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Barro-Becker with Credit Frictions

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Abstract

The Barro-Becker model of fertility has three controversial predictions: (i) fertility and schooling are independent of family income; (ii) children are a net financial burden to society; and (iii) individual consumption is negatively associated to individual income. We show that introducing credit frictions into the model helps overturn these predictions. In particular, a negative relationship between fertility and individual wage income can be obtained when the intertemporal elasticity of substitution is larger than one. The credit constrained model can also explain the quantity-quality trade-off: individuals with higher wage income choose more schooling and fewer children.

Keywords: parental altruism, elasticity of intertemporal substitution
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1 Introduction

Modeling the economic choices of the family unit using dynamic macro-style frameworks has become central in analyzing a range of issues and policies. One of the key decisions of a family is the fertility choice. Deciding the number of children and the amount of resources to invest in each of them has a long lasting impact on the economic outcomes of all family members. The use of dynamic altruistic models of fertility choice was pioneered by the influential work of Becker and Barro (1988) and Barro and Becker (1989).[1] Although the Barro-Becker model (BB model henceforth) is widely used in macroeconomics, some of its predictions are either counterintuitive or hard to reconcile with the data. Specifically, the following are three controversial predictions of the model: (i) fertility is independent of family income; (ii) children are a net financial burden to society; and (iii) individual consumption is negatively associated to individual income.

A well-known empirical regularity is that fertility declines with income (see Jones, Schoonbroodt and Tertilt, 2008). A negative association is evident when looking at the cross-country data on average fertility and per-capita income. Similar evidence is obtained when looking at a cross-section of individuals within a country. For example, Becker (1960) finds a negative fertility-income relationship in the 1910, 1940 and 1950 Censuses, and the Indianapolis survey for the 1900s. In addition, using US Census data a far back as 1826, Jones and Tertilt (2006) estimate an income elasticity of fertility of about -0.38 suggesting the importance of income as a determinant of fertility in within-country micro data. However, the BB model is inconsistent with this evidence. In the BB model fertility is independent of wage income, family income or in general, any level variable. Instead, fertility depends positively on the interest rate, the degree of altruism and the growth of child survival probabilities, while it depends negatively on the rate of technical progress and the growth rate of social security. Although interest rates may play a role in explaining cross-country fertility differences (Manuelli and Seshadri, 2009), this mechanism may not be relevant to explain fertility differences within a country. The reason why in the BB model fertility is independent of income is that it is fully determined by the intergenerational version of the Euler equation rather than by the optimality condition for fertility. As resources can be freely reallocated across generations through unconstrained bequests, the Euler equation pins down the value of fertility as a function of the interest rate, but independently of income.

The BB model also predicts that individual consumption is proportional to the net financial cost of children. The intuition for this result is that when children are costly to produce then the optimal "utilization rate," as given by children's consumption, must be proportional to their net cost (Becker and Barro, 1988, p. 10). Therefore, in order for consumption to be positive, children must be a net financial burden to parents in particular, and to society in general, meaning that the financial costs of raising a child must be larger than the financial benefits. Becker and Barro (1988) acknowledge that this prediction may not hold in many countries, specially in modern ones. Below we provide evidence showing that children, even those who end up as unskilled workers, are a financial gain rather than a financial burden to society. Finally, the BB model predicts

[1] For a summary of the literature after Becker and Barro (1988) see Jones, Schoondbroot and Tertilt (2008).

a negative association between individual consumption and individual income. This is because the net financial cost of a child is negatively associated to the future earnings of the child. This prediction runs counter to standard consumption theory and a variety of evidence suggesting a positive association between lifetime income and lifetime consumption.[2]

This paper shows that introducing credit frictions into the Barro-Becker model helps overturn various controversial predictions of the frictionless version of the model. A critical assumption underlying the predictions of the BB model is that the "bequest" or transfers parents give to children is unconstrained, so the possibility of negative bequests is not ruled out. We start the analysis by characterizing a model identical to BB except for the presence of a non-negative bequest constraint, a constraint that is quite natural and plausible given that parents cannot legally impose debt obligations on their children. We find that when the bequest constraint binds individual income becomes a determinant of fertility, individual consumption increases with income, and children are not necessarily a financial burden to society. Specifically, in the constrained model optimal fertility is determined by the comparison between marginal benefits and marginal costs of children, rather than by the intergenerational Euler equation. A negative fertility-wage relationship is obtained when wages have a larger impact on the marginal cost of children than on the marginal benefits. Moreover, a binding bequest constraint also implies that parents cannot fully control the "utilization rate" or consumption of their children: if children's consumption is too high parents cannot reduce it by leaving negative bequests. As a result, individual consumption is not proportional to the net cost of raising children, consumption can be positive even if children are a net financial benefit to society, and lifetime individual consumption responds positively to lifetime individual income.

We generalize our benchmark economy to a full life-cycle model with endogenous fertility, bequests, and schooling choices, as well as two different types of credit frictions: bequest constraints and borrowing constraints for students. The main purpose of the more general model is to show that various types of credit frictions, not necessarily bequest constraints, can give rise to similar results. We analyze the properties of the life-cycle model in two steps. First, in order to discuss how the relationship between income and fertility is affected by the elasticity of intertemporal substitution (EIS), we keep schooling exogenous. To focus on the fertility decision, we assume that the interest rate is sufficiently low to induce the bequest constraint to bind along a steady state. The resulting model resembles the Samuelson-Diamond OLG economy because bequests are absent, but it differs from it in that the parent's altruism determines fertility and dynasty size endogenously. We find that an EIS larger than one is needed for the model to predict a negative fertility-wage relationship. To see why, notice that since parents care about the utility of children, an increase in wages (income) directly increases the marginal benefit of having children through the increase in the utility of the child. Regarding the marginal cost of having children, an increase in wages has two opposing effects: it increases the opportunity cost of foregone labor income as there is a time-cost of raising children, but it decreases the marginal utility of income. For fertility to be a negative function of income, it must be the case that the increase in the opportunity cost of foregone labor

[2] For example, standard consumption theory predicts that if the income of all individuals in a dynasty doubles then the consumption of all individuals in the dynasty will increase. In the BB model, steady state consumption falls.

income is strong enough to offset the decrease in the marginal utility of income. It turns out that the larger the EIS, the more likely a negative fertility-income relationship would hold.

In a second step we allow the endogenous determination of schooling as well as fertility. In this case the bequest constraint is not binding because parents need to finance the consumption of their children during the schooling years, but the borrowing constraint is binding. The model captures the idea that children depend on the resources of their parents and parents cannot legally charge the children for the resources transferred to them.[3] Although clear-cut analytical results cannot be obtained for the general case, we derive some analytical insights and provide a numerical simulation. We verify that a high EIS is still needed to generate a negative fertility-income relationship when schooling choices are also endogenous. We also show that there is a quantity-quality trade-off: high wage income individuals choose low fertility and high schooling. In sum, the key message of the paper is that dynamic altruistic models of fertility with credit frictions give rise to more plausible predictions than their frictionless counterparts.

Our paper is related to a large literature that uses dynamic altruistic models in order to study fertility choices in macroeconomic models. Papers include Becker, Murphy and Tamura (1990), Alvarez (1999), Boldrin and Jones (2002), Barro and Sala-i-Martin (2004), Doepke (2004, 2005), Jones and Schoonbroodt (2010), Manuelli and Seshadri (2009), Bar and Leukhina (2010) among others. The main issue in this literature is that, as in the BB model, steady state fertility is mainly determined by the interest rate and the growth rate of the economy, but not by income or wages. Again, even though interest rates and growth rates may play a role explaining cross-country differences in fertility, differences in wages appear to be also relevant in explaining the within-country cross-sectional evidence on fertility. The contribution of our paper is to characterize the conditions under which dynamic altruistic models of fertility choice in the tradition of Barro and Becker are able to generate a negative fertility-income relationship. It turns out that credit frictions and a high EIS are essential.

A parallel and complementary literature in macroeconomics studies fertility in non-altruistic settings (i.e., those in which parents care about either the number of children, or their human capital, but not directly about the utility of the children). Notable examples in this category of papers include Galor and Weil (2000) and Greeenwood and Seshadri (2002), among others. In these non-altruistic models, optimal fertility is determined from a comparison of marginal costs and marginal benefits of children. As discussed before, this is not the case in the BB model, where fertility is ultimately linked to the interest rate, and not the cost of raising children. The dynamic altruistic model with credit frictions we propose provides a link to non-altruistic models in the sense that the implied theory of fertility is similar in both. However, there is an important distinction between the two models: while the presence of altruism complicates our model, it also endogenizes the value that parents give to the investments made on their children. This value is not endogenous, but ad hoc, in non-altruistic settings and treated as a free parameter. The altruistic

[3] In the frictionless version of the model both fertility and schooling depend on the interest rate but are independent of parental or individual income. The frictionless model could explain both high fertility and low schooling choices through high interest rates but also all individuals in the economy will choose the same fertility and schooling since both choices are independent of individual income.

model reduces the number of free parameters and provides added discipline. Furthermore, altruistic models in which the value of children's investments to the parent is endogenous are useful tools to study situations in which one cannot plausibly assume parents exogenously value their children's investment differently.

The remainder of the paper is organized as follows. Section 2 revisits the BB model, introduces non-negative bequest constraints, and derives the main results of the paper regarding fertility and consumption. Section 3 extends the benchmark model in Section 2 to a life-cycle model. It examines both the role of the EIS in determining whether the fertility-income relationship is positive or negative, as well as the presence of a quality-quantity trade-off. Concluding comments are provided in Section 4 and the Appendix includes the details of the solutions to the different models presented in the text.

2 The Becker-Barro model revisited

Becker and Barro (1988) study fertility choices within a simple dynastic model in which individuals live one period and leave bequests to their children. Parents have children for altruistic reasons: they care about the utility of their children. In this section we set up a slightly generalized version of the BB framework, review the main findings of the original BB model and highlight the following controversial predictions of the model: (i) fertility is independent of family income; (ii) children are a net financial burden to society; and (iii) individual consumption is negatively associated to individual income.

2.1 The model

Consider the problem of an individual, a parent, who lives for one period, period t, and derives utility from consumption, c_t, the number of children, $n_t \in [0, N]$, and the utility of the children, $V_{t+1} \geq 0$. The parental utility, V_t, satisfies

$$V_t = U(c_t) + \Phi(n_t)V_{t+1} \text{ for } t = 0, 1, ..., \tag{1}$$

where $U(c) \geq 0$ is the utility flow and $\Phi(n_t) \geq 0$ is the weight that parents place on the welfare of their n children. It is assumed that $\Phi(0) = 0$, $\Phi'(n) > 0$ and $\Phi''(n) < 0$ meaning that parents are altruistic toward children but at a decreasing rate. The restriction $\Phi(N) < 1$ is required for utility to be bounded. Define $\beta(n) \equiv \Phi(n)/n$ as average altruism. Notice that $\beta(n)$ decreases with n. Positive fertility choices require that utility flows and stocks (U and V) are non-negative.[4]

[4] If children add negative utility flows to their parents then parents would be better off having no children. Barro and Sala-i-Martin (2004) and Jones and Schoonbroodt (2010) analyze cases with negative utility. See also Cordoba and Ripoll (2011a).

The parent at time t faces the following budget constraint

$$w_t + y_t + (1 + r_t)\, b_t = c_t + \chi(n_t; w_t) + n_t b_{t+1}, \tag{2}$$

where w_t is the wage rate, y_t is non-labor income, b_t are bequests per child, r_t the interest rate, and $\chi(n; w)$ is the total cost of raising n children. The dependence on w reflects the time cost of raising children.[5] If time input is the only cost of raising children then $\chi(n; w) = \chi(n)w$. It is assumed that $\chi_n(n; w) > 0$, $\chi_{nn}(n; w) \geq 0$, and the marginal cost of rasing children increases with wages, $\chi_{nw}(n; w) \geq 0$.[6]

Define net income as $I(n_t; w_t, y_t) \equiv w_t + y_t - \chi(n_t; w_t)$. The following is a recursive representation of the parent's problem at time $t \geq 0$:

$$V_t(b_t) = \max_{b_{t+1} \geq 0,\ N \geq n_t \geq 0} U(I(n_t; w_t, y_t) + (1 + r_t)\, b_t - n_t b_{t+1}) + \Phi(n_t) V_{t+1}(b_{t+1}). \tag{3}$$

This problem incorporates a non-negative bequest constraint and a maximum number of potential children, N. It is natural to introduce non-negative bequest constraint in the parental problem because in practice parents cannot legally impose debt obligations on their children.

The BB framework is a special case of this model. They consider isoelastic functional forms for U and Φ, a linear formulation for $\chi(n; w)$ and $y = 0$. The results below are stated in terms of elasticities in order to provide easy comparison with the BB model. Denote by $\epsilon_x^F(\cdot)$ the elasticity of function F with respect to variable x.

