Shape of the phosphorus 32 beta energy spectrum

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SHAPE OF THE PHOSPHORUS 32 BETA ENERGY SPECTRUM

By
George B. Henton
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December 1957

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Ames, Iowa

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Shape of the Phosphorus 32 Beta Energy Spectrum*

George B. Henton and B. C. Carlson

ABSTRACT

The shape of the beta energy spectrum of $P^{32}$ was determined by making a least squares fit of the experimental data. This was compared with the shape expected when interference between matrix elements of different orders of forbiddenness is assumed.

The experimental shape factor was found to fit a straight line decreasing by two percent over the energy range measured. However, a satisfactory fit was also made to a Fierz interference shape factor which is approximately of the form $1 + r/W$, where $r$ is a small parameter and $W$ is the energy of the beta particle.

Shape factors calculated assuming interference between matrix elements of different orders of forbiddenness were nearly linear, decreasing by eight to twelve percent with increasing energy. If shell model wave functions are assumed, the transition is from a $(\pi s 1/2, \nu d 3/2) J = 1$ configuration to a $(\pi s 1/2)^2 J = 0$ configuration. The interaction was assumed to be purely tensor.

*This report is based on an M.S. thesis by George B. Henton submitted December, 1957, to Iowa State College, Ames, Iowa. This work was done under contract with the Atomic Energy Commission.
(no Fierz interference). The second-forbidden matrix elements were evaluated for the nuclear configuration mentioned above. Since the allowed matrix element $\mathcal{B}$ is zero for this configuration, its small but nonzero value was estimated from the experimental $ft$ value. The slope of the calculated shape factor depends to some extent on the choice of radial wave functions: it decreases by twelve percent if an isotropic harmonic oscillator well is used, and by eight percent in the case of a square well of infinite depth.
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I. ELEMENTARY BETA DECAY THEORY

This section is intended only as an introduction to definitions which will be used in subsequent sections. A person with some background in beta decay may omit this section. Since this material is extensively covered in standard references (1-5), derivations of the equations have been omitted.

In this and in all following sections the so-called natural units are used unless otherwise indicated: all masses are given in units of $m$, the rest mass of the electron; all lengths in units of the reduced Compton wave length of the electron $\hbar/mc$, where $\hbar$ is Planck's constant divided by $2\pi$ and $c$ is the velocity of light; and all times are in units of $\hbar/mc^2$.

The most striking property of beta energy spectra is the fact that the beta particle has a continuous range of energy, from zero kinetic energy to some maximum energy. In order to preserve the conservation of energy (and also angular momentum), Pauli was led to postulate the creation of a neutral particle, called the neutrino, which is emitted at the same time as the beta particle. Then, since there are more than two particles in the final state, the spectrum is expected to be continuous. The neutrino has been experimentally observed only in the inverse beta decay interaction in
which the neutrino is absorbed with the emission of a beta particle.

By quantum mechanical first-order perturbation theory, the transition probability per unit time \( P \) is proportional to the square of the matrix element of the interaction hamiltonian between the initial and final states of the system, multiplied by the density of final states per unit total energy:

\[
P = \frac{2\pi}{\hbar} \left| \langle f | H_{\text{int}} | i \rangle \right|^2 \frac{d\rho}{dE}.
\]

The theory of beta decay was first formulated by Fermi in 1934 by proposing the form of the interaction hamiltonian. The treatment is in analogy to the treatment of electromagnetic interactions. In the electromagnetic case the charged particles which act as the sources of the field are described by a charge-current four-vector \((c\rho, j)\), and the field particles are the photons described by the electromagnetic potentials \((\varnothing, A)\). The interaction hamiltonian is proportional to the scalar product of these vectors, or \(\rho \varnothing - 1/c j \cdot A\).

For the beta decay interaction a four-vector (or other tensor) formed from the initial and final nucleon wave functions acts as the source of the field, and a four-vector (or other tensor) formed from the electron and neutrino wave functions take the place of the electromagnetic potential. Because of the small mass of the electron and (probably) zero
mass of the neutrino, combined with the high energy which these particles must share, it is necessary to treat the problem relativistically. The electron, proton, and neutron are assumed to be Dirac particles, described by four-component spinor wave functions. In the last year, it has been found that the neutrino is probably not a Dirac particle. However, it may be described by a four-component spinor even though only two components may be strictly necessary (6). The common assumption is that the interaction hamiltonian should be relativistically invariant; i.e., a scalar under Lorentz transformations. With this assumption there are five possible bilinear combinations of the wave functions of the two light particles, having the transformation properties of a scalar, vector, anti-symmetric tensor of second rank, axial vector, and pseudoscalar. Each of these may then be contracted with the corresponding covariant formed from the heavy particle wave functions to make a scalar. Recently it has been shown that the interaction hamiltonian is a scalar under proper Lorentz transformations but not under improper ones involving space inversion. Thus the interaction hamiltonian must also contain terms such as a pseudoscalar formed from the light particle wave functions contracted with a scalar formed from the heavy particle wave functions (7). However, in the shape of the beta spectrum only the squares of such terms are observable; there is no interference between terms
which are invariant under space inversion and those which are not. For this reason the shape is unchanged if we specify only the terms which are invariant under space inversion. These are conveniently written in terms of Dirac matrices as given below:

\[(\text{Scalar}) \quad H_S = (\psi^\beta \psi)(\psi^* \gamma^5 \bar{\Phi}) \quad (2)\]

\[(\text{Vector}) \quad H_V = (\psi^\dagger \psi)(\psi^* \gamma^5 \bar{\Phi}) - (\psi^\dagger \gamma^5 \psi)(\psi^* \gamma^5 \bar{\Phi}) \]

\[(\text{Tensor}) \quad H_T = (\psi^\beta \bar{\gamma} \psi)(\psi^* \gamma^5 \bar{\Phi}) + (\psi^\beta \bar{\gamma} \psi)(\psi^* \gamma^5 \bar{\Phi}) \]

\[(\text{Axial Vector}) \quad H_A = (\psi^\dagger \bar{\gamma} \psi)(\psi^* \gamma^5 \bar{\Phi}) - (\psi^\dagger \gamma \psi)(\psi^* \gamma^5 \bar{\Phi}) \]

\[(\text{Pseudoscalar}) \quad H_P = (\psi^\dagger \gamma^5 \psi)(\psi^* \gamma^5 \bar{\Phi}) \cdot \]

Here \(\psi\) and \(\bar{\psi}\) are the electron and neutrino wave functions, respectively, and \(\Phi\) and \(\bar{\Phi}\) are the initial and final nuclear wave functions. In the nuclear part of the interaction there is understood to be an isotopic spin operator changing a neutron in the initial state into a proton. The interaction hamiltonian is actually a linear combination of these invariants in Equation 2 each multiplied by a constant \(G_x\) called a coupling constant, where \(x\) takes the values \(S, V, T, A,\) and \(P\).

The distinction between what are known as the allowed and forbidden transitions is made in the following way. The
electron and neutrino will have fairly long deBroglie wave lengths compared to the nuclear radius. Also, in computing the matrix elements, the nucleon wave functions become negligible beyond the nuclear radius. This effectively cuts off the contributions from the light particle wave functions at the nuclear radius. If we neglect Coulomb corrections, the electron and neutrino wave functions \( \psi^* \) and \( \psi \) may be treated as plane waves proportional to \( e^{-ip\cdot r} \) and \( e^{-iq\cdot r} \) where \( p \) and \( q \) are the electron and neutrino momenta. Thus the product \( \psi^*\psi \) becomes, when expanded, \( 1 - i(p+q)\cdot r - \frac{1}{2}[(p+q)\cdot r]^2 \) + ... . Since \( |p+q| \) is limited to a few mc units and \( r \) is at most equal to the nuclear radius (less than \( 1/40 \) in natural units), successive terms are smaller at least by a factor of ten and usually by a factor of 100. For this reason it is customary to neglect all except the first term, one, and call this the allowed approximation. (In the presence of the nuclear Coulomb field, allowed transitions correspond to emission of the electron and neutrino with no orbital angular momentum.) When this term vanishes, as in some transitions, then the second term must be retained and the transition is correspondingly slower by a factor ranging from 100 to 10,000. These are called the first-forbidden transitions. Also it is customary to omit all terms involving the operators \( \not A \) and \( \gamma_5 \) in the allowed approximation. These operators are velocity dependent; the velocities of the nucleons are
of the order of .1c and these terms are therefore small. If Coulomb corrections are neglected, the matrix elements in the allowed approximation are now only integrals over the nucleon states and are energy independent. As a notational convenience it is customary to write $\int \beta$ for a matrix element such as that of $\psi*\beta \phi$ and similarly for the rest of the operators. The degree of forbiddenness is equal to the number of powers of $r$ in the matrix element plus one if the operators $\vec{\sigma}$ or $\gamma_5$ occur. E.g. $\int \beta \vec{\alpha} x r$ and $\int (\beta \vec{\alpha} \cdot \vec{r}) r$ are both second forbidden.

