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Methods For Planning Accelerated Repeated Measures Degradation Tests

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Keywords
nondestructive degradation, mixed effects linear models, accelerated degradation testing, general equivalence theorem

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Methods For Planning Accelerated Repeated Measures Degradation Tests

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Abstract

Accelerated repeated measures degradation tests can sometimes be used to assess product or component reliability when one would expect few or even no failures during a study. Such tests can be used to estimate the lifetime distributions of highly reliable items. This paper describes methods for selecting a single-variable accelerated repeated measures degradation test plan when the (possibly transformed) degradation is linear in (possibly transformed) time and unit-to-unit variability is described by a random effect. To find optimum test plans, we use a criterion based on a large-sample approximation to the estimation precision of the quantile of the failure-time distribution at use conditions. We also discuss how to find compromise test plans that satisfy practical constraints. We use the general equivalence theorem to verify that a test plan is globally optimum. The resulting optimized plans are also evaluated using simulation and compared with other test plans.

Keywords: Nondestructive Degradation, Mixed Effects Linear Models, Accelerated Degradation Testing, General Equivalence Theorem

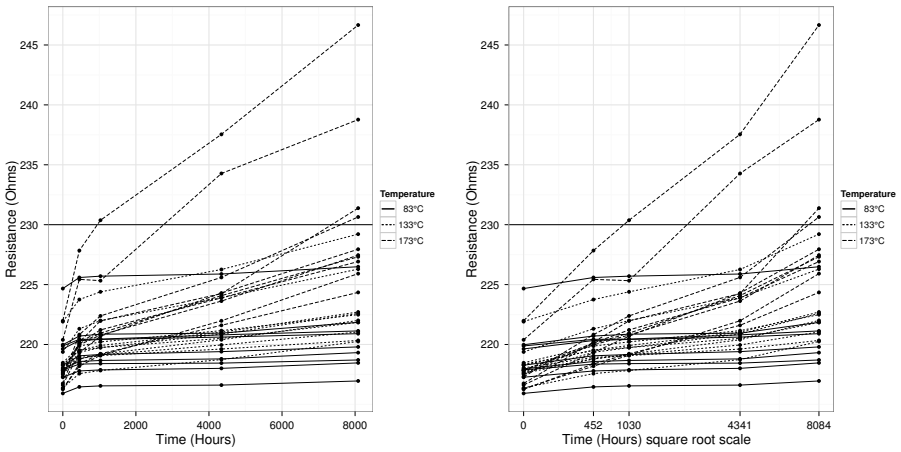
1 Introduction

1.1 Motivating Examples

Engineers often need to quantify the lifetime distribution of highly reliable items. Traditional life tests where the response is time to failure typically yield few or no failures, even with acceleration. Instead engineers can, in some applications, use methods that measure the degradation of an item, providing more information about reliability than traditional life tests. In some situations, however, the degradation rate is so low that noticeable degradation will not be observed during the test. To address this issue, engineers will expose test items to accelerated conditions, such as higher use rate, voltage, temperature, or humidity. Under these accelerated conditions, measurements of degradation are made and the relationship between the accelerating variables and the degradation rate can be modeled and unknown parameters in the model can be estimated. This relationship can then be used to extrapolate to estimate lifetimes at normal use conditions. When planning a repeated measures accelerated degradation test (RMADT), the engineers need to specify the levels of the accelerating variable at which test units will be exposed and the number of units that should be allocated to each level.

We use two examples to motivate this work. The first example is based on an experiment described by Shiomi and Yanagisawa (1979). The engineers measured the resistance of carbon-film resistors at particular points in time. At the beginning of the experiment, the resistance values of nominal resistors varied between 215.92 Ω and 224.7 Ω. The resistors were exposed to three different levels of temperature (83°C, 133°C, and 173°C) in order to accelerate the degradation process. Suppose that the engineers defined degradation to be the amount of increase in resistance over time and we suppose that failure occurs when resistance exceeds 230 Ω. Figure 1a presents the original data and Figure 1b shows the data plotted versus the square root of time (i.e., on a square root axis). We suppose that the engineers wanted to estimate a quantile of the failure-time distribution at the nominal use condition 50°C.

The second example is the sliding metal wear application found in Chapter 21 of Meeker and Escobar (1998). The experiment was conducted to test the wear resistance of a particular metal alloy. The engineers applied a range of different weights (in grams) to a piece of metal which was then slid over another piece of metal. The engineers then measured the widths of scars (in microns) that formed on the metal piece at different points in time. The engineers defined a failure to be a scar width of 50 microns. The purpose of the experiment was to study the effect of weight on wear rate and to gain a better understanding of the wear mechanism. Figure 2 presents the original data (Figure 2a) as well as the data plotted on log-log axes (Figure 2b).



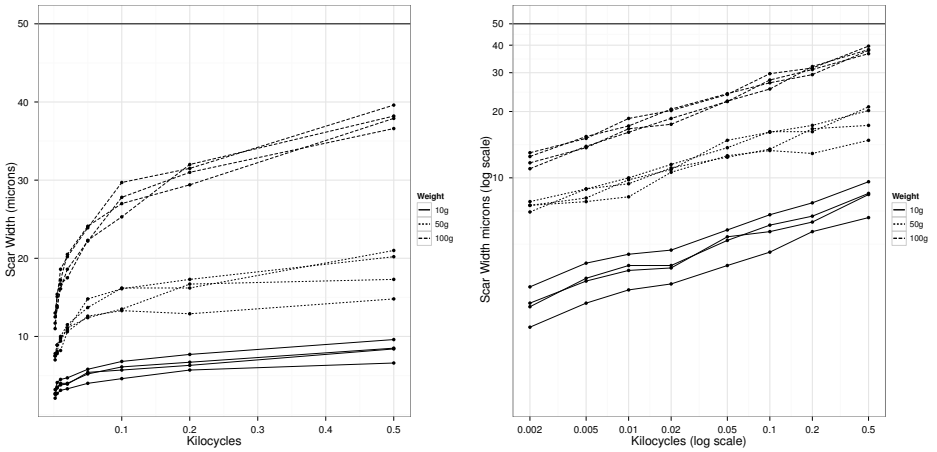
(a) (b)

Figure 1: Carbon-Resistor data from Shiomi and Yanagisawa (1979) plotted on the original scale and a square root of time scale. The horizontal line is the point for which a failure is declared (resistance = 230 Ω).

1.2 Related Work

Nelson (1990) provides much information about accelerated testing and accelerated test models. Additionally, Nelson (2005a) and Nelson (2005b) give an extensive list of references related to accelerated test planning. Boulanger and Escobar (1994) discuss experiment design for RMADTs where the amount of degradation over time levels off to a plateau. Section 22.5 of Meeker and Escobar (1998) provides an example where evaluation by simulation is used to plan an RMADT. Yu and Tseng (1998) provide a procedure for determining when to terminate an RMADT. Wu and Chang (2002) illustrate, using a nonlinear degradation path model with a single random effect, how to find an RMADT that minimizes the variance of an estimator for a failure-time quantile subject to a cost constraint. Marseguerra, Zio and Cipollone (2003) under a model similar to that used in Wu and Chang (2002), use genetic algorithms for finding degradation tests subject to a cost constraint. Yu and Tseng (2003) describe designing an RMADT under a model with a single random effect, assuming a lognormal distribution for the slope of transformed time, under a cost constraint. Yu (2006) finds an optimum RMADT when the degradation rate is assumed to follow a reciprocal Weibull distribution. Li and Kececioglu (2006) use simulation for test planning using a similar dataset, model, and assumptions as Yu and Tseng (2003). Shi, Escobar, Meeker (2009) present methods for test planning for accelerated *destructive* degradation tests (i.e., only one measurement per item). Lim and Yum (2011) find optimum RMADTs assuming a Weiner process model. Lastly, Guan and Tang (2013) find D and V optimal tests for a gamma-process degradation model.