Example 1: BB model $\epsilon_n^{\Phi}(n) = \epsilon_n^{\Phi}$, $\epsilon_c^U(c) = \epsilon_c^U$, $\epsilon_n^{\chi}(n) = 1$ and $y = 0$. In particular, $\Phi(n_t) = \alpha n^{\theta}$, $\chi(n; w) = n(a + dw)$ where a is the goods cost of having children and $0 \leq d < 1$ is the time cost.

The more general model above includes non-labor income, y_t, and allows for general functional forms for U, Φ and $\chi(n; w)$. As discussed in Becker and Barro (1998), $\epsilon_n^{\Phi}(n) > \epsilon_c^U(c)$ is required for the existence of an interior solution for fertility when bequest are unconstrained. We assume this restriction holds. The following assumption is also convenient in order to obtain an interior solution in the constrained case, as we discuss below.

Assumption 1. $\frac{\Phi'(n)}{1-\Phi(n)}$ is decreasing in n for $n \in [0, N]$.

Two formulations of $\Phi(n)$ that satisfy Assumption 1 as well as the restriction $0 \leq \Phi(n_t) < 1$ for any $n \geq 0$ are $\Phi(n) = \alpha\,(1 - e^{-\mu n})$ and $\Phi(n) = \frac{\alpha \mu n}{1+\mu n}$ for $0 < \alpha < 1$ and $\mu > 0$. The BB formulation $\Phi(n) = \alpha n^{\theta}$ needs an added restriction on the maximum number of children in order to satisfy Assumption 1 as well as the restriction $0 \leq \Phi(n) < 1$.

[5] For example, if l is the time cost of raising children and x the goods cost, then $\chi(n; w) = \min_{l,x} \{wl + x\}$ subject to $n \geq z l^{\psi} x^{1-\psi}$. In this case, $\chi(n; w) = \widehat{z} n w^{\psi}$.

[6] Assumption $\chi_{nn}(n, w) \geq 0$ is not necessary, but it will turn out to be sufficient in the Propositions below.

2.2 Optimal decisions

The optimality condition for bequests is given by the following intergenerational version of the standard Euler Equation:

$$U'(c_t) \geq U'(c_{t+1})\,(1 + r_{t+1})\,\beta(n) \;\; \text{with equality if } b_{t+1} > 0. \tag{4}$$

The condition is similar to an Euler equation, except that the discount factor $\beta(n)$ is endogenous and corresponds to the average degree of altruism toward children, while the consumptions correspond to those of the parent and the child. For given interest rate, equation (4) with equality implies a sort of quantity-growth trade off: higher n_t is associated with a reduction in consumption growth as c_{t+1} falls relative to c_t.

The optimality condition for fertility is given by:[7]

$$U'(c_t)\,(b_{t+1} + \chi_n(n_t; w_t)) = \Phi'(n_t)V_{t+1}(b_{t+1}). \tag{5}$$

The left hand side of equation (5) is the marginal cost of children. It includes the bequest to the child plus the cost of raising a child both multiplied by the marginal utility of consumption. The marginal benefit on the other hand is the welfare of the child, V_{t+1}, times the parental weight associated to the marginal child, Φ'. It is instructive to rewrite (5) as:

$$U'(c_t) = U'(c_{t+1})\left(1 + r^n_{t+1}\right)\Phi'(n_t), \tag{6}$$

where

$$1 + r^n_{t+1} \equiv \frac{V_{t+1}(b_{t+1})/U'(c_{t+1})}{b_{t+1} + \chi_n(n_t; w_t)}$$

is the gross return of having a child. To see this last statement, notice that $V_{t+1}/U'(c_{t+1})$ is the value of a life in terms of goods and $b_{t+1}+\chi_n(n_t; w_t)$ is the marginal cost of creating a life. Comparing (6) to (4) makes clear that fertility decisions are analogous to investment decisions, as pointed out by Alvarez (1999), and that the returns on children and bequest must satisfy the arbitrage condition:

$$\left(1 + r^n_{t+1}\right)\Phi'(n_t) \geq (1 + r_{t+1})\,\Phi(n_t)/n_t.$$

Since $\Phi(n_t)$ is concave, this condition implies that in an interior solution $1 + r^n_{t+1} > 1 + r_{t+1}$, meaning that returns to having children need to be larger than the returns to assets. This is because additional savings allow parents to provide each child with higher bequests which increase the utility of the parent in proportion to their average degree of altruism, $\Phi(n_t)/n_t$; however, having an extra child, holding bequests constant, increases the utility of the parent only in proportion to

[7] We focus the discusion on cases in which optimal fertility is interior and consider interior or corner solutions for bequest. A sufficient, but not necessary, condition for positive fertility is that $lim_{n\to 0}\Phi'(n) = \infty$.

the marginal degree of altruism, $\Phi'(n_t)$.

2.3 Steady state

We now characterize the steady state implications of equations (3), (4) and (5).[8] Depending on parameter values, the bequest constraint may bind or not. We first discuss the case when bequest are positive (unconstrained case), and then the case when bequest are zero (constrained case).

2.3.1 Unconstrained allocation

Let n, c and b denote the unconstrained steady state values of fertility, consumption and bequests respectively. The unconstrained solution is characterized by the following equations (see details in Appendix A):

$$1 = (1+r)\,\beta(n), \tag{7}$$

$$c = \frac{\epsilon_c^U(c)}{\epsilon_n^\Phi(n) - \epsilon_c^U(c)} \left[(1+r)\,\chi_n(n;w) - w - y - \chi\,(n;w)\,(\epsilon_n^\chi(n;w) - 1)\right], \tag{8}$$

and

$$b = \frac{c - (w + y - \chi\,(n;w))}{1+r-n}. \tag{9}$$

Equation (7) is the key result of the unconstrained model: it states that steady state fertility is a sole and positive function of the interest rate. In particular, fertility is independent of level variables such as w or y.[9] Therefore, the unconstrained allocation does not generate a negative relationship between fertility and income, contrary to what the data suggests. The reason is that steady state fertility is fully determined by the intergenerational Euler equation, equation (4). In the standard neoclassical growth (Ramsey) model with exogenous fertility and a given interest rate, equation (4) determines the balanced growth rate of individual consumption. However, when fertility is endogenous as in the BB model, the discount factor is endogenous and the role of the intergenerational Euler equation is to pin down the value of the discount factor, and therefore the underlying value of fertility. Notice that the first order condition for fertility plays no role in determining steady state fertility in the unconstrained solution. Therefore, considerations regarding the costs and benefits of having children do not affect fertility choices but instead they determine consumption choices, as we discuss next.[10]

Once fertility is determined by the interest rate, equation (8) determines steady state consumption. Since $\epsilon_n^\chi(n;w) \geq 1$, due to the assumed convexity of marginal costs, consumption is positive

[8] The original Barro-Becker model does not have transitional dynamics. Our more general model may have some transitional dynamics depending on specific functional forms.

[9] More generally, equation (4) states that fertility depends positively on the interest rate and negatively on the rate of consumption growth, but it does not depend on level variables such as y or w.

[10] Becker and Barro (1988) argue that this prediction arises from the neglect of human capital investments in their model. If these investments were included, then the prediction of their model would be that the fixed cost of raising a child (the one that excludes human capital investments) is larger than the fixed component of earnings (earnings independent of human capital). Although Becker and Barro argue that this prediction more reasonable, they do not provide a more systematic defense of it.

only if the financial cost of raising a child, $(1+r)\,\chi_n(n;w)$, is larger than the financial benefit, $w+y$, meaning that children are a net financial burden to parents and to society. This is a problematic prediction of the model, particularly for rich countries, as recognized by Becker and Barro (1988, p. 9). They also argue that this prediction arises from the neglect of human capital investments in their model. If these investments were included, then the prediction of their model would be that the fixed cost of raising a child, one that excludes human capital investments, is larger than the fixed component of earnings, or earnings of unskilled workers. The available data however suggests than the present value of earnings of unskilled individuals is significantly larger than the cost of raising a child. According to the USDA (2010), the typical total cost of raising a child from birth to age 17 for a family of four in the lowest income group was $163,440 in 2010.[11] These expenses include direct parental expenses made on children through age 17 such as housing, food, transportation, health care, clothing, child care, and private expenses in education. If an individual works 40 hours a week 48 weeks a year from age 18 to age 65 earning the federal minimum legal wage of $7.25 dollars per hour, then the present value of earnings is $696,000. Both of these figures are incomplete, but the large gap between benefits and costs is unlikely to disappear. For example, the cost of rasing a child ignores the opportunity cost of parental time spend on children, a cost that may be substantial but not enough to overcome the benefits. Furthermore, the benefits assume only minimum wage earnings for unskilled workers during the entire life but wages are typically increasing due to experience and technological progress.

Finally, another counterintuitive prediction of the unconstrained model is that consumption is negatively related to non-labor income y, and possible to labor income too, according to equation (8). In standard consumption theory a permanent increase of income for all generations increases consumption for all of them. However, when fertility is endogenous, optimality now requires that the utilization rate of an offspring, which is given by his/her consumption, be proportional to the cost of having the child. Since non-labor income reduces the cost of having the child then consumption must fall with non-labor income.

Given fertility and consumption, equation (9) provides optimal bequests. Equation (7) implies that $1+r-n = (1+r)\,(1-\Phi(n)) > 0$. Therefore, steady state bequests are non-negative if $c+\chi(n;w) > w+y$. We now turn to discuss the constrained allocation which occurs when this condition is violated.

2.3.2 Constrained allocation

Consider now the constrained solution. It is important to point out that although in this section we analyze the steady state properties of the model in which bequests are zero, a more general life cycle model with other types of credit frictions delivers the same implications we discuss here (section 3). As we show below, when children cannot borrow and depend on the resources of their parents, then bequests are positive, but the determinants of fertility are similar to the ones discussed here. What is essential is the presence of binding credit frictions, not necessary of a binding non-negative

[11] The USDA calculation assumes a zero interest rate but the main argument here is robust to various plausible interest rates.

bequest constraint.

The following is a necessary and sufficient condition for the bequest constraint to bind in steady state.

Assumption 2. $\frac{\epsilon_c^U(c)\epsilon_n^\chi(n;w)}{\epsilon_n^\Phi(n)}\frac{\chi(n;w)}{w+y-\chi(n;w)} < \frac{n}{1+r-n}$ where n solves $1 = (1+r)\,\beta(n)$.

The condition in Assumption 2 follows after algebraically transforming condition $c + \chi(n;w) < w + y$, which as shown above induces zero bequests. To better understand this condition, consider $\chi(n;w) = n(a + dw)$ as in the BB model. Assume further that $a/w \approx 0$ which describes a rich country in which w is high relative to a. In that case, the condition in Assumption 2 becomes

$$\frac{\epsilon_c^U}{\epsilon_n^\Phi}\frac{dn}{1-dn} < \frac{n}{1+r-n} = \frac{\Phi(n)}{1-\Phi(n)}.$$

Since $\epsilon_c^U/\epsilon_n^\Phi < 1$, this condition states that the weight that parents give to their children, $\Phi(n)$, needs to be larger than the total fraction of time spent raising children, dn. According to this condition, the bequest constraint binds if $(\epsilon_c^U\epsilon_n^\chi)/\epsilon_n^\Phi$, the interest rate and the cost of raising children are sufficiently low, or if wages are sufficiently high.[12]

While Assumption 2 is a necessary and sufficient condition, the following is a sufficient condition for the bequest constraint to bind.

Assumption 3. $\lim_{n\to 0}(1+r)\,\beta(n) < 1$.

If Assumption 3 is satisfied then $(1+r)\,\beta(n) < 1$ for all feasible n since $\beta(n)$ decreases with n. Therefore, the intergenerational Euler equation (4) with equality would imply that, for a constant interest rate and any fertility rate, consumption is falling toward zero. Therefore, Assumption 3 guarantees that in any steady state with constant consumption the bequest constraint must be binding so that (4) does not hold with equality. In words, when the interest rate is low and parents do not attach enough weight to their descendants ($\beta(n)$ is low for any n) then it is optimal for parents to leave negative bequest.

The advantage of Assumption 3 is its simplicity. It is satisfied, for example, by the functions $\Phi(n) = \alpha\left(1 - e^{-\mu n}\right)$ or $\Phi(n) = (\alpha\mu n)/(1+\mu n)$ under the restriction $(1+r)\,\mu < 1$. However, Assumption 3 is not satisfied by the BB function $\Phi(n) = \alpha n^\theta$. In this case, Assumption 2 can be utilized.