Under rotations and inversions of three-dimensional space, the operators $1$ and $\beta$ are ordinary scalar operators. For scalar operators the initial and final nuclear wave functions cannot differ in parity or angular momentum. This is sometimes expressed by saying that these operators are of type $[0^+]$, since a scalar is a tensor of rank zero and positive parity. The operators $\vec{\sigma}$ and $\beta \vec{\gamma}$, being vectors but still being of even parity since $\vec{\sigma}$ transforms like an angular momentum, are called operators of type $[1^+]$. For the matrix element of a particular operator to be nonvanishing, the tensor rank and the initial and final nuclear $J$ values must be able to form a closed triangle of integral perimeter. This results in different selection rules for operators of different ranks. Transitions that occur because of the matrix elements of scalars $1$ or $\beta$ are called 'allowed' by
Fermi selection rules and transitions that occur because of the presence of vectors $\vec{\sigma}$ or $\beta \vec{\sigma}$ are called 'allowed' by Gamow-Teller selection rules. Both types of transitions are found to occur.

The density of final states, $d\mathcal{P}/dE$, is found by considering the volume in momentum space that is available to the electron and neutrino. The result is proportional to $p^2q^2$.

We now have all the components of the transition probability $P$ and can now write the expression for the counting rate $N(p)$. For simplicity we will assume an allowed tensor transition.

$$N(p) = AIF(Z,p) \ G_T^2 \ L_0 \ |\int \beta \vec{\sigma}^2| ^2 \ p^2 q^2$$

(3)

Here $A$ is an undetermined constant which depends, among other things, on the source strength. The spectrometer current $I$, which is proportional to the electron momentum $p$, enters because of the transmission characteristics of a magnetic spectrometer. The Fermi function, $F(Z,p)$, comes from the square of the interaction hamiltonian and contains the essential Coulomb corrections. The $Z$ is that of the daughter nucleus. The terms $G_T^2 L_0 |\int \beta \vec{\sigma}|^2$ also come from the interaction hamiltonian and are commonly referred to as the shape factor. The function $L_0$ is a small additional Coulomb correction. If the neutrino is assumed to have zero rest mass,
then its momentum $q$ is numerically equal to its energy and this is equal to the maximum possible energy of the beta particle minus the actual energy of the beta particle (neglecting the recoil energy of the nucleus). Thus $q = (W_o - W)$.

By dividing both sides of Equation 3 by the energy dependent terms $I_p^2 F(Z, p) L_0$ and taking the square root, we have a linear equation in the energy which can be used to find the maximum beta energy or end-point energy. Such a treatment of an experimental spectrum is referred to as a Kurie plot. If the decay is not allowed, then deviations from a Kurie plot are expected due to additional energy dependence of the shape factor.

It has been shown by Fierz (8) in 1937 that mixtures of two interactions with the same selection rules should also be expected to give an energy dependent shape factor. The only criterion for determining if such mixtures occur is by comparison with experimental data. Such comparisons have indicated that one of the coupling constants involved is considerably smaller than the other (9-11). The form of the shape factor for a pure Gamow-Teller interaction as would be expected for $P^{32}$ is

$$\left( G_T^2|\beta \vec{\sigma}|^2 + G_A^2|\vec{\sigma}|^2 \right) L_0 C \ ,$$

where

$$C = (1 \pm r/W) \quad (4)$$
and

\[
  r = -2 \sqrt{1 - \left( \frac{Z}{137} \right)^2} \frac{g_T \beta \gamma g_A \gamma}{\frac{g_T^2}{c^2} |\beta \gamma|^2 + \frac{g_A^2}{c^2} |\beta \gamma|^2} \approx + 2 \frac{g_A}{g_T} .
\]  

(5)

Here we have included only the allowed matrix elements. The upper sign is taken for electron emission. Angular correlation between the emitted beta particle and neutrino indicates that the tensor interaction is dominant (12). The scalar and vector interactions lead to a shape factor of similar form.
II. BETA DECAY OF PHOSPHORUS 32

The beta spectrum of phosphorus 32 has been extensively measured. The end-point energy of 1.71 Mev is sufficiently high to permit accurate measurements over much of the spectrum and the half-life of 14 days is conveniently long. The decay has been shown to be simple. This is indicated by looking for the nuclear gamma rays that would be emitted if there were two negative beta groups and by looking for positive beta particles. The spectrum has been found to fit a linear Kurie plot in the upper energy region, and by some authors to an energy as low as 100 Kev. Large low energy deviations reported by some authors can be attributed to a superposition of the low energy beta spectrum of $^33P$ with an end-point energy of approximately 250 Kev.

The conveniently high end-point energy and long half-life lead, however, to a log ft value (13) of 7.9. This led early observers, on the basis of the Sargent diagrams, to classify the spectrum as first or even second forbidden. The strongly energy dependent matrix elements for second-forbidden decay could never be fitted to the apparently allowed and therefore energy independent shape of the spectrum (14, 15). Furthermore, the spin of $^32P$ has very recently been measured (16) to be one; since the measured spin of $^32S$ is zero, the decay cannot be second forbidden. An attempt was made to explain the decay as a first-forbidden decay with
allowed shape (17). Such a transition would require a change of parity between the initial and final states of the nucleus. The nuclear shell model predicts positive parities for both of these states. Since measured parities in this region of the periodic table have always confirmed the shell model predictions, the assumption of a first-forbidden decay must be excluded. This leads to the conclusion that the decay is allowed in spite of its high ft value. Since the spectrum is a convenient one to work with experimentally, it has been used to determine the mixture of tensor and axial vector interactions through Fierz interference.

The most recent experimental work at Argonne National Laboratory (18, 19) by Porter, Wagner, and Freedman, has shown strong indications of a small deviation from the strictly allowed shape.\(^1\) This was determined by measuring both Na\(^{24}\) and P\(^{32}\) on a double lens spectrometer at two per cent resolution. The Na\(^{24}\) showed the allowed shape within a half of a per cent and P\(^{32}\) gave a deviation from the allowed shape with a linear shape factor decreasing by about three per cent over the energy range measured. They concluded that there was a definite deviation from the allowed shape in at least one of the two spectra, and with a lower degree of confidence that

\(^1\)We are extremely grateful to this group for making a preprint of their paper on this work available to us.
it was the $P^{32}$ that had the deviation from the allowed shape factor $L_0$. ($L_0$, a small correction to the Fermi function $F(Z,p)$, changes from .993 at low energies to .986 at high energies.) The treatment of their experimental data was as follows. In taking the spectrum they took every other point going up in electron energy and every other point going down in order to determine if there was any hysteresis effect. This, plus the fact that they took several short runs, insured that there was no error introduced in the half-life corrections. Counting rates were sufficiently slow so that no counter dead time corrections were necessary. Resolution corrections were needed only within five per cent of the end-point energy. The end-point energy was determined by a weighted fit to Kurie plots of points with $E > 2/3 E_0$. Extra points were taken at high energies so that this was feasible. Weights were on the basis of the total number of counts. An experimental shape factor was made for each run and the final results found by taking the average of the individual runs weighted according to the number of counts in the given run. The deviation found in $P^{32}$ was not the effect of source thickness since the Na$^{24}$ sources were the thicker.

In the next section are the results of a statistical analysis indicating that deviations as reported by the Argonne group are present in other $P^{32}$ data. This statistical analysis was made independently of the Argonne results and had
established (20) the presence of such a positive deviation prior to our knowledge of the Argonne results.

This report also contains a possible explanation of such an effect as being that which occurs when the leading matrix element $\hat{E} \hat{j} \sigma$ is $l$-forbidden and therefore has appreciable interference with normally second-forbidden matrix elements. The nuclear shell model predicts that the transition is that of a $d_{3/2}$ neutron into an $s_{1/2}$ proton. Since the operator connecting such states differing by two units of orbital angular momentum must be a tensor of second rank in the orbital coordinates, only second-forbidden matrix elements would be expected to occur. The $ft$ value indicates however that this is not the case. Thus $\hat{E} \hat{j} \sigma$ must contribute to the transition. This could occur only if there were other nucleon configurations contributing to the transition. However the configuration is expected to be reasonably pure, in which case $\hat{E} \hat{j} \sigma$ is appreciably smaller than for an allowed decay while the second-forbidden matrix elements are of normal size. In such a case interference between $\hat{E} \hat{j} \sigma$ and the second-forbidden matrix elements should be observable in the spectrum. Such an interpretation of the decay gives a shape factor having the same type of linear dependence on energy as is observed in the experimental data. If this interpretation is correct, then this is the first known ex-
ample of observed interference between beta decay matrix elements of different orders of forbiddenness.
III. STATISTICAL ANALYSIS OF THE DATA

This section presents calculations showing that deviations as reported by the Argonne group can also be found in some experimental data taken here by Pohm, Waddell, and Jensen (21). These data were taken on both an intermediate-image spectrometer and a thin-lens spectrometer. The latter instrument is iron free.

The intermediate-image data were the average of four runs. Each run was corrected separately for background without a source present, dead time of the counter, half-life of the sample, and spectrometer resolution. An additional small correction was made on the mean values for scattering background with the source present. Of these only the first background correction and the resolution correction were appreciable. The background correction without a source present was as large as three per cent, but this should give no difficulty since it was measured accurately. While the resolution correction (applied as a multiplicative factor approximately equal to unity) changed by five per cent over the spectrum, it is believed to be accurate to within one per cent.

