Idempotents in plenary train algebras

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Abstract
In this paper we study plenary train algebras of arbitrary rank. We show that for most parameter choices of the train identity, the additional identity \((x^2 - w(x)x)^2 = 0\) is satisfied. We also find sufficient conditions for \(A\) to have idempotents.

Keywords
Plenary train algebras, Idempotent element

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Comments

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Idempotents in Plenary Train Algebras

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Abstract

In this paper we study plenary train algebras. We show that for most parameter choices of the train identity, the additional identity 
\((x^2 - \omega(x)x)^2 = 0\) is satisfied. In this case we prove that it has idempotents.

1 Introduction

Plenary powers are defined inductively by \(x^{(1)} = x\) and \(x^{(n+1)} = (x^{(n)})^2\). The pair \((A, \omega)\) is called a baric algebra if \(\omega : A \to K\) is a nontrivial homomorphism. If a baric algebra \((A, \omega)\) satisfies an identity of the form

\[x^{(n)} = \alpha_1 \omega(x)^{(2n-1)}x + \alpha_2 \omega(x)^{(2n-2)}x^2 + \cdots + \alpha_{n-1} \omega(x)^{(2n-1)}x^{(n-1)}\]  (1)

then we call it a plenary train algebra. We will further assume that \(A\) is commutative.

An important question in nonassociative algebras in general and in train algebras in particular is the existence of idempotents.

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2 Main Section

Lemma 1. Let $A$ be any baric algebra with weight function $\omega$. If $A$ satisfies the identity

$$ (x^2 - \omega(x)x)^2 = 0 \tag{2} $$

then, for any integers $i, j > 0$ and for any element $x$ of weight 1:

$$ (x^{(i)} - x^{(j)})^2 = 0 \tag{3} $$

Proof. We proceed by induction on $n = |i - j|$. The case $n = 0$ is obvious. The case $n = 1$ is a direct consequence of (2). We start by expanding and linearizing (2):

$$ 4(xx)(xy) - 2\omega(y)x(xx) - 2\omega(x)y(xx) - 4\omega(x)(xy) + 2\omega(x)\omega(y)(xx) + 2\omega(x)\omega(x)(xy) = 0 $$

When $\omega(x) = \omega(y) = 1$ this shortens to

$$ (4x^2 - 4x)(xy) + 2xy - 2x^2y - 2xx^2 + 2x^2 = 0 \tag{4} $$

Our inductive hypothesis is that (3) holds for all $x$ of weight 1 and for all $i, j$ such that $|i - j| < n$:

$$ 2x^{(i)}x^{(j)} = x^{(i+1)} + x^{(j+1)} \tag{5} $$

Replacing $y = x^{(n)}$ in (4) we get:

$$ (4(x^2 - 4x)(xx^{(n)}) + 2xx^{(n)} - 2x^2x^{(n)} - 2xx^2 + 2x^2 = 0 $$

using (5) on the first occurrence of $xx^{(n)}$

$$ (2(x^2 - 2x)(x^2 + x^{(n+1)}) + 2xx^{(n)} - 2x^2x^{(n)} - 2xx^2 + 2x^2 = 0 $$

again using (5) where appropriate

$$ 2x^{(3)} + (x^{(3)} + x^{(n+2)}) - (x^2 + x^{(3)}) - 2xx^{(n+1)} + x^2 + x^{(n+1)} $$

$$ - (x^{(3)} + x^{(n+1)}) - (x^2 + x^{(3)}) + 2x^2 = 0 $$

collecting similar terms

$$ x^{(n+2)} - 2xx^{(n+1)} + x^2 = (x^{(n+1)} - x)^2 = 0 $$

finally substituting $x^{(i)}$ for $x$ we get (3) for $|i - j| = n$. \qed
Theorem 2. Let $A$ be a plenary train algebra of rank $n$ with defining identity:

$$x^{(n)} = \alpha_1 \omega(x)^{(2^n-1)} x + \alpha_2 \omega(x)^{(2^n-2)} x^2 + \cdots + \alpha_{n-1} \omega(x)^{(2^n-1)} x^{(n-1)}$$ (6)

Let

$$\lambda = \sum_{i=1}^{n-1} (n-i) \alpha_i$$

Assume $A$ satisfies $(x^2 - \omega(x)x)^2 = 0$. If $\lambda \neq 0$ then $A$ has idempotents.