Denote by n^* and c^* the steady state solutions in the constrained case. Equations (5) and (2) in steady state become:

$$U'(w+y-\chi(n^*;w))\chi_n(n^*;w) = \frac{\Phi'(n^*)}{1-\Phi(n^*)}U(w+y-\chi(n^*;w)). \tag{10}$$

[12] A sufficient condition for this condition to hold is $(1+r)\,d < 1$. In that case, it follows, using (7), that

$$\frac{\Phi(n)}{1-\Phi(n)} = \frac{1}{1/\Phi(n)-1} = \frac{1}{(1+r)/n-1} > \frac{1}{1/(dn)-1} > \frac{\epsilon_c^U}{\epsilon_n^\Phi}\frac{dn}{1-dn}.$$

$$c^* = w + y - \chi(n^*; w) \tag{11}$$

The left hand side of (10) is the marginal cost of raising children while the right hand side is the marginal benefit. According to (10) the solution for n^* in the constrained model will depend in general on w and y but not on r, contrary to the unconstrained version.

We now characterize key properties of the constrained allocation and contrast them with the ones obtained for the unconstrained case. First, are children still a net financial burden to parents? To answer this question consider again the case $\chi(n, w) = n(a + dw)$ as in the BB model. Recall that in the unconstrained case positive consumption requires $(1 + r)(a + dw) > w + y$ meaning that children are a financial burden. In the constrained case instead, equation (11) implies $c^* = w + y - (a + dw)n^* \geqslant 0$. Therefore, a sufficient condition for children to be a net financial benefit is $n^* > (1 + r)$ which requires a relatively low interest rate. Such condition can be satisfied because r is not a determinant of n^*. More generally, for any function $\chi(n, w)$ and $\Phi(n) = (\alpha\mu n)/(1 + \mu n)$ it can be shown that a necessary condition for children to be a net financial benefit is

$$n^* > \frac{\left(1 - \epsilon_c^U(c^*)\right)}{\epsilon_c^U(c^*)}(1 + r),$$

which is an easier condition to satisfy as long as $\epsilon_c^U(c^*)$ is not much lower than one.

It is convenient to rewrite (10) as:

$$c^* = \epsilon_c^U(c^*)\chi_n(n; w)\frac{1 - \Phi(n^*)}{\Phi'(n^*)}. \tag{12}$$

Equations (11) and (12) can be used to obtain a graphical solution for c^* and n^*. Equation (11) provides a negative relationship between consumption and fertility, while (12) provides a positive relationship, as long as $\epsilon_c^U(c^*)$ does not decrease too strongly with consumption, i.e., $\partial\epsilon_c^U(c^*)/\partial c > -1$. Figure 1 illustrates the determination of c^* and n^* for the BB functional forms.[13] It is easy to check graphically that $\partial c^*/\partial y > 0$, $\partial c^*/\partial w > 0$, $\partial n^*/\partial y > 0$ while $\partial n^*/\partial w$ is undetermined. Remember that in the frictionless case $\partial c/\partial y < 0$, and $\partial n/\partial y = \partial n/\partial w = 0$ while $\partial c/\partial w < 0$ if $(1 + r)d < 1$ or $\partial c/\partial w > 0$ otherwise. The constrained version provides more standard and plausible comparative statics. In particular, it predicts a positive response of consumption to both labor and non-labor income which overcomes one limitation of the frictionless version. Regarding the response of fertility to non-labor income, children are normal goods in both the constrained and unconstrained models, but in the unconstrained model higher y for all generations translates into more children only for the initial generation while in the constrained version fertility increase for all generations.

It is interesting to notice the similarity between equation (10) and the determinants of fertility

[13] The figure assumes $w + y - \chi(N; w) > 0$ and N such that $\Phi'(n^*)/(1 - \Phi(n^*))$ is decreasing. As mentioned before, if N is too large problems may arise becuase either $\Phi(N)$ may be larger than 1 or $\Phi'(n^*)/(1 - \Phi(n^*))$ may start increasing. Moreover, if $w + y - \chi(N; w) < 0$ then one can redefine N, the maximum number of children, as the one that satisfies $w + y - \chi(N; w) = 0$.

in non-altruistic models. For instance, in Greenwood and Seshadri (2002, p. 156) fertility is determined by equalizing marginal costs and marginal benefits of children. Marginal costs include the opportunity wage cost of children, and the benefits are a function of the future wage of the child. This stands in sharp contrast with the unconstrained allocation of the BB model, where fertility is not determined from such comparison of costs and benefits of children. Although there is a connection between the determinants of fertility in the constrained allocation of our altruistic model and those in a non-altruistic model, there is an important distinction between the two. As in the BB model, our model endogenizes the value of the children to the parent, while this value is ad hoc in non-altruistic models.

Consider now the sign of $\partial n/\partial w$ in the constrained model. According to equation (10), the marginal benefit of children increases with wages because higher wages increase the welfare of offsprings making children more valuable for altruistic parents. This effect alone would imply a positive fertility-income relationship, contrary to the data. To obtain a negative relationship the marginal cost of children must not only increase with wages, but increase more than the marginal benefit. Whether the marginal cost of children increase with wages or not depends on the result of two opposite effects. On the one hand, the marginal cost increases with w because $\chi_{nw}(n; w) > 0$ as this term captures the larger opportunity cost of parents' time invested in children. On the other hand, the marginal utility of consumption $U'(c)$ falls because consumption increase with wages. Therefore, the marginal cost of children increase with wages only if the first effect dominates the second, which ultimately limits the curvature of the utility function.

Equation (10) can be written using elasticities and expressing the marginal cost and benefit in terms of the level of U as follows

$$\epsilon_c^U(c^*)\epsilon_n^\chi(n^*, w)\frac{\chi(n^*; w)}{w + y - \chi(n^*; w)} = \epsilon_n^\Phi(n^*)\frac{\Phi(n^*)}{1 - \Phi(n^*)}. \tag{13}$$

Equation (13) is useful because w and y only affect the left hand side of the equation, the relative marginal cost, but not the right hand side, the relative marginal benefit. Moreover, given that the relative marginal benefit is decreasing in n, due to Assumption 1, a negative fertility wage relationship occurs when the relative marginal cost increases with wages. The following two propositions characterize the relationship between fertility and wages implied by equation (13) for CRRA preferences $U(c) = \frac{c^{1-\sigma}}{1-\sigma} + A$. The constant $A > 0$ allows to consider values of σ larger than 1.[14]

The first proposition states that high wage individuals choose maximum fertility if the elasticity of substitution is less than one; otherwise, they may choose less than the maximum.

Proposition 1. Let $U(c) = \frac{c^{1-\sigma}}{1-\sigma} + A \geq 0$ where $\sigma \geq 0$ and suppose $\lim_{w\to\infty} \chi(n^*; w)/w = \chi(n^*) > 0$. Then, $\lim_{w\to\infty} n^*(w) \to N$ if $\sigma > 1$ and $\lim_{w\to\infty} n^*(w) \to \widetilde{n}^*$ if $\sigma < 1$ where $\widetilde{n}^*$ solves the

[14] $A > 0$ is needed when $\sigma > 1$ for utility to be positive. Otherwise the utility flow, $\frac{c_t^{1-\sigma}}{1-\sigma}$, would be negative when $\sigma > 1$, and parent would have no children to avoid adding negative utility flows to their own utility. Furthermore, when $\sigma > 1$ consumption has to be above $[(\sigma - 1)A]^{\frac{1}{1-\sigma}}$ to guarantee $U(c) \geq 0$. Notice that A corresponds to the maximum utility flow when $\sigma > 1$. The minimum utility flow of zero. These issues do not emerge when $0 < \sigma < 1$, and A can be set to zero.

equation:

$$(1-\sigma)\,\epsilon_n^{\chi}(\widetilde{n}^*)\frac{\chi(\widetilde{n}^*)}{1-\chi(\widetilde{n}^*)} = \epsilon_n^{\Phi}(\widetilde{n}^*)\frac{\Phi(\widetilde{n}^*)}{1-\Phi(\widetilde{n}^*)}. \tag{14}$$

Proof. See Appendix B.

The following is the intuition behind Proposition 1. When $\sigma > 1$ the marginal value of income $U'(c) = c^{-\sigma}$ falls at a faster rate as $w \to \infty$ than the rate at which the opportunity cost of raising children increases. As a result, the marginal cost of raising children decreases toward zero while the marginal benefit remains bounded above zero making maximum fertility optimal. A similar result is obtained by Hall and Jones (2007) in the context of longevity rather than fertility. They use a similar specification to show that when $\sigma > 1$, longevity is a superior good because the marginal utility of consumption falls with income, while the marginal utility of longevity does not. Although such feature of preferences is convenient in the case of longevity and health spending, it is problematic in context of fertility because it also implies that fertility should increase with wages.

When $\sigma \in (0,1)$, fertility approaches a value defined by (14). For the BB formulation, expression (14) becomes $\frac{(1-\sigma)d}{1-d\widetilde{n}} = \frac{\alpha\theta\widetilde{n}^{\theta-1}}{1-\alpha\widetilde{n}^{\theta}}$. This equation has a well-defined solution if d is large enough and/or α is low.[15] If the altruistic function $\Phi(n) = \alpha\,(1-e^{-\mu n})$ is used instead, the equation becomes $\frac{(1-\sigma)d}{1-d\widetilde{n}} = \frac{\alpha\mu}{(1-\alpha)e^{\mu\widetilde{n}}+\alpha}$ which has a unique solution if $(1-\sigma)\,d < \alpha\mu$. In both cases, fertility for high wage individuals depends negatively on the time cost of raising children, d, and positively on the degree of altruism, α.

At first glance Proposition 1 is a negative result because an elasticity below one, or $\sigma > 1$, is the common value used in most applied work in macro.[16] However, for fertility choices such elasticity entails the counterfactual implication that the most productive individuals would have the highest fertility. Intuitively, this is because a low elasticity of substitution increases the value providing positive consumption to all possible descendants as wages increase. In contrast, if the elasticity is larger than 1 then high parental consumption can substitute for low or zero consumption of descendants. However, this intuition also reveals that the key parameter is not the intertemporal elasticity of substitution, which controls intra-personal consumption smoothing, but the intergenerational elasticity of substitution, which controls inter-personal consumption smoothing. In separable models, such as the ones in this paper, both elasticities are equal to $1/\sigma$. In a companion paper, Cordoba and Ripoll (2011a) show that if these two elasticities are disentangled using non-separable models, then the negative result above does not hold: the most productive individuals do not need to have the highest fertility even if the intertemporal elasticity of substitution is low. In the context of separable models as the current one, the practical implication is that when analyzing fertility choices it is better to interpret $1/\sigma$ as the intergenerational rather than as the intertemporal elasticity of substitution.

The following Lemma provides a necessary and sufficient condition for fertility to decrease with

[15] An interior solution may not exist as corner solutions are possible. Besides, as mentioned above, the BB altruistic function does not satisfy Assumption 1 nor $\Phi(n) < 1$ for all n. As a result, there may be multiple solutions or no solutions to equation (14) for that formulation.

[16] See discussion in Jones, Schoonbroodt and Tertilt (2008).

wages for all wage levels.

Lemma 2. Let $U(c) = \frac{c^{1-\sigma}}{1-\sigma} + A \geq 0$ where $\sigma \geq 0$. Then $\frac{\partial n}{\partial w} < 0$ if and only if

$$\epsilon_w^{\chi_n}(n^*; w)\left(1 + \frac{y}{w}\right) + \left(\Psi(c^*)\epsilon_w^{\chi}(n^*; w) - \epsilon_w^{\chi_n}(n^*; w)\right)\frac{\chi(n^*; w)}{w} > \Psi(c^*) \tag{15}$$

where $\Psi(c) = \sigma + \epsilon_c^U(c)$. For the case when $\epsilon_n^{\chi}(n^*; w) = \epsilon_n^{\chi}$ for all $(n^*; w)$ condition (15) simplifies to:

$$\epsilon_w^{\chi}(n^*; w)\left(\left(1 + \frac{y}{w}\right) + (\Psi(c^*) - 1)\frac{\chi(n^*; w)}{w}\right) > \Psi(c^*). \tag{16}$$

Proof. See Appendix B.

The following proposition uses Lemma 2 to typify two polar cases in which fertility decreases with wages. While in both cases $\sigma \in (0, 1)$ is needed, the first case also requires strictly positive non-labor income while the second requires non-homothetic preferences.

Proposition 3. Let $\epsilon_n^{\chi}(n^*; w) = \epsilon_n^{\chi}$. Then fertility decreases with wages in the following cases:

(i) non-labor income case: $A = 0$, $0 < \sigma < 1$ and $\varepsilon_w^{\chi}(n^*; w)(1 + y/w) > 1$;

(ii) non-homothetic preferences: $\chi(n^*; w) = \chi(n^*)w$, $y = 0$, $A > 0$ and $0 < \sigma < 1$.

Proof. See Appendix B.