In this paper we illustrate methods for finding optimum RMADT when the degradation path is adequately modeled using a linear mixed-effects models, a model that was not considered by the



(a) (b)

Figure 2: Scar width data of Meeker and Escobar (1998) plotted on the original scale and on a log-log scale. The horizontal line represents the point of degradation where a failure is declared (scar width = 50 microns).

perviously mentioned authors. In particular, we find the levels of the accelerating variable that minimize a failure-time quantile. Optimality is then proved using the general equivalence theorem.

1.3 Overview

The rest of the paper is organized as follows. Section 2 introduces the assumed general form of the degradation model and the corresponding Fisher information matrix. Section 3 discusses the relationship between the degradation model and the failure-time distribution, the failure-time distribution quantile, and the large-sample approximate variance of the maximum likelihood estimator for the failure-time distribution quantile. Section 4 describes RMADTs. Section 5 discusses the criterion for finding optimum test plans. Section 6 discusses other types of RMADT plans. Section 7 presents two different examples of single-variable RMADT planning. Section 8 shows how to use simulation to compare different types of test plans. Section 9 provides concluding remarks and areas for future research.

2 Degradation Models

2.1 Accelerated Repeated Measures Degradation Models

The actual degradation level for an observational unit at time t and accelerating variable(s) is denoted by $\mathcal{D} = \mathcal{D}(\tau, x_1, x_2, \boldsymbol{\theta})$ where $\tau = h_t(t)$ is a monotone increasing transformation of time, x_1 and x_2 are scalars of (possibly transformed) accelerating variables, and $\boldsymbol{\theta}$ is the unknown parameter vector.

The (possibly transformed) observed degradation Y at some (possibly transformed) time point t for a fixed value(s) of the (possibly transformed) accelerating variables x_1 and x_2 is

$$\begin{aligned} Y &= h_d(\mathcal{D}) + \varepsilon = \mu(\tau, x_1, x_2) + \varepsilon \\ &= x_1\gamma_1 + x_2\gamma_2\tau + b_0 + b_1\tau + \varepsilon, \end{aligned} \tag{1}$$

so that $\mu(\tau, x_1, x_2) = h_d(\mathcal{D})$ is a location parameter for the distribution of Y that depends on the unknown parameters $\boldsymbol{\theta}$, $\varepsilon \overset{iid}{\sim} \text{N}(0, \sigma^2)$ is a random variable that describes the within unit variation, and h_d is a monotone increasing transformation of $\mathcal{D}$. In (1), the term $x_1\gamma_1$ describes how the intercept or initial value of degradation changes as a function of the accelerating variable, $x_2\gamma_2$ describes how the degradation slope or rate changes as a function of the accelerating variable, and b_0 and b_1 describe the unit-to-unit variability in the degradation intercepts and slopes. Note that x_1 and x_2 will sometimes be exactly the same variables (but in general they do not have to be).

We assume that the variability in the linear regression parameters b_0 and b_1 can be described by a bivariate normal distribution

$$(b_0, b_1)' \sim \text{BVN}(\boldsymbol{\beta}, \mathbf{V})$$

where $\boldsymbol{\beta} = (\beta_0, \beta_1)'$ is the mean vector and

$$\mathbf{V} = \begin{pmatrix} \sigma_{b_0}^2 & \rho\sigma_{b_0}\sigma_{b_1} \\ \rho\sigma_{b_0}\sigma_{b_1} & \sigma_{b_1}^2 \end{pmatrix}$$

is the covariance matrix. We further assume that $(b_0, b_1)'$ is independent of ε and that the ε values are independent over time.

For the applications described in Section 1.1, the items were measured under well-controlled laboratory conditions. Under such conditions, the primary sources of variability are unit-to-unit differences (represented by the random intercept b_0 and random slope b_1, respectively), and from the measurement system (represented by ε). Additionally, because the test environment is well controlled, autocorrelation is less important when observations on experimental units are not taken too close together. Such applications are adequately modeled with a mixed-effects model. For applications where there is a substantial amount of within unit variability and autocorrelation, a time series or stochastic processes model such as gamma process or Weiner process model might be required.

2.2 Approximate Variance-Covariance Matrix of the Maximum Likelihood Estimators

Let $\boldsymbol{\theta} = (\gamma_1, \gamma_2, \boldsymbol{\beta}, \boldsymbol{\vartheta})'$ be the r-element parameter vector where $\boldsymbol{\vartheta} = (\sigma_{b_0}, \sigma_{b_1}, \rho, \sigma)'$. The large-sample approximate covariance matrix of the maximum likelihood (ML) estimators is

$$\text{AVar}(\widehat{\boldsymbol{\theta}}) = [\boldsymbol{\mathcal{I}}(\boldsymbol{\theta})]^{-1} \tag{2}$$

where $\boldsymbol{\mathcal{I}}(\boldsymbol{\theta}) = -\text{E}\left(\partial^2 \mathcal{L}/\partial \boldsymbol{\theta}^2\right)$ is the Fisher Information matrix and $\mathcal{L}$ is the total loglikelihood. The derivation of the information matrix is given in the appendix. $\text{AVar}(\widehat{\boldsymbol{\theta}})$ can be estimated by evaluating (2) at the ML estimates $\widehat{\boldsymbol{\theta}}$. We denote this estimator by $\widehat{\text{AVar}}(\widehat{\boldsymbol{\theta}})$.

3 Failure-Time Distribution for Degradation Models

This section describes the relationship between the degradation model and the induced failure-time model. If a degradation model and definition of failure are given, a failure-time distribution is implied (e.g., Chapter 13 of Meeker and Escobar 1998).

3.1 Failure-Time Cumulative Distribution Function

This section shows how to derive the failure-time distribution from the degradation model in (1). A degradation process with soft failures is assumed. A soft failure occurs when the degradation level of a unit, $\mathcal{D}$, reaches a pre-specified degradation level $\mathcal{D}_\text{f}$. This is equivalent to $h_d(\mathcal{D}) \geq \mu_\text{f}$ where $\mu_\text{f} = h_d(\mathcal{D}_\text{f})$. The failure time for a unit is the time at which it reaches the degradation level $\mathcal{D}_\text{f}$. Let T define the random variable associated with the item's time to failure. If $b_0 + b_1\tau \sim \text{N}(\beta_0 + \tau\beta_1, \sigma_{b_0}^2 + \tau^2\sigma_{b_1}^2 + 2\tau\rho\sigma_{b_0}\sigma_{b_1})$, then for increasing $\mathcal{D}$ (i.e., assuming that $\Pr(b_1\tau < 0)$ is negligible)

$$\begin{aligned}\Pr(T \leq t) = \Pr(\mathcal{D} \geq \mathcal{D}_\text{f}) = \Pr(h_d(\mathcal{D}) \geq \mu_\text{f}) = \Pr(b_0 + b_1\tau \geq \mu_\text{f} - x_1\gamma_1 - x_2\tau\gamma_2) \\ = 1 - \Pr(b_0 + b_1\tau \leq \mu_\text{f} - x_1\gamma_1 - x_2\tau\gamma_2) = 1 - \Phi_\text{nor}(\kappa)\end{aligned} \tag{3}$$

where

$$\kappa = \frac{\mu_\text{f} - x_1\gamma_1 - x_2\tau\gamma_2 - b_0 - b_1\tau}{\sqrt{\sigma_{b_0}^2 + \tau^2\sigma_{b_1}^2 + 2\tau\rho\sigma_{b_0}\sigma_{b_1}}}$$

and Φ_nor is the CDF of the standard normal distribution. Similarly, for decreasing $\mathcal{D}$, a failure happens when $\mathcal{D} \leq \mathcal{D}_\text{f}$ and

$$F(t; x) = \Pr(\mathcal{D} \leq \mathcal{D}_\text{f}) = \Phi_\text{nor}(\kappa). \tag{4}$$

Note that the failure-time distribution is defined by the true degradation $\mathcal{D}$ and not the observed

degradation, given in Equation 1. This is because it is the true degradation process that is causing the items to fail in our population, regardless if we have observed the process or not. This derivation also assumes that when the degradation of an item is measured that the measurements do not in any way change the degradation paths of our items.

