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Generalized alternative and Malcev algebras

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GENERALIZED ALTERNATIVE AND MALCEV ALGEBRAS

I.R. HENTZEL AND H.F. SMITH

1. Introduction. As observed in [1], both alternative algebras and Malcev algebras satisfy the flexible law

\[(x, y, x) = 0,\]

and

\[(zx, x, y) = -x(z, y, x),\]

where the associator \((a, b, c) = (ab)c - a(bc)\). Algebras satisfying in addition to (1) and (2) the identity

\[(xz, x, y) = -(z, y, x)x\]

were studied initially by Filippov [1], who showed that a prime algebra of this sort (with characteristic \(\neq 2, 3\)) must be either alternative, Malcev, or a Jordan nil-algebra of bounded index 3. In this paper we shall consider algebras (with characteristic \(\neq 2, 3\)) which satisfy only (1) and (2). (Note that algebras opposite to these satisfy instead (1) and \((*)\).) We shall prove that in this variety nil-semisimple algebras are alternative, and that prime algebras are either alternative or nil of bounded index 3. We shall also establish for finite-dimensional algebras the standard Wedderburn principal theorem.

To begin with, there are some elementary consequences of (1) and (2) which need to be noted. We first set

\[T(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z.\]

It can be verified by simply expanding the associators that in any algebra \(T(w, x, y, z) = 0\). Also, the linearized form of (2) is

\[(zx, w, y) + (zw, x, y) = -x(z, y, w) - w(z, y, x),\]

so that

\[F(z, x, w, y) = (zx, w, y) + (zw, x, y) + x(z, y, w) + w(z, y, x) = 0.\]
Then $0 = F(x, x, y, x) - T(x, y, x, x) = \{x, y, z, x\} = 0.$ Using repeatedly the flexible law and its linearization $(x, y, z) = (y, x, z)$, we arrive at

\[ (x^2, y, x) = 0. \]

Algebras which satisfy (1) and (3) are called noncommutative Jordan. In particular, provided the characteristic $\neq 2$, such algebras are power-associative [5], and as in [2] they satisfy the identity

\[ (y, x^2, z) = x \circ (y, x, z), \]

where the symmetric product $a \circ b = ab + ba$.

Next (2) and flexibility imply $(x^2, x, y) = -x(x, y, x) = 0$, that is

\[ (x^2, x, y) = 0. \]

Then using (3) and flexibility we have $0 = T(x, x, y, x) = (x, x, yx) - (x, x)yx$. Hence flexibility, (2), and (4) give $0 = -(yx, x, x) + (y, x, x)x = (y, x^2, x)$. Thus we also have

\[ (x^2, x^2, y) = 0. \]

We now use linearizations of (5) and (3) to obtain $(x^2, y, y) = -(x \circ y, x, y) = (y^2, x, x)$, so that

\[ (x^2, y, y) = (y^2, x, x). \]

Using linearized (7), we then see $2(x^3, y, y) = (x^2 \circ x, y, y) = (y^2, x^2, x) + (y^2, x, x^2) = 0$ by (5), (6), and flexibility. Thus we arrive at

\[ (x^3, y, y) = 0. \]

Finally, let $A$ be a noncommutative Jordan algebra (with characteristic $\neq 3$), and let $N(A)$ denote the linear span of the set $\{x^3 | x \in A\}$. Then as in [1] $N(A)$ is an ideal of $A$. Hence from (8) actually follows

\[ (z x^3, y, y) = 0 = (x^3 z, y, y). \]

2. Main Section. Let $A$ be an algebra which satisfies (1) and (2), and denote by $B(A)$ the linear span of the set $\{(y, x, x) | x, y \in A\}$. We shall now establish two identities that imply $B(A)$ is an ideal of $A$.

**Proposition 1.** If $A$ is a flexible algebra (with characteristic $\neq 2$) which satisfies (2), then the following identities hold in $A$:

\[ z(y, x, x) = -(yz, x, x) - \{(yx, z, x) + (yx, x, z)\} \]

\[ - \{(z, x, yx) + (z, yx, x)\} + \{(xz, y, x) + (xz, x, y)\} \]

\[ + \{(x, xz, y) + (x, y, xz)\} + \{(x, xy, z) + (x, z, xy)\}, \]
(11) \((z, x, x)y = 2\{(x, xz, y) + (x, y, xz)\} + 2\{(x, xy, z) + (x, z, xy)\} + \{(y, z, x^2) + (y, x^2, z)\} - (yz, x, x) - \{(yx, z, x) + (yx, x, z)\} - \{(z, x, yx) + (z, yx, x)\} + \{(xz, y, x) + (xz, x, y)\} - \{(y, z, x^2) - (y, x^2, z)\} - (yz, x, x).\)

**Proof.** First (2) implies \(x(y, x, x) = -(yx, x, x).\) Then linearization of this identity gives

\[(i) \quad z(y, x, x) = -(yz, x, x) - \{(yx, z, x) + (yx, x, z)\} - \{(x(y, z, x) + x(y, x, z)\}.
\]

