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A Dominance Relation for Unconditional Multi-Attribute Preferences

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1 A Language for Unconditional Preferences

Let \( X = \{ X_i \} \) be a set of variables, each with a domain \( D_i \). An outcome \( \alpha \in O \) is a complete assignment to all the variables, denoted by the tuple \( \alpha := \langle \alpha(X_1), \alpha(X_2), \ldots, \alpha(X_m) \rangle \) such that \( \alpha(X_i) \in D_i \) for each \( X_i \in X \). The set of all possible outcomes is given by \( O = \prod_{X_i \in X} D_i \). We consider a preference language \( L \) for specifying: (a) unconditional intra-variable preferences \( \succ_i \) that are strict partial orders (i.e., irreflexive and transitive relations) over \( D_i \); and (b) unconditional relative importance preferences that are strict partial orders over \( X \).

\[ L \text{ includes unconditional preference statements of the form } x \succ_i x'[Z] \text{ such that } x, x' \in D_i; \text{ and } X_i \not\in Z. \text{ Here, the set } Z \text{ of variables are relatively less important than } X_i, \text{ i.e., } X_i \succ X_j \text{ for each } X_j \in Z. \text{ However, the language } L \text{ does not include statements specifying conditional relative importance, i.e., if } x \succ_i x'[Z_1] \text{ and } x_1 \succ_i x'_1[Z_2] \text{ in } L \text{ then } Z_1 = Z_2. \text{ Additionally, because } \succ \text{ is assumed to be a binary (strict partial order) relation, } |Z| = 0 \text{ or } 1. \text{ We now compare the expressiveness of } L \text{ to that of some well known preference languages.}\]

1.1 Expressiveness

CP-nets [1] use a compact graphical model to specify conditional intra-variable preferences \( \succ_i \) over a set of variables \( X \). Each node \( i \) in the graph corresponds to a variable \( X_i \in X \), and each edge \((i, j)\) in the graph captures the fact that the intra-variable preference \( \succ_j \) with respect to variable \( X_j \) is conditioned (or dependent) on the valuation of \( X_i \). For any variable \( X_j \) (corresponding to node \( j \)), the set of variables \( \{ X_i : (i, j) \text{is an edge} \} \) that influence \( \succ_j \) are called the parent variables, denoted \( Pa(X_j) \). Each node \( i \) in the graph is associated with a conditional preference table (CPT) (defining \( \succ_i \) conditionally) that maps all possible assignments to the parents \( Pa(X_i) \) to a total order over \( D_i \). An acyclic CP-net is one that does not contain any
dependency cycles. We denote the language of conditional preferences specified by CP-nets as $\mathcal{L}_{CP}$.

TCP-nets [2] generalize CP-nets by allowing additional edges $(i,j)$ to be specified in the graph describing the relative importance among variables ($X_i \triangleright X_j$). Each relative importance edge could be either unconditional or conditioned on a set of selector variables (analogous to parent variables in the case of intra-variable preferences). Each edge $(i,j)$ describing conditional relative importance is undirected and is associated with a table (analogous to the CPT) mapping each assignment of the selector variables to either $X_i \triangleright X_j$ or vice versa. We denote the language of conditional preferences specified by TCP-nets as $\mathcal{L}_{TCP}$.

An extended preference language due to Wilson [7, 6] (denoted $\mathcal{L}_{Ext}$) allows arbitrary preference statements of the form $y : x \succ_i x'[Z]$ where $X \in X$, $x, x' \in D_X$, $y \in Y \subseteq X \backslash \{X\}$, $Z \subseteq X \backslash \{Y\} \backslash \{X\}$.

We make the following observations:

- $\mathcal{L}$ is neither more expressive nor less expressive compared to $\mathcal{L}_{CP}$. $\mathcal{L}$ allows the expression of relative importance while $\mathcal{L}_{CP}$ does not; and $\mathcal{L}_{CP}$ allows the expression of conditional intra-variable preferences while $\mathcal{L}$ does not.

- $\mathcal{L}$ is less expressive than $\mathcal{L}_{TCP}$ because it does not allow the expression of conditional intra-variable preferences and relative importance.

- When $\mathcal{L}_{TCP}$ is restricted to unconditional intra-variable and unconditional relative importance preferences, its expressiveness is the same as that of $\mathcal{L}$.

- $\mathcal{L}_{Ext}$ is more expressive than $\mathcal{L}_{CP}$ and $\mathcal{L}_{TCP}$ [7, 6], and therefore is more expressive than $\mathcal{L}$ as well.