Although the failure time is a function of the true, unobserved degradation, our modeling for estimation of parameters is based on the observed and possibly transformed degradation. $F_T(t;x)$ can be estimated by evaluating (3) or (4) at the ML estimates of $\boldsymbol{\theta}$. Meeker and Escobar (1998), page 330, describes numerical integration and simulation based approaches for more complicated models where a closed form solution does not exist.

3.2 Failure-Time Quantiles

From (3) or (4) the p quantile of the failure-time distribution, is $t_p = h_t^{-1}(\tau_p)$ where

$$\tau_p = \frac{-\left[k\rho\sigma_{b_0}\sigma_{b_1} + (\mu_{\mathrm{f}} - x_1\gamma_1 - \beta_0)(x_2\gamma_2 + \beta_1)\right] \pm \sqrt{\psi}}{\left[k\sigma_{b_1}^2 - (x_2\gamma_2 + \beta_1)^2\right]}. \tag{5}$$

Here $k = \left[\Phi_{\mathrm{nor}}^{-1}(1-p)\right]^2$ or $k = \left[\Phi_{\mathrm{nor}}^{-1}(p)\right]^2$ depending on whether a failure is declared when $\mathcal{D} \geq \mathcal{D}_{\mathrm{f}}$ or $\mathcal{D} \leq \mathcal{D}_{\mathrm{f}}$, respectively, $k\sigma_{b_1}^2 \neq (x_2\gamma_2 + \beta_1)^2$, and

$$\psi = \left[k\rho\sigma_{b_0}\sigma_{b_1} + (\mu_{\mathrm{f}} - x_1\gamma_1 - \beta_0)(x_2\gamma_2 + \beta_1)\right]^2 - \left[k\sigma_{b_1}^2 - (x_2\gamma_2 + \beta_1)^2\right]\left[k\sigma_{b_0}^2 - (\mu_{\mathrm{f}} - x_1\gamma_1 - \beta_0)^2\right].$$

The derivation of (5) is given in the appendix. If $0 < p < 0.5$, then t_p is the root where the radical in (5) is added. If $0.5 < p < 1$, t_p is given by the root where the radical in (5) is subtracted. Notice that when $p = 0.50$, then $\psi = 0$ (because $k = 0$), giving only one root.

3.3 Approximate Variance for the Maximum Likelihood Estimator of a Failure-Time Quantile

Let $\mathbf{a}$ be a vector with elements $a_i = \partial t_p / \partial \theta_i,\ \ i = 1, \ldots, r$. Then by the delta method, the large-sample approximate variance of $\widehat{t_p}$ is

$$\mathrm{AVar}(\widehat{t_p}) = \mathbf{a}'\mathrm{AVar}(\widehat{\boldsymbol{\theta}})\mathbf{a}. \tag{6}$$

$\mathrm{Var}(\widehat{t_p})$ can be estimated by evaluating (6) at $\widehat{\boldsymbol{\theta}}$.

4 Repeated Measures Accelerated Degradation Test Planning

4.1 Planning Information

RMADT planning requires specification of the form of (1), information (planning values) for the unknown model parameters, a distribution for the variability, the failure-defining critical degradation level $\mathcal{D}_{\mathrm{f}}$, the points in time at which degradation measurements will be made, and a range of the accelerating variables allowed for the test.

4.2 Inspection Schedule Specification

Part of an RMADT plan specification is the inspection schedule (i.e., the points in time that a measurement will be made on a unit). The optimally selected inspection times for our mixed-effects model, puts all measurements at the beginning and the end of the experiment. Such a schedule is not practical in actual applications. Such a two-point inspection schedule would provide no information about the assumed (linear) slope of the degradation if measuring the items changed the items' degradation paths. This could not be detected with just a two-point plan in time. In many applications of RMADT, equal or approximately equal spacing in time is appealing because it is easy to administer, allows periodic assessment of test results, and has good statistical properties. For these reasons, such inspection schedules are commonly used in practice. Therefore, for our examples we will use the original inspection schedules used by the engineers.

4.3 Accelerating Variable Plan Specification

Let $\mathbf{x}_i = (x_{1,i}, x_{2,i})'$ be a factor level combination of the (possibly transformed) accelerated conditions for unit i and let π_i denote the proportion of units that are to be tested at $\mathbf{x}_i$. An RMADT plan will specify the levels of the accelerating variable to test and the proportion of units to be tested at those levels. A test plan with K levels is denoted by

$$\eta = \begin{bmatrix} \mathbf{x}_1, & \pi_1 \\ \mathbf{x}_2, & \pi_2 \\ \vdots & \vdots \\ \mathbf{x}_K, & \pi_K \end{bmatrix},$$

where $\sum_{i=1}^{K} \pi_i = 1$.

4.4 Constraints on the Experimental Region

The acceleration model in (1) adequately describes the underlying failure mechanism only within a certain range of $\mathbf{x}$, say $[\mathbf{x}_U, \mathbf{x}_H]$ where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}_U = (x_{1U}, x_{2U})$ is the value of the accelerating variable at the use conditions, but testing will be limited to the interval $[\mathbf{x}_L, \mathbf{x}_H]$. Testing beyond $\mathbf{x}_H$ would cause the model to break down (e.g., by generating new failure modes) and testing below $\mathbf{x}_L$ would not provide much meaningful degradation information because the degradation rate would be small. Generally, $\mathbf{x}_L > \mathbf{x}_U$. Thus for practical RMADTs, the experimental region is between $\mathbf{x}_L$ and $\mathbf{x}_H$.

5 Optimum RMADT Plan

5.1 Criterion for Choosing an Optimum Plan

Suppose that the purpose of conducting an RMADT is to estimate t_p, a quantile of the failure-time distribution at the use conditions. We use (6), the large-sample approximate variance of the ML estimator of t_p, as our test-plan evaluation criterion.

