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Bayesian Life Test Planning for the Log-Location-Scale Family of Distributions

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Keywords
life data, log-location-scale family, sample size determination, Weibull distribution

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1 Introduction

1.1 Background

Careful planning of life tests in reliability studies is important as performing such experiments can be expensive and the experimental time is often limited. Usually life tests are censored by stopping the experiment after a certain amount of test time has elapsed (time or Type I censoring) or when a certain number of failures have occurred (failure or Type II censoring). Life tests are planned to control the number of sample units, test time (for Type I censoring), and/or number of failures to be observed (for Type II censoring) so that, if possible, a specified precision can be obtained for the estimation of a particular quantity of interest.

In traditional life test planning, "planning values" of the unknown parameters of the lifetime distribution are required as inputs. Then, with the test plan specification (i.e., sample sizes and censoring time/number of failures), one can compute a precision factor of the estimation, such as the confidence interval precision factor described in Meeker and Escobar (1998, Chapter 10). Planning values are chosen based on the experimenter's best knowledge about the underlying failure-time model to allow for planning an efficient experiment. Bayesian methods allow one to combine prior information with data to make inferences and are used in this paper to provide appropriate life test planning tools for situations in which prior information may be used in inference problems.

Log-location-scale distributions, such as the Weibull and lognormal, are commonly used in lifetime studies. In some special cases, such as when the shape parameter of the Weibull distribution is given (e.g., the exponential and Raleigh distributions), a conjugate prior distribution for the unknown scale parameter and closed-form solutions for the Bayesian planning problem are available (see Zhang and Meeker 2005). In other situations, however, the solutions become more complicated because both parameters are unknown and censoring is present. No closed forms exist for these planning problems so numerical methods, simulations, or approximations are required.

In this paper, we develop a general large sample approximation approach that makes it relatively easy to solve the life test planning problem for the log-location-scale family of distributions. We then illustrate the use of simulation-based methods that allow evaluation

of proposed test plans without relying on the large sample approximation.

1.2 Related Literature

In the framework of non-Bayesian life test planning, Meeker and Escobar (1998, Chapter 10) describe general approaches and useful techniques. Gupta (1962), Grubbs (1973), and Narula and Li (1975) describe life test sample size determination to control error probabilities in hypothesis testing. Danziger (1970) provides life test plans for estimating a Weibull hazard rate when the shape parameter is given. Meeker and Nelson (1976, 1977) present large-sample approximate methods and applications in life test planning for estimating functions of Weibull parameters. Meeker, Escobar, and Hill (1992) develop the asymptotic theory and methods for life test planning if a Weibull hazard function is to be estimated and both parameters are unknown.

For Bayesian life test planning, Thyregod (1975) presents a cost-based utility function approach to exponential life testing with Type II censored data. Zaher, Ismail, and Bahaa (1996) provide Bayesian life test plans for Weibull Type I censoring cases when the shape parameter is given, based on a Shannon information criterion, while Zhang and Meeker (2005) describe Bayesian methods for estimating a specified quantile of interest when the Weibull data are Type II censored. Kundu (2008) provides Bayesian test planning methods under progressively censored samples of a Weibull distribution. Hamada et al. (2008) provides a recent treatment of Bayesian estimation in reliability.

Bayesian methods are also used in other sample size determination problems. For example, Polson (1993) develops a general decision-theory approach for a Bayesian accelerated life test design problem. Joseph, Wolfson, and du Berger (1995a, 1995b) develop Bayesian criteria to determine sample sizes. Lindley (1997) describes a decision-theoretic Bayesian approach for the general sample size problem and makes comparisons between the proposed method and other Bayesian criteria based on interval estimation precision. Pham-Gia (1997), Joseph and Wolfson (1997), and Adcock (1997) provide more detailed discussions and comparisons of these two kinds of criteria in Bayesian applications. Zhang and Meeker (2006) describe Bayesian methods for accelerated life test planning with one accelerating variable. Tang and Liu (2010) propose using a sequential approach to plan accelerated life tests. Shi and Meeker

(2012) develop Bayesian methods for accelerated destructive degradation test planning.

1.3 Overview

The remainder of this paper is organized as follows. Section 2 describes the lifetime distribution, the Bayesian life test planning framework, and the large sample approximation (LSA) and simulation approaches for the Bayesian life planning. Section 3 applies the LSA approach to log-location-scale distributions with Type II censoring and prior information on the scale parameter. Section 3 also provides a comparison of the results with the LSA approach. Section 4 provides a numerical example to illustrate the application. Section 5 provides a discussion on test planning under Type I censoring, different prior distributions, and joint informative prior information. Section 6 gives some concluding remarks.

2 The Bayesian Life Test Planning Framework

2.1 The Lifetime Model

The log-location-scale family of distributions is commonly used to describe the distribution of lifetimes. The Weibull and lognormal distributions are frequently used members of this family. The cumulative distribution function (cdf) and probability density function (pdf) of a log-location-scale distribution are

$$F(t;\boldsymbol{\theta}) = \Phi\left[\frac{\log(t)-\mu}{\sigma}\right] \quad \text{and} \quad f(t;\boldsymbol{\theta}) = \frac{1}{\sigma t}\phi\left[\frac{\log(t)-\mu}{\sigma}\right], \tag{1}$$

respectively. Here $\boldsymbol{\theta} = (\mu, \sigma)'$ is the unknown parameter vector, μ is a location parameter, σ is a scale parameter, and Φ and ϕ are the standard cdf and pdf for the location-scale family of distributions (location 0 and scale 1), respectively.

The cdf and pdf of the lognormal distribution are obtained by replacing Φ and ϕ in (1) with Φ_{nor} and ϕ_{nor}, the standard normal cdf and pdf, respectively. The cdf and pdf of the Weibull distribution are obtained by replacing by replacing Φ and ϕ in (1) with $\Phi_{\text{sev}}(z) = 1 - \exp[-\exp(z)]$ and $\phi_{\text{sev}}(z) = \exp[z - \exp(z)]$, which are the standard (i.e., $\mu = 0, \sigma = 1$) smallest extreme value cdf and pdf, respectively. An alternative parametrization for the Weibull distribution is also widely used. In particular, the pdf of the Weibull distribution is

expressed as

$$f(t;\eta,\beta) = \left(\frac{\beta}{\eta}\right)\left(\frac{t}{\eta}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right]$$

where $\eta = \exp(\mu)$ is a scale parameter and $\beta = 1/\sigma$ is a shape parameter.

For a high-reliability component, a quantile in the lower tail of the failure-time distribution provides a meaningful life characteristic, such as the 0.1 quantile. In particular, the p quantile is solved from $p = F(t_p;\boldsymbol{\theta})$. The log of the quantile can be expressed as

$$\log(t_p) = \mu + z_p\sigma = \boldsymbol{c}'\boldsymbol{\theta},$$

where $\boldsymbol{c} = [\,1, z_p\,]'$ and z_p is the p quantile of the standardized log-location-scale distribution.