We next consider several alternative semantics for the unconditional preference language $\mathcal{L}$ in terms of a binary preference relation $\succ$ (dominance) over outcomes, which is derived from the input preferences $\{\succ_i\}$ and $\triangleright$.

## 2 Dominance under Ceteris Paribus Semantics

One of the first formal semantics for preference languages involving conditional intra-variable and relative importance preferences in terms of the *ceteris paribus* interpretation was given by Brafman et al. in [2]. Under this interpretation, the dominance relation $\succ^\circ$ over the set of possible outcomes is defined as any strict partial order that is *consistent* with the input preferences $\{\succ_i\}$ and $\triangleright$ (as given in Definition 6 in [2]). Dominance testing between two outcomes is then cast as a search for an *improving flipping sequence* of outcomes from either outcome to the other. In what follows, we describe dominance testing based on the search for a flipping sequence for the restricted case of language $\mathcal{L}$.

**Definition 1** (Improving flipping sequence: adapted from [2] for the case of unconditional preferences). A sequence of outcomes $\beta = \gamma_1, \gamma_2, \ldots, \gamma_n$, $\gamma_n = \alpha$ such that

$$\alpha = \gamma_n \succ^\circ \gamma_{n-1} \succ^\circ \cdots \succ^\circ \gamma_2 \succ^\circ \gamma_1 = \beta$$
is an improving flipping sequence with respect to a set of preference statements if and only if, for $1 \leq i < n$, either

1. (V-flip) outcome $\gamma_i$ is different from the outcome $\gamma_{i+1}$ in the value of exactly one variable $X_j$, and $\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$.

2. (I-flip) outcome $\gamma_i$ is different from the outcome $\gamma_{i+1}$ in the value of exactly two variables $X_j$ and $X_k$, $\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$, and $X_j \succ_k X_k$.

Note that the notion of an I-flip in this definition revises the one presented in [2] in order to accurately reflect the semantics of $\succ^o$. Furthermore, this definition adapts the original definition to the unconditional case.

The following theorem due to Brafman et al. [2] establishes the equivalence between the existence of a flipping sequence between two outcomes and the dominance relationship with respect to $\succ^o$ between the same outcomes.

**Theorem 1.** [2] Given a set of preference statements $N$ and a pair of outcomes $\alpha$ and $\beta$, we have that $N \models \alpha \succ^o \beta$ if and only if there is an improving flipping sequence with respect to $N$ from $\beta$ to $\alpha$.

The following definition captures the notion of swapping sequence based dominance presented in [7, 6].

**Definition 2** (Worsening swapping sequence : adapted from [7, 6] for the case of unconditional preferences). A sequence of outcomes $\alpha = \gamma_1, \gamma_2, \cdots, \gamma_n = \beta$ such that

$$\alpha = \gamma_1 \succ \gamma_2 \succ \cdots \succ \gamma_{n-1} \succ \gamma_n = \beta$$

is an worsening swapping sequence with respect to a set of preference statements if and only if, for $1 \leq i < n$, either

1. (V-flip) outcome $\gamma_i$ is different from the outcome $\gamma_{i+1}$ in the value of exactly one variable $X_j$, and $\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$.

2. (I-flip) outcome $\gamma_i$ is different from the outcome $\gamma_{i+1}$ in the value of variables $X_j$ and $X_k$, $\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$, and $X_j \succ_k X_k$.

The corresponding theorem relating the existence of a worsening swapping sequence to dominance is as follows.

**Theorem 2.** [7] Given a set of preference statements $N$ and a pair of outcomes $\alpha$ and $\beta$, we have that $N \models \alpha \succ \beta$ if and only if there is a worsening swapping sequence with respect to $N$ from $\alpha$ to $\beta$.

---

1Specifically, Definition 1 relaxes the stronger requirement (see Definition 13 in [2]) that “$\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$ and $\gamma_i(X_k) \succ_k \gamma_{i+1}(X_k)$” to a weaker requirement that “$\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$” – based on a personal communication exchanged by the authors with Ronen Brafman.
3 Our Approach to Dominance Testing

We now present our approach to dominance testing for unconditional intra-variable preferences and relative importance preferences. We define a first order logic formula parameterized by the outcomes $\alpha$ and $\beta$, and preferences $>_i$ and $\triangleright$ such that the satisfiability of the formula determines whether or not $\alpha$ dominates $\beta$. We denote by $>_\ast$ the dominance relation induced by the satisfiability of the formula over outcomes.

