (V_G, E_G)$ be a graph of order at least two. Then

$$\xi(G)^2 + \xi(\overline{G})^2 \leq |G|^2 - |G|.$$  \hspace{1cm} (5)

**Proof.** Let $|G| = n \geq 2$. In the case where either $G$ has no edges or $\overline{G}$ has no edges, $\xi$ will take the value 1 for one of the two graphs and the value $n - 1$ for the other, in which case the result holds. In any other case inequality (4) applies both to $G$ and to $\overline{G}$, giving us two inequalities the sum of whose squares is

$$\xi(G)^2 + \xi(\overline{G})^2 \leq 2|E_G| + 2|E_{\overline{G}}| = |G|^2 - |G|.$$ \hfill $\Box$

**Corollary 3.5.** Let $G = (V_G, E_G)$ be a graph of order at least two. Then

$$\xi(G) + \xi(\overline{G}) \leq \sqrt{2}|G|,$$

and thus $b_\xi \leq \sqrt{2}$.

**Proof.** Let $|G| = n \geq 2$, and by Corollary 3.4 choose $x \geq \xi(G)$ and $y \geq \xi(\overline{G})$ such that $x$ and $y$ lie on the circle $x^2 + y^2 = n^2$. The maximum value of $x + y$ on this circle is $\sqrt{2}n$. \hfill $\Box$
4 Conclusions

In summary, we have established

\[ 1.333 < \frac{4}{3} \leq b_{[\kappa]} \leq b_\nu \leq b_\xi \leq \sqrt{2} < 1.415. \]

We have no evidence that the construction in Section 2 is tight, even for \( b_{[\kappa]} \). On the other hand, the inequality (2) with \( a = 0 \) is known to be tight for some small examples and for complete graphs (it is tight with \( a = 1 \) for \( K_{3,3} \)). For \( \nu \), since a diagonal entry for a vertex of degree at least one cannot be zero, \( a = 0 \) and the inequality (2) becomes \( |E_G| \geq \frac{\nu(G)(\nu(G)+1)}{2} \) for graphs with at least one edge; again this is tight for some small graphs and complete graphs. This leaves open the possibility that Corollaries 3.3 — 3.5 may be asymptotically tight.

**Question 4.1.** Given \( x \) and \( y \) positive with \( x^2 + y^2 = 1 \), does there exist an increasing sequence of graphs \( G_i \) on \( n_i \) vertices such that \( \nu(G_i)/n_i \) approaches \( x \) and \( \nu(G_i)/n_i \) approaches \( y \)? Or such that \( \xi(G_i)/n_i \) approaches \( x \) and \( \xi(G_i)/n_i \) approaches \( y \)?

The particular case of \( x = y = \frac{\sqrt{2}}{2} \) suggests the next question.

**Question 4.2.** Do \( b_\nu \) and \( b_\xi \) take the maximum possible value of \( \sqrt{2} \)?

On the other hand it seems more difficult to construct examples for \( b_\mu \), and the only bounds we know are those from \( h \) (due to Stiebitz [18]) and \( \xi \), i.e.,

\[ 1.2 = \frac{6}{5} = b_h \leq b_\mu \leq b_\xi \leq \sqrt{2} < 1.415. \]

References