2.2 Test Planning and the Criterion

Life tests are often planned to estimate a particular quantile (i.e., t_p) of the lifetime distribution. It is natural to use a criterion for the planning problem that is constructed from some measure of the precision of estimation of $\log(t_p)$ (because the quantile is a positive quantity). In the Bayesian framework, estimation precision is usually specified as a (monotone) function of the posterior variance. For a given test plan D, the posterior variance depends on the data. A reasonable Bayesian criterion for test planning is then the preposterior expectation of the posterior estimation precision function. This criterion is computed by taking an expectation over the marginal distribution of the data to account for all possible outcomes from the experiment.

Denote the data from the experiment by $\boldsymbol{t}$. The criterion for the Bayesian life test planning is

$$C(D) \;=\; \mathrm{E}_{\boldsymbol{t}|D}\left[g\left(\mathrm{Var}_{\boldsymbol{\theta}|\boldsymbol{t},D}[\log(t_p)]\right)\right], \tag{2}$$

which is the preposterior expectation of some function g of the posterior variance of $\log(t_p)$.

The function $g(\cdot)$ should be chosen to provide an interpretable precision measure defined by the experimenter to focus on the quantity of interest for the estimation. Using the function $g(x) = x$ expresses precision in terms of the posterior variance of the quantile in the log scale, which is equivalent to the utility defined from the quadratic loss function. An alternative functional form of g is based on an appropriate credibility interval precision of t_p on the

original scale. Such an interval can be constructed approximately as $[\breve{t}_p/\breve{R}, \quad \breve{t}_p \times \breve{R}]$, where $\breve{t}_p$ is a Bayesian estimator of t_p, $\breve{R}$ is a posterior normal approximate credibility interval precision factor

$$\breve{R} = \exp\left(z_{1-\alpha/2}\sqrt{\mathrm{Var}_{\boldsymbol{\theta}|\boldsymbol{t},D}[\log(t_p)]}\right),$$

and $z_{1-\alpha/2}$ is the $(1-\alpha/2)$ quantile of the standard normal distribution. The preposterior credibility interval precision factor R, defined as $\mathrm{E}_{\boldsymbol{t}|D}(\breve{R})$, can be used as the precision measure and in this case,

$$g(x) = \exp(z_{1-\alpha/2}\sqrt{x}\,). \tag{3}$$

The life test planning problem therefore is to find the most cost/time-effective combination of sample size and censoring time (Type I censoring) or number of failing (Type II censoring) subject to a constraint on expected precision defined by the criterion (2). The key step is to compute the variance of the posterior distribution $\mathrm{Var}_{\boldsymbol{\theta}|\boldsymbol{t},D}(\boldsymbol{\theta})$. Life test data are usually censored and for Bayesian methods, numerical methods are needed to obtain the posterior distribution. The next sections introduce the two commonly used numerical approaches: the large sample approximation (LSA) approach and the simulation approach.

2.3 Prior Distributions

Let $\omega(\boldsymbol{\theta}|\boldsymbol{\beta})$ be a test planning prior distribution for the unknown parameters $\boldsymbol{\theta} = (\mu, \sigma)'$, where $\boldsymbol{\beta}$ is a known vector of hyperparameters. In this paper, we use a diffuse prior distribution for μ (i.e., non-informative prior, such as an improper uniform), but an informative prior distribution for σ, as is commonly done in practice. This is because previous experience with a known failure mode generally provides information primarily about possible values for σ. For example, Nelson (1990) points out the Weibull shape parameter $\beta = 1/\sigma$ can be expected to be between 1.1 and 1.5 for the life time of steel bearings. The gamma distribution is often used as the prior distribution for σ. In general, the prior distributions for the parameters can be obtained from multiple sources such as expert judgment, previous experiments, and historical data.

Let S^{-1} denote the precision matrix of the prior distribution for the unknown parameter

$\boldsymbol{\theta}$. The inference prior precision matrix is obtained as

$$S^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & [\mathrm{Var}(\sigma)]^{-1} \end{bmatrix}. \tag{4}$$

In the Bayesian test planning literature (e.g., Tsutakawa 1972, Etzione and Kadane 1993, and Shi and Meeker 2012), two different types of priors are often used, which are planning priors and inferential priors. In particular, the planning prior is the prior used in the design of experiments and the inferential prior is the prior used in the inference. The justification of using two priors is that the risk of those doing the experiment is different from the risk of those who are concerned with the accuracy of the inference based on data. In general, more informative priors are often used during the test planning stage while non-informative priors are often used during the statistical inference state (e.g., Chaloner and Larntz 1989).

2.4 Large Sample Approximation Approach

For the Bayesian life test planning problem, the LSA provides a simple, useful description. For example, Berger (1985) gives some analytical forms of normal approximations for stable estimation. The variance of the posterior distribution can be expressed as a combination of the inference prior information and the data. In particular, the posterior variance-covariance matrix for $\boldsymbol{\theta}$ is

$$\mathrm{Var}_{\boldsymbol{\theta}|\boldsymbol{t},D}(\boldsymbol{\theta}) \approx \left[S^{-1} + \widehat{I}_{\boldsymbol{\theta}}(D)\right]^{-1}. \tag{5}$$

Here $I_{\boldsymbol{\theta}}(D)$ denotes the Fisher information matrix (FIM) for the proposed test plan D, $\widehat{\boldsymbol{\theta}}$ is an estimator of $\boldsymbol{\theta}$, and $\widehat{I}_{\boldsymbol{\theta}}(D)$ is $I_{\boldsymbol{\theta}}(D)$ evaluated at $\widehat{\boldsymbol{\theta}}$.

The Fisher information matrix $\widehat{I}_{\boldsymbol{\theta}}(D)$ in (5) quantifies the amount of information provided by the proposed experiment. Therefore, the test plan D affects the estimation precision only through the FIM. With larger sample sizes this information from the experimental data will increase and in the limit will dominate the prior information, thus increasing the estimation precision. With experimental cost control, however, the sample size will be chosen as small as possible subject to some precision requirements.

Censoring also plays a role through the FIM. In this paper, we use Type II censoring as an illustration. The test plan is specified by $D = (n, r)$ where n is the sample size and r is

the number of failures. The proportion failing, denoted by $p_c = r/n$, can be used to describe the amount of censoring independently of n.