We proceed by defining a relation $\succeq_i$ (for each variable $X_i$) that is derived from $>_i$.

**Definition 3 ($\succeq_i$).** $\forall u, v \in D_i : u \succeq_i v \iff u = v \lor u >_i v$

Since $>_i$ is a strict partial order, i.e., irreflexive and transitive, the following property holds for $\succeq_i$.

**Proposition 1.** $\succeq_i$ is reflexive and transitive, i.e., a preorder.

We next define the dominance between any pair of outcomes using a logic formula, for unconditional intra-variable ($>_i$, $\succeq_i$) and relative importance ($\triangleright$) preferences.

**Definition 4 (Dominance with Unconditional Preferences).** Given input preferences $\{>_i, \succeq_i\}$ and $\triangleright$, and a pair of outcomes $\alpha$ and $\beta$, we say that $\alpha$ dominates $\beta$, denoted $\alpha >^* \beta$ whenever the following holds.

$$\alpha >^* \beta \iff \exists X_i : \alpha(X_i) >_i \beta(X_i) \land \forall X_k : (X_k \triangleright X_i \lor X_k \sim \triangleright X_i) \Rightarrow \alpha(X_k) \succeq_k \beta(X_k)$$

$X_i$ is called the witness of the relation.

Intuitively, this definition of dominance of $\alpha$ over $\beta$ (i.e., $\alpha >^* \beta$) requires that with respect to at least one attribute, namely the witness, $\alpha$ is preferred to $\beta$. Further, it requires that for all attributes that are relatively more important or indifferent with respect to importance to the witness, $\alpha$ either equals or is preferred to $\beta$. In Example 2, $\alpha >^* \beta$, with $X_1$ serving as the witness.

### 3.1 Properties of Dominance

We now proceed to analyze some properties of $>_\ast$. Specifically, we would like to ensure that $>_\ast$ has two desirable properties of preference relations: irreflexivity and transitivity, which make it a strict partial order. First, it is easy to see that $>_\ast$ is irreflexive, due to the irreflexivity of $>_i$ (since it is a partial order).

**Proposition 2 (Irreflexivity of $>_\ast$).** $\forall \alpha : \alpha \not>_\ast \alpha$.

The above proposition ensures that the dominance relation $>_\ast$ is strict over compositions. In other words, no composition is preferred over itself. Regarding transitivity, we observe that $>_\ast$ is not transitive when $>_i$ and $\triangleright$ are both arbitrary strict partial orders, as illustrated by the following example.
Example 1. Let $\mathcal{X} = \{X_1, X_2, X_3, X_4\}$, and for each $X_i \in \mathcal{X}$ : $D_i = \{a_i, b_i\}$ with $a_i \succ_i b_i$. Suppose that $X_1 \triangleright X_3$ and $X_2 \triangleright X_4$. Let $\alpha = \langle a_1, a_2, b_3, b_4 \rangle$, $\beta = \langle b_1, a_2, a_3, b_4 \rangle$ and $\gamma = \langle b_1, b_2, a_3, a_4 \rangle$. Clearly, we have $\alpha \succ^* \beta$ (with $X_1$ as witness), $\beta \succ^* \gamma$ (with $X_2$ as witness), but there is no witness for $\alpha \succ^* \gamma$, i.e., $\alpha \not\succ^* \gamma$ according to Definition 4.

Because transitivity of preference is a necessary condition for rational choice [5, 4], we proceed to investigate the possibility of obtaining such a dominance relation by restricting $\triangleright$. In particular, we find that $\succ^*$ is transitive when $\triangleright$ is restricted to a special family of strict partial orders, namely interval orders as defined below. We prove that such a restriction is necessary and sufficient for the transitivity of $\succ^*$.

Definition 5 (Interval Order). A binary relation $R \subseteq \mathcal{X} \times \mathcal{X}$ is an interval order iff it is irreflexive and satisfies the ferrers axiom [3]: for all $X_i, X_j, X_k, X_l \in \mathcal{X}$, we have:

$$
(X_i R X_j \land X_k R X_l) \Rightarrow (X_i R X_l \lor X_k R X_j)
$$

We now proceed to establish the transitivity of $\succ^*$ when $\triangleright$ is an interval order. We make use of two intermediate propositions 3 and 4 that are needed for the task.