The thin-lens data were the sum of seven runs. All of the corrections made on the intermediate-image data were also made on the thin-lens data. The background counts without a source were again very high, being of the order of twenty
per cent on the high energy end. This again should not affect the results since this may be determined to arbitrary accuracy by taking extra counts. Again the resolution correction was less than five per cent with an uncertainty of less than one per cent.

Values of the Fermi function were found by graphical interpolation in the National Bureau of Standards tables.

The calibration of the spectrometers was checked by also measuring the beta spectrum of $\gamma^{90}$. This is a first-forbidden spectrum with two units change in angular momentum. The shape factor for such a spectrum is known theoretically and thus deviations from this shape would indicate distortion of the spectrum by the spectrometer. No such deviations were found.

For a more extensive discussion of the experiment see Pohm's Ph.D. thesis (22).

In making the statistical analysis it was desirable to weight the points. This is due to the importance of the end-point energy in determining the shape of the spectrum. With the intermediate-image spectrometer extra counting time had been taken at the high energy points, thus increasing the statistical reliability of these points.

The weights of the points were determined by considering all of the error as being in the total number of counts and the background counting rates. All other corrections includ-
ing the resolution gave negligible contributions to the weights. The contributions from the background counts were small and essentially constant over the spectrum. The weights were computed by the usual statistical model. If \( N \) is the corrected number of counts at a given momentum and \( A \) is the original total number of counts, then \( N \) can be written \( N = f(x_1; A, B_j) \) where the \( x_1 \) are the corrections with negligible error such as the half-life correction and the \( B_j \) are the background corrections. Then the variance of \( N \) is

\[
\sigma^{-2}(N) = \left( \frac{\partial N}{\partial A} \right)^2 \sigma^{-2}(A) + \sum_j \left( \frac{\partial N}{\partial B_j} \right)^2 \sigma^{-2}(B_j) \tag{6}
\]

The weight of \( N \) is equal to the reciprocal of the variance of \( N \). The variance of \( A \) was taken as being equal to the total number of counts, as is customary (23). The variances of the background corrections were in some cases slightly less than the total number of background counts since extra background counting time had been taken.

In treating the experimental data two different shape factors were assumed. The first is the Fierz interference shape factor \( (1 + r/W) \). A fit of the data had previously been made by Pohm using a different statistical analysis than that which will be used here. In his analysis he found that \( r \) was positive for the intermediate image data, in good qualitative agreement with the Argonne results, but for the
thin lens data he found a negative value of \( r \). The second shape factor assumed was \((1 + aW)\) as indicated by the Argonne data and also suggested by the theory to be presented in the following section.

The two equations to be fitted by least squares can then be written in the following form:

\[
\begin{align*}
\eta &= \frac{N}{Ip^2F} = B (W_0 - W)^2 \left(1 + \frac{r}{W}\right) \\
\eta &= \frac{N}{Ip^2F} = B (W_0 - W)^2 (1 + aW).
\end{align*}
\]

The factor \( L_0 \) which might be expected on the right side of these equations has been omitted, thus modifying the parameters \( r \) and \( d \). This was done in Equation 7 so that the values found for \( r \) could be directly compared with those found by Pohm. Since \( L_0 \) depends almost linearly on \( W \), it is convenient also in Equation 8 to include the energy dependence of \( L_0 \) in the parameter \( d \).

It should be noticed that these equations are not linear in the parameters to be determined by least squares. This necessitated the use of a statistical method that requires initial values for the parameters and then yields successive corrections to these initial values. This method is treated in greater detail in the Appendix. The results of all such computations together with the initial values are given in Tables 1, 2, 3 and 4. The errors are probable errors.

**Table 1. Intermediate-image data; \((1 + r/W)\) shape factor**

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<th>Initial estimate</th>
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<tr>
<td>( B )</td>
<td>20.10</td>
<td>20.036</td>
<td>20.03 ± .10</td>
</tr>
<tr>
<td>( W_0 )</td>
<td>4.3500</td>
<td>4.3531</td>
<td>4.3531 ± .0009</td>
</tr>
<tr>
<td>( r )</td>
<td>+ .030</td>
<td>+ .0314</td>
<td>+ .033 ± .010</td>
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Table 2. Thin-lens data; \((1 + r/W)\) shape factor

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<tr>
<td>(B)</td>
<td>364.40</td>
<td>361.00</td>
<td>359.0 ± 3.1</td>
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<tr>
<td>(W_0)</td>
<td>4.3500</td>
<td>4.3522</td>
<td>4.3533± .0016</td>
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<tr>
<td>(r)</td>
<td>0</td>
<td>+ .019</td>
<td>+ .028 ± .018</td>
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\(^1\)Weights taken so that points on a Kurie plot are equally weighted since it was originally thought that this was the choice of weights taken by Pohm.

Table 3. Intermediate-image data; \((1 + aW)\) shape factor

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<tr>
<td>(B)</td>
<td>20.83</td>
<td>20.70 ± .08</td>
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<tr>
<td>(W_0)</td>
<td>4.3544</td>
<td>4.3551± .0010</td>
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<tr>
<td>(a)</td>
<td>-.0100</td>
<td>-.0085± .0017</td>
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Table 4. Thin-lens data; \((1 + aW)\) shape factor

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<th>First revision</th>
<th>Second revision</th>
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<tr>
<td>(B)</td>
<td>370.55</td>
<td>367.00</td>
<td>366.6 ± 2.0</td>
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<tr>
<td>(W_0)</td>
<td>4.3550</td>
<td>4.3540</td>
<td>4.3537± .0020</td>
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<tr>
<td>(a)</td>
<td>-.0085</td>
<td>-.0045</td>
<td>-.0039± .0028</td>
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The intermediate-image data show significant differences in the end-point energy and the slope of the shape factor for the two choices of shape factor. This may indicate a greater sensitivity to the curvature of the shape factor than one
would expect. Plots of this experimental data in Fig. 1 indicate that the data are slightly concave downward. This sensitivity is also reflected in the test of dispersion chi square. Chi square, which is the sum of the weighted squares of the residuals, is 39 for the \((1 + r/W)\) shape factor and 33 for the \((1 + aW)\) shape factor, each with 27 degrees of freedom. This corresponds to a probability that a random sample gives no better fit of .05 and .2 respectively.

The thin-lens data showed no such sensitivity to the choice of shape factor. The reason for this is undoubtedly the fact that the points for the thin-lens data did not go to as high or as low energies as for the intermediate-image data. Over this shorter energy range there is very slight change in the Fierz interference shape factor as can be seen from Fig. 2. Also the fewer number of points and shorter energy range result in a considerably larger uncertainty in the parameters. The value of chi square for both choices of shape factor and the thin-lens data was 15 with 20 degrees of freedom. This is somewhat lower than would be expected and may be partially due to too large an error being assigned to the background counting rate which would make the weights slightly smaller than they should be.

Taking the mean values of the two sets of data, weighted according to the probable errors, yields; \(a = -0.0076 \pm 0.011\) and \(W_0 = 4.3548 \pm 0.007\) which when converted into Mev be-
Fig. 1. Experimental shape factors, determined by statistical analysis of data taken on an intermediate-image spectrometer.
Fig. 2. Experimental shape factors, determined by statistical analysis of data taken on a thin-lens spectrometer.
comes $1.7142 \pm .0004$. Errors quoted on the end-point energy are only the statistical errors. This is a somewhat higher end-point energy than most people find; the Argonne value (19) is $1.711 \pm .002$ and the average of earlier values (24) is $1.708 \pm .004$ Mev. However, a Kurie plot through the higher energy points yields $1.712$ Mev. Since most of the values in the literature are from Kurie plots of the whole spectrum, they would therefore be expected to be lower. The value of $a$ corresponds to a $2.0$ per cent decrease in the shape factor compared to the three per cent decrease found at Argonne, in both cases over the energy range from 270 to 1630 Kev. Replacement of $L_0$ by unity in Equation 8 accounts for a decrease of $0.7$ per cent; thus the observed slope of the shape factor is three times the slope of $L_0$.

A previous fit of this experimental data had been made by Pohm in his Ph.D. thesis, using a different statistical method. The values which he found for $r$ were $-.032 \pm .045$ from the thin-lens data and $+.030 \pm .040$ from the intermediate-image data. This is in disagreement with the results above for the thin-lens data. Accordingly, an attempt was made to determine the cause of this disagreement. Pohm's method, like most of the methods used to determine the extent of Fierz interference, fitted the higher energy points to a straight line and then examined the deviations from this line in the lower energy part of the spectrum.
More precisely Pohm's method of analysis was as follows. The equation for the spectrum assuming Fierz interference is

$$\eta = B (W_0 - W)^2 (1 + \frac{r}{W}) . \quad (7)$$

This equation contains three unknown parameters. A least-squares fit of the higher energy points was made to a linear Kurie plot, thus obtaining the end-point energy and a least-squares slope denoted by $S$. This value of $W_0$ was assumed to be correct and solution was made for $B$ in terms of $r$ and the constant $S$. This was done by setting this numerical value of the slope equal to an algebraic expression for the average slope between the first and the last of the higher energy points, $W_1$ and $W_2$.

$$S = B^{1/2} (W_0 - W_1)(1 + \frac{r}{W_1})^{1/2} - B^{1/2} (W_0 - W_2)(1 + \frac{r}{W_2})^{1/2}$$

$$W_1 - W_2$$

(9)

The square roots containing $r$ were expanded to second order in $r$ and solution was made for $B$ as a function of $r$. This was substituted into Equation 7 and again an expansion was made to second order in $r$. Now the lower energy experimental points were represented by a function of $W$ and a single undetermined parameter $r$. As a numerical convenience the factor $(W_0 - W)^2$ was divided out of the equation. The sum of the squares of the residuals was then formed and differentiated to form the normal equation in $r$. The terms resulting from
the second order in \( r \) were dropped by Pohm although they may affect \( r \) by as much as 20 per cent. However, dropping these terms should have no affect on the sign of \( r \), and is perhaps justified by the large probable error in \( r \). The resulting expression for \( r \) is

\[
r = \frac{\sum \left( \frac{n_i}{(W_o - W_i)^2} - s^2 \right) \left( \frac{1}{W_1} - 2x \right)}{s^2 \sum \frac{1}{W_1} - 2x^2}
\]

(10)

where

\[
2x = \frac{W_o}{W_1 W_2}
\]

The sums are over the lower energy points not previously used to find \( W_o \) and \( S \). The derivation of this equation was checked and found to be correct. The only possible source of trouble is in the assumption that \( S \) represents the average slope. This implies that \( 1/W \) can be represented as a linear function of \( W \), which seems reasonable for a short range of \( W \).