The large sample approximate Bayesian criterion is computed as follows. Under the LSA, the criterion (2) for Bayesian life test planning is

$$\begin{aligned} C(D) &= \mathrm{E}_{\boldsymbol{t}|D}\left[g\left(\mathrm{Var}_{\boldsymbol{\theta}|\boldsymbol{t},D}[\log(t_p)]\right)\right] = \mathrm{E}_{\boldsymbol{t}|D}\left[g\left(\boldsymbol{c}'\mathrm{Var}_{\boldsymbol{\theta}|\boldsymbol{t},D}(\boldsymbol{\theta})\boldsymbol{c}\right)\right] && (6)\\ &\approx \int g\left(\boldsymbol{c}'\left[S^{-1}+\widehat{I}_{\boldsymbol{\theta}}(D)\right]^{-1}\boldsymbol{c}\right) d(p(\widehat{\boldsymbol{\theta}})) && (7)\\ &\approx \int g\left(\boldsymbol{c}'\left[S^{-1}+I_{\boldsymbol{\theta}}(D)\right]^{-1}\boldsymbol{c}\right) d(\omega(\boldsymbol{\theta})). && (8) \end{aligned}$$

The approximation from (6) to (7) is based on the LSA of the posterior variance. From (7) to (8), the prior distribution was used *a preposteriori* to approximate the predictive distribution of $\widehat{\boldsymbol{\theta}}$. The predictive distribution of $\widehat{\boldsymbol{\theta}}$ is a convolution of the test planning prior distribution of $\boldsymbol{\theta}$ and the distribution arising from the estimation of $\boldsymbol{\theta}$ from the data, and will approach the test planning prior distribution as the sample size increases. Similar approximations have also been used for other Bayesian design problems (e.g., Clyde, Müller, and Parmigiani 1995).

2.5 Simulation Approach

An alternative to the LSA for (2) is to use a simulation-based approach, which is computationally intensive. The simulation method, however, provides a tool for visualization and validation as well as for evaluation and improvement of test plans obtained from other methods, without the use of any approximation. To evaluate the criterion $C(D)$ in (2) with a given plan D, we use the following algorithm:

Simulation Algorithm:

1. Simulate m random samples $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m$ from the test planning prior distribution $\omega(\boldsymbol{\theta}|\boldsymbol{\beta})$. Note here $\boldsymbol{\theta} = (\mu, \sigma)'$ and $\boldsymbol{\beta}$ denotes the parameters for the prior distributions. We use a diffuse prior on μ. The prior information on σ can be specified as a gamma distribution.

2. For each $\boldsymbol{\theta}_i$, simulate random samples $\boldsymbol{t}_i$, with appropriate size and censoring defined by plan D, from the lifetime distribution in (1).

3. For each $\boldsymbol{t}_i$, compute the posterior variance of $\log(t_p)$ and the posterior precision measure $g\left(\mathrm{Var}_{\boldsymbol{\theta}|\boldsymbol{t}_i,D}[\log(t_p)]\right)$.

4. Compute the criterion $C(D)$ as the sample mean of the simulated posterior precision measure values. That is,

$$C(D) = \sum_i g\left(\mathrm{Var}_{\boldsymbol{\theta}|\boldsymbol{t}_i,D}[\log(t_p)]\right)/m.$$

In Step 3, a Markov Chain Monte Carlo (MCMC) method is used to obtain the samples from the posterior distribution. The posterior variance $\mathrm{Var}_{\boldsymbol{\theta}|\boldsymbol{t}_i,D}[\log(t_p)]$ is approximated by the sample variance of those posterior samples. The Metropolis algorithm is used to simulate the Markov chain process. More detailed and general methods and applications for MCMC methods can be found in a wide range of literature, for example, in Gelman et al. (1995, Chapter 11).

3 Test Planning for Log-Location-Scale Distributions

3.1 The Setup

This section applies the LSA approach to the log-location-scale distribution with Type II censoring case when prior information is available on the scale parameter σ. It is common that the experimenter needs a $100(1-\alpha)\%$ interval estimate of t_p. Because the precision is specified in terms of the width of the credibility interval for t_p, the functional form of $g(\cdot)$ is as in (3). The resulting Bayesian criterion under the LSA in (8) or (4) is referred to as the large sample approximate preposterior precision factor (LSAPPF). In the LSA approach, the life test planning problem requires finding the most cost-effective combination of (n, r) subject to a specified LSAPPF.

The Type II censored experiment is planed in terms of sample size n and p_c.

$$I_{\boldsymbol{\theta}}(D) = \frac{n}{\sigma^2}\mathcal{F}(p_c), \tag{9}$$

$\mathcal{F}(p_c)$ is the scaled FIM, the elements of which can be directly computed with the LSINF algorithm in Escobar and Meeker (1994) or similar formulas given in Hong, Ma, and Meeker (2010). Under Type II censoring, the FIM is only a function of the unknown scale parameter σ (given n and r) and not a function of μ. This means the prior information on μ will not affect the elements of the FIM. Thus the integral in (8) reduces to a one-dimensional integral over the informative prior information on σ.

The LSAPPF is

$$C(D) \approx \int \exp\left[z_{1-\alpha/2} \left(\boldsymbol{c}' \left[S^{-1} + \frac{n}{\sigma^2} \mathcal{F}(p_c) \right]^{-1} \boldsymbol{c} \right)^{\frac{1}{2}} \right] d(\omega(\sigma)), \tag{10}$$

where $\boldsymbol{c} = [1, z_p]'$ and $\omega(\sigma)$ is the marginal planning prior distribution for σ (i.e., the gamma distribution).

3.2 Prior Information and Effects

In this section we illustrate, under the LSA, how inference prior information affects the estimation precision of the quantile of interest. By (9) and (4), the scaled asymptotic posterior variance factor of $\log(t_p)$ is

$$\frac{n\mathrm{Var}_{\boldsymbol{\theta}|\boldsymbol{t},D}[\log(t_p)]}{\sigma^2} \approx \boldsymbol{c}' \left[r_\sigma \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \mathcal{F}(p_c) \right]^{-1} \boldsymbol{c}, \tag{11}$$

where the *prior precision ratio* r_σ, defined as

$$r_\sigma = \frac{\sigma^2/n}{\mathrm{Var}(\sigma)} \tag{12}$$

gives a measure of the amount of prior information with respect to the information resulting from the experiment. From (11), we have the following results.

- When $r_\sigma \to 0$ (not much inference prior information for σ or μ), information about t_p is mainly from the data and results will behave in a manner that is similar to a non-Bayesian approach such as ML. Furthermore, the right hand side of (11) approaches the scaled asymptotic variance factor of the ML estimators.

- When $r_\sigma \to \infty$, implying that the inference prior knowledge of σ is approaching perfect information, with no inference prior information for μ, results will be similar to the non-Bayesian methods where σ is given (e.g. Zhang and Meeker 2005). Furthermore, the right hand side of (11) approaches a constant of $1/f_{11}(p_c)$, proportional to the asymptotic variance of μ with σ given, where $f_{11}(p_c)$ is the $(1,1)$ element of the scaled FIM.