In Proposition 3, we prove that if an attribute $X_i$ is relatively more important than $X_j$, then $X_i$ is not more important than a third attribute $X_k$ implies that $X_j$ is also not more important than $X_k$. This will help us prove the transitivity of the dominance relation. Figure 1 illustrates the cases that arise.

Proposition 3. \forall X_i, X_j, X_k : X_i \triangleright X_j \Rightarrow

$$
(X_k \triangleright X_i \lor X_k \sim \triangleright X_i) \Rightarrow (X_k \triangleright X_j \lor X_k \sim \triangleright X_j)
$$

The proof follows from the fact that $\triangleright$ is a partial order.

Proof.

1. $X_i \triangleright X_j$ \hspace{0.5cm} (Hyp.)
2. $X_k \triangleright X_i \lor X_k \sim \triangleright X_i$ \hspace{0.5cm} (Hyp.) Show $X_k \triangleright X_j \lor X_k \sim \triangleright X_j$
   2.1. $X_k \triangleright X_i$. By transitivity of $\triangleright$ and (1.); see Figure 1(a)
Figure 2: $X_i \sim \sqsupset X_j$

2.2. $X_k \sim \sqsupset X_i \Rightarrow X_k \triangleright X_j \lor X_k \sim \sqsupset X_j$

i. $X_k \sim \sqsupset X_i$ (Hyp.)

ii. $(X_k \triangleright X_j) \lor (X_j \triangleright X_k) \lor (X_k \sim \sqsupset X_j)$ Always; see Figure 1(b,c)

iii. $X_j \triangleright X_k \Rightarrow X_i \triangleright X_k$ (1) Contradiction!

iv. $X_k \triangleright X_j \lor X_k \sim \sqsupset X_j$ (2.ii., iii.)

3. $X_i \triangleright X_j \Rightarrow \left( (X_k \triangleright X_i \lor X_k \sim \sqsupset X_i) \Rightarrow (X_k \triangleright X_j \lor X_k \sim \sqsupset X_j) \right)$ (1., 2.1, 2.2)

Proposition 4 states that if attributes $X_i, X_j$ are such that $X_i \sim \sqsupset X_j$ then at least one of them, $X_u$, is such that with respect to the other, $X_v$, there is no attribute $X_k$ that is less important while at the same time $X_k \sim \sqsupset X_u$. This result is needed to establish the transitivity of the dominance relation.

**Proposition 4.** If $\triangleright$ is an interval order, then
\[ \forall X_i, X_j, u \neq v, X_i \sim \sqsupset X_j \Rightarrow \exists X_u, X_v \in \{X_i, X_j\}, \exists X_k : (X_u \sim \sqsupset X_k \land X_v \triangleright X_k). \]

**Proof.** Let $X_i \sim \sqsupset X_j$, and $X_i'$ and $X_j'$ be variables that are less important than $X_i$ and $X_j$, respectively (if any). Figure 2 illustrates all the possible cases that arise. Figure 2(a, b, c, d, e) illustrates the cases when at most one of $X_i'$ and $X_j'$ exists, and in each case the claim holds trivially. For example, in the cases of Figure 2(a, b, c), both $X_u = X_i; X_v = X_j$ and $X_u = X_j; X_v = X_i$ satisfy the implication, and in the cases of Figure 2(d, e), the corresponding satisfactory assignments to $X_u$ and $X_v$ are shown in the figure. The final case (Figure 2(f)) corresponds to $\triangleright$ not being an interval order (see Definition 5). Hence, the proposition holds in all cases.

The above proposition reflects the interval order property of the $\triangleright$ relation, and relates to Example 1 in which $\succ \ast$ was shown to be intransitive when $\triangleright$ is not an interval
order. In fact, if relative importance was defined as a strict partial order instead, it is easy to see that the above proof does not hold. Given that \( \alpha \succ \beta \) with witness \( X_i \) and \( \beta \succ \gamma \) with witness \( X_j \), the above proposition guarantees that one among \( X_i \) and \( X_j \) can be chosen as a potential witness for \( \alpha \succ \gamma \) so that the conditions demonstrated in Example 1 are avoided. Using the propositions 3 and 4, we are now in a position to prove the transitivity of \( \succ \) in Proposition 5.

**Proposition 5** (Transitivity of \( \succ \)). \( \forall \alpha, \beta, \gamma, \alpha \succ \beta \land \beta \succ \gamma \Rightarrow \alpha \succ \gamma \) when \( \succ \) is an interval order.