Consider a situation where n available units are to be tested at K different points in the experimental space $\mathbf{x}$. Based on the assumption of independence across units, the Fisher information matrix can be expressed as

$$\boldsymbol{\mathcal{I}}(\boldsymbol{\theta}, \eta) = \sum_{i=1}^{n} \boldsymbol{\mathcal{I}}_i(\boldsymbol{\theta}, \eta) = \sum_{i=1}^{K} n_i \boldsymbol{\mathcal{I}}_i(\boldsymbol{\theta}, \mathbf{x}_i) = n \sum_{i=1}^{K} \pi_i \boldsymbol{\mathcal{I}}_i(\boldsymbol{\theta}, \mathbf{x}_i) \tag{7}$$

where n_i, π_i, and $\boldsymbol{\mathcal{I}}_i(\boldsymbol{\theta})$ are the number of units, proportion of units, and the Fisher information matrix at condition i, respectively. Taking (7) and replacing it in (6) gives

$$\begin{aligned} \Psi(\eta) = \text{Avar}(\hat{t}_p) = \mathbf{a}' \boldsymbol{\mathcal{I}}^{-1}(\boldsymbol{\theta}, \eta) \mathbf{a} &= \frac{1}{n} \mathbf{a}' \left[\sum_{i=1}^{K} \pi_i \boldsymbol{\mathcal{I}}_i(\boldsymbol{\theta}, \mathbf{x}_i) \right]^{-1} \mathbf{a} \\ &\propto \mathbf{a}' \left[\sum_{i=1}^{K} \pi_i \boldsymbol{\mathcal{I}}_i(\boldsymbol{\theta}, \mathbf{x}_i) \right]^{-1} \mathbf{a} \end{aligned} \tag{8}$$

and (8) will be used as the optimization criterion for evaluating test plans. It should be noted that the test-plan designs we calculate will be continuous designs. These designs can be approximated using discrete designs, however, which we do in the examples when approximate variances (or standard errors) are calculated for a given test plan.

5.2 Checking the Initial Optimum Plan

We can use a numerical optimization algorithm to minimize (8) for given test-planning inputs. Whittle's (1973) general equivalence theorem (GET) can be used to check for global optimality of test plans. The GET can also be used to check whether an optimum test plan is unique or not (see Shi, Escobar, and Meeker, 2009 for example).

For an optimality criterion $\Psi(\cdot)$, the directional derivative, Λ, at a test plan η and in the direction of another test plan ν is defined as

$$\Lambda(\eta, \nu) = \lim_{\delta \to 0^+} \frac{\Psi[(1-\delta)\eta + \delta\nu] - \Psi(\eta)}{\delta}.$$

For a criterion similar to (8), Shi, Escobar, and Meeker (2009) give the following expression as the directional derivative

$$\Lambda(\eta, \nu) = \mathbf{a}'[\boldsymbol{\mathcal{I}(\eta)}]^{-1}\boldsymbol{\mathcal{I}(\nu)}[\boldsymbol{\mathcal{I}(\eta)}]^{-1}\mathbf{a} - \mathbf{a}'[\boldsymbol{\mathcal{I}(\eta)}]^{-1}\mathbf{a} \tag{9}$$

and this expression also applies to our optimization problem. Shi, Escobar, Meeker (2009) in their appendix also show how (9) satisfies all the requirements needed to apply the GET and the same requirements are satisfied for our model. Therefore, Whittle's GET can be used to determine if a test plan is optimum.

According to the GET, a test plan η^* will be optimum if and only if

$$\sup_{\mathbf{x}} \Lambda_\Psi(\eta^*, \eta_{\mathbf{x}}) = 0$$

for each singular test plan $\eta_{\mathbf{x}}$ (i.e., a test plan where all units are tested at one factor-level combination $\mathbf{x}$), and the set of conditions $\mathbf{x}_i^*$ in the optimal plan η^* are a subset of the conditions satisfying $\Lambda_\phi(\eta^*, \eta_{\mathbf{x}}) = 0$.

6 Other RMADT Plans

Optimum accelerated test plans generally have practical deficiencies. Optimum plans are, however, useful in that they provide insight and heuristics for finding practically appealing compromise test plans that have good statistical properties.

6.1 Traditional Plans

As described in Chapter 6 of Nelson (1990), a traditional test plan is one that uses equally spaced levels of the accelerating variable and equal allocation of units to those levels. In testing situations

that require extrapolation, as in RMADT, traditional plans may not be statistically efficient, yielding less precise estimates than the alternative test plans described here.

6.2 Compromise RMADT

Although optimum test plans have the smallest value of the large-sample approximate variance of the ML estimator of the failure-time quantile at the use conditions, these test plans have practical deficiencies and are not robust to deviations from model assumptions and other inputs. Traditional test plans will have a larger standard error than an optimum test plan, but tend to be more robust to departures from assumptions. Compromise test plans also tend to be robust to deviations from specified inputs, but have statistical properties that are better than traditional plans. A compromise test plan can be formed by starting with an optimum test plan structure and adding an additional point in the middle of the test space. A fixed proportion of units (say 10%) will then be tested at this additional point while optimization is performed over the other proportions. By fixing the proportion of units at the additional point, the compromise test plan can be optimized, subject to the constraint, and will not degenerate to an optimum test plan.

7 Examples of Planning RMADT

This section gives specific examples of planning RMADTs. For each example the planning information will be given, an optimum test will be found and verified, and the optimum test will be compared to compromise and traditional RMADTs.

7.1 Test Plan to Estimate the Lifetime of Carbon-Film Resistors

This first example is based on the resistor data in Figure 1b. The square root transformation of time was chosen to make the sample paths approximately linear. Let y_{ijk} be the observed degradation for unit i, $i = 1, \ldots, 19$, at time point j, $j = 1, \ldots, 5$ for level k of the accelerating variable, $k = 1, 2, 3$, and let x_k denote the Arrhenius transformation of temperature where $x = -11605/(\text{Temp}^\circ\text{C} + 273.15)$. The assumed linear accelerated degradation random effects model is

$$y_{ijk} = \gamma x_k \tau_{ij} + b_0 + b_1 \tau_{ij} + \varepsilon_{ijk}, \tag{10}$$

where $\tau = \sqrt{\text{Time}}$. Notice that (10) is a special case of (1) with $x_1 = 0$, $x_2 = x$ and $\gamma_2 = \gamma$. The coefficient -11605 in the Arrhenius transformation is used so that the coefficient γ can be interpreted as the effective activation energy in the commonly used units eV/K where eV denotes electron volts and K denotes kelvin temperature.

For this example, we will consider two different types of test plans. The first will be a test

plan that tests some units at the nominal use conditions. These plans will be called Plan type 1. The second type of plan will only test units within the experimental region $[x_L, x_H]$ where $x_L > x_U$. These plans will be called Plan type 2. Engineers may want to test some units at the nominal use conditions to check for differences in behavior relative to degradation at the accelerated conditions (e.g., analytical chemical measurements could be used to assure that increased levels of the accelerating variables have not changed the nature of the chemical reactions).

The rest of the planning information for this example is as follows:

- The failure definition is $\mathcal{D}_{\text{f}} = 230\ \Omega$.
- The test will run for 8084 hours.
- All units will have the same measurement schedule that was used in the original test with the inspections being made at 0, 452, 1030, 4341, and 8084 hours.
- 173°C is the maximum allowable testing temperature.
- 83°C is the minimum testing temperature.
- The use temperature is 50°C.
- The proportion of units tested at the nominal use conditions, denoted by π_U, is 5%.

We suppose that the goal is to develop a test plan that will evaluate the failure-time distribution of new lower-cost resistors that are otherwise believed to have degradation properties that are similar to those in the previous test. The test plan properties will depend on the unknown model parameters $\boldsymbol{\theta}$. The planning values for $\boldsymbol{\theta}$ are $\beta_0^{\square} = 218.4$, $\beta_1^{\square} = 0.53$, $\gamma^{\square} = 0.016$, $\sigma_{b_0}^{\square} = 2.181$, $\sigma_{b_1}^{\square} = 0.00038$, $\rho^{\square} = 0.628$, and $\sigma^{\square} = 0.59$ (these are based on the ML estimates from the original data).