Dr. Erling Jensen, in going over Pohm's numerical work, noted that the values of \( I \) used to calculate \( \eta \) were incorrect for the intermediate-image data. Correction of this error should have decreased Pohm's value of \( r \) by .01. No other significant numerical error was found in the calculation of \( r \). However, \( r \) is a fairly sensitive function of \( x \), which in turn depends on the choice of points used in evalu-
ating $S$ and $W_0$ through the energy values $W_1$ and $W_2$. It seemed possible that quite different values of $r$ would be found by varying this choice of points. Numerical work showed this to be at least partially the case although the results found were not of a systematic nature, thus indicating that $r$ was not sensitive directly to $\alpha$ but rather to the scatter of the experimental high energy points. The numerical results are summarized in Table 5.

Table 5. Sensitivity of Pohm's method to experimental scatter (Thin-lens data)

<table>
<thead>
<tr>
<th>Number of points used for $S^2$ and $W_0$</th>
<th>$S^2$</th>
<th>$W_0$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>362.33</td>
<td>4.3515</td>
<td>+ .06</td>
</tr>
<tr>
<td>9</td>
<td>363.63</td>
<td>4.3521</td>
<td>+ .01</td>
</tr>
<tr>
<td>7</td>
<td>360.70</td>
<td>4.3536</td>
<td>+ .06</td>
</tr>
</tbody>
</table>

It was found however that in the conversion of $I$ to the energy values, Pohm and the authors differed by one in approximately half of the fifth figure values. This was apparently round-off error. In the least-squares fit of the higher energy points, the sum of the energy values was a seven-digit number. In the least-squares solution for $S$ there were two subtractions, each with a loss of two significant figures. For this reason the errors in the energy values
cause a disagreement in the third figure of $S$. With the authors' values of $W$, $S^2$ was found to be 363.6 as compared to 366.6, both with a probable error of 1.6. This decrease in $S^2$ by one per cent corresponds to an increase in $r$ by .10. This error in the energy values of about one part in 60,000 is smaller than the uncertainty in the energy values. This seemed a sufficient reason for discounting the previous results of Pohm. Another disadvantage of his statistical method was that it was not a convenient method for working with weighted points. All of the work using Pohm's method was done with unweighted points. Therefore the effect of weighting the points, if any, was not determined. Since the main advantage of weighting the points is an improved estimate of the end-point energy which in turn improves the reliability of the other parameters, and since Pohm's method yields a fairly good end-point energy, the effects due to weighting the points should be small.

A further check which indicated that $r$ should be positive was by use of modified Kurie plots. If a value is assumed for $r$ in Equation 7, and the factor $(1 + \frac{r}{W})$ is divided out, then on taking the square root of both sides the equation becomes an ordinary linear equation which may be solved by ordinary least squares. The best value of $r$ would be that value which minimizes the sum of the squares of the deviations. Such computations are worthless for predicting prob-
able errors of the parameters since they do not make allowance
for the covariances. They do however give good estimates of
the parameters. Three such (unweighted) fits were made of
the thin-lens data. The results which appear in the table
below indicate that \( r \) is positive. In this computation
Pohm's \( W \) values were used.

Table 6. Modified Kurie plots

<table>
<thead>
<tr>
<th>Value of ( r ) assumed</th>
<th>Sum of assumed residuals</th>
<th>( W_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>- .032</td>
<td>.1328</td>
<td>4.3467</td>
</tr>
<tr>
<td>0</td>
<td>.1160</td>
<td>4.3499</td>
</tr>
<tr>
<td>+ .028</td>
<td>.1036</td>
<td>4.3532</td>
</tr>
</tbody>
</table>

The above sensitivity of Pohm's method to small errors
in the energy values raises the question of whether the Taylor
series method used in the authors' statistical analysis is
also sensitive to such small errors. This was checked by
repeating the Taylor series method with the different \( W \)
values. The same first estimates were used for the three
parameters. The results are given in Table 7. No such sen-
sitivity is indicated.
Table 7. Sensitivity of Taylor series method to errors in $W$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>First estimate</th>
<th>With Pohm's $W$'s</th>
<th>With the authors' $W$'s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.019</td>
<td>0.0295 ± 0.019</td>
<td>0.028 ± 0.018</td>
</tr>
<tr>
<td>$W_0$</td>
<td>4.3522</td>
<td>4.3533</td>
<td>4.3533</td>
</tr>
<tr>
<td>$B$</td>
<td>361.0</td>
<td>358.7</td>
<td>359.0</td>
</tr>
</tbody>
</table>
IV. THE THEORETICAL SHAPE FACTOR

A. Preliminary Survey

The preceding section shows that the observed deviations could be accounted for by Fierz interference between the tensor and a weak axial vector interaction. Previous experimental limits are not sufficiently narrow to exclude this possibility. However, there is a possibility of explaining the deviations with only the tensor interaction. The possibility of doing this rests on interference between the allowed tensor matrix element and second-forbidden tensor matrix elements which are ordinarily neglected. In order to calculate the shape factor including this interference, it is necessary to have numerical values for $\beta^2$ and the second-forbidden matrix elements. The latter will be obtained by calculation with shell-model wave functions for the initial and final nuclear states. Because the transition is $\ell-$forbidden, the simplest shell-model wave functions give a zero value for $\beta^2$; hence the experimental fit value must be used to determine its small but non-vanishing value.

The shell model (25) is a nuclear model that assigns to each nucleon a set of quantum numbers $(n, l, j, m_j)$ in analogy to the case of atomic electrons. That shell structure effects exist in nuclear spectra has been experimentally demonstrated. The magic numbers corresponding to closed shells
of nucleons (2, 8, 20, 28, 50, 82, 126) are different from those found for the atomic case. A model that gives reasonably good agreement with such a level sequence is a perturbed three-dimensional isotropic harmonic oscillator potential. The unperturbed three-dimensional isotropic harmonic oscillator is highly degenerate. The energy eigenvalues are \( E = \hbar \omega (2n + l + 3/2) \) where \( n \) is the number of radial nodes, not including the ones at zero and infinity required by the boundary conditions, and \( l \) is the angular momentum quantum number. Thus there is degeneracy between the states of different \( n \) and \( l \). Since two units change in \( l \) is equivalent to one unit change in \( n \), the degeneracy is between states of the same parity. The \( j \) values for each state are \( j = l \pm 1/2 \).

Since \( m_j \) changes by units of one from \( +j \) to \( -j \), there are \( 2j + 1 \) states of a given \( j \). Of the low lying states there are two \( s \) states, six \( p \) states, ten \( d \) states, and a second occurrence of two \( s \) states. Thus the unperturbed harmonic oscillator well has associated with it the magic numbers 2, 8, and 20. To get the higher magic numbers Mayer and Jensen postulated a large \( l \cdot s \) term in the Hamiltonian (possibly the effect of tensor forces) such that \( j = l + 1/2 \) states lie lower and the magnitude of the splitting increases with \( l \).

When this model is applied to \( \text{P}^{32} \) then the proton and neutron configurations are as follows.
Proton configuration: \((s_{1/2})^2(p_{3/2})^4(p_{1/2})^2(d_{5/2})^6(s_{1/2})^1\).

Neutron configuration: \((s_{1/2})^2(p_{3/2})^4(p_{1/2})^2(d_{5/2})^6
\hspace{1cm}(s_{1/2})^2(d_{3/2})^1\).

The beta transition involves the change of the \(d_{3/2}\) neutron into an \(s_{1/2}\) proton, thus making a closed subshell configuration. Both of these single particle states have even parity, and they differ by two units of orbital angular momentum. Thus the operator in the matrix elements must be of type \([2+]\) in the orbital coordinates. Since \(Jp'\sigma'\) is of type \([0+]\) in the orbital coordinates it cannot connect two such states. This is true even though \(J\) and the total nuclear spin \(J\) each change by one unit, which would lead one to believe that \(Jp'\sigma'\) would be an allowed matrix element for the decay. Such a matrix element is said to be \(L\)-forbidden.