The proof proceeds by considering all possible relationships between \( X_i, X_j \), the respective attributes that are witnesses of the dominance of \( \alpha \) over \( \beta \) and \( \beta \) over \( \gamma \). Lines 5, 6, 7 in the proof establish the dominance of \( \alpha \) over \( \gamma \) in the cases \( X_i \succ X_j, X_j \succ X_i \) and \( X_i \sim X_j \) respectively. In the first two cases, the more important attribute among \( X_i \) and \( X_j \) is shown to be the witness for \( \alpha \succ \gamma \) with the help of Proposition 3; and in the last case we make use of Proposition 4 to show that at least one of \( X_i, X_j \) is a witness for \( \alpha \succ \gamma \).

**Proof.**

1. \( \alpha \succ \beta \) (Hyp.)
2. \( \beta \succ \gamma \) (Hyp.)
3. \( \exists X_i : \alpha(X_i) \succ_{i} \beta(X_i) \) (1.)
4. \( \exists X_j : \beta(X_j) \succ_{i} \gamma(X_j) \) (2.)
   Three cases arise: \( X_i \succ X_j \) (5.), \( X_j \succ X_i \) (6.) and \( X_i \sim X_j \) (7.).
5. \( X_i \succ X_j \Rightarrow \alpha \succ \gamma \)
6. \( X_i \succ X_i \Rightarrow \alpha \succ \gamma \) (1.)
7. \( X_j \succ X_j \Rightarrow \alpha \succ \gamma \)
8. \( X_i \sim X_j \Rightarrow \alpha \succ \gamma \) (1.)

1. This is true by symmetry of \( X_i, X_j \) in the proof of (5.); in this case, it can easily be shown that \( \alpha(X_j) \succ_{i} \gamma(X_j) \) and \( \forall X_k : (X_k \succ X_j \land X_k \sim X_j) \Rightarrow \alpha(X_k) \succ_{i} \gamma(X_k) \).
Figure 3: A $2 \oplus 2$ substructure, not an Interval Order

7.2. $\exists X_u, X_v \in \{X_i, X_j\} : X_u \neq X_v \land \exists X_k : (X_u \not\succ X_k \land X_v \succ X_k)$ \quad (7.1., Proposition 4)

7.3. Without loss of generality, suppose that $X_u = X_i, X_v = X_j$ \quad (Hyp.)

7.4. $\beta(X_i) \geq \gamma(X_i)$ \quad (2., 7.1.)

7.5. $\alpha(X_i) \succ \gamma(X_i)$ \quad (3., 7.4.)

7.6. $\forall X_k : X_k \not\succ X_i \Rightarrow \alpha(X_k) \not\geq \gamma(X_k)$.

i. $X_k \not\succ X_i$ \quad (Hyp.)

ii. $\alpha(X_k) \not\geq \beta(X_k)$ \quad (1., 7.6.i.)

iii. $X_k \not\succ X_j \lor X_k \not\sim X_j$ \quad Because $X_j \not\succ X_k$ Contradicts (7.1., 7.6.i.)!

iv. $\beta(X_k) \not\geq \gamma(X_k)$ \quad (2., 7.6.iii.)

v. $\alpha(X_k) \not\geq \gamma(X_k)$ \quad (7.6.ii., 7.6.iv.)

7.7. $\forall X_k : X_k \not\sim X_i \Rightarrow \alpha(X_k) \not\geq \gamma(X_k)$

i. $X_k \not\sim X_i$ \quad (Hyp.)

ii. $\alpha(X_k) \not\geq \beta(X_k)$ \quad (1., 7.7.i.)

iii. $X_k \not\succ X_j \lor X_k \not\sim X_j$ \quad Because $X_j \not\succ X_k$ Contradicts (7.2., 7.3.)!

iv. $\beta(X_k) \not\geq \gamma(X_k)$ \quad (2., 7.7.iii.)

v. $\alpha(X_k) \not\geq \gamma(X_k)$ \quad (7.7.ii., 7.7.iv.)

7.8. $\forall X_k : X_k \not\succ X_i \land X_k \not\sim X_i \Rightarrow \alpha(X_k) \not\geq \gamma(X_k)$ \quad (7.6., 7.7.)

7.9. $X_i \not\sim X_j \Rightarrow \alpha \not\succ \gamma$ \quad (7.5., 7.8.)