Notice that since $\varepsilon_w^{\chi} \leq 1$, the first part of Proposition 3 requires the share of labor income over total income, $w/(w + y)$, to be below ε_w^{χ} to obtain a negative fertility-income relationship. If labor is the only input in the production of children then $\varepsilon_w^{\chi} = 1$, the condition is satisfied for all wages, and $\partial n^*/\partial w \to 0$ as $w \to \infty$. A similar result can be obtained even if ε_w^{χ} is not constant but $\varepsilon_w^{\chi} \to 1$ as $w \to \infty$. This occurs, for example, when χ is a CES function with low elasticity of substitution. The second part of Proposition 3 assumes $\varepsilon_w^{\chi} = 1$, lets $y = 0$ but set $A > 0$. The non-homotheticity in the utility function generates a relative marginal benefit of children that is decreasing with w.

The cases highlighted in Proposition 3 hint at the importance of the presence of either non-labor income or non-homothetic utility in order to generate a negative fertility-income relationship under the constrained allocation. Although this is indeed the case here, we have verified in ongoing work that the presence of non-labor income or non-homothetic utility are not required in generalizations of the constrained model to settings with uncertainty. In those settings, what is essential to obtain the negative fertility-income relationship is the credit friction.

The analysis in this section reveals that the constrained model can overcome three key shortcomings of the unconstrained model. In the constrained allocation fertility can decrease with wages, consumption is increasing in labor and non-labor income, and children do not need to be a net financial burden to parents.

3 A life cycle model of fertility and the quantity-quality trade-off

This section extends the benchmark model by incorporating life cycle features, human capital accumulation, and alternative sources of financial frictions. We study the ability of the extended model to predict a negative fertility-wage relationship. We consider two types of financial frictions: non-negative bequests and borrowing constraints for students. We show that key results of the previous section hold. In particular, absent financial frictions, fertility and income are unrelated, but if financial frictions are binding, a negative relationship between fertility and wages can be obtained. As in the benchmark, this is the case if the elasticity of intertemporal substitution is larger than one and preferences, or technologies, are non-homothetic. The result is obtained even when bequests are positive but individuals are constrained during their schooling years. We also study the ability of the model to generate the so called quantity-quality trade-off: a negative association between fertility and schooling years which is typically observed in cross-country data as well as across individuals within a country. We show that the model can generate such pattern: as wages increase, fertility decreases and schooling increases.

3.1 Individual's problem

Individuals live for T periods, go to school from age 0 to age s, work from age s to age T, and have n children at age F. They choose a life-cycle consumption profile $C = [c_0, c_T]$, number of children $n \in [0, N]$, schooling years $s \in [0, F]$, and transfers to each child $b' \geq 0$, in order to maximize their life-time utility. Lifetime resources are composed of non-negative parental transfers b, non-labor income $[y_0, y_T]$, and lifetime labor income $wH(s)$, where w is the wage rate per unit of human capital and $H(s)$ is life-time human capital. $H(s)$ depends positively on years schooling. The lifetime cost of having children is given by $\chi(n, s; w)$.[17] Finally, we assume that children are unable to borrow during schooling years. This has the natural and plausible implication that parental transfers are the only source of income during the first s years.[18]

The following is the recursive formulation of the individual's problem:

$$V(b) = \max_{C=[c_0,c_T],a\geq 0,b'\geq 0,n\in[0,N],0\leq s\leq F} U(C) + \Phi(n)V(b')$$

[17]The main results presented below only depend on the general properties of $H(s)$ and $\chi(n, s; w)$. They could be defined more precisely as $H(s) = \int_0^T e^{-rt}h_t(s)dt$ and $\chi(n, s; w) = \int_0^T e^{-rt}\chi_t(n, s; w)dt$ where $h_t(s)$ is the human capital at age t and $\chi_t(n, s; w)$ is the cost of children at age t.

[18]Notice that in this model individuals decide their own schooling and consumption. They are linked to parents only through the transfers they give them at birth. In Cordoba and Ripoll (2011b) we show that this simpler model yields almost identical steady state predictions to one in which children live with the parent during schooling years, become independent upon finishing school and receive a non-negative transfer from the parent upon independence. The latter is a more complicated model with additional state variables. In the analysis that follows, parental transfers to children are the channel by which the quality-quantity trade-off takes place.

subject to:

$$b + \int_0^s e^{-rt} y_t dt = \int_0^s e^{-rt} c_t dt + e^{-rs} a; \text{ and}$$

$$e^{-rs} a + e^{-rs} w H(s) + \int_s^T e^{-rt} y_t dt = \int_s^T e^{-rt} c_t dt + e^{-rs} \chi(n, s; w) + n e^{-rF} b',$$

In this representation $V(b)$ is the lifetime welfare of the individual, and $U(C)$ is an utility index, or composite good, associated to the consumption profile $C = [c_0, c_T]$. The function $\Phi(n) \geq 0$ is the weight that parents place on the welfare of their n children, which is assumed to have the same properties as before and to satisfy Assumption 1. The constraints of the problem are the two budget constraints: one for the first s year of life and one for the remaining years, both in present value at age 0. Individuals can also choose the amount of assets at age s, a. The two financial frictions imbedded in the problem are the non-negativity of lifetime parental transfers to their children, $b' \geq 0$, and the non-negativity of assets at the end of schooling years, $a \geq 0$. We call these constraints the bequest and borrowing constraints respectively. Let $Y(s) \equiv \int_s^T e^{-r(t-s)} y_t dt$ and $I(n, s; w) \equiv wH(s) + Y(s) - \chi(n, s; w)$ be the present value at age s of non-labor and total income net of child costs after age s respectively.

We focus on the CRRA utility function

$$U(C) = \int_0^T e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt + A,$$

where ρ is the parent's subjective discount rate, and $1/\sigma$ is the elasticity of intertemporal substitution. The constant $A > 0$ allows to consider values of σ larger than 1.

The solution of this model is summarized in Appendix C. In what follows, we present the main steady state insights regarding fertility and schooling. The following proposition considers fertility for the case in which financial constraints are not binding.

Proposition 4. Suppose the non-negative bequest constraint $b' \geq 0$ does not bind. Then steady state optimal fertility n is given by the solution to the equation

$$\frac{\Phi(n)}{n} = e^{-rF}$$

Proof. See Appendix C.

The key message of Proposition 4 is that in a frictionless dynamic altruistic model with endogenous fertility and schooling, steady state fertility is determined by the interest rate. Thus, the main insight from the basic model still holds. As discussed in the previous section, this stems from the fact that with no financial constraints fertility is determined from the bequest optimality condition, rather than from the fertility optimality condition.

We now study the predictions under financial frictions. The analysis below proceeds in two steps. First, we assume $H(0) > 0$ and $H_s(s) = 0$ in order to abstract from schooling choices. In this case it is optimal to choose $s = 0$ and $b = a \geq 0$ so that borrowing constraints are not binding but the bequest constraint may bind. A sufficient, but not necessary, condition for the bequest constraint to bind is given by Assumption 4 below, a condition analogous to Assumption 1. We show that the model can be reduced to a model analogous to that of Section 2 and therefore Proposition 1 and 3 as well as Lemma 2 still apply. In a second step we consider the case $H(0) = 0$ and $H_s > 0$. In this case the bequest constraint is not binding because otherwise children's consumption would be zero. On the other hand, the borrowing constraint may bind. We show that Assumption 2 provides a sufficient condition for the borrowing constraint to bind. Given the added complication of choosing schooling years, s, the analytical characterization is more difficult. However, we confirm through numerical simulations that the results of the simpler model still hold. We also show that the model can produce a quantity-quality trade-off: as wages increase, fertility falls but schooling years increase.

3.2 Case 1: exogenous human capital

Suppose $H(0) > 0$ and $H_s(s) = 0$. In this case it is optimal to pick $s = 0$ and $b = a \geq 0$. Let $H \equiv H(0)$ be human capital. The individual's problem can then be written as:

$$V(b) = \max_{n \in [0,N], b' \geq 0, C = [c_0, c_T]} U(C) + \Phi(n)V(b')$$

subject to

$$b + I(n; w) = \int_0^T e^{-rt} c_t dt + ne^{-rF} b'.$$

We are interested in cases when the bequest constraint binds. The following assumption, analogous to Assumption 2, guarantees that the bequest constraint binds in the steady state. As before, let $\beta(n) \equiv \Phi(n)/n$ denotes the average level of altruism.

Assumption 4. $\lim_{n \to 0} \beta(n) e^{rF} \leq 1$.

To gain some intuition on Assumption 4, suppose that $\beta(0) = e^{-\rho F}$ so that a parent with no children discounts the utility of his first dn children only by the date of their birth using the parent's own time discount rate, ρ. In this case Assumption 4 becomes $e^{-(\rho - r)F} \leq 1$ or $r \leq \rho$ which is a standard assumption for bequest constraints to bind. This is because if the interest rate is below the rate of time preference, parents would like to borrow rather than save or leave bequests. Assumption 4 provides a sufficient but not necessary condition for the bequest to bind.

Proposition 5. Let Assumption 4 hold. Then the bequest constraint binds in steady state.

Proof. See Appendix D.

Optimal fertility Given Assumption 4, optimal fertility can be derived focusing on the steady state situation $b = b' = 0$. It is convenient to solve the problem in two steps: first, find the optimal consumption path given n, and second, solve for n. In absence of bequests and for given n, the optimal consumption plan solves the subproblem:

$$U^*\left(I\left(n;w\right)\right) = \max_{C=[c_0,c_T]} \int_0^T e^{-\rho t}\frac{c_t^{1-\sigma}}{1-\sigma}dt + A \text{ subject to } I(n;w) \geq \int_0^T e^{-rt}c_t dt.$$

The optimality condition is given by the standard equation $c_t = c_0 e^{\frac{r-\rho}{\sigma}t}$ and therefore $I(n;w) = c_0\frac{1-e^{\left(\frac{r-\rho}{\sigma}-r\right)T}}{r-(r-\rho)/\sigma}$. Moreover,

$$U^*(I(n;w)) = \frac{\Theta}{1-\sigma}I(n;w)^{1-\sigma} + A, \tag{20}$$

where $\Theta = \left(\frac{1-e^{-(r-(r-\rho)/\sigma)T}}{r-(r-\rho)/\sigma}\right)^{\sigma} > 0$. Notice that $\partial U^*/\partial I = \Theta I^{-\sigma} > 0$, which implies that the marginal utility of income is decreasing at a rate that depends on σ. Thus, the higher the σ, or the lower the elasticity of intertemporal substitution, the lower the utility value of each additional dollar. Once $U^*\left(I\left(n;w\right)\right)$ is solved for then the parental problem can be recasted as one of choosing only the number of children:

$$V(b) = \max_{n\in[0,N]} U^*\left(I\left(n;w\right)\right) + \Phi(n)V(b'). \tag{21}$$

Assuming an interior solution, the optimality condition for fertility is given by

$$\left(-\frac{\partial I(n;w)}{\partial n}\right) \times \frac{\partial U^*\left(I\left(n;w\right)\right)}{\partial I} = \Phi'(n)V(b'). \tag{22}$$

The left-hand side of equation (22) is the marginal cost and the right-hand side is the marginal benefit of children both measured in personal utils (or composite good). Consider the effects of wages on the marginal benefits and costs. The marginal benefit of children increases with wages because it increases the utility that children enjoy which also increase the utility that altruistic parents derive from children. This effect alone would imply a positive fertility-wage relationship, contrary to the data. To obtain a negative relationship the marginal cost of children must not only increase with wages but increases more than the marginal benefit. Whether the marginal cost of children increases with wages depends on the result of two opposite effects. On one hand, the marginal cost increases with w because the opportunity cost of forgone labor income by the parents increases. That is, $-(\partial I(n;w)/\partial n)$ increases with w for any given number of children. On the other hand, the marginal utility of income, $\partial U^*\left(I\left(n;w\right)\right)/\partial I$ may decrease with wages. In a well-behaved model the first effect should dominate the second effect so that the marginal cost of children increases with wages. As we confirm next, this requires $\sigma > 1$.

Along a steady state equation (22) can be written, using (21) and the definition of $I(n;w)$, as:

$$\chi_n(n;w)\Theta I(n)^{-\sigma} = \frac{\Phi'(n)}{1-\Phi(n)} U^*\left(I\left(n\right)\right).$$

Finally, using (20), this equation can be written in a way analogous to (13):

$$\epsilon_I^{U^*}\left(n^*;w\right)\epsilon_n^{\chi}\left(n^*;w\right)\frac{\chi(n^*,w)}{wH+Y-\chi(n^*,w)} = \epsilon_n^{\Phi}(n^*)\frac{\Phi(n^*)}{1-\Phi(n^*)}. \tag{23}$$

Since (13) and (23) are qualitatively the same, it is easy to check that Propositions 1 and 3 and Lemma 2 still apply.