Next using (2) and flexibility we obtain \(0 = x(y, x, x) - (yx, x, x) - T(x, y, x, z) = \{x(y, z, x) + x(y, x, z)\} + \{(yx, x, z) + (x, yx, z)\} - (xy, x, z) + (xz, y, x).\) From this by flexibility one has

\[(ii) \quad 0 = \{x(y, z, x) + x(y, x, z)\} - \{(z, x, yx) + (z, yx, x)\} + \{(xz, y, x) + (xz, x, y)\} - (xy, x, z).\]

Now by flexibility \(0 = T(x, y, x, y) = (xy, x, y) - (x, xy, y) + (x, y, xy) = (xy, x, y) - (x, y \circ x, y) + \{(x, xy, y) + (x, y, xy)\}.\) But \(0 = (y, x^2, y) = -(x, y \circ x, y)\) by flexibility and linearized (6). Substituting this in the preceding equation, we have

\(0 = (xy, x, y) + \{(x, xy, y) + (x, y, xy)\}.\)

Linearization of this last identity gives

\[(iii) \quad 0 = (xz, x, y) + (xy, x, z) + \{(x, xz, y) + (x, y, xz)\} + \{(x, xy, z) + (x, z, xy)\}.
\]

If we now add equations (i)–(iii), we arrive at (10).

To prove (11), we first use linearized (1) repeatedly to show

\((z, x, x)y = (z, x, x)y + T(z, x, x, y)\)

\[= (zx, x, y) - (z, x^2, y) - (xy, x, z) + z(y, x, x)\]

\[= (zx, x, y) + (x^2, z, y) - (xy, x, z) - \{(x^2, z, y) + (z, x^2, y)\} + z(y, x, x)\]

\[= (zx, x, y) + (x^2, z, y) - (xy, x, z) + \{(y, z, x^2) + (y, x^2, z)\} + z(y, x, x).\]

Now by linearized (5) \((zx, x, y) + (x^2, z, y) = -(xz, x, y).\) Substituting this in the preceding equation gives

\[(iv) \quad (z, x, x)y = -(xz, x, y) - (xy, x, z) + \{(y, z, x^2) + (y, x^2, z)\} + z(y, x, x).\]
Then adding equations (iii) and (iv) we obtain

\[(v) \quad (z, x, x)y = \{(x, xz, y) + (x, y, xz)\} + \{(x, xy, z) + (x, z, xy)\}
+ \{(y, z, x^2) + (y, x^2, z)\} + z(y, x, x).\]

If we now use the already established (10) to substitute for \(z(y, x, x)\) in (v), we arrive at (11).

**COROLLARY.** If \(A\) is a flexible algebra (with characteristic \(\neq 2\)) which satisfies (2), then \(B(A)\) is an ideal of \(A\).

**PROOF.** This follows immediately from (10) and (11), since

\[\{(a, b, c) + (a, c, b)\} = (a, b + c, b + c) - (a, b, b) - (a, c, c).\]

**PROPOSITION 2.** If \(A\) is a flexible algebra (with characteristic \(\neq 2, 3\)) which satisfies (2), then the following identity holds in \(A\):

\[(12) \quad y^3 \circ (z, x, x) = 0.\]

**PROOF.** To facilitate notation let us set \(t = y^3\). We shall now proceed to justify a series of equations whose sum will imply (12). First, from (11) we have

\[(vi) \quad -(z, x, x)t = -2\{(x, xz, t) + (x, t, xz)\} - 2\{(x, xt, z) + (x, z, xt)\}
- \{(t, z, x^2) + (t, x^2, z)\} + (tz, x, x)
+ \{(tx, z, x) + (tx, x, z)\} + \{(z, x, tx) + (z, tx, x)\}
- \{(xz, t, x) + (xz, x, t)\}.\]

We also need

\[(vii) \quad 0 = F(z, t, x, x) - T(x, x, t, z)
= (zt, x, x) + (zx, t, x) + t(z, x, x) + x(z, x, t)
- (x^2, t, z) + (x, xt, z) - (x, x, tz) + x(x, t, z) + (x, x, t)z.\]

By (2) one has

\[(viii) \quad 0 = \{(zx, x, t) + x(z, t, x)\} - \{(tx, x, z) + x(t, z, x)\}.\]

From linearized (3) we obtain

\[(ix) \quad 0 = -\{(x^2, t, z) + (xz, t, x) + (xz, t, x)\}
- 2\{(x^2, z, t) + (xt, z, x) + (tx, z, x)\},\]

and from linearized (6)

\[(x) \quad 0 = \{(t, x^2, z) + (x, tx, z) + (x, xt, z)\}
+ \{(z, x^2, t) + (x, zx, t) + (x, xz, t)\}.\]
Linearized (7) implies

(xi) \[ 0 = 2\{(x^2, z, t) + (x^2, t, z) - (tz, x, x) - (zt, x, x)\}. \]

Since \( t = y^3 \), from (8) and its linearization we have

(xii) \[ 0 = (t, x, x)z + \{(t, x, x) + (t, zx, x)\} \]
\[ + \{x(t, x, z) + x(t, z, x)\} - \{(t, x, x) + (t, xz, x)\} \]
\[ + \{(t, x^2, z) + (t, x^2, x)\}. \]

Analogously from (9) follows

(xiii) \[ 0 = (zt, x, x) + \{(tx, x, z) + (tx, z, x)\} \]

If we now add equations (vi)–(xiii) and use the linearized flexible law to make repeated cancellations in addition to immediate ones, we arrive at \(- (z, x, x)t = t(z, x, x)\) which is (12).