Figure 1 provides a visualization of these effects of prior information on the posterior variance factor of the quantile of interest for the Weibull distribution. From Figure 1 we can see that, when the prior precision ratio $r_\sigma = 0$, as the proportion of failures p_c and quantile of

interest p vary, the posterior variance factor of $\log(t_p)$ behaves the same as that from a non-Bayesian approach (e.g., in Figure 10.5 of Meeker and Escobar 1998). The minimum value of the posterior variance factor for each curve with fixed p_c is $1/f_{11}(p_c)$, implying that the estimation precision for a quantile is limited by the estimation variation of μ. As the prior precision ratio r_σ increases, the posterior variance factor for any quantile gradually decreases to this minimum value. The potential improvement of estimation precision is especially significant for heavily censored data (i.e., a small amount of prior information induces a large decrease in the posterior variance factor for small p_c's), in which cases the posterior distribution is mainly determined by the prior information. The relationship between the prior information and the posterior variance gives an assessment of how much improvement in estimation precision one can obtain for a given amount of prior information.

3.3 Test Plan Solutions

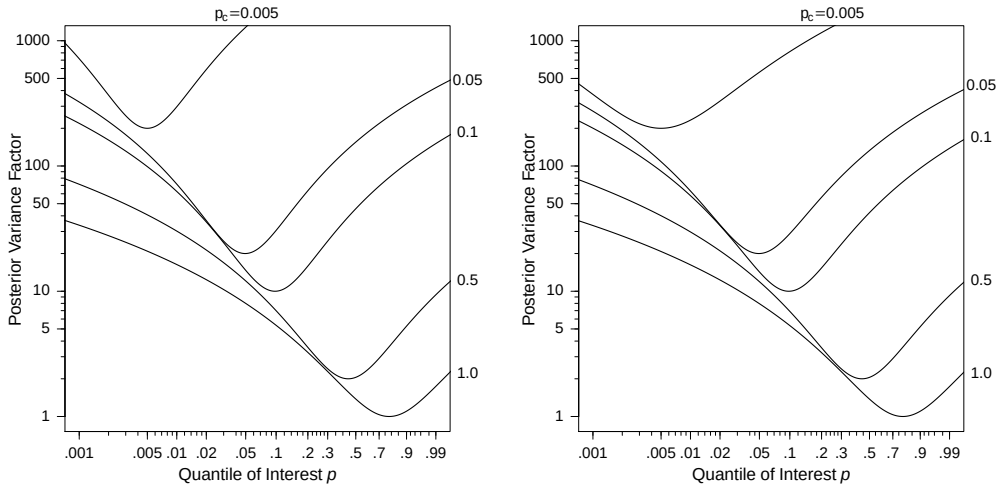
The test plan solution (n, r) to give the desired value of the LSAPPF criterion can be determined in the following way. Let $r_\sigma^\triangle$ denote the prior-predicted precision ratio, obtained by substituting $\mathrm{E}(\sigma)$ for σ in (12). That is

$$r_\sigma^\triangle = \frac{[\mathrm{E}(\sigma)]^2/n}{\mathrm{Var}(\sigma)}.$$

To illustrate the effects of the mean of the prior distribution $\mathrm{E}(\sigma)$ and the prior-predicted precision ratio $r_\sigma^\triangle$ on the LSAPPF criterion, we consider a scenario of test planning for estimating $t_{0.1}$ for the Weibull distribution. Figure 2 shows the relationship between the sample size n and the LSAPPF criterion value for various combinations of values of $\mathrm{E}(\sigma)$, $r_\sigma^\triangle$ and p_c. Note that $\mathrm{E}(\sigma) = 10$ indicates a large amount of variation in the failure-time process. Although this amount of variation is unlikely in real applications, we use it here as an illustration of an extreme case. The following gives a discussion of the results in Figure 2.

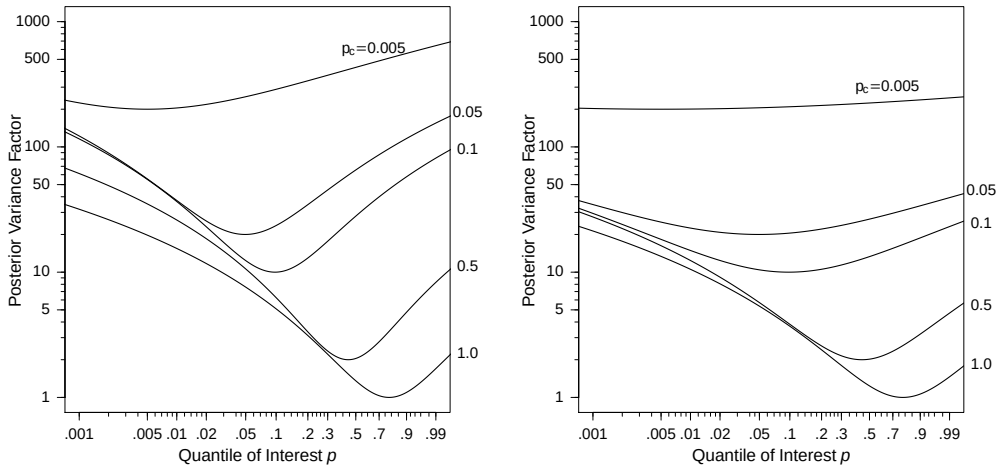
- For the effect of proportion of failures p_c, we find a larger n is needed for smaller p_c (heavier censoring) to attain a specified estimation precision. The appropriate combination of (n,r) can be determined by the experimenter for given cost and time constraints.

- With more prior precision $r_\sigma^\triangle$, we find that less experimental resources (i.e., smaller n or smaller p_c) are needed to provide a given amount of posterior precision. For a small p_c, a



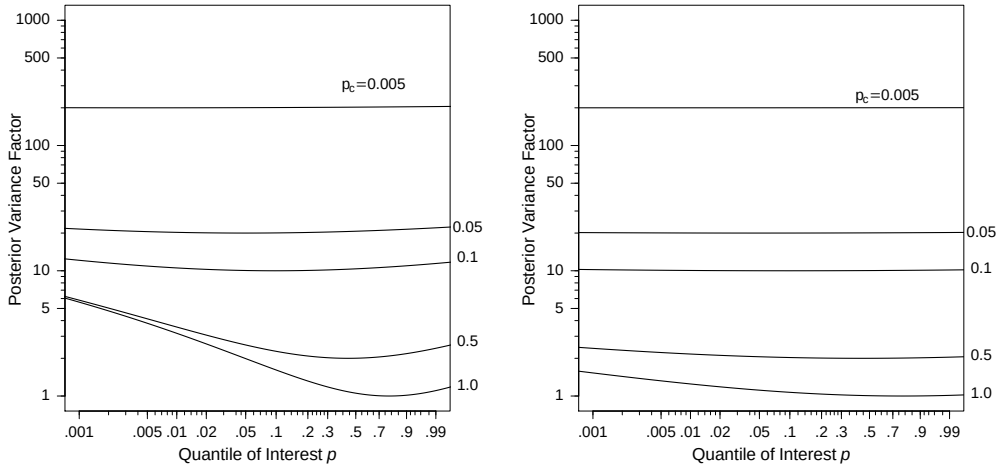
(a) Prior precision ratio =0

(b) Prior precision ratio =0.01



(c) Prior precision ratio =0.1

(d) Prior precision ratio =1



(e) Prior precision ratio =10

(f) Prior precision ratio =100

Figure 1: The effect of prior precision ratio on estimation precision of a quantile of interest p, for different amounts of censoring p_c, under the Weibull distribution.