Proof. Let $x$ be any weight one element of $A$ and let

$$b_k = \sum_{i=1}^{k} \alpha_i \quad b = \sum_{k=1}^{n-1} b_k x^{(k)}$$

Notice that $\sum b_k = \lambda$ and that $b_{n-1} = 1$. Next we calculate $b^2$:

$$b^2 = \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} b_k b_j x^{(k)} x^{(j)}$$

$$= \frac{1}{2} \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} b_k b_j (x^{(k+1)} + x^{(j+1)} - (x^{(k)} - x^{(j)}))^2$$

using Lemma 1, $(x^{(k)} - x^{(j)})^2 = 0$,

$$b^2 = \frac{1}{2} \left( \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} b_k b_j x^{(k+1)} + \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} b_k b_j x^{(j+1)} \right)$$

relabeling the indices of the second sum and using that $\sum b_k = \lambda$,

$$b^2 = \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} b_k b_j x^{(j+1)} = \lambda \sum_{j=1}^{n-1} b_j x^{(j+1)}$$

from the plenary identity and noticing that $b_{n-1} = 1$,

$$b^2 = \lambda \left( \sum_{j=1}^{n-2} b_j x^{(j+1)} + \sum_{k=1}^{n-1} \alpha_k x^{(k)} \right)$$
collecting terms and using the definition of the \(b_k\),

\[
b^2 = \lambda \left( \alpha_1 x + \sum_{k=2}^{n-1} (b_{k-1} + \alpha_k)x^{(k)} \right) = \lambda \left( \sum_{k=1}^{n-1} b_k x^{(k)} \right) = \lambda b
\]

We conclude that \(e = \frac{b}{\lambda}\) is an idempotent in \(A\).

We may notice that in the previous proof the hypothesis \((x^2 - \omega(x)x)^2 = 0\) is not fully used. A sufficient condition would be \(\sum_{k<j<n} b_k b_j (x^{(k)} - x^{(j)})^2 = 0\) where the \(b_k\) are defined as in the proof of the theorem.

**Lemma 3.** Let \(A\) be a baric algebra. If all weight one elements \(x \in A\) satisfy the equation:

\[
x^{(k)} = \sum_{i=1}^{n-1} \beta_i x^{(i)}
\]

where \(k \geq n\) and \(\sum \beta_i = 1\), then they also satisfy

\[
\sum_{1 \leq i < j < n} \beta_i \beta_j (x^{(i)} - x^{(j)})^2 = 0
\]

**Proof.** Let

\[
S = 2 \sum_{1 \leq i < j < n} \beta_i \beta_j (x^{(i)} - x^{(j)})^2
\]

We can turn (8) into a full double sum by adding some trivially zero terms where \(i = j\):

\[
S = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i \beta_j (x^{(i)} - x^{(j)})^2
\]

expanding the squared terms

\[
S = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i \beta_j (x^{(i)})^2 + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i \beta_j (x^{(j)})^2 - 2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i \beta_j x^{(i)} x^{(j)}
\]

changing the summation order and factoring the sums

\[
S = \sum_{j=1}^{n-1} \beta_j \sum_{i=1}^{n-1} \beta_i x^{(i+1)} + \sum_{i=1}^{n-1} \beta_i \sum_{j=1}^{n-1} \beta_j x^{(j+1)} - 2 \sum_{i=1}^{n-1} \beta_i x^{(i)} \sum_{j=1}^{n-1} \beta_j x^{(j)}
\]
using (7) and that \( \sum \beta_i = 1 \)

\[
S = 2 \sum_{j=1}^{n-1} \beta_j x^{(j+1)} - 2(x^{(k)})^2
\]

using (7) again for \( x^2 \) in place of \( x \)

\[
S = 2x^{(k+1)} - 2x^{(k+1)} = 0
\]

\[
\blacksquare
\]

**Lemma 4.** Let \( A \) be a plenary train algebra of rank \( n \) with defining identity:

\[
x^{(n)} = \sum_{i=1}^{n-1} \alpha_i \omega(x)^{2^{n-2}} x^{(i)}
\]

Consider an element \( x \in A \) of weight one and let its plenary powers up to \( x^{(n-1)} \) be the basis of a vectorspace where \( x^{(i)} = (0 \ldots 1 \ldots 0) \) has a one in the \( i \)th position. Then we can express \( x^{(k+1)} \) in this basis by \((1, 0, 0, 0 \ldots 0)A^k\) where

\[
A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
& & & \ddots & & \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{n-2} & \alpha_{n-1}
\end{pmatrix}
\]