The fact that the configurations of the initial and final states are not completely pure is expected on theoretical grounds and is indicated directly by the \(\beta\) value. If \(Jp'\sigma'\) were zero, then the \(\beta\) value would be a few orders of magnitude larger than its actual value of \(8 \times 10^7\) seconds. The size of \(Jp'\sigma'\) can be estimated by assuming that it is the only matrix element present. Since \(\log_{10} \beta\) is of the order of five for a normal allowed transition and of the order of eight for this transition, there is a difference of three. Since the \(\beta\) value is inversely proportional to the square
of the matrix element, $\int \beta \hat{\sigma}$ is smaller by a factor of 30 than for a normal allowed transition. With such a small value for $\int \beta \hat{\sigma}$ it is likely that in the shape factor there are appreciable contributions from interference between $\int \beta \hat{\sigma}$ and second-forbidden matrix elements. Ordinarily such terms are expected to be three orders of magnitude smaller than the leading term $L_0 \int \beta \hat{\sigma}^2$. First-forbidden matrix elements do not contribute because they have odd parity and therefore cannot contribute when the initial and final states are of the same parity.

The shape factor for such a decay, including the cross terms with the second-forbidden matrix elements, is, for a pure tensor interaction (26),

$$
\mathcal{G}_T^2(\beta \hat{\sigma}^2) \left\{ L_0 \int \beta \hat{\sigma}^2 + \frac{2q}{3}(N_0 - \frac{1}{2}qL_0) \int \beta \hat{\sigma} \hat{r}^2 - \frac{4qN_0}{3} \int (\beta \hat{\sigma} \cdot \hat{r}) \hat{r}^2 + 2(N_0 - \frac{qL_0}{3}) \int \beta \hat{\sigma} \hat{x} \hat{r} \right\}. \quad (11)
$$

All of the symbols used here have been defined elsewhere except $N_0$, which is a tabulated function of energy.

A pure tensor interaction is assumed for the following reasons. In order to produce a transition between states of spin one and spin zero, we must have a vector operator. The axial vector interaction is ruled out by explicit assumption, as explained in the first paragraph of this section. Combinations of the scalars $\beta$ or 1 with two powers of $r$ to form a
vector are impossible. Thus the only possibilities are $\int \bar{\alpha} x^r$ (from vector coupling) and $\int \beta \gamma_5^r$ (from pseudoscalar coupling). Both of these will be shown to be $J$-forbidden and therefore negligible compared to the second-order tensor terms, provided the coupling constants involved are not excessively large.

The Dirac operators involved in the matrix elements operate on a four-component spinor space. However, since the conventional shell model assumes that the nucleons are non-relativistic, the shell-model wave functions are two-component spinors. In order to use such wave functions in calculating the matrix elements, it is necessary to take the non-relativistic limit of all operators so that they become operators in a two-component spinor space (27). In transforming from a four-component spinor space to the two-component space, $\beta$ becomes -1 and the Dirac matrix $\bar{\sigma}$ becomes the ordinary Pauli spin matrix in all matrix elements not involving the operators $\bar{\alpha}$ or $\gamma_5$. The non-relativistic limit for those velocity dependent terms which are of interest is as follows:

$$1 \int \beta \bar{\alpha} x^r \rightarrow \frac{1}{M} \left[ 1 \int \bar{\sigma} - \int \bar{\sigma} (r \cdot p) + \int (\bar{\sigma} \cdot r \cdot \frac{1}{p}) \right], \quad (12)$$

*This differs in sign (28) from reference (27).*
\[ \int \vec{a} \cdot \vec{r} \rightarrow \frac{1}{M} \left[ \int \vec{\sigma} \cdot \vec{r} x \vec{p} \right] = \frac{1}{M} \left[ \int \vec{\sigma} + \int \vec{p} \right], \quad (13) \]
\[ \int \vec{\sigma} \cdot \vec{r} \rightarrow \frac{1}{M} \int \vec{\sigma}. \quad (14) \]

Here \( M \) is the nucleon mass in units of the electron mass.

The matrix elements involved are first computed for single particle states. Since the initial state is a d state and the final state is an s state, the operator must be a tensor of second rank in the orbital coordinates if the matrix element is not to vanish. Thus, of the operators involved in Equations 12, 13, and 14, only \( \int (\vec{\sigma} \cdot \vec{r}) \vec{p} \) is not \( \lambda \)-forbidden.

In order to calculate the second-forbidden matrix elements in Equation 11, it is therefore necessary to calculate \( \int (\vec{\sigma} \cdot \vec{r}) \vec{r} \) and \( \int (\vec{\sigma} \cdot \vec{r}) \vec{p} \). The third integral, \( \int \vec{\sigma} r^2 \), is \( \lambda \)-forbidden as well as second-forbidden.

**B. Calculation with Harmonic Oscillator Model**

To compute the matrix elements involved, it is necessary to have some knowledge of the nuclear wave functions. A convenient set of wave functions, as indicated by the shell model, are those of the isotropic harmonic oscillator. These wave functions may be separated into a spherical harmonic times a radial function times a two-component spinor. Such states are characterized by the quantum numbers \( n, m_l, m_s \). For \( \text{P}^{32} \) it is appropriate to assume \( jj \) coupling. To get the
wave functions for such a coupling scheme, which are characterized by the quantum numbers \( n \ell j m_j \), one takes a linear combination of these states each multiplied by the appropriate Clebsch-Gordan coefficient. Before doing this, it will first be shown without the use of explicit wave functions that the matrix element \( \int (\mathbf{\hat{r}} \cdot \mathbf{\hat{r}}) \mathbf{\hat{p}} \) (and therefore \( \int \mathbf{\hat{\alpha}} \mathbf{\hat{\alpha}} \times \mathbf{\hat{r}} \)) vanishes because of a special selection rule for the isotopic harmonic oscillator.

This can conveniently be shown by use of annihilation and creation operators for harmonic oscillator quanta of excitation. The operators \( \mathbf{r} \) and \( \mathbf{p} \) may be written as follows:

\[
\mathbf{r} = \frac{\mathbf{A} + \mathbf{A}^*}{\sqrt{2}}, \quad \mathbf{p} = \frac{\mathbf{A} - \mathbf{A}^*}{\sqrt{2}},
\]

where \( \mathbf{A} \) is the annihilation and \( \mathbf{A}^* \) is the creation operator. Then the product in which we are interested \( \langle \mathbf{r}_i \mathbf{p}_j \rangle \) becomes

\[
\langle \mathbf{r}_i \mathbf{p}_j \rangle = \frac{1}{2i} \left( \mathbf{A}_i \mathbf{A}_j - \mathbf{A}_i^* \mathbf{A}_j^* + \mathbf{A}_i^* \mathbf{A}_j - \mathbf{A}_i \mathbf{A}_j^* \right),
\]

which may be written as

\[
\frac{1}{2i} \left( \mathbf{A}_i \mathbf{A}_j - \mathbf{A}_i^* \mathbf{A}_j^* + \mathbf{A}_i^* \mathbf{A}_j - \mathbf{A}_i \mathbf{A}_j^* \right) - \delta_{ij}
\]

since the commutator of \( \mathbf{A}_i \) and \( \mathbf{A}_j^* \) is \( \delta_{ij} \). The first two terms, being products of two-creation or two-annihilation operators, must vanish unless the two states differ by two
quanta of excitation, which is not the case here. The last term \( \delta_{1j} \) is clearly \( \ell \)-forbidden. The other two terms cancel if \( i \) equals \( j \) and together form a component of \( \vec{r} \) or \( \vec{\ell} \) for \( i \) different from \( j \). Since \( \vec{\ell} \) is an operator that does not change the angular momentum of the state, matrix elements of \( \vec{\ell} \) must vanish.

Evaluation of the matrix element \( \int (\vec{\sigma} \cdot \hat{r}) \hat{r} \) is greatly simplified by using spherical components of the vector operators (29, 30):

\[
x^1 = -\frac{x + iy}{\sqrt{2}} = -\frac{r \sin \theta \text{e}^\phi}{\sqrt{2}},
\]

\[
x^0 = z = r \cos \theta,
\]

\[
x^{-1} = \frac{x - iy}{\sqrt{2}} = \frac{r \sin \theta \text{e}^{-\phi}}{\sqrt{2}},
\]

\[
\sigma^{-1} = -\frac{\sigma_x + i\sigma_y}{\sqrt{2}},
\]

\[
\sigma^0 = \sigma_z,
\]

\[
\sigma^{-1} = \frac{\sigma_x - i\sigma_y}{\sqrt{2}}.
\]

The \( \sigma^{-1} \) and \( \sigma^{-1} \) are ordinary step-up and step-down operators for spin except for normalization. The product \( \vec{\sigma} \cdot \hat{r} \) becomes \( \sigma^0 x^0 - \sigma^1 x^{-1} - \sigma^{-1} x^1 \). The use of such components is
required to make all of the matrix elements real. Also, such coordinates make possible the use of the Wigner-Eckart theorem for a vector operator. The essential result of this theorem is that the matrix elements of a vector operator can be written as the product of a reduced matrix element, which is independent of the initial and final \( m_j \) values as well as of \( M \) (labeling the spherical components of the operator), and a Clebsch-Gordan coefficient, which takes account of all of the \( m_j \) and \( M \) dependence of the matrix element. The Clebsch-Gordan coefficient is independent of the dynamical nature of the vector involved.