8. $(X_i \not\succ X_j \land X_i \not\sim X_j \not\succ X_j) \Rightarrow \alpha \not\succ \gamma$ \quad (5., 6., 7.)

9. $\alpha \not\succ \beta \land \beta \not\succ \gamma \Rightarrow \alpha \not\succ \gamma$ \quad (1., 2., 8.)

From Propositions 2 and 5, we have the first main result of this paper as follows.

**Theorem 3.** $\succ$ is a strict partial order when intra-attribute preferences $\succ$, are arbitrary strict partial orders and relative importance $\succ$ is an interval order.

The above theorem applies to all partially ordered intra-variable preferences and a wide range of relative importance preferences including total orders, weak orders and semi orders [3] which are all interval orders. Having seen in Example 1 that the transitivity of $\succ^*$ does not necessarily hold when $\succ$ is an arbitrary partial order, a natural question that arises here is whether there is a condition weaker than the interval order restriction on $\succ$ that still makes $\succ^*$ transitive. The answer turns out to be negative, which we show next. We make use of a characterization of interval orders by Fishburn in [3], which states that $\succ$ is an interval order if and only if $2 \oplus 2 \not\subseteq \succ$, where $2 \oplus 2$ is a relational structure shown in Figure 3. In other words, $\succ$ is an interval order if and only if it has no restriction of itself that is isomorphic to the partial order structure shown in Figure 3.
Theorem 4. For arbitrary partially ordered intra-attribute preferences $\succ^\bullet$ is transitive only if relative importance $\succ$ is an interval order.

Proof. Assume that $\succ$ is not an interval order. This is true if and only if $2 \oplus 2 \subseteq \succ$. However, we showed in Example 1 that in such a case $\succ^\bullet$ is not transitive. Hence, $\succ^\bullet$ is transitive only if relative importance $\succ$ is an interval order. \hfill \square

4 Semantics: Relationship Between $\succ^\circ$, $\succ^\bullet$ & $\succ^\bullet$

We now proceed to investigate the relationship between the classical semantics ($\succ\circ$), our semantics ($\succ\bullet$), and the revised, extended semantics proposed by Wilson ($\succ^\bullet$) for the language $\mathcal{L}$. The main results that we will establish are:

a) $\succ^\bullet \subseteq \succ$

b) $\succ^\bullet = \succ$ when $\succ$ is an interval order

c) $(\succ^\bullet)^* = \succ^\bullet$, where $(\succ^\bullet)^*$ is the transitive closure of $\succ^\bullet$

d) $\succ^\bullet \not\subseteq \succ$ and $\succ \not\subseteq \succ^\bullet$ in general; but $\succ^\circ \subseteq \succ$ when $\succ$ is an interval order

Theorem 5. $\succ^\bullet \subseteq \succ^\bullet$.

Proof. We will show that $\alpha \succ^\bullet \beta \Rightarrow \alpha \succ^\bullet \beta$ for any pair of outcomes $\alpha, \beta$.

Assume $\alpha \succ^\bullet \beta$. By Definition 4, there is a witness $X_i \in \mathcal{X}$ such that:

$\alpha \succ^\bullet \beta \Leftrightarrow \exists X_i : \alpha(X_i) \succ_i \beta(X_i) \land \forall X_k : (X_k \triangleright X_i \lor X_k \bowtie X_i) \Rightarrow \alpha(X_k) \lesssim_k \beta(X_k)$

Define the sets $L = \{X_i : X_i \triangleright X_1\}$, $M = \{X_i : (X_i \triangleright X_1 \lor X_i \bowtie X_1) \land \alpha(X_i) \succ_i \beta(X_i)\}$, and $M' = \{X_i : (X_i \triangleright X_1 \lor X_i \bowtie X_1) \land \alpha(X_i) = \beta(X_i)\}$. Clearly, the sets $\{X_i\}$, $L$, $M$, $M'$ form a partition of $\mathcal{X}$. Let $X_{i1}, X_{i2}, \ldots X_{in}$ be an enumeration of $M$.