small amount of prior information (i.e., $r_\sigma^\triangle = 0.1$ versus $r_\sigma^\triangle = 1$) will reduce significantly the needed amount of experimental resources. Further increases of prior precision (i.e., $r_\sigma^\triangle = 1$ versus $r_\sigma^\triangle = 10$), however, will only slightly increase the estimation precision. The experimental resources in the latter case are needed primarily to estimate μ, because the prior for μ is diffuse.

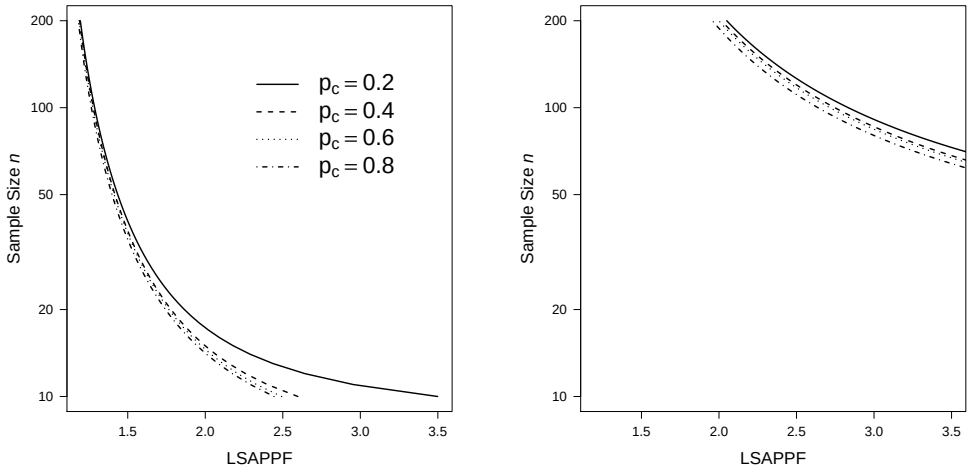
- In Figure 2 the above relationships are similar for different prior means $\mathrm{E}(\sigma)$, with more experimental resources needed when the prior mean $\mathrm{E}(\sigma)$ is larger. Because σ represents the variation of time to failure in the log scale, a larger value implies more relative variation in the lifetime distribution, so more experimental resources are needed to achieve a certain estimation precision.

We have discussed the planning solutions when prior information is given in terms of $r_\sigma^\triangle$. This is meaningful when comparing the effects of prior information on the experiment to be performed. Note that $r_\sigma^\triangle$ also depends on sample size n. In practice, the prior information available *a priori* is fixed and independent of n. Alternatively, the prior distribution for σ can be specified in terms of $\mathrm{E}(\sigma)$ and the prior standard deviation $\mathrm{Stdev}(\sigma)$. That is, the prior precision can be represented by $r_{\sigma,1}^\triangle = [\mathrm{E}(\sigma)]^2/\mathrm{Var}(\sigma)$, the prior-predicted precision ratio for $n = 1$. Figure 3 gives the planning solutions for the same scenario as in Figure 2 with prior information specified in terms of $\mathrm{E}(\sigma)$ and $r_{\sigma,1}^\triangle$ (and equivalent to $\mathrm{Stdev}(\sigma)$). The results are similar to Figure 2.

3.4 Comparison of the Two Approaches

In this section, we provide comparisons of the LSA and simulation approaches. We consider a scenario that is the same as in Section 3.2 and use the following specifications in the simulation. We use a random sample of $m = 1000$ from the prior distribution. For each simulated data set, the Metropolis algorithm provides 1000 thinned samples from the posterior distribution of $\boldsymbol{\theta}$. Those 1000 samples are obtained by selecting every 100th sample after removing the burn-in of 1000 samples based on a graphical check of the MCMC sample traces. The 1000 thinned samples are sufficient to provide a stable estimate for the posterior variance.

Figure 4 compares the relationship between the planning specifications and the criterion values obtained from LSA and the simulation approach, when $\mathrm{E}(\sigma) = 1$, $r_{\sigma,1}^\triangle = 1$, and $p_c =$

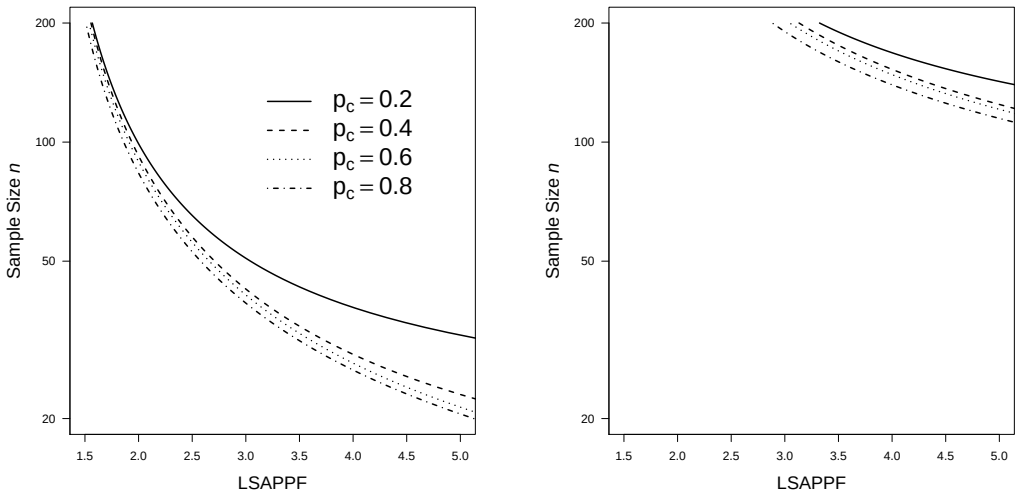


(a) $\mathrm{E}(\sigma) = 0.5$, $r_\sigma^\Delta = 0.1$ (b) $\mathrm{E}(\sigma) = 2$, $r_\sigma^\Delta = 0.1$