**Proof.** The proof goes by induction on \( k \). For \( k = 0 \) there is nothing to prove. So we assume

\[
x^{(k)} = \sum_{i=1}^{n-1} \beta_i x^{(i)} = (\beta_1, \beta_2, \beta_3, \ldots, \beta_{n-2}, \beta_{n-1}) = (1, 0, 0, 0 \ldots 0)A^{k-1}
\]

Replacing \( x \) by \( x^2 \) we have

\[
x^{(k+1)} = \sum_{i=1}^{n-1} \beta_i x^{(i+1)} = \sum_{i=2}^{n-1} \beta_{i-1} x^{(i)} + \beta_{n-1} \sum_{i=1}^{n-1} \alpha_i x^{(i)}
\]

\[
= (0, \beta_1, \beta_2, \ldots, \beta_{n-3}, \beta_{n-2}) + \beta_{n-1}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-2}, \alpha_{n-1})
\]

\[
= (\beta_1, \beta_2, \beta_3, \ldots, \beta_{n-2}, \beta_{n-1})A
\]

\[
= (1, 0, 0, 0 \ldots 0)A^k
\]

\[
\blacksquare
\]
Theorem 5. Let $A$ be a plenary train algebra of rank $n$ with defining identity:

$$x^{(n)} = \sum_{i=1}^{n-1} \alpha_i \omega(x)^{(2^{n-2^i-1})} x^{(i)}$$

Let $\lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of the matrix $A$ defined in lemma 4 (the $\lambda_k$ are the nonzero roots of the associative polynomial $x^n - \sum \alpha_i x^i$). If all the products $\lambda_i \lambda_j$ are distinct then $A$ satisfies $(x^2 - \omega(x)x)^2 = 0$ and $A$ has idempotents.

Proof. Using lemma 3 and lemma 4 we get identities

$$\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i^k \beta_j^k (x^{(i)} - x^{(j)})^2 = 0$$

where

$$(\beta_1^k, \beta_2^k, \beta_3^k, \ldots, \beta_n^k, \beta_{n-1}^k) = e_1 A^{k-1}$$

and $k$ is any positive integer. So we have a homogeneous system of identities satisfied by the squares $(x^{(i)} - x^{(j)})^2$. In matrix form this can be written as:

$$\left\langle (e_1 A^{k-1})^T e_1 A^{k-1}, U \right\rangle = 0$$

Where $U$ is the symmetric matrix such that $U_{ij} = (x^{(i)} - x^{(j)})^2$, and the angled brackets stand for the Hadamard product of the matrices. Now consider $v_1, \ldots, v_{n-1}$ eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ of $A$. and write $e_1 = \sum c_i v_i$ as a linear combination of them. Since $e_k = e_1 A^{k-1} = \sum \lambda_i^{k-1} c_i v_i$ we notice that the $c_i v_i$ also form a basis of eigenvectors for $A$, so we may assume that $c_i = 1$ for every $i$. Then

$$0 = \left\langle (e_1 A^k)^T e_1 A^k, U \right\rangle = \left\langle \left( \sum_{i=1}^{n-1} \lambda_i^k v_i \right)^T \sum_{i=1}^{n-1} \lambda_i^k v_i, U \right\rangle$$

$$= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (\lambda_i \lambda_j)^k v_i^T v_j, U \right\rangle$$

$$= \sum_{1 \leq i \leq j < n} (\lambda_i \lambda_j)^k \left\langle (v_i^T v_j + v_j^T v_i), U \right\rangle$$

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since this holds for all \( k \), the Vandermonde determinant says that for each \( 1 \leq i \leq j < n \) we have

\[
\left\langle (v_i^T v_j + v_j^T v_i) , U \right\rangle = 0
\]

Using the symmetry of \( U \),

\[
2 \left\langle (v_i^T v_j) , U \right\rangle = 0
\]

Since the \( v_i \) form a basis for the \((n-1)\) dimensional rowspace, the matrices \( v_i^T v_j \) form a basis for the space of all \((n-1)\times(n-1)\) matrices. To verify this, it suffices to show that they are linearly independent. In fact, if \( \sum r_{ij} v_i^T v_j = 0 \) then multiplying by any \( v_k \) on the left we get \( \sum_j (\sum_i r_{ij} v_k v_i^T) v_j \). Since the \( v_j \) are linearly independent, \( \sum_i r_{ij} v_k v_i^T = 0 \) for every \( k, j \). Now since the \( v_k \) form a basis \( \sum_i r_{ij} v_i^T = 0 \), and finally since the \( v_i^T \) are linearly independent, \( r_{ij} = 0 \) for every \( i, j \).