We now construct a sequence of outcomes $\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}$ as follows. Construct outcome $\gamma_{i1} = \langle \gamma_{i1}(X_1), \gamma_{i1}(X_2), \ldots, \gamma_{i1}(X_m) \rangle$ such that $\gamma_{i1}(X_{i1}) = \alpha(X_{i1})$ and $\forall X_j \in \mathcal{X} - \{X_i\} : \gamma_{i1}(X_j) = \beta(X_j)$. Similarly construct outcomes $\gamma_{i2}$ corresponding to each $X_{i1}$ as follows:

$\gamma_{i2} = \langle \gamma_{i2}(X_1), \gamma_{i2}(X_2), \ldots, \gamma_{i2}(X_m) \rangle$ such that $\gamma_{i2}(X_{i1}) = \alpha(X_{i1})$; and $\forall X_j \in \mathcal{X} - \{X_{i1}\} : \gamma_{i2}(X_j) = \gamma_{i1-1}(X_j)$.

Now, we make use of Definition 2 to compare the constructed outcomes with respect to $\succ^\bullet$. $\gamma_{i1} \succ^\bullet \beta$ because $\gamma_{i1}(X_{i1}) = \alpha(X_{i1}) \succ_i \beta(X_{i1})$ with $\gamma_{i1}$ and $\beta$ being equal in all variables other than $X_{i1}$. Also $\gamma_{i+1} \succ^\bullet \gamma_{i1}$ because $\gamma_{i+1}(X_{i+1}) = \alpha(X_{i+1}) \succ_i \gamma_{i+1}(X_{i+1}) = \beta(X_{i+1})$, with $\gamma_{i+1}$ and $\gamma_{i1}$ being equal in variables other than $X_{i1}$. At the end of the sequence of constructed outcomes, we have $\alpha \succ^\bullet \gamma_{in}$ because $\alpha(X_1) \succ_i \gamma_{in}(X_1) = \beta(X_1)$ and $\forall X_j \in M \cup M' : \alpha(X_j) = \gamma_{in}(X_j)$, regardless of the assignments to variables $X_j \in L$ (they are less important than $X_i$). Therefore, $\alpha \succ^\bullet \gamma_{in} \succ^\bullet \ldots \succ^\bullet \gamma_{i} \succ^\bullet \beta$.

By the transitivity of $\succ^\bullet$ [7, 6], we have $\alpha \succ^\bullet \beta$ as required. \hfill \square
The above theorem establishes that $\succ^\bullet$ is included in $\succ$. We now investigate whether the other side of inclusion holds.

**Example 1 (continued).** Recall that $\alpha = (a_1, a_2, b_3, b_4)$, $\beta = (a_1, a_2, a_3, b_4)$ and $\gamma = (b_1, b_2, a_3, a_4)$ with $\alpha \succ^\bullet \beta$ (with $X_1$ as witness), $\beta \succ^\bullet \gamma$ (with $X_2$ as witness), but $\alpha \not\succ^\bullet \gamma$ according to Definition 4. However, there exists a sequence of worsening swaps from $\alpha$ to $\gamma$, namely $\alpha, \beta, \gamma$ according to Definition 2. Hence, $\alpha \not\succ^\bullet \gamma$.

This example shows that $\succ^\bullet \subseteq \succ$ does not hold in general. However, observe that $\succ^\bullet$ holds for each consecutive pair of outcomes in the swapping sequence. Hence, $\succ^\bullet$ is transitive, it must be possible to show that $\succ^\bullet \subseteq \succ^\bullet$. The following theorem proves this result using Theorem 3, which relates the interval order property of $\succ$ to the transitivity of $\succ^\bullet$.

**Theorem 6.** $\succ^\bullet \subseteq \succ^\bullet$ when $\succ$ is an interval order.

**Proof.** We show that given a set of conditional variable preferences $\succ_i$ and relative importance $\succ$, $\alpha \succ^\bullet \beta \implies \alpha \succ^\bullet \beta$ when $\succ$ is an interval order.

Let $\alpha \succ^\bullet \beta$. According to Definition 2, there exists a set of outcomes $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n$ such that $\alpha = \gamma_n \succ \gamma_{n-1} \succ \cdots \succ \gamma_2 \succ \gamma_1 \succ \beta$ such that for all $1 \leq i < n$ there is either an improving V-flip or I-flip from $\gamma_i$ to $\gamma_{i+1}$.

**Case 1:** (V-flip) $\gamma_i$ and $\gamma_{i+1}$ differ in the value of exactly one variable $X_j$ and $\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$. With $X_j$ as the witness, the first clause in the definition of $\gamma_{i+1} \succ^\bullet \gamma_i$ is satisfied ($\gamma_{i+1}(X_j) \succ_i \gamma_i(X_j)$). Because $\gamma_{i+1}(X_k) = \gamma_i(X_k)$ for all $X_k \in X - \{X_j\}$, we have $\forall X_k : (X_k \succ X_j \lor X_k \sim X_j) \implies \gamma_{i+1}(X_k) \geq \gamma_i(X_k)$ by Definition 3. Therefore, we have $\gamma_{i+1} \succ^\bullet \gamma_i$ with $X_j$ as the witness.