From optimum experiment design theory, we generally expect that an optimum test plan will spread the units out to the boundary of the experimental region. This leads us to consider a two-point optimum plan where a proportion π_L of the units should be at $\mathbf{x}_L$ (the lowest allowable accelerating conditions of the test) and $1 - \pi_L$ of the units should be at $\mathbf{x}_H$ (the highest allowable accelerating conditions of the test). Then (8) reduces to

$$\text{Avar}(\hat{t}_p) = \mathbf{a}' \left[\pi_U \boldsymbol{\mathcal{I}}_U(\boldsymbol{\theta}, x_U) + \pi_L \boldsymbol{\mathcal{I}}_L(\boldsymbol{\theta}, x_L) + (1 - \pi_U - \pi_L)\boldsymbol{\mathcal{I}}_H(\boldsymbol{\theta}, x_H)\right]^{-1} \mathbf{a} \tag{11}$$

for tests under Plan type 1 and for tests under Plan type 2

$$\text{Avar}(\hat{t}_p) = \mathbf{a}' \left[\pi_L \boldsymbol{\mathcal{I}}_L(\boldsymbol{\theta}, x_L) + (1 - \pi_L)\boldsymbol{\mathcal{I}}_H(\boldsymbol{\theta}, x_H)\right]^{-1} \mathbf{a}. \tag{12}$$

Table 1 gives the optimum RMADT (and other kinds of test plans) under Plan type 1 and Table 2 gives the optimum RMADT under Plan type 2 for the planning information given above.

	Plan type 1		
Plan	Temperature°C	Allocation	ASE($\hat{t}_p$)
Optimum	(50, 83, 173)	(.05, .711, .239)	21808
Compromise	(50, 83, 111.5, 173)	(0.05, 0.625, 0.1, 0.225)	22162
Original	-	-	-
Traditional	(50, 83, 113, 143, 173)	(.05, .2375, .2375, .2375, .2375)	24189

Table 1: Optimum, compromise, and traditional test plans as well as the large-sample approximate standard error under Plan 1 for the resistor example and assuming $n = 29$ test units (as in the original experiment).

	Plan type 2		
Plan	Temperature°C	Allocation	ASE($\hat{t}_p$)
Optimum	(83, 173)	(.748, .252)	22561
Compromise	(83, 111.5, 173)	(.662, .1, .238)	22853
Original	(83, 133, 173)	(.33, .33, .33)	26289
Traditional	(83, 113, 143, 173)	(.25, .25, .25, .25)	26578

Table 2: Optimum, compromise, the original plan from the study, and traditional test plans as well as the large-sample approximate standard error under Plan 2 for the resistor example and assuming $n = 29$ test units (as in the original experiment).

In order to use the GET when some units are constrained to be tested at the use conditions, a modification to the theorem must be made, as described by Ying and Meeker (2010). Instead of considering singular tests where all the units are tested at a single level of temperature, the alternative plan must also put the same proportion of units at the use conditions and the remaining proportion at a single test condition. Figure 3 shows (9) at η_1^* as a function of x_L. Notice that at the end points of the test design space, the directional derivative reaches its maximum and is zero showing that this plan is indeed optimum. The corresponding figure for η_2^* is similar and is omitted to save space.

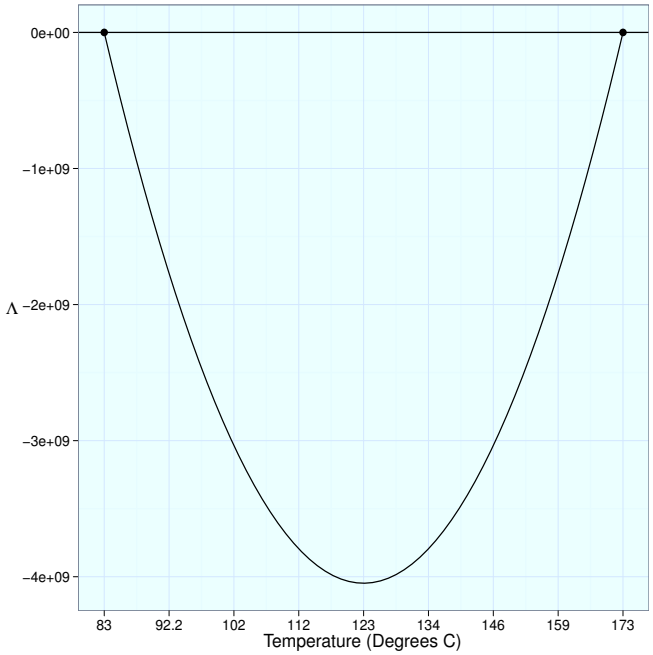


Figure 3: Plot of the directional derivative in (9), evaluated at the optimum plan in Section 5 versus lowest level of temperature. The points in this figure correspond to the temperatures selected for the optimized plan.

We next consider other RMADT test plans, beginning with a traditional test plan. Recall that a traditional test plan places an equal proportion of units to equally spaced levels of the accelerating variable. Traditional test plans for Plans 1 and 2 for this example are given in Tables 1 and 2, respectively.

A compromise between the optimum and traditional test plans will select a third, middle level in between the lowest and the highest levels of x in the optimized test plan. For this example, the midpoint is $x_M = -30.17$ (the Arrhenius transformation of 111.5°C). A constraint must be put on π_M; otherwise the compromise test plan will degenerate to a two-point optimum plan. We use the constraint $\pi_M = 0.10$. The optimized compromise test plan is found by selecting the values of x_L and π_L, subject to the center-level constraint that minimizes (14). For this example, the compromise test plan for both plans 1 and 2 are also given in Tables 1 and 2, respectively.

Notice that the value of ASE($\hat{t}_p$) for the optimum test plans given in Tables 1 and 2 are rather different. This difference comes from the additional points being tested at the use conditions in the Plan 1 optimum plan. Generally for RMADTs, ASE($\hat{t}_p$) can be improved by testing more units at the use conditions. This is because these test selection procedures assume that the given model is correct and additionally there is complete data (i.e., no censoring). With this model assumption, in order to best estimate t_p at use conditions, all one needs to do is test all units at these conditions. This, however, is not a practical strategy because generally highly reliable items will have little

meaningful degradation, and thus the needed inferences about lifetime will implicitly require an unreasonable amount of extrapolation in time.

7.2 Test Plan to Assess Scar Width Growth

This example is based on the scar-width data presented in Figure 2b. Let $y_{ijk} = \log(\mathcal{D}_{ijk})$ be the observed, transformed degradation for unit i, $i = 1, \ldots, 12$, at time point $t_{i,j}$, $j = 1, \ldots, 8$ and for accelerating level k, $k = 1, 2, 3$, where $\mathcal{D}$ was defined to be scar width in microns. A model that can describe these data is

$$y_{ijk} = \gamma_1 x_{1i} + \gamma_2 x_{2i} \tau_{ij} + b_0 + b_1 \tau_{ij} + \varepsilon_{ijk} \tag{13}$$

where $\tau_{ij} = \log(t_{ij})$, $x_{1i} = x_{2i} = \log(\text{weight}_i)$ where weight is in grams, and $\varepsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$.

The rest of the planning information is as follows:

- The failure definition is $\mathcal{D}_\text{f} = 50$ microns ($y_\text{f} = \log(50) = 3.912023$).
- The test will run for 500 cycles.
- All units will have the same measurement schedule where inspections were made after 2, 5, 10, 20, 50, 100, 200, and 500 cycles.
- 100g is the maximum allowable testing weight.
- 10g is the lowest allowable testing weight.
- The use weight is 5g.