If it is assumed that the initial and final states both have \( m_j \) equal to one half, then the wave functions are as follows:

\[
\Psi_i = \left( \sqrt{\frac{3}{2}} Y_2^0 \alpha - \sqrt{\frac{3}{2}} Y_2^1 \beta \right) R_i(r),
\]

\[
R_i(r) = \mathcal{N}^{-1/4} a^{-3/2} \sqrt{16 \over 15} \left( \frac{r}{a} \right)^2 e^{-1/2 \left( \frac{r}{a} \right)^2},
\]

\[
\Psi_f = (Y_0^0 \alpha) R_f(r),
\]

\[
R_f(r) = \mathcal{N}^{-1/4} a^{-3/2} \sqrt{8 \over 3} \left[ \frac{3}{2} - \left( \frac{r}{a} \right)^2 \right] e^{-1/2 \left( \frac{r}{a} \right)^2}.
\]

In these equations the \( Y_{m_i}^k \) are the spherical harmonics. \( \alpha \) is a spin function denoting spin-up, and \( \beta \) is the spin func-
tion for spin-down. The radial functions depend only on the quantum numbers \( n \) and \( l \). The parameter \( a \) may be related to the frequency of the oscillator \( \omega \):

\[
a = \sqrt{\frac{\hbar \omega}{s}} = \sqrt{\frac{\hbar}{m} \omega}.
\]

(24)

In this equation \( s \) is the spring constant of the oscillator.

Since the mean square radius of the charge distribution is known (31) for \( S^{32} \), it is convenient to eliminate the unknown parameter \( a \) by expressing it in terms of this radius. To do this, one uses the following relationships:

\[
\frac{1}{2} s \left\langle r^2 \right\rangle_n = \frac{1}{2} \hbar \omega (2n + l + 3/2), \tag{25}
\]

\[
s a^2 = \hbar \omega, \tag{26}
\]

so that

\[
\frac{\left\langle r^2 \right\rangle_n}{a^2} = 2n + l + 3/2. \tag{27}
\]

To express this relationship in terms of the mean square radius of the charge distribution, one evaluates Equation 27 for each combination of \( n \) and \( l \) occurring in the proton configuration and then one takes the mean weighted by the number of protons of the given \( n \) and \( l \). When this is done for \( S^{32} \), then \( a^2 \) is found to be
\[ a^2 = \frac{8}{23} \overline{r}^2 = \frac{8}{23} \left(3.19 \text{ fermi}\right)^2 = 3.54 \left(\text{fermi}\right)^2. \]  

The only nonvanishing terms of \( \int (\vec{\sigma} \cdot \vec{r})r \) for the choice of states above are \( \int (\sigma^0 x^0)x^0 \) and \( -\int (\sigma^{-1} x^{-1})x^0 \). The matrix elements may be separated into a radial and an angular integral. The angular integration gives \( +\sqrt{2/3} \), and the radial part of the matrix element, which is the same for all components of the matrix element, is

\[ \int R_f r^2 R_i r^2 dr = -\sqrt{10} a^2 = -11.19 \left(\text{fermi}\right)^2 = -7.5 \times 10^{-5} \left(\frac{\hbar}{\text{mc}}\right)^2. \]

The reduced matrix element may be found by use of the Wigner-Eckart theorem. The Clebsch-Gordan coefficient involved is \( C(3/2 \ 1 1/2; 1/2 0) = -1/\sqrt{3} \). This is of the form \( C(j_1 L j_f; m_1 M) \), where the subscripts \( i \) and \( f \) refer to the initial and final states, \( L \) stands for the tensor rank of the operator involved, and \( M \) labels its spherical component. The reduced matrix element, which is the ratio of the matrix element to the coefficient, is

\[ \left( s \right| (\vec{\sigma} \cdot \vec{r}) x \left| d \right) = -\frac{\sqrt{2}}{3} R_f r^2 R_i r^2 dr = 6.13 \times 10^{-5} \left(\frac{\hbar}{\text{mc}}\right)^2. \]

In the above calculation it was assumed that the transition was completely described as a single-particle transition from a \( d_{3/2} \) state to an \( s_{1/2} \) state. This is not quite correct
since in the initial state the \( d_{3/2} \) neutron must be coupled to the extra \( s_{1/2} \) proton in such a way that the initial nuclear spin, \( J \), is one. In addition, the use of wave functions that include isotopic spin variables is implicit in the definition (30) of the beta-decay matrix elements. Isotopic spin, which is a charge variable, is so named because its algebraic properties in what is called isotopic spin space are closely analogous to the properties of spin in ordinary space. Isotopic spin states are usually denoted by the symbol \( \gamma \) for a neutron and \( \Pi \) for a proton. The quantum number \( T_s \) is defined by the fact that \( T_s \) is \(-1/2\) for a proton and \(+1/2\) for a neutron. Then \( T \) is called the total isotopic spin and must be equal to or greater than \( |T_s| \), just as \( S \) must be equal to or greater than \( |M_s| \). The ground states of light nuclei satisfy the condition (32) \( T = |T_s| \). Thus the ground state of \( p^{32} \) has \( T \) of one and the ground state of \( s^{32} \) has \( T \) of zero.

Because subshells that are filled with protons and also with neutrons contribute neither to the spin nor to the isotopic spin, the initial and final nuclear states can be written in the form

\[
\psi_i = \frac{1}{\sqrt{4!}} \left\{ a \left| S \right. \frac{1}{2} \Pi \right. d^{1/2} \gamma \frac{1}{2} \gamma s^{1/2} \gamma s^{-1/2} \gamma \left| + \right. b \left| S \right. -1/2 \Pi \right. d^{3/2} \gamma \frac{1}{2} \gamma s^{1/2} \gamma s^{-1/2} \gamma \left| \right. \right\}, \tag{31}
\]
Here the initial state was chosen with $M_J$ equal to one. The single particle states written between bars stand for the totally antisymmetric Slater determinant of the four particles in these states. The single particle $j$ values of 1/2 for an s state and 3/2 for a d state have been omitted. It may be shown that $T$ is one for the initial state by applying the step-up operator $T_+ = \sum_{i=1}^{4} \tau_i^+ + 1$, which changes $T_0$ to two. $\tau_+^i$ annihilates a neutron or steps a proton up to a neutron, in which case the determinant vanishes. Thus $T_+^i$ annihilates the initial state, which must therefore have $T = T_0 = 1$. Similarly one can assure that $J$ is one by applying the operator $J_+ = \sum_{i=1}^{4} j_i^+ + 1$ and requiring the result to vanish.

Recall that $J_+^i$ acting on a state with quantum numbers $J$ and $M$ equals $(J-M)(J+M+1)$ times a state with $J$ and $M+1$. Thus $J_+^i$ changes the initial state to

\[
(a \sqrt{3} + b) \left| S^{1/2} \pi d^{3/2} \nu s^{1/2} \nu^{-1/2} \nu \right| = 0.
\]  

The further condition that $a^2 + b^2 = 1$ is satisfied by $a = 1/2$ and $b = -\sqrt{3}/2$. Since the beta-decay operator is a sum of single-particle operators, the matrix element between two Slater determinants may be written, according to Condon and Shortley (29, p. 171), as

\[
\Psi_f = \frac{1}{\sqrt{4!}} \left| S^{1/2} \pi s^{-1/2} \pi s^{1/2} \nu s^{-1/2} \nu \right|.
\]

(32)
\[ \frac{1}{n!} \left| \psi_{j_1} \cdots \psi_{j_n} \right| \sum_{i} \left| \psi_{k_1} \cdots \psi_{k_n} \right| d\tau = \pm \left( \psi_{k_1} \right| f \left| \psi_k \right) . \] (34)

The final state must be the same as the initial state except for the single-particle state \( \psi_{k_1} \) replacing \( \psi_k \). The + or - sign is taken according to whether the initial and final wave functions differ by an even or an odd permutation of the single-particle states. The matrix element of interest to us is therefore

\[ \int (\vec{\sigma} \cdot \vec{r}) x^{-1} = a(S^{-1/2} \left| (\vec{\sigma} \cdot \vec{r}) x^{-1} \right| d^{1/2}) - b(S^{1/2} \left| (\vec{\sigma} \cdot \vec{r}) x^{-1} \right| d^{3/2}) \]
\[ = (0 \left| (\vec{\sigma} \cdot \vec{r}) x \right| 1) C(110; 1-1). \] (35)

From this one finds that the reduced matrix element between nuclear states is just \( \sqrt{2} \) times the reduced matrix element between single-particle states and is therefore

\[ (0 \left| (\vec{\sigma} \cdot \vec{r}) x \right| 1) = \sqrt{2} (S \left| (\vec{\sigma} \cdot \vec{r}) x \right| d) = 8.67 \times 10^{-5} \left( \frac{\hbar}{mc} \right)^2 . \] (36)

An alternate calculation can be made without the use of the isotopic spin formalism. The matrix element, according to Nordheim and Yost (33), may be written in the form

\[ M = (N_1 Z_f)^{1/2} \sum_{1,2,\ldots,N-1; N,\ldots,N+Z} \mathcal{O}(N) \mathcal{Y}_+ (1,2,\ldots,N; N+1,\ldots,N+Z) . \] (37)

The wave functions used here do not include isotopic spin but are assumed to be totally antisymmetric in the neutrons.
and the protons separately. This equation gives the same reduced matrix element as was found above with the isotopic spin formalism if closed subshells are omitted in both the wave functions and the coefficient.