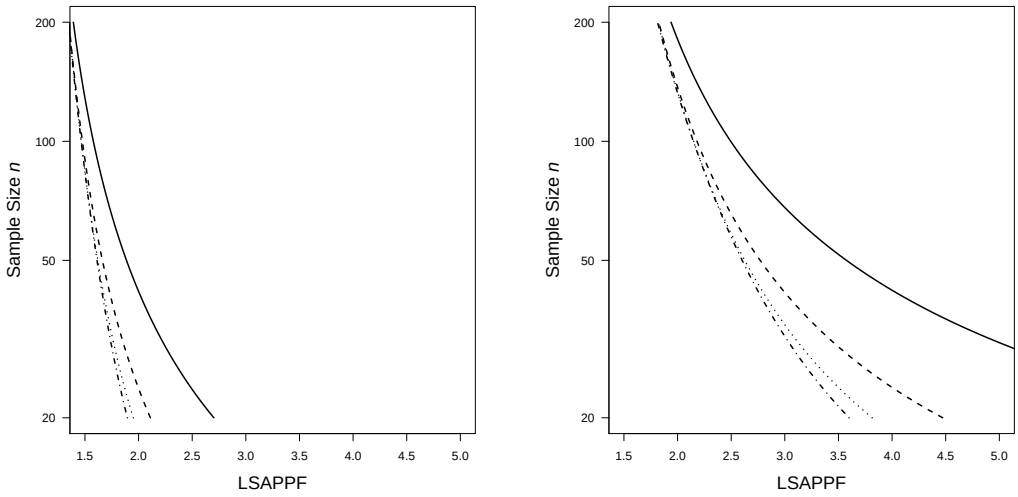
(c) $\mathrm{E}(\sigma) = 0.5$, $r_\sigma^\Delta = 1$ (d) $\mathrm{E}(\sigma) = 2$, $r_\sigma^\Delta = 1$

(e) $\mathrm{E}(\sigma) = 0.5$, $r_\sigma^\Delta = 10$ (f) $\mathrm{E}(\sigma) = 2$, $r_\sigma^\Delta = 10$

Figure 2: Needed sample size and proportion of failures as a function of LSAPPF for $t_{0.1}$ when prior information on σ is specified by $\mathrm{E}(\sigma)$ and $r_\sigma^\triangle$.



(a) $\mathrm{E}(\sigma) = 1$, $r^{\Delta}_{\sigma,1} = 1$ (b) $\mathrm{E}(\sigma) = 2$, $r^{\Delta}_{\sigma,1} = 1$



(c) $\mathrm{E}(\sigma) = 1$, $r^{\Delta}_{\sigma,1} = 100$ (d) $\mathrm{E}(\sigma) = 2$, $r^{\Delta}_{\sigma,1} = 100$

Figure 3: Needed sample size and proportion of failures as a function of LSAPPF for $t_{0.1}$ when prior information for σ is fixed and is specified by $\mathrm{E}(\sigma)$ and $r^{\triangle}_{\sigma,1}$.

$0.2, 0.4, 0.6, 0.8$. Figure 5 shows results from a similar setting but with $r^{\triangle}_{\sigma,1} = 100$. Overall, the results from the two approaches are fairly consistent, indicating that the LSA approach provides reasonably accurate approximations.

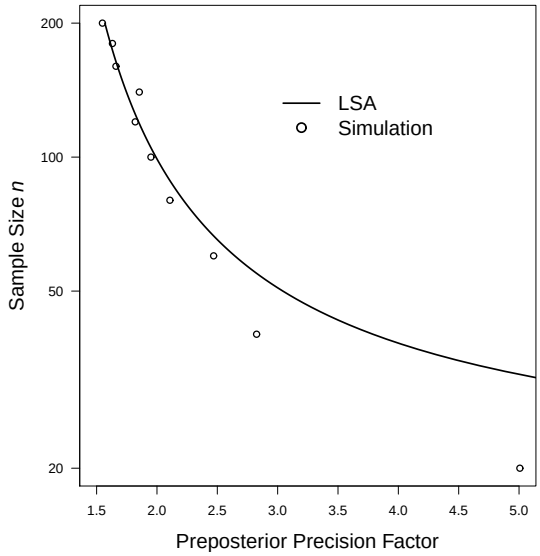
- For larger n, the differences of the two approaches are small in terms of the criterion value of the same plan, when $n > 100$. The approaches agree very well when there is a large amount of prior information on σ (e.g., $r^{\triangle}_{\sigma,1} = 100$).
- For larger p_c, the differences are small for small n. The differences are even smaller with a large amount of prior information.
- When r is small, the two approaches differ from each other because the normal approximation for the posterior distribution becomes inadequate.

There is also an effect of rounding from a continuous plan (r is considered to be continuous in the LSA approach) to a discrete plan. Thus the sampling error in simulations becomes larger. In situations involving small r, the simulation approach can be used as a validation tool after an initial plan is obtained from the easy-to-be-computed LSA. This allows the experimenter to assess how much difference is expected in the estimation precision from the experiment and adjust the sample sizes as needed.

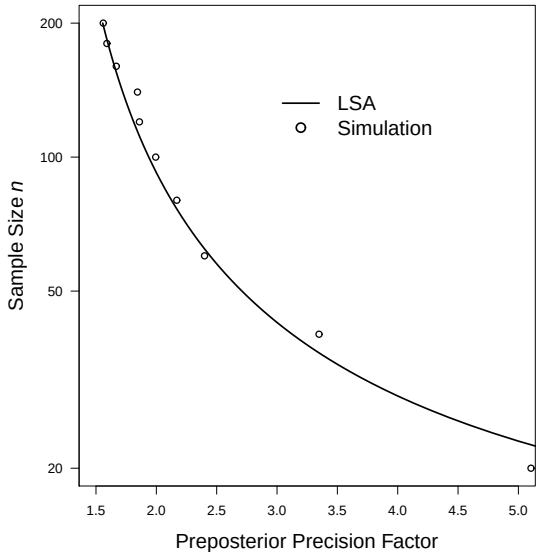
4 A Life Test Planning Application

4.1 Solution by the Large Sample Approximation Approach

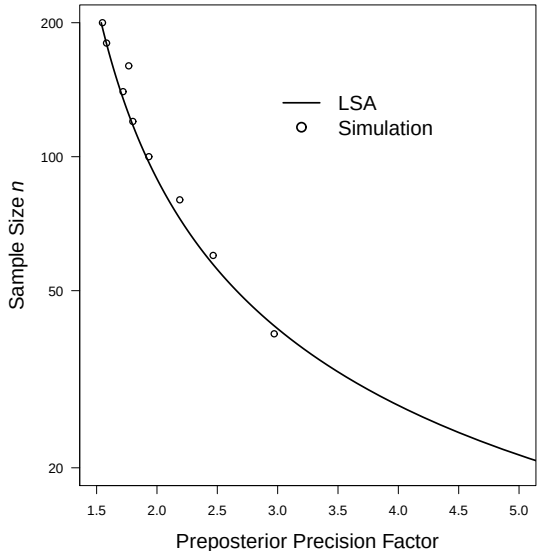
Here we extend Example 10.7 in Meeker and Escobar (1998) by including the use of prior information. Suppose that a manufacturer wants to estimate the 0.1 quantile of the lifetime distribution of a newly developed insulation. Tests will be run until 20% of the units fail ($p_c = 0.2$). To allow a shorter test time, the experiment will be done at a higher than usual level of voltage stress, using a given acceleration factor. The engineers suggest that the Weibull distribution should be appropriate for the lifetime distribution of the insulation specimens. Also, based on previous experiments of similar insulation lifetime studies, the engineers state that prior information on the Weibull shape parameter β is available and $\sigma(= 1/\beta)$ is expected to be around 1.2, with a standard deviation of 0.1 (i.e., $\mathrm{E}(\sigma) = 1.2$, $\mathrm{stdev}(\sigma) = 0.1$). Using