Suppose that the goal is to develop a test plan that will evaluate the wear resistance of new lower cost metal plates that are believed to be equally durable and otherwise similar to those that were tested previously. The test plan will depend on the unknown model parameters $\boldsymbol{\theta}$. The planning values for $\boldsymbol{\theta}$ are $\beta_0^\square = 2.089$, $\beta_1^\square = 0.183$, $\gamma_1^\square = 0.018$, $\gamma_2^\square = 0.00014$, $\sigma_{b_0}^\square = 0.117$, $\sigma_{b_1}^\square = 0.019$, $\rho^\square = -0.252$, and $\sigma^\square = 0.048$ (these are again based on the ML estimates from the original data).

Next we find an optimum test plan for this example. As in Section 7.1, the optimum test plan is expected to spread the units out to the boundary of the experimental region. This leads us to consider a proposed two-point optimum plan where a proportion π_L of the units should be tested at $\mathbf{x}_L$ (the lowest allowable accelerating conditions of the test) and $1 - \pi_L$ of the units should be tested at $\mathbf{x}_H$ (the highest allowable accelerating conditions of the test). This again reduces (8) to

$$\text{Avar}(\hat{t}_p) = \mathbf{a}' \left[\pi_L \boldsymbol{\mathcal{I}}_L(\boldsymbol{\theta}, \mathbf{x}_L) + (1 - \pi_L)\boldsymbol{\mathcal{I}}_H(\boldsymbol{\theta}, \mathbf{x}_H)\right]^{-1} \mathbf{a}. \tag{14}$$

The optimum plan for this example is given in Table 3 along with a compromise test plan using the same structure and constraints used in the resistor example, a traditional test plan, and the

plan used in the original study. We verified optimality numerically by using the GET but omit the plot as it is similar to Figure 3. The compromise test plan meets practical constraints and provides useful improvement in statistical precision, relative to the original and traditional plans.

Test Plan	Weight in grams	Allocation	ASE($\hat{t}_p$) in kilocycles
Optimum	(10, 100)	(.95, .05)	1795
Compromise	(10, 55, 100)	(.855, .1, .045)	1837
Original	(10, 50, 100)	(.33, .33, .33)	2086
Traditional	(10, 40, 70, 100)	(.25, .25, .25, .25)	2269

Table 3: Optimum, compromise, the original plan from the study, and a traditional test plan, comparing the large-sample approximate standard error for the metal wear example and assuming $n = 12$ test units (as in the original experiment).

8 Monte Carlo Simulation To Evaluate a RMADT Plans

Simulation is a powerful tool to help visualize the variability involved in the test-planning and data analysis processes and allows evaluation of the properties of accelerated test plans without having to use large-sample approximations. As such, simulation and analytical evaluation are complimentary tools for test planning. Additionally, simulation can be used to evaluate the adequacy of large-sample approximations. We illustrate the use of simulation in test planning using the scar-width example.

A simulation trial was conducted for each of the RMADT plans listed in Table 3. A simulation trial consisted of allocating 24 units to the accelerating levels based on the different test plans. The planning information (i.e., assumed values of the model parameters) were used to generate data at each of the different accelerating variable levels. ML estimates were then obtained from the simulated data and used to estimate the lifetime quantile. The simulation was repeated 10,000 times for each of the four test plans.

Figure 4 shows histograms of the ML estimates of the 0.10 quantile of the failure-time distribution at use conditions for the different plans listed in Table 3. The vertical line in Figure 4 is the actual failure-time quantile according to the planning information. First notice that the shape of the histograms are similar for the optimal and compromise test plans. The reason for this is that with a relatively small sample size, the test plans are very similar in the way the units are allocated to the accelerating levels. That is, for the optimum test plan, the allocation to the low and high levels are 23 and 1 units respectively. For the compromise test plan, the allocation is 21, 2, and 1 units allocated to the lowest level, the middle level, and the highest level of the accelerating variable, respectively. Additionally the plot shows the increased variability in the original and the traditional test plans.

Figure 5 shows the sample standard deviation computed from the simulated values in Figure 4 as well as the large-sample approximate standard errors for other test plans using $n =$

$12, 24, 36, 60, 120, 240$ and 300. The large sample approximate variance tends to be less than the actual variance. Because the ratio between the approximate and actual variances do not differ importantly across different kinds of test plans, the approximate variance provides an effective method for optimizing, but simulation should be used to evaluate the actual precision and chosen sample size. As the sample sizes increases, the large-sample and the simulation-based values have better agreement. For $n = 240$ the two approaches produce similar values.

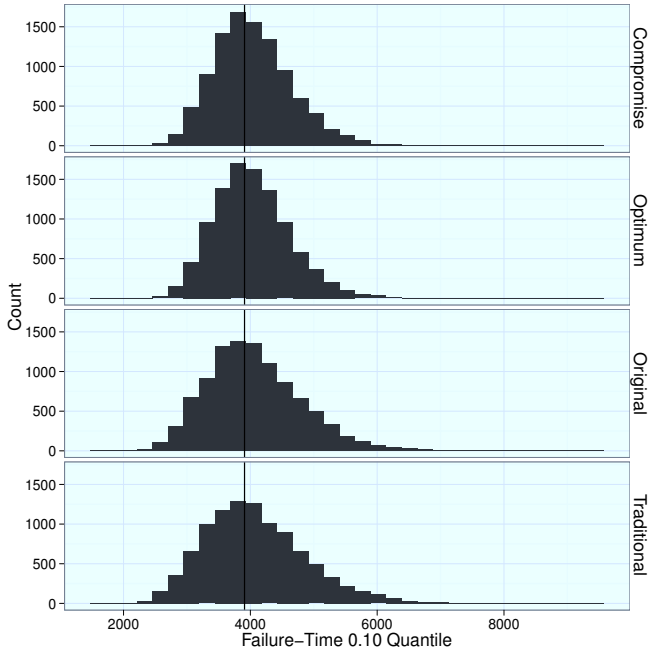


Figure 4: Histograms of 10,000 simulated ML estimates of failure-time distribution quantiles at use conditions (50°C), for the different scar-width test plans given in Table 3 with $n = 120$. The vertical line indicates the failure-time quantile at the use conditions, evaluated at the planning information.