**COROLLARY.** If \( A \) is a flexible algebra (with characteristic \( \neq 2,3 \)) which satisfies (2), then \( B(A) \) is a nil ideal of bounded index 4.

**PROOF.** Of course \( B(A) \) is an ideal by the corollary to Proposition 1. Let

\[ a = \sum_{i=1}^{n} \alpha_i(y_i, x_i, x_i) \in B(A). \]

Then

\[ 2a^4 = a^3 \circ a = a^3 \circ \sum_{i=1}^{n} \alpha_i(y_i, x_i, x_i) = \sum_{i=1}^{n} \alpha_i(a^3 \circ (y_i, x_i, x_i)) = 0 \]

by (12), whence \( a^4 = 0 \).

We can now prove the following theorems.

**THEOREM 1.** Let \( A \) be an algebra (with characteristic \( \neq 2, 3 \)) which satisfies (1) and (2). If \( A \) is without nonzero nil ideals, then \( A \) is alternative.

**PROOF.** Since by the preceding corollary \( B(A) \) is a nil ideal of \( A \), \( B(A) = (0) \) by our assumption. Hence \((y, x, x) = 0 = (x, x, y)\) using flexibility, and such an algebra is alternative by definition.

**THEOREM 2.** Let \( A \) be an algebra (with characteristic \( \neq 2, 3 \)) which satisfies (1) and (2). If \( A \) is prime, then \( A \) is either alternative or nil of bounded index 3.

**PROOF.** First let \( C \) be any ideal such that \((C, y, y) = 0\) for all \( y \in A \). Linearizing this identity, for any ideal \( D \) one has

(xiv) \[ (cd)a = -(ca)d + c(a \circ d), \]
where \( c \in C, d \in D, a \in A \). This shows \( CD \) is a right ideal of \( A \). Since also \( (y, y, C) = 0 \) by flexibility, analogously \( DC \) is a left ideal. In particular, by (8) we can let \( C = N(A) \) and \( D = B(A) \). Then since by Proposition 2 the elements of \( N(A) \) and \( B(A) \) anti-commute, it follows that \( K = N(A)B(A) \) is an ideal of \( A \).

Now \( K \subseteq N(A) \), so also \( (K, y, y) = 0 \) for all \( y \in A \). Thus likewise \( K^2 \) is an ideal of \( A \). Furthermore, the elements of \( K \) also anti-commute. Thus if \( c, d, a \in K \), then \( (cd)a = -(ca)d \) by (xiv). In particular, \( (dc)a = -(cd)a = (ca)d = -d(ca) \), which shows the elements of \( K \) anti-associate. But then for \( x, y, z, w \in K \) we have

\[
(xy)(zw) = -x[y(zw)] = x[(yz)w] = -[x(yz)]w = [(xy)z]w = -(xy)(zw).
\]

Hence it follows that \( K^2 \) is an ideal which squares to zero. Since \( A \) is prime, this means \( K^2 = (0) \), and so in turn \( K = (0) \). Thus either \( B(A) = (0) \) or \( N(A) = (0) \), which completes the proof of the theorem.

**Theorem 3 (Wedderburn principal theorem).** Let \( A \) be a finite-dimensional algebra which satisfies (1) and (2) over a field with characteristic \( \neq 2, 3; \) and let \( N \) be the nil radical of \( A \). If \( A/N \) is separable, then \( A = S + N \) (vector space direct sum) where \( S \) is a subalgebra of \( A \) such that \( S \cong A/N \).

**Proof.** If \( A \) is without nonzero nil ideals, then by Theorem 1 \( A \) is alternative. Thus for such a nil-semisimple \( A \) we know \( A = A_1 \oplus \cdots \oplus A_n \), where each \( A_i \) is simple with an identity element [6]. Also, since \( A \) is noncommutative Jordan, from [3] we know that if \( e \) is an idempotent in \( A \) then \( A_i(i)A_i(i) \subseteq A_i(i) \) for \( i = 0, 1 \). This means that by [4] we can now reduce consideration to the case when \( A \) itself has an identity element 1. But then \( 0 = F(1, x, x, y) = 2(x, x, y) \), whence \( (x, x, y) = 0 = (y, x, x) \). Thus \( A \) is alternative, and so the result follows in this case from [6].

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**References**


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