3.3 Case 2: endogenous human capital

We now consider the case of endogenous schooling. In particular, suppose now that $H(0) = 0$ and $H_s(s) > 0$. Given these assumptions and the Inada condition for utility, it is optimal for parents to leave positive bequests. Otherwise children's consumption would be zero for a positive time interval, which is suboptimal. Furthermore, optimal schooling is now positive. The model with endogenous human capital adds two complications to the analysis as optimal bequest and schooling need to be endogenously determined in addition to fertility. To simplify the analysis, it is convenient to assume $r = \rho$, $T = \infty$ and $Y = 0$. It is also convenient to postulate specific functional forms. In particular, suppose $\Phi(n) = \frac{\mu n}{1+\mu n}$ and $\chi(n,s;w) = wH(s)\left(1-f(n)\right)$ where $f(n)$ is a function satisfying $f(0) = 1$ and $f'(n) < 0$. This formulation of the cost of children $\chi(n,s;w)$ implies that labor costs are the main costs. As a result of this formulation, lifetime income is given by $I(n,s;w) = wf(n)H(s)$ where $f(n)$ is the effective lifetime labor supply of parents, a decreasing function of the number of children.

It is straightforward to show that if $r = \rho$ then $c_t = c^S$ for $t \leq s$ and $c_t = c^W$ for $t \geq s$. In the absence of borrowing constraints it would be optimal to perfectly smooth consumption by setting $c^S = c^W$. This is not the case under Assumption 4, as the following proposition summarizes.

Proposition 6. Let Assumption 4 hold. Then the borrowing constraint binds in steady state.

Proof. See Appendix D.

We assume that Assumption 4 holds in the remainder of this section. As shown in Appendix C, the first order condition with respect to fertility is given by:

$$\left(-e^{-rs}I_n(n,s;w) + e^{-rF}b'\right)u'\left(c^W\right) = \Phi'(n)V\left(b'\right)$$

This condition is similar to (22) but now both s and b' are not zero and need to be determined.

Appendix C shows that fertility and schooling are jointly determined by the following two equations:

$$-\frac{f'(n)}{f(n)} = \frac{\pi(s,n)\,\beta(n)/r}{1-\sigma}\left[\left(\beta(n)e^{rF}\right)^{(1-\sigma)/\sigma}(e^{rs}-1)\,\sigma+1\right]+(\pi(s,n))^{\sigma}\,(I(n,s;w))^{\sigma-1}\,\beta(n)e^{rs}A, \tag{24}$$

$$\frac{H'(s)}{H(s)} = r+\frac{\sigma}{1-\sigma}\left(1-(\beta(n)e^{rF})^{(1-\sigma)/\sigma}\right)\pi(s,n)\,, \tag{25}$$

where

$$\pi(s,n) \equiv \frac{c^W}{I(n,s;w)} = \frac{r}{1+ne^{-rF}\left(\beta(n)e^{rF}\right)^{1/\sigma}(e^{rs}-1)}.$$

Consider first the fertility decision. The left-hand-side of equation (24) is the marginal cost of children as a fraction of lifetime labor. The right-hand-side is the relative marginal benefit of children. First notice that if $A=0$ then income $I(n,s;w)$ is not a determinant of fertility, nor of schooling. Next, if $A>0$ and $\sigma\in(0,1)$ then the relative marginal benefit of children falls with income as in the previous sections. Consider now the quantity-quality trade-off between fertility and schooling implied by equation (24). First, since schooling increases lifetime income, then fertility is a negative function of schooling through this channel when $A>0$ and $\sigma\in(0,1)$. The second key term involved in the quantity-quality trade-off is $\pi(s,n)$, the fraction of lifetime income consumed by parents. Notice that $\pi_s(s,n)<0$ which means that schooling further reduces the marginal benefit of children through this channel. To understand why, notice that $1+ne^{-rF}\left(\beta(n)e^{rF}\right)^{1/\sigma}(e^{rs}-1)$ is the "effective" life span of a parent. Absent children, this effective life span is 1. Term $(e^{rs}-1)$ is the life span of a child while at school. The longer children stay at school, the longer the parent needs to finance their consumption which explains why $\pi(s,n)$ falls with s.

Consider now the schooling decision. The left-hand-side of equation (25) represents the returns to schooling measured as the rate at which lifetime income increases with additional schooling; the right-hand-side is the opportunity cost of schooling. To better understand this equation, suppose for a moment that $\beta(n)e^{rF}=1$ which means that credit frictions are not binding. In that case the opportunity cost of schooling is just the rate of returns to savings. If credit frictions are binding, $\beta(n)e^{rF}<1$, the opportunity cost of schooling is larger because the implicit shadow price of credit is larger (see Cordoba and Ripoll (2011b) for more discussion).

Whether the opportunity cost of schooling increases or decreases with fertility depends on two potentially offsetting effects. On the one hand, higher fertility increases the shadow price of credit through the term $\sigma(1-\sigma)^{-1}\left(1-(\beta(n)e^{rF})^{(1-\sigma)/\sigma}\right)$ which is associated to the consumption jump that occurs at time individuals become workers. On the other hand, term $\pi(s,n)$ typically decreases with the number of children, although the sign of $\pi_n(s,n)$ is not fully determined, because parents need to reduce their consumption to finance schooling for a larger number of children. In the simulations below we find that the first effect always dominates the second effect because the main component of the effective life span of the parent is the life span of the parent, 1, rather than the life span associated to their children, $ne^{-rF(s)}\beta(n)^{1/\sigma}(e^{rs}-1)$.

Numerical simulation Although a fully calibrated version of the model is beyond the scope of this paper, here we provide a numerical simulation to illustrate the ability of the model to replicate the quality-quantity trade-off. The functional forms assumed are $H(s) = s^{\gamma}$, $f(n) = 1 + \kappa^{\theta} - (\delta n + \kappa)^{\theta}$ and $\Phi(n) = \frac{\mu n}{1+\mu n}$. The following are the parameter values used: $r = 3\%$, $F = 30$, $\sigma = 0.25$, $\mu = 0.40$, $\gamma = 0.8$, $\theta = 0.225$, $\kappa = 230$ and $\delta = 140$. This parametrization implies lifetime labor supplies associated to having 0, 0.5 and 1 child of 1, 0.79 and 0.61 respectively. Keep in mind that 0.5 children per parent means a child per couple. Moreover, the degree of altruism implied is $\Phi(0) = 0$, $\Phi(0.5) = 0.16$ and $\Phi(1) = 0.28$ which implicitly includes time discounting due to the fact that children are born at age 30. Finally, the implied returns to schooling at age 16 are 5%.

Figure 2 shows the predictions of the model for schooling and fertility for different wage levels. The relevant scale of the model is $A/w^{1-\sigma}$. For sufficiently high wages this scale is zero so that fertility and schooling are independent of wages. In the figure, schooling for high wage, or high ability, individuals is close to 23 years while fertility is close to 0.5 children. For low wage individuals, schooling is close to 19.5 year while fertility is just below 3.5 children. This simulation confirms that the model can generate the following results: (i) fertility decreases with wages; (ii) schooling increases with wages; (iii) a quantity-quality trade-off as fertility decreases as schooling increases.

We can also check if children are a net benefit or cost to society in the sense that the present value of their earnings are larger than the present value of their costs. The marginal cost of a child for a parent (when the parent is age 0) is $e^{-rs}\chi_n(n, s; w) = -e^{-rs}wH(s)f'(n)$. Since the child is born at time F and starts working at age s, the present value of the child earnings (when the parent is age-0) is $e^{-r(F+s)}wH(s)$. Therefore, the net benefit to society of a newborn is $e^{-rs}wH(s)\left(e^{-rF} + f'(n)\right)$. This surplus is positive if $e^{-rF} > -f'(n)$, which holds for this numerical simulation for all wages as shown in Figure 3.

4 Concluding comments

The Barro-Becker model is important because it ultimately corresponds to the neoclassical growth model with endogenous fertility. However, endogenizing fertility within this framework turns out to generate predictions that overturn some of the most appealing implications of neoclassical growth such as the existence of a unique steady state, the dependence of consumption growth on the interest rate and the rate of time preference, or the positive relationship between consumption and income. We have shown that introducing credit frictions into the Barro-Becker model helps restoring some of the appealing predictions of the neoclassical model as well as providing new plausible predictions for fertility and schooling.

Regarding fertility choices, the BB model predicts that fertility depends mainly on the interest rate and on the degree of altruism, but it is independent of income. One of the implications of this theory is that if we were to explain the cross-sectional distribution of fertility within the same country, one would have to attribute it to either differences in altruism across parents, or to different interest rates parents face. None of these explanations seems appealing. We show that introducing credit frictions in the BB model makes income a determinant of fertility. This implies that within the same country, parents with different levels of income would choose different fertility rates. In fact, using micro US data, Jones and Tertilt (2006) estimate an income elasticity of fertility of about -0.38.

This paper has focused on explaining the pervasive negative fertility-income relationship apparent in the data. Hazan and Zoabi (2012) have recently provided some evidence that this relationship may actually be non-monotonic, generally decreasing with wages but slightly increasing for highly educated workers. Models in this paper could explain that pattern. Consider for example Proposition 2 with $\varepsilon_w^\chi < 1$ and $y > 0$. For relatively low wages $\varepsilon_w^\chi (1 + y/w) > 1$ is satisfied which implies that fertility decreases with wages. However, for sufficiently high wages $\varepsilon_w^\chi (1 + y/w) < 1$ which implies that fertility increases with wages.

One of the most interesting findings here is that even though income becomes a determinant of fertility in the presence of credit frictions, obtaining the negative fertility-income relationship is challenging, specially if the elasticity of intertemporal substitution is low, as it is standard in quantitative macro. In a companion paper, Cordoba and Ripoll (2011a), we explore other types of preferences that may help generate a fertility-income relationship consistent with the data without making assumptions on parameters that run contrary to other evidence used in quantitative macroeconomics.

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A Solution of benchmark model

The individuals' problem for $t \geq 0$ is:

$$V_t(b_t) = \max_{b_{t+1} \geq 0,\ N \geq n_t \geq 0} U(I_t(n_t; w_t, y_t) + (1 + r_t)\, b_t - n_t b_{t+1}) + \Phi(n_t) V_{t+1}(b_{t+1}), \tag{26}$$

where $I_t(n_t; w_t, y_t) \equiv w_t + y_t - \chi(n_t; w_t)$. Optimality conditions for the bequest and fertility choices are:

$$U'(c_t) \geq U'(c_{t+1})\,(1+r_{t+1})\,\Phi(n_t)/n_t, \tag{27}$$

and

$$U'(c_t)\,(b_{t+1} + \chi_n(n_t; w_t)) = \Phi'(n_t) V_{t+1}(b_{t+1}), \tag{28}$$

where

$$c_t = w_t + y_t + (1+r_t)\,b_t - \chi(n_t, w_t) - n_t b_{t+1}. \tag{29}$$

Unconstrained solution Equation (27) can be written as

$$\epsilon_c^U(c_t)\frac{U(c_t)}{c_t} = \epsilon_c^U(c_{t+1})\frac{U(c_{t+1})}{c_{t+1}}\,(1+r_{t+1})\,\frac{\Phi(n_t)}{n_t}. \tag{30}$$

In this steady state this equation becomes

$$1 = (1+r)\,\frac{\Phi(n)}{n}.$$

On the other hand, using (30) and (29), equation (28) can be written as:

$$\begin{aligned}
\Phi(n_t)V_{t+1}(b_{t+1}) &= (n_t b_{t+1} + n_t \chi_n(n_t; w_t))\,\frac{\epsilon_c^U(c_t)}{\epsilon_n^\Phi(n_t)}\frac{U(c_t)}{c_t} \\
&= (w_t + y_t + (1+r_t)\,b_t - \chi(n_t; w_t) + n_t\chi_n(n_t; w_t) - c_t)\,\frac{\epsilon_c^U(c_t)}{\epsilon_n^\Phi(n_t)}\frac{U(c_t)}{c_t} \\
&= (w_t + y_t + (1+r_t)\,b_t + \chi(n_t; w_t)\,(\epsilon_n^\chi(n_t; w_t) - 1) - c_t)\,\frac{\epsilon_c^U(c_t)}{\epsilon_n^\Phi(n_t)}\frac{U(c_t)}{c_t}.
\end{aligned} \tag{31}$$

Lagging (31) one period and using (30) results in:

$$\begin{aligned}
V_t(b_t) &= \frac{n_{t-1}\,(b_t + \chi_n(n_{t-1}; w_{t-1}))}{\Phi(n_{t-1})}\frac{1}{\epsilon_n^\Phi(n_{t-1})}\frac{\epsilon_c^U(c_{t-1})U(c_{t-1})}{c_{t-1}} \\
&= \frac{n_{t-1}\,(b_t + \chi_n(n_{t-1}; w_{t-1}))}{\Phi(n_{t-1})}\frac{\epsilon_c^U(c_t)}{\epsilon_n^\Phi(n_{t-1})}\frac{U(c_t)}{c_t}\,(1+r_t)\,\frac{\Phi(n_{t-1})}{n_{t-1}} \\
&= (1+r_t)\,(b_t + \chi_n(n_{t-1}; w_{t-1}))\,\frac{\epsilon_c^U(c_t)}{\epsilon_n^\Phi(n_{t-1})}\frac{U(c_t)}{c_t}.
\end{aligned} \tag{32}$$