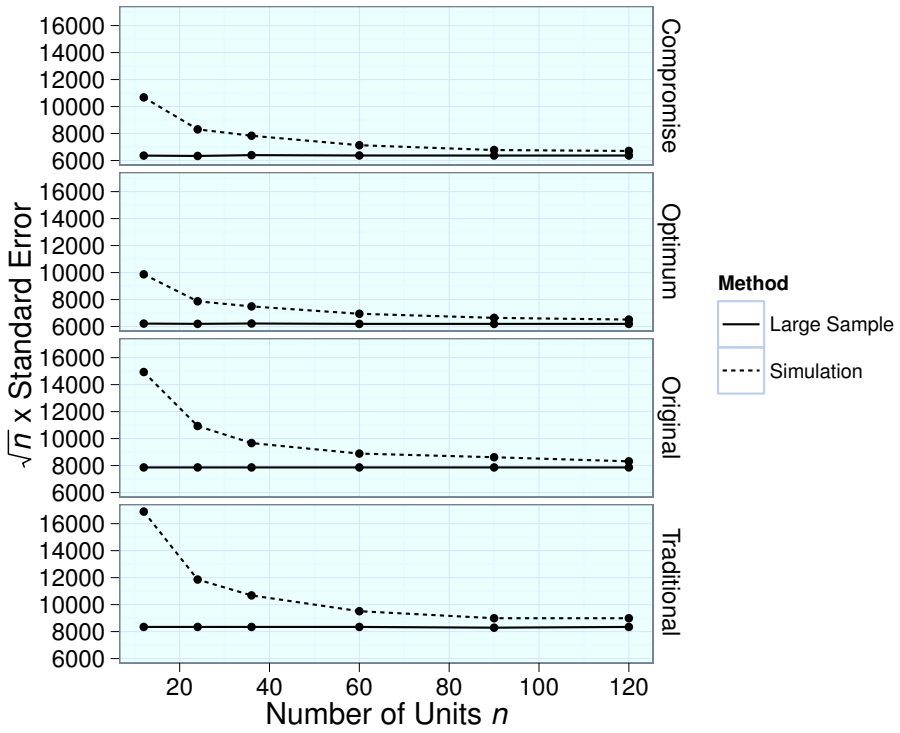


Figure 5: Comparison of values of $\sqrt{n}\text{ASE}(\hat{t}_p)$ and simulation estimates of $\sqrt{n}\text{ASE}(\hat{t}_p)$ for different RMADT plans and different sample sizes for the scar-width example.

9 Conclusions and Areas for Future Research

Nondestructive repeated measures degradation tests are useful for studying material or performance degradation of a product over time. It is important to plan these tests properly in order to achieve the desired level of precision while working within resource (time, number of units, and number of measurements) constraints. The methodology presented in this paper can be extended to more complicated situations. The following extensions are areas for future research:

- The examples used in this paper only considered a single-accelerating variable. Although they are not commonly used, some accelerated tests use two accelerating variables (e.g., temperature and humidity). A natural extension to this paper would be to develop test planning methods for two accelerating variables.

- It would be possible to extend the work in this paper to handle models that cannot be transformed to linear, such as the model with an asymptote used in Meeker, Escobar, and Lu (1998). For an example of test planning for such a model that has an asymptote see Section 22.5 of Meeker and Escobar (1998).

- Allow more complicated covariance structures to describe the within-unit variability (such as autocorrelations that might be expected with smaller spacing between measurements).
- Extend to models where the accelerating variables can also influence the variability in the degradation slopes and intercepts.
- In our motivating applications, like most accelerated tests, the lengths of the tests were fixed according to a product development schedule. It would be possible to extend our work to test planning where optimization is performed over both the accelerating variable levels and length of the test. See Yu and Tseng (2003) for some work in this direction.
- Extend to competing risks models where there are multiple failure-causing degradation mechanisms. Shi and Meeker (2011) consider such a situation for an application involving destructive degradation testing.
- One could use Bayesian methodologies for test planning so that prior knowledge about the model parameters can be incorporated into both planning and analysis, similar to what has been done for accelerating life testing in Zhang and Meeker (2006) and accelerated degradation testing in Liu and Tang (2010).

10 Acknowledgments

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11 Appendix

11.1 Derivation of the Information Matrix

For the derivation of the Fisher information matrix, we make (1) a little more concrete. Suppose that n items are to be tested and item i will be measured m_i times at factor-level combination j of the (possibly transformed) levels of the accelerating variables $\mathbf{x}_j = (x_{1j}, x_{2j})$. Then (1) can be rewritten as

$$Y_{ijk} = \mathbf{x}_j\boldsymbol{\gamma}' + \mathbf{z}_i\mathbf{b}' + \varepsilon_{ik},$$

for $i = 1, \ldots, n$, $j = 1, \ldots, K$, and $k = 1, \ldots, m_i$.

Collecting into $\mathbf{Y}_i = (Y_{ij1}, \ldots, Y_{ijm_i})'$ the observations from unit i, an equivalent expression for the linear degradation model in (1) is

$$\mathbf{Y}_i = \mathbf{X}_i\boldsymbol{\gamma}' + \mathbf{Z}_i\mathbf{b}_i' + \boldsymbol{\varepsilon}_i \tag{15}$$

where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$, $(b_0, b_1)' \sim BVN(\boldsymbol{\beta}, \mathbf{V})$, $\mathbf{X}_i$ and $\mathbf{Z}_i$ are matrices of explanatory variables

$$\mathbf{X}_i = \begin{pmatrix} x_{1j} & \tau_{i1} x_{2j} \\ \vdots & \vdots \\ x_{1j} & \tau_{im_i} x_{2j} \end{pmatrix}, \quad \mathbf{Z}_i = \begin{pmatrix} 1 & \tau_{i1} \\ \vdots & \vdots \\ 1 & \tau_{im_i} \end{pmatrix},$$

and $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \ldots, \varepsilon_{im_i})'$.

We assume independence between $\boldsymbol{\varepsilon}_i$ and $\mathbf{b}_i$ and that the components of $\boldsymbol{\varepsilon}_i$ are independent and jointly normal distributed (a reasonable assumption when spacing between measurements is not too small), expressed as, $\boldsymbol{\varepsilon}_i \sim \text{MVN}(0, \sigma^2 \mathbf{I}_i)$ where $\mathbf{I}_i$ is an $m_i \times m_i$ identity matrix. It follows that $\mathbf{Y}_i \sim \text{MVN}(\mathbf{X}_i \boldsymbol{\gamma} + \mathbf{Z}_i \boldsymbol{\beta}, \Sigma_i)$ where

$$\Sigma_i = \text{Var}(\mathbf{X}_i \boldsymbol{\gamma}' + \mathbf{Z}_i \mathbf{b}_i' + \boldsymbol{\varepsilon}_i) = \mathbf{Z}_i \mathbf{V} \mathbf{Z}_i' + \sigma^2 \mathbf{I}_i. \tag{16}$$

The log-likelihood for observational unit i is

$$\mathcal{L}_i = -\frac{1}{2} \log \left[\det(\Sigma_i)\right] - \frac{1}{2} (\mathbf{Y}_i - \mathbf{X}_i \boldsymbol{\gamma} - \mathbf{Z}_i \boldsymbol{\beta})' \Sigma_i^{-1} (\mathbf{Y}_i - \mathbf{X}_i \boldsymbol{\gamma} - \mathbf{Z}_i \boldsymbol{\beta}).$$

The total log-likelihood for n units is

$$\mathcal{L} = \sum_{i=1}^{n} \mathcal{L}_i = -\frac{1}{2} \sum_{i=1}^{n} \log \left[\det(\Sigma_i)\right] - \frac{1}{2} \sum_{i=1}^{n} (\mathbf{Y}_i - \mathbf{X}_i \boldsymbol{\gamma} - \mathbf{Z}_i \boldsymbol{\beta})' \Sigma_i^{-1} (\mathbf{Y}_i - \mathbf{X}_i \boldsymbol{\gamma} - \mathbf{Z}_i \boldsymbol{\beta}).$$