**Case 2:** (I-flip) $\gamma_i$ and $\gamma_{i+1}$ differ in the value of variables $X_j$ and $X_{k_1}, X_{k_2}, \ldots, X_{k_n}$, and $X_j \succ X_{k_1}, X_j \succ X_{k_2}, \ldots, X_j \succ X_{k_n}$, such that $\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$. With $X_j$ as the witness, the first clause in the definition of $\gamma_{i+1} \succ^\bullet \gamma_i$ is satisfied ($\gamma_{i+1}(X_j) \succ_i \gamma_i(X_j)$).

By Definition 2, $\gamma_{i+1}(X_k) = \gamma_i(X_k)$ for all $X_k \in X - \{X_j, X_{k_1}, X_{k_2}, \ldots, X_{k_n}\}$. In particular, $\gamma_{i+1}(X_k) = \gamma_i(X_k)$ for all $X_k$ such that $X_k \succ X_j \lor X_k \sim X_j$, which means that $\forall X_k : (X_k \succ X_j \lor X_k \sim X_j) \implies \gamma_{i+1}(X_k) \geq \gamma_i(X_k)$ by Definition 3. Therefore, we have $\gamma_{i+1} \succ^\bullet \gamma_i$ with $X_j$ as the witness by Definition 4.

From Cases 1 and 2, $\gamma_{i+1} \succ^\bullet \gamma_i$ for every pair of consecutive outcomes $\gamma_i$ and $\gamma_{i+1}$. Using the fact that $\succ^\bullet$ is transitive when $\succ$ is an interval order (Theorem 3), we have $\alpha \succ^\bullet \beta$ (by Definition 4) when $\succ$ is an interval order. Hence, $\succ^\bullet \subseteq \succ^\bullet$ when $\succ$ is an interval order.

From the fact that $\succ^\bullet$ holds for each pair of consecutive outcomes in a swapping sequence supporting $\alpha \succ^\bullet \beta$, we make the following observation.

**Observation 1.** $(\succ^\bullet)^+ = \succ^\bullet$, where $(\succ^\bullet)^+$ is the transitive closure of $\succ^\bullet$.

Note that this observation holds even when $\succ$ is not an interval order. However, it does not yield a computationally efficient algorithm for dominance testing in general because computing the transitive closure of $\succ^\bullet$ is in itself an expensive operation.
Example 2. Let $X = \{X, Y, Z\}$ and $D_X = \{x_1, x_2\}; D_Y = \{y_1, y_2\}; D_Z = \{z_1, z_2\}$. Suppose that the intra-variable preferences are given by $x_1 \succ_X x_2, y_1 \succ_Y y_2$ and $z_1 \succ_Z z_2$, and the relative importance among the variables is given by $X \triangleright Y$ and $X \triangleright Z$. Given two outcomes $\alpha = (x_1, y_2, z_2)$ and $\beta = (x_2, y_1, z_1)$, there is no improving flipping sequence from $\alpha$ to $\beta$ or vice versa with respect to Definition 1. Therefore, $\alpha \not\succ^\circ \beta$ and $\beta \not\succ^\circ \alpha$.

We now investigate the relationship between $\succ^\circ$ and $\succ^\bullet$. In Example 1, $\gamma$, $\beta$, $\alpha$ forms an improving flipping sequence from $\gamma$ to $\alpha$, resulting in $\alpha \succ^\circ \gamma$ by Definition 1. However, $\alpha \not\succ^\bullet \gamma$. Since $\succ^\bullet$ holds for each pair of consecutive outcomes in a flipping sequence supporting a dominance $\alpha \succ^\circ \beta$, we have $\succ^\circ \subseteq \succ^\bullet$ when $\succ^\bullet$ is transitive. The other side of the inclusion is negated by Example 2, where $\alpha \succ^\bullet \beta$ but $\alpha \not\succ^\circ \beta$. This leads us to the following observation.

Observation 2. $\succ^\bullet \subseteq \succ^\circ$ and $\succ^\circ \subseteq \succ^\bullet$ in general; but $\succ^\circ \subseteq \succ^\bullet$ when $\triangleright$ is an interval order.

References