Substituting (31) and (32) into (26) yields:

$$\begin{aligned}
&(1+r_t)\,(b_t + \chi_n(n_{t-1}; w_{t-1}))\,\frac{\epsilon_c^U(c_t)}{\epsilon_n^\Phi(n_{t-1})}\frac{U(c_t)}{c_t} \\
=\ & U(c_t) + (w_t + y_t + (1+r_t)\,b_t + \chi(n_t; w_t)\,(\epsilon_n^\chi(n_t; w_t) - 1) - c_t)\,\frac{\epsilon_c^U(c_t)}{\epsilon_n^\Phi(n_t)}\frac{U(c_t)}{c_t},
\end{aligned}$$

or

$$\begin{aligned}
&(1+r_t)\,(b_t + \chi_n(n_{t-1}; w_{t-1}))\,\frac{\epsilon_c^U(c_t)}{\epsilon_n^\Phi(n_{t-1})} \\
=\ & \left(\frac{\epsilon_n^\Phi(n_t) - \epsilon_c^U(c_t)}{\epsilon_n^\Phi(n_t)}\right) c_t + (w_t + y_t + (1+r_t)\,b_t + \chi(n_t; w_t)\,(\epsilon_n^\chi(n_t; w_t) - 1))\,\frac{\epsilon_c^U(c_t)}{\epsilon_n^\Phi(n_t)}.
\end{aligned}$$

Finally, collecting terms we obtain:

$$c_t = \frac{\epsilon_c^U(c_t)}{\epsilon_n^\Phi(n_t) - \epsilon_c^U(c_t)} \times \tag{33}$$
$$\left[(1+r_t)\left(b_t + \chi_n(n_{t-1}; w_{t-1})\right) \frac{\epsilon_n^\Phi(n_t)}{\epsilon_n^\Phi(n_{t-1})} - \left(w_t + y_t + (1+r_t)\, b_t + \chi\left(n_t; w_t\right)\left(\epsilon_n^\chi(n_t; w_t) - 1\right)\right) \right].$$

This provides the solution for consumption. A simplification arises when functions U and Φ are isoelastic, as in BB. In this case the previous equation becomes:

$$c_t = \frac{\gamma}{\theta - \gamma}\left[(1+r_t)\,\chi_n(n_{t-1}; w_{t-1}) - \left(w_t + y_t + \chi\left(n_t; w_t\right)\left(\epsilon_n^\chi(n_t; w_t) - 1\right)\right)\right].$$

BB further assume $\chi\left(n_t; w_t\right) = \chi\left(w_t\right) n_t$, so that $\epsilon_n^\chi(n_t) = 1$, and $y = 0$ which further simplifies the previous equation to:

$$c_t = \frac{\gamma}{\theta - \gamma}\left[(1+r_t)\,\chi(w_{t-1}) - w_t\right].$$

In the steady state equation (33) becomes:

$$c = \frac{\epsilon_c^U(c)}{\epsilon_n^\Phi(n) - \epsilon_c^U(c)}\left[(1+r)\,\chi_n(n; w) + \chi\left(n; w\right)\left(1 - \epsilon_n^\chi(n)\right) - w - y\right],$$

which corresponds to equation (8) in the text. Next, the budget constraint in steady state reads:

$$(1 + r - n)\, b = c - \left(w + y - \chi\left(n; w\right)\right) = c - I(n; w, y).$$

Plugging in steady state consumption and simplifying:

$$(1 + r - n)\left(\epsilon_n^\Phi(n) - \epsilon_c^U(c)\right) b = \epsilon_c^U(c)\left[(1+r)\,\chi_n(n; w) - \epsilon_n^\chi(n; w)\chi\left(n; w\right)\right] - I\left(n; w, y\right)\epsilon_n^\Phi(n),$$

and since $(1 + r - n)\left(\epsilon_n^\Phi(n) - \epsilon_c^U(c)\right) > 0$, then a necessary and sufficient condition for $b \geq 0$ is, using (27) in steady state:

$$\begin{aligned}
& \epsilon_c^U(c)\left[(1+r)\,\chi_n(n; w) - \epsilon_n^\chi(n; w)\chi\left(n; w\right)\right] \\
= \; & \epsilon_c^U(c)\epsilon_n^\chi(n; w)\chi(n; w)\left(\frac{1+r}{n} - 1\right) \\
= \; & \epsilon_c^U(c)\epsilon_n^\chi(n; w)\chi(n; w)\left(\frac{1 - \Phi(n)}{\Phi(n)}\right) \\
> \; & I\left(n; w, y\right)\epsilon_n^\Phi(n) = \left(w + y - \chi(n; w)\right)\epsilon_n^\Phi(n),
\end{aligned}$$

or

$$\frac{\epsilon_c^U(c)\epsilon_n^\chi(n; w)}{\epsilon_n^\Phi(n)}\frac{1 - \Phi(n)}{\Phi(n)} > \frac{w + y - \chi(n; w)}{\chi(n; w)}$$

which provides the basis for Assumption 2 in the text.

Constrained solution The following equations describe the constrained solution:

$$V_t = U(c_t) + \Phi(n_t)V_{t+1},$$

$$c_t = w_t + y_t - \chi\left(n_t; w_t\right),$$

$$U'(w_t + y_t - \chi(n_t; w_t))\chi_n(n_t; w_t) = \Phi'(n_t)V_{t+1},$$

where the last equation can be written as:

$$\epsilon_c^U(c_t)\epsilon_n^\chi(n_t; w_t)\frac{U(c_t)}{c_t}\frac{\chi(n_t; w_t)}{n_t} = \epsilon_n^\Phi(n_t)\frac{\Phi(n_t)}{n_t}V_{t+1}.$$

In the steady state this equation reads:

$$\epsilon_c^U(c^*)\epsilon_n^\chi(n^*; w)\frac{U(c^*)}{c^*}\chi(n^*; w) = \epsilon_n^\Phi(n^*)\frac{\Phi(n^*)}{1-\Phi(n^*)}U(c^*),$$

$$\epsilon_c^U(c^*)\epsilon_n^\chi(n^*; w)\frac{\chi(n^*; w)}{c^*} = \epsilon_n^\Phi(n^*)\frac{\Phi(n^*)}{1-\Phi(n^*)},$$

$$\epsilon_c^U(c^*)\epsilon_n^\chi(n^*; w)\frac{\chi(n^*; w)}{w + y - \chi(n^*; w)} = \epsilon_n^\Phi(n^*)\frac{\Phi(n^*)}{1-\Phi(n^*)},$$

which corresponds to equation (13) in the text.

B Proofs - section 2

This appendix includes the proofs of all propositions and lemmas for the benchmark model (section 2).

Proof of Proposition 1 Suppose $w \to \infty$. From equation (11) it follows that $c \to \infty$ since $n^* \leq N$. Furthermore, notice that $\epsilon_c^U(c^*) = \frac{1}{\frac{1}{1-\sigma} + Ac^{\sigma-1}} > 0$. Consider first the case $\sigma > 1$. In that case, $\lim_{w\to\infty} \epsilon_c^U(c^*) = 0$. Thus, the limit of the relative marginal cost, in equation (13), is zero while the relative marginal benefit is positive for any $n \leq N$. As a result, maximum fertility, $n^* = N$, is optimal. For the case $\sigma < 1$, $\lim_{w\to\infty} \epsilon_c^U(c^*) = 1 - \sigma$ and equation (13) can be written as (14).

Proof of Lemma 2 First, write (10) as

$$\ln \chi_n(n; w) - \sigma \ln c - \ln\left(\frac{1}{1-\sigma}c^{1-\sigma} + A\right) = \ln \Phi'(n) - \ln(1 - \Phi(n)).$$

Totally differentiating this equation around the steady state one obtains

$$\frac{\chi_{nw}(n; w)dw + \chi_{nn}(n; w)dn}{\chi_n(n; w)} - \sigma\frac{dc}{c} - \frac{c^{-\sigma}dc}{\frac{1}{1-\sigma}c^{1-\sigma} + A} = \frac{\Phi"}{\Phi'}dn + \frac{\Phi'}{1-\Phi}dn,$$

or

$$\frac{\chi_{nw}(n; w)dw}{\chi_n(n; w)} - \Psi dc/c = \left[\frac{\Phi"}{\Phi'} + \frac{\Phi'}{1-\Phi} - \frac{\chi_{nn}(n; w)}{\chi_n(n; w)}\right]dn,$$

where

$$\Psi(c) = \sigma + \frac{c^{1-\sigma}}{\frac{1}{1-\sigma}c^{1-\sigma} + A} = \sigma + \epsilon_c^U(c) > 0.$$

Since (11) implies $dc = -\chi_n(n; w)dn + (1 - \chi_w(n; w))\,dw$, then the previous equation can be written as

$$\left[\frac{\chi_{nw}(n; w)}{\chi_n(n; w)} - \frac{\Psi}{c}(1 - \chi_w(n; w))\right]dw = \left[\frac{\Phi"}{\Phi'} + \frac{\Phi'}{1-\Phi} - \frac{\chi_{nn}(n; w)}{\chi_n(n; w)} - \chi_n(n; w)\frac{\Psi}{c}\right]dn.$$

Next, notice from (10) that

$$\frac{\Phi'(n)}{1-\Phi(n)} = \frac{\chi_n(n;w)}{c}\left(\Psi-\sigma\right),$$

and thus

$$\frac{dn}{dw} = \frac{-\frac{\chi_{nw}(n;w)}{\chi_n(n;w)} + \frac{\Psi}{c}\left(1-\chi_w(n;w)\right)}{-\frac{\Phi"}{\Phi'} + \sigma\frac{\chi_n(n;w)}{c} + \frac{\chi_{nn}(n;w)}{\chi_n(n;w)}}.$$

Since by assumption $\chi_{nn}(n;w) \geqslant 0$, then the denominator is always positive, then $dn/dw < 0$ if and only if $-\frac{\chi_{nw}(n;w)}{\chi_n(n;w)} + \frac{\Psi}{c}\left(1-\chi_w(n;w)\right) < 0$, or $(w+y-\chi)\,\epsilon_w^{\chi_n} > \Psi\left(w - \epsilon_w^{\chi}\chi\right)$.[19] This expression can be written as

$$\epsilon_w^{\chi_n}\left(n^*;w\right)\left(1+\frac{y}{w}\right) + \left(\Psi(c^*)\epsilon_w^{\chi}\left(n^*;w\right) - \epsilon_w^{\chi_n}\left(n^*;w\right)\right)\frac{\chi\left(n^*;w\right)}{w} > \Psi(c^*),$$

which corresponds to (15) in the Lemma. In addition, when $\epsilon_n^{\chi}\left(n^*;w\right) = \epsilon_n^{\chi}$ then $\epsilon_w^{\chi}\left(n^*;w\right) = \epsilon_w^{\chi_n}\left(n^*;w\right)$. To see this, notice that in this case $\chi_n(n;w) = \epsilon_n^{\chi}\chi(n;w)/n$ for all (n,w). Therefore, $\chi_{nw}(n;w) = \epsilon_n^{\chi}\frac{\chi_w(n^*;w)}{n}$ and

$$\begin{aligned}
\epsilon_w^{\chi_n}\left(n^*;w\right) &= \chi_{nw}\left(n^*;w\right)\frac{w}{\chi_n\left(n^*;w\right)} = \epsilon_n^{\chi}\left(n^*;w\right)\frac{\chi_w\left(n^*;w\right)}{n}\frac{w}{\chi_n\left(n^*;w\right)} \\
&= \epsilon_n^{\chi}\chi_w\left(n^*;w\right)\frac{w}{\chi\left(n^*;w\right)}\frac{\chi\left(n^*;w\right)/n}{\chi_n\left(n^*;w\right)} = \epsilon_n^{\chi}\left(n^*;w\right)\epsilon_w^{\chi}\left(n^*;w\right)\frac{\chi\left(n^*;w\right)/n}{\chi_n\left(n^*;w\right)} \\
&= \epsilon_n^{\chi}\left(n^*;w\right)\epsilon_w^{\chi}\left(n^*;w\right)\frac{1}{\epsilon_n^{\chi}\left(n^*;w\right)} = \epsilon_w^{\chi}.
\end{aligned}$$

Making $\epsilon_w^{\chi}\left(n^*;w\right) = \epsilon_w^{\chi_n}\left(n^*;w\right)$ in (15) results in (16).

Proof of Proposition 3 (i) In this case $\Psi(c^*) = 1$ and equation (16) simplifies to $\varepsilon_w^{\chi}\left(n^*;w\right)(1+y/w) > 1$; (ii) If $0 < \sigma < 1$ and $A > 0$ then $\Psi(c^*) < \Psi(\infty) = 1$. In addition, equation (16) can be written as $\left(1-\Psi(c^*)\right)\left(1-\chi\left(n^*\right)/w\right) > 0$ which is satisfied because both factors are positive.