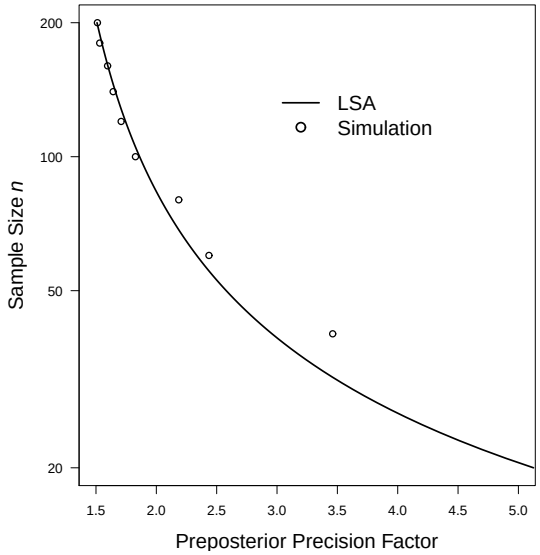
(a) $\mathrm{E}(\sigma) = 1$, $r^{\Delta}_{\sigma,1} = 1$, $p_c = 0.2$



(b) $\mathrm{E}(\sigma) = 1$, $r^{\Delta}_{\sigma,1} = 1$, $p_c = 0.4$

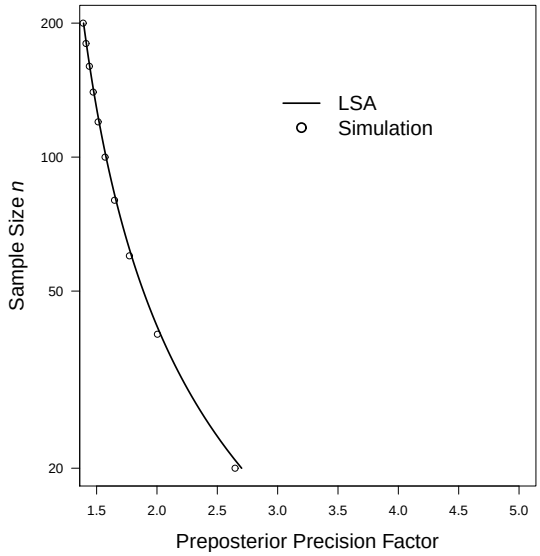


(c) $\mathrm{E}(\sigma) = 1$, $r^{\Delta}_{\sigma,1} = 1$, $p_c = 0.6$

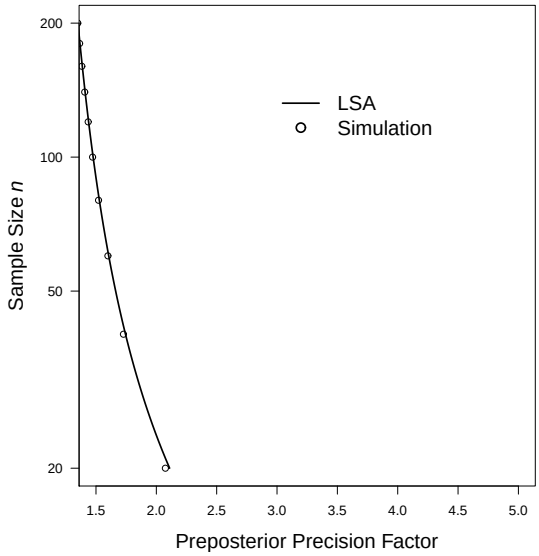


(d) $\mathrm{E}(\sigma) = 1$, $r^{\Delta}_{\sigma,1} = 1$, $p_c = 0.8$

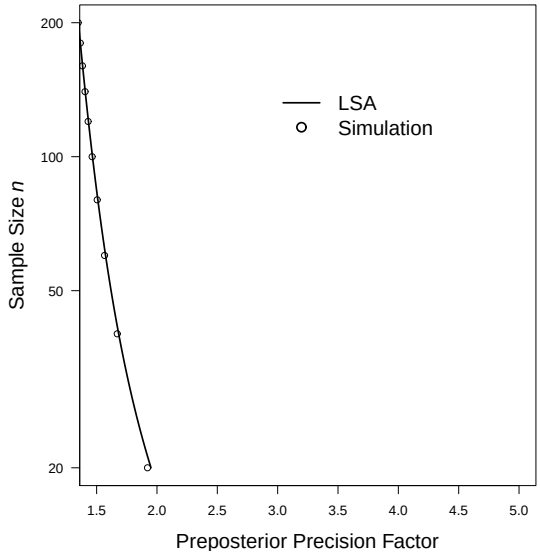
Figure 4: Comparison of the planning solutions between the simulation approach and the LSA approach, for smaller amount of prior information.



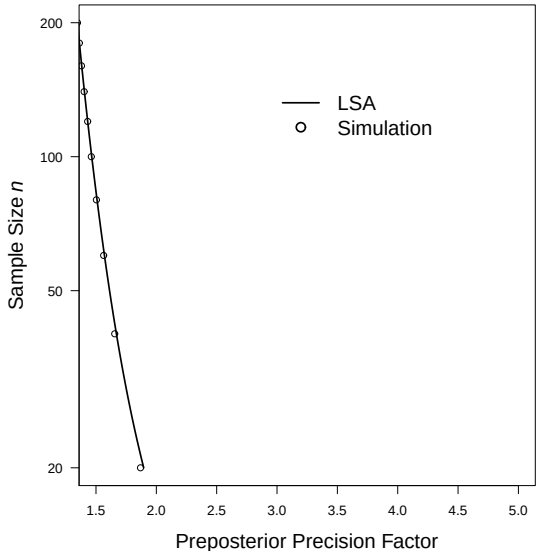
(a) $\mathrm{E}(\sigma) = 1$, $r^{\Delta}_{\sigma,1} = 100$, $p_c = 0.2$



(b) $\mathrm{E}(\sigma) = 1$, $r^{\Delta}_{\sigma,1} = 100$, $p_c = 0.4$



(c) $\mathrm{E}(\sigma) = 1$, $r^{\Delta}_{\sigma,1} = 100$, $p_c = 0.6$



(d) $\mathrm{E}(\sigma) = 1$, $r^{\Delta}_{\sigma,1} = 100$, $p_c = 0.8$

Figure 5: Comparison of the planning solutions between the simulation approach and the LSA approach, for larger amount of prior information.