To simplify notation, collect into $\boldsymbol{\beta}^* = (\boldsymbol{\gamma}, \boldsymbol{\beta})$, the fixed effects model parameters, and collect into $\boldsymbol{\vartheta} = (\sigma_{b_0}^2, \sigma_{b_1}^2, \rho, \sigma^2)$, the model variance component parameters. Using Equation (4) of Jenrich and Schluchter (1986), it can be shown that the Hessian Matrix, $\mathbf{H}_i$, for unit i and its expected value are given by

$$\mathbf{H}_i = \begin{pmatrix} \mathbf{H}_{\boldsymbol{\beta}^*\boldsymbol{\beta}^*,i} & \mathbf{H}_{\boldsymbol{\beta}^*\boldsymbol{\vartheta},i} \\ \mathbf{H}_{\boldsymbol{\vartheta}\boldsymbol{\beta}^*,i} & \mathbf{H}_{\boldsymbol{\vartheta}\boldsymbol{\vartheta},i} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2 \mathcal{L}_i}{\partial \boldsymbol{\beta}^* \partial \boldsymbol{\beta}^*} & \dfrac{\partial^2 \mathcal{L}_i}{\partial \boldsymbol{\beta}^* \partial \boldsymbol{\vartheta}} \\ \dfrac{\partial^2 \mathcal{L}_i}{\partial \boldsymbol{\vartheta} \partial \boldsymbol{\beta}^*} & \dfrac{\partial^2 \mathcal{L}_i}{\partial \boldsymbol{\vartheta} \partial \boldsymbol{\vartheta}} \end{pmatrix}$$

$$\text{E}(\mathbf{H}_i) = I_i(\boldsymbol{\theta}) = \begin{pmatrix} \mathbf{X}_i' \Sigma_i^{-1} \mathbf{X}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_i \end{pmatrix}, \tag{17}$$

where $\mathbf{M}_i$ is a 4×4 symmetric matrix with elements

$$M_{jk}^i = \frac{1}{2} \text{tr}(\Sigma_i^{-1} \dot{\Sigma}_{ij} \Sigma_i^{-1} \dot{\Sigma}_{ik}), \; j = 1, \ldots, 4; \; k = 1, \ldots, 4,$$

and $\dot{\Sigma}_{ij} = \partial\Sigma_i/\partial\vartheta_j,\ j = 1, \ldots, 4$. From (16), it follows that

$$\begin{aligned}
\dot{\Sigma}_{i1} &= \frac{\partial\Sigma_i}{\partial\sigma_{b_0}} = \mathbf{Z}_i \begin{pmatrix} 2\sigma_{b_0} & \rho\sigma_{b_1} \\ \rho\sigma_{b_1} & 0 \end{pmatrix} \mathbf{Z}_i', \\
\dot{\Sigma}_{i2} &= \frac{\partial\Sigma_i}{\partial\sigma_{b_1}} = \mathbf{Z}_i \begin{pmatrix} 0 & \rho\sigma_{b_0} \\ \rho\sigma_{b_0} & 2\sigma_{b1} \end{pmatrix} \mathbf{Z}_i', \\
\dot{\Sigma}_{i3} &= \frac{\partial\Sigma_i}{\partial\rho} = \mathbf{Z}_i \begin{pmatrix} 0 & \sigma_{b1}\sigma_{b_0} \\ \sigma_{b1}\sigma_{b_0} & 0 \end{pmatrix} \mathbf{Z}_i', \\
\dot{\Sigma}_{i4} &= \frac{\partial\Sigma_i}{\partial\sigma} = 2\sigma\mathbf{I}_i.
\end{aligned}$$

Then the information matrix for all n units is $\mathbf{I}(\boldsymbol{\theta}) = \sum_{i=1}^{n} I_i(\boldsymbol{\theta})$.

11.2 Derivation of t_p Given in Section 3.2

Suppose that a failure occurs at the first point in time where $\mathcal{D} > \mathcal{D}_{\mathrm{f}}$. Let F denote the CDF of the random variable T and let $\sigma_{b_0b_1} = \rho\sigma_{b_0}\sigma_{b_1}$. Then

$$\begin{aligned}
F(t_p) = 1 - \Phi_{\mathrm{nor}}\left(\frac{y_{\mathrm{f}} - x_1\gamma_1 - x_2\tau_p\gamma_2 - \mathbf{z}\boldsymbol{\beta}'}{\sqrt{\sigma_{b_0}^2 + \tau_p^2\sigma_{b_1}^2 + 2\tau_p\sigma_{b_0b_1}}}\right) &= p \\
\Phi_{\mathrm{nor}}\left(\frac{y_{\mathrm{f}} - \beta_0 - x_1\gamma_1 - (\beta_1 + x_2\gamma_2)\tau_p}{\sqrt{\sigma_{b_0}^2 + \tau_p^2\sigma_{b_1}^2 + 2\tau_p\sigma_{b_0b_1}}}\right) &= 1 - p \\
\frac{y_{\mathrm{f}} - \beta_0 - x_1\gamma_1 - (\beta_1 + x_2\gamma_2)\tau_p}{\sqrt{\sigma_{b_0}^2 + \tau_p^2\sigma_{b_1}^2 + 2\tau_p\sigma_{b_0b_1}}} &= \Phi_{\mathrm{nor}}^{-1}(1-p) \\
\frac{\left[y_{\mathrm{f}} - \beta_0 - x_1\gamma_1 - (\beta_1 + x_2\gamma_2)\tau_p\right]^2}{\sigma_{b_0}^2 + \tau_p^2\sigma_{b_1}^2 + 2\tau_p\sigma_{b_0b_1}} &= \left[\Phi_{\mathrm{nor}}^{-1}(1-p)\right]^2 \qquad (18) \\
k\left(\sigma_{b_0}^2 + \tau_p^2\sigma_{b_1}^2 + 2\tau_p\sigma_{b_0b_1}\right) &= (h - r\tau_p)^2 \\
k\sigma_{b_1}^2\tau_p^2 + 2\sigma_{b_0b_1}k\tau_p + k\sigma_{b_0}^2 &= h^2 - 2hr\tau_p + r^2\tau_p^2 \\
\tau_p^2\left(k\sigma_{b_1}^2 - r^2\right) + 2\tau_p\left(k\sigma_{b_0b_1} + hr\right) + \left(k\sigma_{b_0}^2 - h^2\right) &= 0
\end{aligned}$$

where $k = \left[\Phi_{\mathrm{nor}}^{-1}(1-p)\right]^2$, $h = y_{\mathrm{f}} - \beta_0 - x_1\gamma_1$, and $r = \beta_1 + x_2\gamma_2$.

Let $a = \left(k\sigma_{b_1}^2 - r^2\right)$, $b = 2\left(k\sigma_{b_0b_1} + hr\right)$, $c = \left(k\sigma_{b_0}^2 - h^2\right)$. This equation is of the form:

$$a\tau_p^2 + b\tau_p + c = 0 \qquad (19)$$

with solutions for τ_p

$$\begin{aligned}
\tau_p &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-2\left(k\sigma_{b_0b_1} + hr\right) \pm \sqrt{4\left(k\sigma_{b_0b_1} + hr\right)^2 - 4(k\sigma_{b_1}^2 - r^2)\left(k\sigma_{b_0}^2 - h^2\right)}}{2(k\sigma_{b_1}^2 - r^2)} \\
&= \frac{-\left(k\sigma_{b_0b_1} + hr\right) \pm \sqrt{\left(k\sigma_{b_0b_1} + hr\right)^2 - (k\sigma_{b_1}^2 - r^2)\left(k\sigma_{b_0}^2 - h^2\right)}}{k\sigma_{b_1}^2 - r^2}.
\end{aligned}$$

The derivation when failure is defined as the first time at which $\mathcal{D} < \mathcal{D}_{\mathrm{f}}$ is similar, starting with

$$F(t_p) = \Phi_{\mathrm{nor}}\left(\frac{y_{\mathrm{f}} - x_1\gamma_1 - x_2\tau_p\gamma_2 - \mathbf{z}\boldsymbol{\beta}'}{\sqrt{\sigma_{b_0}^2 + \tau_p^2\sigma_{b_1}^2 + 2\tau_p\sigma_{b_0b_1}}}\right).$$

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