C Solution of model with fertility and schooling

Consider the individual's problem in the text:

$$V\left(b\right) = \max_{C=[c_0,c_T],a\geq 0,b'\geq 0,n\in[0,N],0\leq s\leq F} U(C) + \Phi(n)V\left(b'\right),$$

subject to

$$b + \int_0^s e^{-rt}y_t dt = \int_0^s e^{-rt}c_t dt + e^{-rs}a, \tag{34a}$$

$$e^{-rs}a + e^{-rs}I(n,s;w) = \int_s^T e^{-rt}c_t dt + ne^{-rF}b', \tag{35a}$$

where $I(n,s;w) \equiv wH(s) + \displaystyle\int_s^T e^{-r(t-s)}y_t dt - \chi(n,s;w)$.

[19] Notice that $\chi_{nn}(n;w) \geqslant 0$ is sufficient but not necessary for the result. We could have $\chi_{nn}(n;w) < 0$ as long as it is small in absolute value.

C.1 General solution

The Lagrangian associated to the problem above is given by:

$$\mathcal{L} = \int_0^s e^{-\rho t} u(c_t)\, dt + \int_s^T e^{-\rho t} u(c_t) dt + A + \Phi(n) V(b') + \lambda_1 \left[b + \int_0^s e^{-rt} y_t dt - \int_0^s e^{-rt} c_t dt - e^{-rs} a \right]$$
$$+\lambda_2 \left[e^{-rs} a + e^{-rs} I(n,s;w) - \int_s^T e^{-rt} c_t dt - n e^{-rF} b' \right] + \lambda_3 a + \lambda_4 b',$$

and the first order necessary conditions with respect to c_t, s, n, b' and a are respectively:

$$\begin{cases} e^{-\rho t} u'(c_t) = \lambda_1 e^{-rt} \text{ for } t \leq s \\ e^{-\rho t} u'(c_t) = \lambda_2 e^{-rt} \text{ for } t > s \end{cases},$$

$$-\Delta u_s e^{-\rho s} - \lambda_1 \left[-r e^{-rs} a - e^{-rs} y_s + c_s^S e^{-rs} \right]$$
$$+\lambda_2 \left[-r e^{-rs} (a + I(n,s;w)) + e^{-rs} I_s(n,s;w) + c_s^W e^{-rs} \right] = 0,$$
$$\Phi'(n) V(b') = \lambda_2 \left[e^{-rF} b' - e^{-rs} I_n(n,s;w) \right],$$
$$\Phi(n) V'(b') + \lambda_4 = \lambda_2 e^{-rF} n, \tag{36}$$

and

$$e^{-rs} (\lambda_1 - \lambda_2) = \lambda_3,$$

where

$$\Delta u_s \equiv u(c_s^W) - u(c_s^S),$$

c_s^W denotes the consumption at age s as worker, and c_s^S the consumption at age s as student. If the borrowing constraint between ages 0 and s binds, so that $a = 0$ and $\lambda_3 > 0$, then $\lambda_1 \neq \lambda_2$ and $c_s^W \neq c_s^S$. If the borrowing constraint does not bind, then $c_s^W = c_s^S$ and $\Delta u_s = 0$, so there is perfect consumption smoothing. In addition to the optimality conditions above, notice that the following envelope condition holds:

$$V'(b) = \lambda_1. \tag{37}$$

C.2 Special case with $r = \rho$

As in section 3.3., consider the special case in which $r = \rho$ and $y_t = 0$ for all t. In this case the individual problem can be written as

$$V(b) = \max_{n \in [0,N], c^S, c^W, b' \geq 0, s \in [0,F]} u(c^S) \frac{1 - e^{-rs}}{r} + u(c^W) \frac{e^{-rs} - e^{-rT}}{r} + A + \Phi(n) V(b'),$$

subject to

$$b = c^S \left(1 - e^{-rs}\right)/r,$$
$$e^{-rs} I(n,s;w) = c^W \left(e^{-rs} - e^{-rT}\right)/r + n e^{-rF} b',$$

where $I(n,s;w) \equiv wH(s) - \chi(n,s;w)$. Recall that $\chi(n,s;w) = wH(s)(1 - f(n))$ so that $I(n,s;w) = wH(s)f(n)$. As discussed in section 3.3, in this case it is optimal for parents to give a positive bequest, as otherwise student's consumption would be zero while at school. It would also be the case that $a = 0$. Let λ_1 and λ_2 be the multipliers on the two budget constraints.

Consumption and bequests The first-order necessary conditions with respect to c^S, c^W and b' and the envelope condition are:

$$u'\left(c^S\right)=\lambda_1; \qquad u'\left(c^W\right)=\lambda_2; \qquad \Phi(n)V'\left(b'\right)=\lambda_2 n e^{-rF}; \qquad V'(b)=\lambda_1.$$

These equations can be written as $\Phi(n)\lambda_1=\lambda_2 n e^{-rF}$, or in steady state:

$$c^S=c^W\psi(n)^{1/\sigma},$$

where $\psi(n)\equiv\frac{\Phi(n)}{n}e^{rF}=\beta(n)e^{rF}<1$ which guarantees that the borrowing constraint binds. The two budget constraints and this condition can be used to solve for c^S, c^W and b as:

$$c^W=\pi(s,n)\,I(n,s;w),$$

$$c^S=\pi(s,n)\,\psi(n)^{1/\sigma}I(n,s;w),$$

and

$$b=\pi(s,n)\,\psi(n)^{1/\sigma}\left(1-e^{-rs}\right)I(n,s;w)/r,$$

where

$$\pi(s,n)=\frac{r}{1-e^{-r(T-s)}+ne^{-rF}\psi(n)^{1/\sigma}\left(e^{rs}-1\right)}.$$

In addition, in the steady state:

$$\begin{aligned} V(b) &= \frac{u\left(c^S\right)\left(1-e^{-rs}\right)/r+u\left(c^W\right)\left(e^{-rs}-e^{-rT}\right)/r+A}{1-\Phi(n)} \\ &= \frac{1}{1-\sigma}\frac{\left(c^W\right)^{1-\sigma}\left[\psi(n)^{(1-\sigma)/\sigma}\left(1-e^{-rs}\right)+\left(e^{-rs}-e^{-rT}\right)\right]/r}{1-\Phi(n)}+\frac{A}{1-\Phi(n)}. \end{aligned}$$

Fertility The first order condition with respect to fertility is:

$$\Phi'(n)V\left(b'\right)=\lambda_2\left(-e^{-rs}I_n(n,s;w)+e^{-rF}b'\right).$$

The left-hand-side is the marginal benefit while the right-hand-side is the marginal cost. In steady state, using the solutions for V and $b:$

$$\begin{aligned} &\left[\frac{1}{1-\sigma}\left(c^W\right)^{1-\sigma}\left[\psi(n)^{(1-\sigma)/\sigma}\left(1-e^{-rs}\right)+\left(e^{-rs}-e^{-rT}\right)\right]/r+A\right]\frac{\Phi'(n)}{1-\Phi(n)} \\ =\; & \lambda_2 e^{-rs}\chi_n(n,s;w)+\lambda_2 e^{-rF}\pi(s,n)\,\psi(n)^{1/\sigma}\left(1-e^{-rs}\right)I(n,s;w)/r. \end{aligned}$$

Suppose $\Phi(n)=\frac{\mu n}{1+\mu n}$. Then

$$\begin{aligned} \Phi'(n) &= \frac{\mu}{1+\mu n}-\frac{\mu^2 n}{(1+\mu n)^2}=\frac{\mu n}{1+\mu n}\frac{1}{n}-\frac{\mu^2 n^2}{(1+\mu n)^2}\frac{1}{n} \\ &= \frac{\Phi(n)}{n}\left(1-\frac{\mu n}{1+\mu n}\right)=\frac{\Phi(n)}{n}\left(1-\Phi(n)\right)=\psi(n)\left(1-\Phi(n)\right)e^{-rF}, \end{aligned}$$

so that $\frac{\Phi'(n)}{1-\Phi(n)} = \psi(n)e^{-rF}$. In this case the first order condition can be written as

$$
\begin{aligned}
& \lambda_2 \left[e^{-rs}\chi_n(s,n;w) + e^{-rF}\pi(s,n)\,\psi(n)^{1/\sigma}\left(1-e^{-rs}\right) I(n,s;w)/r \right] \\
= & \left[\frac{1}{1-\sigma}\left(c^W\right)^{1-\sigma}\left[\psi(n)^{(1-\sigma)/\sigma}\left(1-e^{-rs}\right) + \left(e^{-rs}-e^{-rT}\right)\right]/r + A \right] \frac{\Phi'(n)}{1-\Phi(n)} \\
= & \frac{e^{-rF}}{1-\sigma} c^W \lambda_2 \left[\psi(n)^{(1-\sigma)/\sigma}\psi(n)\left(1-e^{-rs}\right) + \psi(n)\left(e^{-rs}-e^{-rT}\right)\right]/r + \psi(n)e^{-rF}A \\
= & \frac{e^{-rF}\lambda_2}{1-\sigma}\pi(s,n)\,I(n,s;w)\left[\psi(n)^{1/\sigma}\left(1-e^{-rs}\right) + \psi(n)\left(e^{-rs}-e^{-rT}\right)\right]/r + \psi(n)e^{-rF}A,
\end{aligned}
$$

or collecting terms

$$
\begin{aligned}
e^{-rs}\chi_n(n,s;w) = & \frac{e^{-rF}/r}{1-\sigma}\pi(s,n)\,I(n,s;w)\left[\psi(n)^{1/\sigma}\left(1-e^{-rs}\right)\sigma + \psi(n)\left(e^{-rs}-e^{-rT}\right)\right] \\
& + \left(\pi(s,n)\,I(n,s,w)\right)^{\sigma}\psi(n)e^{-rF}A,
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{\chi_n(n,s;w)}{I(n,s;w)} = & \frac{e^{-rF}\pi(s,n)/r}{1-\sigma}\left[\psi(n)^{1/\sigma}\left(e^{rs}-1\right)\sigma + \psi(n)\left(1-e^{-r(T-s)}\right)\right] \\
& + \left(\pi(s,n)\right)^{\sigma}\left(I(n,s;w)\right)^{\sigma-1}\psi(n)e^{-r(F-s)}A.
\end{aligned}
$$

In order to simplify the expression above, consider the case when $T=\infty$. In this case the expression simplifies to

$$
\frac{\chi_n(n,s;w)}{I(n,s;w)} = \frac{e^{-rF}\pi(s,n)/r}{1-\sigma}\left[\psi(n)^{1/\sigma}\left(e^{rs}-1\right)\sigma + \psi(n)\right] + \left(\pi(s,n)\right)^{\sigma}\left[I(n,s;w)\right]^{\sigma-1}\psi(n)e^{-r(F-s)}A
$$

which corresponds to equation (24) in the text.

Schooling The first order condition with respect to schooling is:

$$
\left(u\left(c^S\right) - u\left(c^W\right)\right)e^{-rs} + \left(\lambda_2 c^W - \lambda_1 c^S\right)e^{-rs} + \lambda_2\frac{\partial\left[e^{-rs}I(n,s;w)\right]}{\partial s} = 0.
$$

Note that without credit friction the previous equation becomes $\lambda_2\left[\partial\left(e^{-rs}I(n,s;w)\right)/\partial s\right] = 0$ so that optimal schooling is the one that maximizes present value income. The equation above can be written as:

$$
\begin{aligned}
\lambda_2\frac{\partial\left[e^{-rs}I(n,s;w)\right]}{\partial s} & = \lambda_2\left[-re^{-rs}I(n,s;w) + e^{-rs}I_s(n,s;w)\right] \\
& = \left(u\left(c^W\right) - u\left(c^S\right)\right)e^{-rs} + \left(\lambda_1 c^S - \lambda_2 c^W\right)e^{-rs} \\
& = \left(u\left(c^W\right) - u\left(c^S\right)\right)e^{-rs} + (1-\sigma)\left(u\left(c^S\right) - u\left(c^W\right)\right)e^{-rs} \\
& = \sigma\left(u\left(c^W\right) - u\left(c^S\right)\right)e^{-rs},
\end{aligned}
$$

or

$$
\left[\frac{I_s(n,s;w)}{I(n,s;w)} - r\right] I(n,s;w) = \sigma\frac{u\left(c^W\right) - u\left(c^S\right)}{u'\left(c^W\right)} = \frac{\sigma}{1-\sigma}\left(1 - \frac{u\left(c^S\right)}{u\left(c^W\right)}\right)c^W,
$$

or

$$\underbrace{\frac{I_s(n,s;w)}{I(n,s;w)}}_{\text{returns to schooling}} = \underbrace{r + \frac{\sigma}{1-\sigma}\left(1 - \frac{u\left(c^S\right)}{u\left(c^W\right)}\right)\pi\left(s,n\right)}_{\text{adjusted returns to savings}},$$

which corresponds to equation (25) in the text.