There is a convenient relationship between the matrix element and the reduced matrix element for a spin change of one to zero. If $V$ stands for any vector operator then

$$\mid \sum_M |SV^M|)^2 = \sum_M \mid \langle 00|V^M|1 M_1 \rangle \mid^2 =$$

$$1/3 \ (0 \parallel V \parallel 1)^2 \sum_M \sum_{M_1} c(110; M_1 M)^2 =$$

$$1/3 \ (0 \parallel V \parallel 1)^2 \sum_M \sum_{M_1} c(110; -M M)^2 = 1/3 \ (0 \parallel V \parallel 1)^2.$$  \hspace{1cm} (38)

Here the $V^M$ are the spherical components of $V$ and $1/3 \sum_{M_1}$ is an average over initial states.

In order to determine the shape factor, it is still necessary to find the value of $\int \sigma$. This could be determined in two ways. The best way in principle would be to determine what other configurations contribute to the transition and compute $\int \sigma$ for these configurations. However the amount of configuration mixing is difficult to estimate. A second method which should give a reasonably reliable estimate of $\int \sigma$ is by the use of the $ft$ value. If the higher order terms are neglected for the present, then (34)

$$ft = \frac{4300}{\int |\sigma|^2}.$$  \hspace{1cm} (39)
In this equation the square of the tensor coupling constant has been absorbed into the constant. For $P^{132} \log_{10} ft$ is 7.9 and thus

$$|\frac{1}{\sqrt[3]{3}}(0||s|| 1) = \pm 7.8 \times 10^{-3} .$$

This method fails to give the sign of the matrix element.

Now that all of the matrix elements of interest have been determined, we may write the shape factor in much simpler form:

$$G_T^2 |\frac{1}{\sqrt[3]{3}}|^2 \left\{ L_0 - \frac{4qN_0}{3} \sqrt{\frac{(\sigma \cdot r)^2}{\sigma}} \right\} .$$

The factor $G_T^2 |\frac{1}{\sqrt[3]{3}}|^2$ is of no particular interest here, but only the energy-dependent part, since it is the change of the shape factor over the spectrum, not the absolute value, that is determined from experimental data. Since the matrix element and the reduced matrix element differ by a constant which is independent of the vector involved, the ratio of the two matrix elements may be written as the ratio of the reduced matrix elements, as below:

$$\frac{(0||s|| 1)}{(0||s|| 1)} = \frac{8.67 \times 10^{-5}}{\sqrt{3} \times 7.8 \times 10^{-3}} = \pm 6.42 \times 10^{-3} \left(\frac{\hbar}{mc}\right)^2 .$$

Values were put into the energy-dependent part of Equation 41. The values of the functions $L_0$ and $N_0$ were taken from the tables of Rose, Perry, and Dismuke (35). If the
relative sign of the matrix elements \( \int \vec{r} \) and \( \int (\vec{r} \cdot r) \vec{r} \) is taken to be positive then the shape factor will be approximately linear with downward slope. A plot of this shape factor is given in Fig. 3. The change of this shape factor over the energy range plotted, which is approximately the same as the measured energy range, is 12 per cent. This is considerably larger than the two or three per cent indicated by the experimental data.

One factor that may contribute to the disagreement with experiment is the choice of radial wave functions. The matrix elements might be sensitive to these functions since the initial nucleon state, being the first d state, has no radial nodes while the final state, being the second s state, has one radial node. This means that the relative sign of the two radial functions changes with \( r \) and there is cancellation in the radial integral. The cancellation amounts to only a few per cent with harmonic-oscillator wave functions, but might be larger with a different model. Also, it was a property of the harmonic-oscillator wave functions that made the term \( \int (\vec{r} \cdot r) \vec{r} \) zero so that the matrix element \( \int \beta \hat{a} \vec{x} \vec{r} \) was negligible. Ordinarily matrix elements with the velocity-dependent operator \( \hat{a} \) are larger than those with an extra power of \( r \). For this reason one might have expected \( \int \beta \hat{a} \vec{x} \vec{r} \) to have been larger than \( \int (\beta \vec{r} \cdot r) \vec{r} \). Fortunately, the slope of the shape factor is not sensitive to the value of \( \int \beta \hat{a} \vec{x} \vec{r} \).
Fig. 3. Theoretical shape factor calculated with isotropic harmonic-oscillator wave functions
because $\int (\vec{\beta} \cdot \vec{r}) \vec{r}$ is the dominant term in determining this slope. This may be seen in Fig. 4 where a plot has been made of the two coefficients of interest. The coefficient of $\int \vec{\beta} \cdot \vec{r}$ has only a slight variation with energy.

C. Calculation with Infinite Square Well Model

In view of the numerical discrepancy between the calculations and the experimental results, it seemed desirable to get some idea of the sensitivity of the computed shape factor to the details of the radial wave functions. For this reason the radial integrals were recalculated with the wave functions appropriate to a square well of infinite depth. This choice of potential also gives the low lying magic numbers 2, 8, and 20 of the nuclear shell model. The only difference between these wave functions and those previously used is the radial functions; the radial functions for the square well are

$$R_i = \left[ \left( \frac{3}{(Kr)^3} - \frac{1}{Kr} \right) \sin Kr - \frac{3}{(Kr)^2} \cos Kr \right] \frac{2}{1.6555b^{3/2}}, \quad \text{(43)}$$

$$R_f = \frac{\sin kr}{kr} \cdot k \sqrt{\frac{2}{b}}. \quad \text{(44)}$$

These are spherical Bessel functions. The parameter $K$ is determined by requiring that $Kb$, $b$ being the radius of the well, is the first zero of the function. Likewise $k$ is determined by requiring that $kb$ be the second zero of the final
Fig. 4. Comparison of the energy dependence of the coefficient of $\int (\mathbf{B} \cdot \mathbf{r})^2$, namely $4/3 qN_o$, with the coefficient of $\int \beta \alpha x^2$, namely $2(N_o - qL_o/3)$.
radial function. This must be the second zero since the s state must have one radial node. The normalization is chosen so that \( \int_{0}^{b} R^2 r^2 \, dr = 1 \), the radial functions being zero beyond this interval as is required by an infinite potential.

When the radial integral that occurs in \( \int (\mathbf{\tilde{r}} \cdot \mathbf{\hat{r}}) r \), namely \( \int_{0}^{b} R_f r_2 R_1 r_2 \, dr \), is recomputed with these radial functions, its value is

\[
\int_{0}^{b} R_f r_2 R_1 r_2 \, dr = -0.308 b^2 = -5.52 \times 10^{-5} \left( \frac{\hbar}{m c} \right)^2. \tag{45}
\]

The numerical value of \( b \) may be found in terms of the electromagnetic radius in the same manner as was determined for the harmonic oscillator. The value found was (36)

\[
b^2 = \frac{r_2^2}{3804} = 1.79 \times 10^{-4} \left( \frac{\hbar}{m c} \right)^2. \tag{46}
\]

With this value for the radial integral the reduced matrix element is

\[
\langle 0 \| (\mathbf{\tilde{r}} \cdot \mathbf{\hat{r}}) x \| 1 \rangle = 6.37 \times 10^{-5} \left( \frac{\hbar}{m c} \right)^2. \tag{47}
\]

The matrix element \( \int (\mathbf{\tilde{r}} \cdot \mathbf{\hat{r}}) p \) must also be evaluated for these wave functions. The initial and final \( m_j \) values are again chosen to be 1/2 so that the angular part of the wave functions in Equations 20 and 22 may be used again. The non-
vanishing term must be the term with $p_z$ which is proportional to $\int (\vec{p} \cdot \hat{r}) \frac{\partial}{\partial z}$, with

$$\frac{\partial}{\partial z} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}$$

(48)
in spherical coordinates. The nonvanishing terms will be

$$\int \left( \sigma^{0} \sigma^{0} - \sigma^{-1} \sigma^{-1} \right) \frac{\partial}{\partial z}.$$  

(49)

When the angular parts are separated and integrated first, Equation 49 becomes

$$\frac{\alpha r^2}{3} \int R_1(r \frac{d}{dr} R_1) r^2 dr + \sqrt{2} \int R_1 R_2 r^2 dr.$$  

(50)

The radial integrals may be evaluated in terms of simple trigonometric functions and the tabulated function $\text{Si}(y) = \int_0^y \sin(x/x) dx$. The value of the first integral was found to be 1.848 and the second -0.4554. This yields for the reduced matrix element of $\int (\vec{p} \cdot \hat{r}) \frac{\partial}{\partial z}$

$$\langle 0 \parallel (\vec{p} \cdot \hat{r}) \text{Grad.} \parallel 1 \rangle = -2(\frac{1}{2}) \langle \sigma^{1/2} \parallel (\vec{p} \cdot \hat{r}) \text{Grad.} \parallel \sigma^{3/2} \rangle = -0.556.$$  

(51)