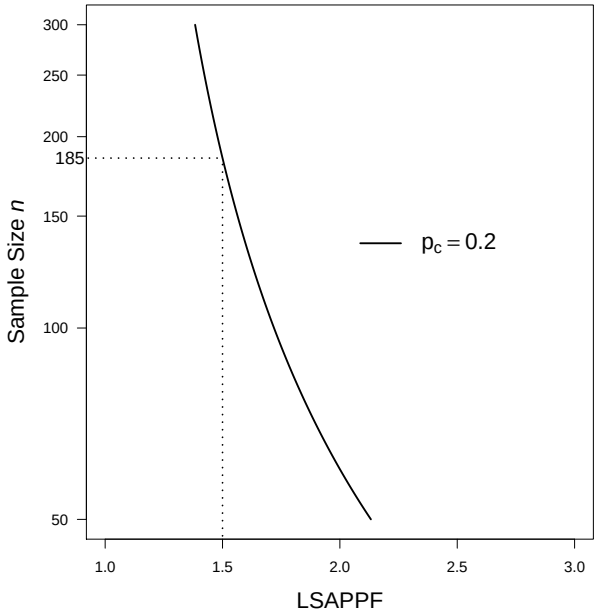


Figure 6: The relationship between the needed sample size and the LSAPPF criterion value for the example in Section 4.1. Here, $\mathrm{E}(\sigma) = 1.2$ and $\mathrm{stdev}(\sigma) = 0.1$.

the information, the engineers want to know how many samples are needed if a 95% posterior interval for the quantile of interest will have endpoints that are approximately 50% away from the posterior mode of the quantile of interest (so that LSAPPF $= 1.5$).

From the LSAPPF in (10), the relationship between the criterion value and the sample n can be computed, as given in Figure 6. Therefore, to satisfy the estimation precision requirement in terms of LSAPPF$= 1.5$, the number of specimens under test should be at least $n = 185$. The sample size determined from a non-Bayesian approach of the problem, similar to the calculations in Example 10.7 of Meeker and Escobar (1998), is $n = 245$. Although the results are not directly comparable because point planning values are used instead of prior distributions for planning in the non-Bayesian approach, the results do indicate the possibility of using a smaller sample size with prior information.

4.2 Solution by the Simulation Approach

For an illustration, consider the example in Section 4.1. From the LSA approach, $n = 185$ is planned and the LSAPPF is 1.5001 with the proposed plan. From the simulation approach,

this sample size gives the preposterior precision factor equal to 1.4928. Therefore, the proposed plan is expected to provide the estimation precision, on average, that is only 0.49% lower than the required level. For the simulation approach, a sample size of 180 gives the preposterior factor equal to 1.5003. Considering the fraction failing is $p_c = 0.2$, the LSA and simulation only differ by 1 in the effective sample size $n \times p_c$. This result shows that the approaches agree well.

5 Discussions for Test Planning

5.1 Solution for Type I Censoring

We illustrated the development of life test plans for Type II censored experiments, where the length of the experiment is controlled by the number of failures (or the proportion of failures given the sample size n). The LSA approach can also give planning solutions for Type I censored experiments. The main difference is that the scaled FIM for Type I censoring is a function of the standardized log censoring time $[\log(t_c) - \mu]/\sigma$, which is a function of both parameters μ and σ. Thus, instead of the one-dimensional integration for the Type II censoring case in (10), the integration of the LSAPPF criterion for the Type I censoring case is a two-dimensional integration, depending also on the prior distribution of μ.

5.2 Shape of the Prior Distribution

A gamma distribution, because of its flexible shape, is used in the numerical investigations as the prior distribution of σ. The lognormal distribution and the inverted gamma distribution, which have somewhat different shapes, were also used for the prior distribution of (the positive quantity) σ. The results (not shown here) are almost identical to those shown in Section 3.3 using a gamma prior distribution, if the same prior specifications (in terms of prior means and prior-predicted precision ratios) are provided. The planning solutions from these examples are found insensitive to the shape of the prior distribution of σ.

5.3 Joint Informative Prior Information

In some circumstances, the prior distribution for μ can also be informative. In the LSA approach, the major change is in the prior precision matrix S^{-1} (for Type I censoring, the prior distribution for μ will also be involved in the integration). A prior precision ratio can be defined for each parameter and the competing behaviors between the prior precision ratios for the two parameters can be studied similarly. The linear separation of prior information and information from the experiment to be performed in the large sample approximate posterior variance makes the use of precision ratios clear and the analysis and planning based on them meaningful. One more consideration for cases with both priors being informative is the independence specification of prior parameters. Based on past experience with censored data studies, σ and $\log(t_{p_1})$ for some lower quantile p_1 are usually approximately independent of each other. Thus, this parameterization is appropriate for independent prior specification (which, with censoring, implies that μ and σ are correlated). When the prior information for μ is diffuse, as in this example, specifying independent μ and σ will reduce to the same as specifying independent marginal distributions for $\log(t_{p_1})$ (diffuse) and σ (possibly informative).

6 Concluding Remarks

We have presented Bayesian planning methods for life testing problems. A Bayesian criterion based on the estimation precision of a quantile of interest is provided. Two numerical approaches are described to evaluate the Bayesian criterion and solve the planning problem, which are the LSA approach and the simulation approach. These approaches are shown to be valid for fairly general situations in life test planning when prior information is available and when it is appropriate to apply the Bayesian method. For the LSA approach, the criterion is easy to compute and it gives quick yet valid and easy-to-interpret life test solutions. In the simulation approach, the computation is relatively intensive, but the criterion can be computed without approximation. The simulation approach can be used as a validation tool after a preliminary life test plan is obtained from the LSA.

In the future, it will be interesting to consider Bayesian test planning under multiple failure

modes (e.g., Suzuki, Nakamoto, and Matsuo 2010). In some applications, units can fail due to multiple causes such as crack, wear-out, and sudden death. It is usually more challenging to plan tests under competing risks but the use of prior information can be expected to bring in additional strength. Robustness of the test planning will also be an interesting topic to look into. That is, how will the optimum test plan be changed if the prior information is not correctly specified? In the current planning, only statistical precision is incorporated in the optimum criterion. It will also be useful to consider other constrains such as cost and testing conditions (e.g., Pascual and Meeker 1998) into the optimum criterion in the future.

Acknowledgments

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