Finally, this shows that \( U \) is orthogonal to a basis for the space of all matrices, so \( U = 0 \) and in particular \((x^{(i)} - x^{(j)})^2 = 0 \) for every \( i, j \). Finally, to use theorem 2 we need to check that \( \lambda = \sum (n-i) \alpha_i \neq 0 \). We will show that this just means that 1 is not a repeated eigenvalue of \( A \) and so it is part of the hypothesis. We want to factor the plenary polynomial:

\[
x^n - \sum_{i=1}^{n-1} \alpha_i x^i = \sum_{i=1}^{n-1} \alpha_i (x^n - x^i) = \sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \alpha_i (x^{k+1} - x^k) = (x - 1) \sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \alpha_i x^k
\]

Evaluating the right factor at \( x = 1 \) we get

\[
\sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \alpha_i = \sum_{i=1}^{n-1} (n-i) \alpha_i
\]

So \( A \) has idempotents.

As an illustration we consider some small cases:

**Example 1** (n=3). Let \( A \) be a plenary train algebra satisfying:

\[
x^{(3)} = \alpha x + (1-\alpha)x^2
\]

The nonzero roots of the polynomial \( x^3 - (1-\alpha)x^2 - \alpha x \) are 1 and \( \alpha \) so by theorem 5 we can guarantee that \( A \) has an idempotent as long as 1, \( \alpha, \alpha^2 \) are
all different, that is \( \alpha \not\in \{0, 1, -1\} \). Furthermore, for every \( x \) of weight 1, we now an idempotent to be

\[
\frac{1}{\alpha + 1} \left( \alpha x + x^2 \right)
\]

Notice that when \( \alpha = 0 \), \( x^2 \) is an idempotent and that when \( \alpha = 1 \) we do not find an idempotent in this way but it is known (Etherington...) that there are idempotents.

**Example 2** (n=4). Let \( A \) be a plenary train algebra satisfying:

\[
x^{(4)} = \alpha x + \beta x^2 + \gamma x^{(3)}
\]

where \( \alpha + \beta + \gamma = 1 \). Lets assume that \( 1, \lambda, \mu \) are the nonzero roots of \( x^4 - \gamma x^3 - \beta x^2 - \alpha x = 0 \) so that \( \alpha = \lambda \mu \), \( \beta = -(\lambda \mu + \lambda + \mu) \), \( \gamma = \lambda + \mu + 1 \). Theorem 5 says that \( A \) has an idempotent as long as \( 1, \lambda, \mu, \lambda \mu, \lambda \mu^2, \mu^2 \) are all distinct, that is \( \lambda \mu (\lambda^2 - 1)(\mu^2 - 1)(\lambda - \mu^2)(\lambda^2 - \mu) \neq 0 \).

Furthermore, in this case, we now an idempotent to be

\[
\frac{1}{3\alpha + 2\beta + \gamma} \left( \alpha x + (\alpha + \beta)x^2 + (\alpha + \beta + \gamma)x^{(3)} \right)
\]

One may notice again that the given condition is not really necessary since to really answer the question, we need to solve a linear algebra problem. We need to know whether the vector

\[
\begin{pmatrix}
\alpha(\alpha + \beta) & \alpha(\alpha + \beta + \gamma) & (\alpha + \beta)(\alpha + \beta + \gamma)
\end{pmatrix}
\]

is in the rowspace of the following matrix:

\[
\begin{pmatrix}
\alpha\beta & \alpha\gamma & (\alpha + \beta)\gamma \\
\alpha(\alpha + \beta) & \alpha(\alpha + \beta + \gamma) & \alpha(\alpha + \beta + \gamma) \\
\alpha(\alpha + \beta + \gamma) & \alpha(\alpha + \beta + \gamma + \beta + \gamma^2) & \alpha(\alpha + \beta + \gamma + \beta + \gamma^2)
\end{pmatrix}
\]

It turns out that this is the case as long as \( (\beta - 1)(\alpha - 1) \neq 0 \). Which in terms of the eigenvalues leaves the final condition as \( (\lambda^2 - 1)(\mu^2 - 1)(\lambda \mu - 1) \neq 0 \).

This result was obtained recently by Labra and Suazo (see...).