From Equation 12 the reduced matrix element of $\int \beta \frac{\partial}{\partial \hat{r}}$ is

$$\langle 0 \parallel \beta \frac{\partial}{\partial \hat{r}} \parallel 1 \rangle \rightarrow -\frac{1}{M} \langle 0 \parallel (\vec{p} \cdot \hat{r}) \text{Grad.} \parallel 1 \rangle = +3.03 \times 10^{-4}.$$  

(52)

The size of this matrix element necessitates reevaluating $\int \sigma^z$. The correct expression for $\int \sigma^z$ including the cross
\[ ft = \frac{4800}{\langle L_0 \rangle \int \sigma^2 \left| 2 \left( N_0 - \frac{qL_0}{3} \right) \right\| \int \sigma^2 \| + 2 \left( N_0 - \frac{qL_0}{3} \right) \int \sigma^2 \| \| \sigma^2 \|} \]

where the upper signs should be taken if \( \int (\sigma \cdot r) r / \int \sigma^2 \) is positive. The brackets around the energy-dependent terms indicate that these are to be evaluated at the average beta energy which is approximately 1/3 the end-point energy. There is one sign change in this equation from that of the shape factor in Equation 11, introduced by replacing \( \beta \) by -1 wherever possible. When numerical values are substituted \( |\int \sigma^2| \) is found to be \( 6.20 \times 10^{-3} \) for the upper signs or \( 8.96 \times 10^{-3} \) for the lower signs. The first value is to be taken if the shape factor is to have a negative slope. The reduced matrix element then becomes

\[ (0 \| \sigma \| 1) = \sqrt{3} |\int \sigma^2| = 10.74 \times 10^{-3}. \] (54)

The shape factor, expressed in the form,

\[ L_0 - \frac{4qN_0}{3} \int (\sigma \cdot r) r - 2(N_0 - \frac{qL_0}{3}) \int \sigma^2 \| \int \sigma^2 \] (55)

has been plotted in Fig. 5. This shape factor changes by eight per cent over the energy range plotted. The matrix elements \( \int (\sigma \cdot r) r \) and \( \int \sigma^2 \| \) have the same relative sign so that a slight cancellation in slope occurs between these two terms.
Fig. 5. Theoretical shape factor calculated with infinite square-well wave functions
V. COMPARISON WITH EXPERIMENT

The theoretical shape factor as computed in the preceding section does not give good quantitative agreement with the experimental results. It does however give good qualitative agreement with experiment, in the sense that the cross terms between the allowed matrix element $\mathbf{\sigma}_\mathbf{g}$ and the second-forbidden matrix elements of a pure tensor interaction are clearly capable of causing the observed deviations. If they are in fact the cause, then the $\mathbf{p}^{32}$ spectrum is the first known example of observed interference between matrix elements of different orders of forbiddenness. The experimental data analyzed indicated that the shape factor should be linear and decreasing by two per cent over the energy range measured; this is in good agreement with the three per cent decrease found by the Argonne group. The theory also predicts a linear shape factor but with a somewhat larger change of eight to 12 per cent.

There are several factors which may contribute to the disagreement between experiment and theory. A factor which may explain the disagreement is configuration mixing. Indications from the evaluation of allowed matrix elements are that the matrix element predicted by the shell model is generally somewhat larger than the observed value (37). The interpretation given to this is that there is considerable configuration mixing, the matrix element being forbidden for the other
configurations. Indications are that the amount of configuration mixing is generally small near a closed subshell configuration such as $s^{32}$. The amount of configuration mixing required is not so large as to be unreasonable, however. The calculations of allowed matrix elements on which these conclusions are based are independent of the radial functions and thus do not depend critically on the choice of potential.

A second factor is the possibility of a nonspherical charge distribution. This would lead to collective motions of the nucleons so that no individual nucleon model such as the nuclear shell model would be strictly valid. Such a charge distribution in one of the states would be expected to decrease the calculated matrix element. The reason for this is that the degree of overlap between the initial and final nuclear wave functions is decreased; hence matrix elements between them would also be expected to be smaller.

A third factor to be considered is the possibility that the shape factor used is not sufficiently complete. The matrix element $\int \beta \hat{\alpha} x^\tau$ was smaller than $\int \beta \delta$ by only a factor of 40 when computed with infinite square-well wave functions. This is sufficiently large so that the term involving $|\int \beta \hat{\alpha} x^\tau|^2$ could affect the shape of the spectrum if the coefficient of this term is very large.

Other less likely alternatives which could change the theoretical shape factor are the presence of Fierz inter-
ference with a small value for the axial vector coupling constant or the presence of the pseudoscalar interaction with a large pseudoscalar coupling constant. The quantitative uncertainties in the tensor part of the shape factor indicated above make it impossible to draw any strong conclusions as to the existence of these other beta decay interactions.
The least-squares treatment of equations that are non-linear in the parameters to be estimated, although not new (38), is not readily available to physicists. For this reason this section has been added on the statistical method used.

The treatment of such a problem involves solving a set of normal equations which are non-linear in the unknown parameters. In order to do this, the normal equations are linearized after expansion in Taylor series about approximate values of the parameters, and corrections to these values are computed from the set of linearized equations.

Assume the original equations are of the form $y_i = (x_1; \theta_1, \theta_2... \theta_n) + e_i$. Here $y_i$ is the value taken by an observed quantity, $y$, when another observable, $x$, takes the value $x_i$. The observable $x$ is regarded as being exact. $A$ is a known function of the parameters to be determined, $\theta_1, \theta_2... \theta_n$. When the function $A$ is evaluated at the true values of the $\theta$'s the value of the function may still differ from the corresponding $y$ because of errors in the measurement of the $y$'s. This is indicated by the error $e_i$. Since the true values of the $\theta$'s are generally not known, it is necessary to find the most likely values of the parameters $\theta$. This may be done by least squares if it is assumed that the $e$'s are random variables with zero mean, normal distribution, and variance $w_i^{-1}$. 
When the errors are assumed to have these properties then they are identical with what is more commonly called the residuals. In the process of least squares the maximum likelihood values of the parameters are obtained by requiring that the sum of the squares of the residuals be a minimum. The first step is to form the weighted sum of the errors,

$$\sum_{i} w_i e_i^2 = \sum_{i} w_i (y_i - A(x_1; \theta_1 \ldots \theta_n))^2 = \sum_{i} w_i (y_i - A_1)^2 \quad (56)$$

and then differentiate this with respect to each of the parameters, setting each derivative equal to zero. It is at this point that it is necessary to linearize the normal equations. For this purpose a Taylor series expansion is made to first order about initial estimates:

$$\frac{\partial \sum_{i} w_i e_i^2}{\partial \theta \lambda} \Bigg|_{\theta_1 \ldots \theta_n} = \frac{\partial \sum_{i} w_i e_i^2}{\partial \theta \lambda} \Bigg|_{\theta_1^o \ldots \theta_n^o} +$$

$$\sum_{\mu=1}^{n} \left( \theta_\mu - \theta_\mu^o \right) \frac{\partial^2 \sum_{i} w_i e_i^2}{\partial \theta \lambda \partial \theta \lambda} \Bigg|_{\theta_1^o \ldots \theta_n^o} = 0 \quad (57)$$

where

$$\frac{\partial \sum_{i} w_i e_i^2}{\partial \theta \lambda} \Bigg|_{\theta_1^o \ldots \theta_n^o} = - 2 \sum_{i} w_i (y_i - A_1) \frac{\partial A_1}{\partial \theta \lambda} \Bigg|_{\theta_1^o \ldots \theta_n^o} \quad (58)$$
The use of these equations may be simplified by use of a matrix notation. The column vectors $Q$ and $\Delta$ and the symmetric matrices $T$ and $U$ are defined by:

\[ Q_{\lambda} = \sum_{1} w_{i}(y_{1} - A_{1}) \frac{\partial A_{1}}{\partial \theta_{\lambda}}, \]

\[ \Delta_{\mu} = (\theta_{\lambda} - \theta_{\lambda}^{0}), \]

\[ T_{\lambda \mu} = \sum_{1} w_{i} \frac{\partial A_{1}}{\partial \theta_{\lambda}} \frac{\partial A_{1}}{\partial \theta_{\mu}}, \]

\[ U_{\lambda \mu} = \sum_{1} w_{i}(y_{1} - A_{1}) \frac{\partial^{2} A_{1}}{\partial \theta_{\lambda} \partial \theta_{\mu}}. \]

The evaluation of $A_{1}$ and its derivatives at the first estimates of the parameters is here understood. The matrix equation then is $(T-U) \Delta = Q$. The corrected parameters are $\theta_{\lambda}^{0} + \Delta_{\lambda} = \theta_{\lambda}$. These values may still not be correct because of the approximation involved in terminating the Taylor series expansion. In general several approximations are necessary to get the correct $\theta$'s. Usually it is sufficient
to stop when $\theta_\lambda$ is less than its standard deviation for all $\lambda$. The standard deviations are found by considering the matrix $T^{-1}$ as the variance matrix as noted by Kendall (39) in equation 17.116. Then the squares of the standard deviations are the diagonal elements and the off diagonal elements are the covariances. The matrix $T^{-1}$ is normally multiplied by the sum of the weighted squares of the residuals divided by the number of degrees of freedom (defined as the number of equations minus the number of parameters to be determined). This multiplication converts the "internal" error into the "external" error.
VI. REFERENCES CITED


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