D Proofs - section 3

This appendix includes the proofs of all propositions for the model with fertility and schooling (section 3).

Proof of Proposition 4 Consider first the determinants of fertility in the frictionless case. As shown in Appendix C, when the borrowing and the bequest constraints do not bind, then $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4 = 0$ so that using (37) into (36) we obtain

$$\Phi(n)V'\left(b'\right) = V'\left(b\right)e^{-rF}n.$$

Along a steady state in which $b = b'$, the equation above determines optimal fertility n.

Proof of Proposition 5 Consider a marginal reallocation of consumption from the parent to his n children. The parent reduces his age-0 consumption c_0 in one unit and, in exchange, increases the age-0 consumption of each of his n children c_0' in the amount e^{rF}/n. This reallocation can be obtained by adjusting bequests. The optimality condition for bequests therefore must satisfy the condition $\partial V/\partial c_0 \geq \left(e^{rF}/n\right) \times \partial V/\partial c_0'$, with equality if $b' > 0$. In the steady state,

$$\frac{\partial V/\partial c_0}{\partial V/\partial c_0'} = \frac{\partial V/\partial U \times \partial U/\partial c_0}{\Phi(n) \times \partial V/\partial V' \times \partial V'/\partial U' \times \partial U'/\partial c_0'} = \frac{1}{n\beta(n)},$$

which implies that the bequest constraint is binding if $1 > \beta(n)e^{rF}$. Since $\beta(n)$ is strictly decreasing in n then a sufficient condition for the bequest constraint to bind for any n is $1 \geq \beta(0)e^{rF}$.

Proof of Proposition 6 Consider a marginal reallocation of consumption from the parent to his n children. The optimality condition is $u'\left(c^W\right) = \Phi(n,F)u'\left(c^S\right)e^{rF}/n$ which holds with equality because optimal bequest are positive. On the other hand, the optimal choice of savings, a, implies $u'\left(c^S\right) \geq u'\left(c^W\right)$ which holds with equality if $a > 0$. These two conditions together imply that the borrowing constraint binds if $1 > \Phi(n,F)e^{rF}/n$. This is the case under Assumption 4.

Figure 1. Consumption and fertility under constrained allocation

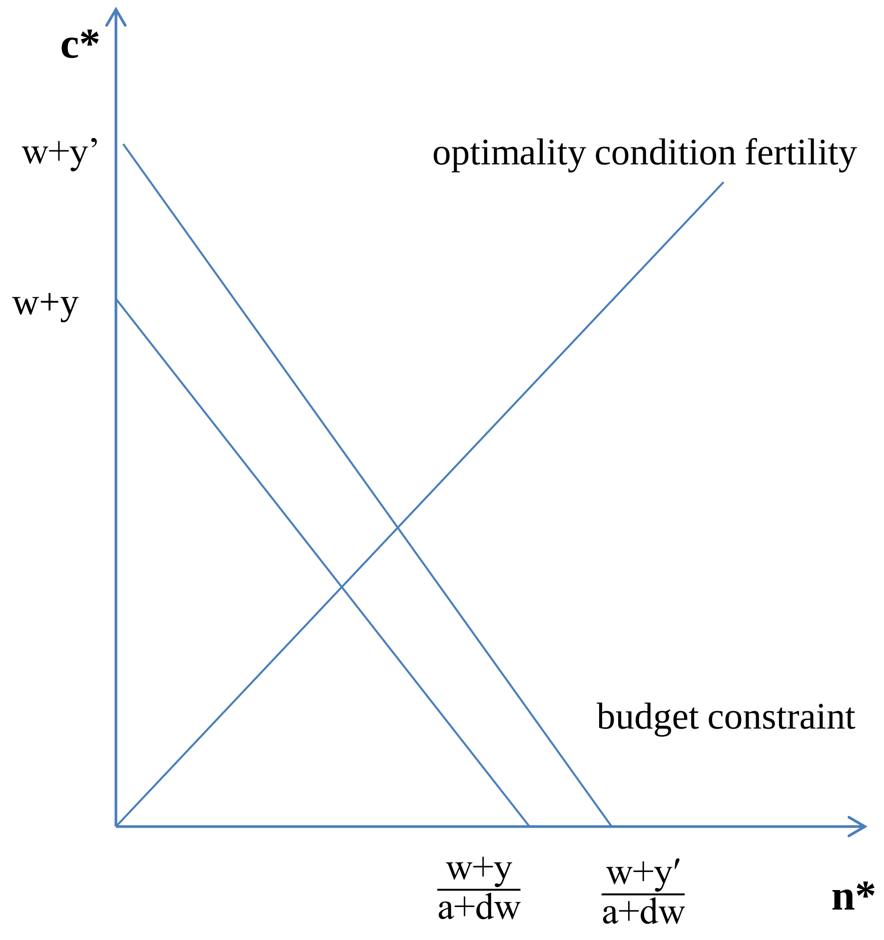
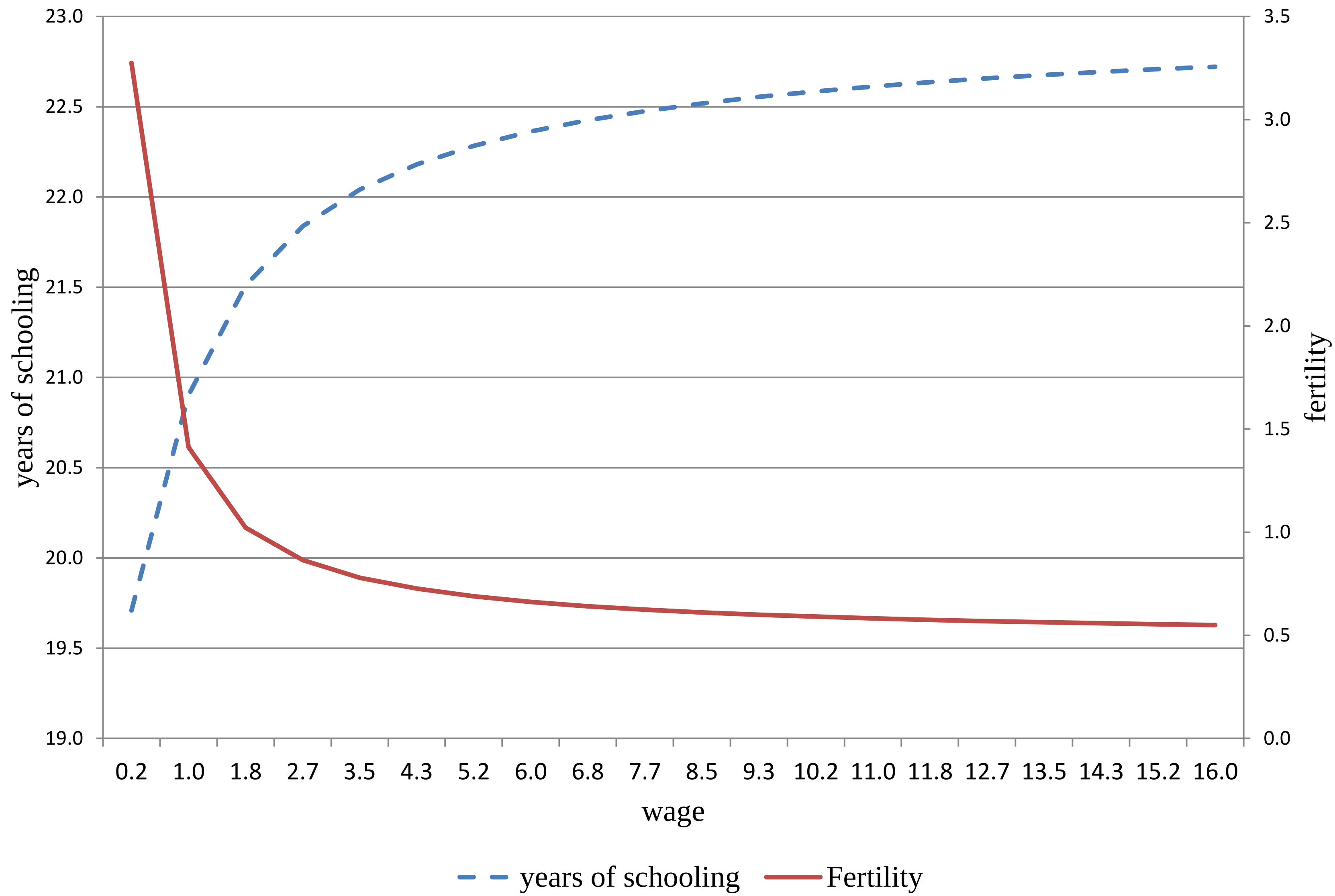


Figure 2. Schooling and fertility for various wage levels



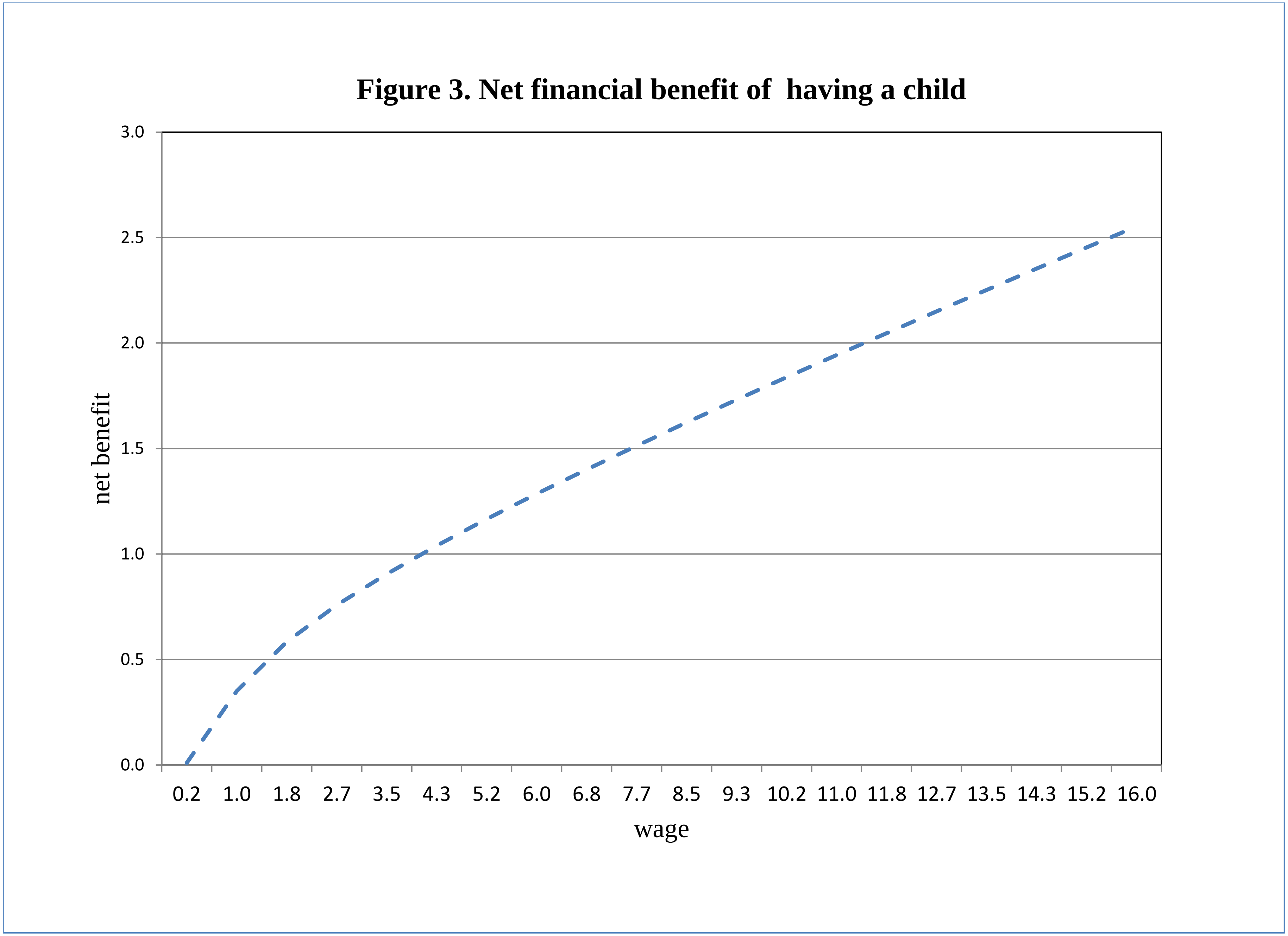


Figure 3. Net financial benefit of having a child
net benefit
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.2 1.0 1.8 2.7 3.5 4.3 5.2 6.0 6.8 7.7 8.5 9.3 10.2 11.0 11.8 12.7 13.5 14.3 15.2 16.0
wage