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# A Dominance Relation for Unconditional Multi-Attribute Preferences

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## 1 A Language for Unconditional Preferences

Let $\mathcal{X} = \{X_i\}$ be a set of variables, each with a domain $D_i$. An outcome $\alpha \in \mathcal{O}$ is a complete assignment to all the variables, denoted by the tuple $\alpha := \langle \alpha(X_1), \alpha(X_2), \ldots, \alpha(X_m) \rangle$ such that $\alpha(X_i) \in D_i$ for each $X_i \in \mathcal{X}$. The set of all possible outcomes is given by $\mathcal{O} = \prod_{X_i \in \mathcal{X}} D_i$. We consider a preference language $\mathcal{L}$ for specifying: (a) unconditional intra-variable preferences $\succ_i$ that are strict partial orders (i.e., irreflexive and transitive relations) over $D_i$; and (b) unconditional relative importance preferences that are strict partial orders over $\mathcal{X}$.

$\mathcal{L}$ includes unconditional preference statements of the form $x \succ_i x'[\mathcal{Z}]$ such that $x, x' \in D_i$; and $\{X_i\} \notin \mathcal{Z}$. Here, the set $\mathcal{Z}$ of variables are relatively less important than $X_i$, i.e., $X_i \rhd X_j$ for each $X_j \in \mathcal{Z}$. However, the language $\mathcal{L}$ does not include statements specifying conditional relative importance, i.e., if $x \succ_i x'[\mathcal{Z}_1]$ and $x_1 \succ_i x_1'[\mathcal{Z}_2]$ in $\mathcal{L}$ then $\mathcal{Z}_1 = \mathcal{Z}_2$. Additionally, because $\rhd$ is assumed to be a binary (strict partial order) relation, $|\mathcal{Z}| = 0$ or $1$. We now compare the expressiveness of $\mathcal{L}$ to that of some well known preference languages.

### 1.1 Expressiveness

CP-nets [1] use a compact graphical model to specify conditional intra-variable preferences $\succ_i$ over a set of variables $\mathcal{X}$. Each node $i$ in the graph corresponds to a variable $X_i \in \mathcal{X}$, and each edge $(i, j)$ in the graph captures the fact that the intra-variable preference $\succ_j$ with respect to variable $X_j$ is conditioned (or dependent) on the valuation of $X_i$. For any variable $X_j$ (corresponding to node $j$), the set of variables $\{X_i : (i, j) \text{is an edge}\}$ that influence $\succ_j$ are called the *parent* variables, denoted $Pa(X_j)$. Each node $i$ in the graph is associated with a *conditional preference table* (CPT) (defining $\succ_i$ conditionally) that maps all possible assignments to the parents $Pa(X_i)$ to a total order over $D_i$. An *acyclic* CP-net is one that does not contain any dependency cycles. We denote the language of conditional preferences specified by CP-nets as $\mathcal{L}_{CP}$.

TCP-nets [2] generalize CP-nets by allowing additional edges $(i, j)$ to be specified in the graph describing the relative importance among variables ($X_i \rhd X_j$). Each relative importance edge could be either unconditional or conditioned on a set of *selector* variables (analogous to parent variables in the case of intra-variable preferences). Each edge $(i, j)$ describing conditional relative importance is undirected and is associated with a table (analogous to the CPT) mapping each assignment of the selector variables to either $X_i \rhd X_j$ or vice versa. We denote the language of conditional preferences specified by TCP-nets as $\mathcal{L}_{TCP}$.

An extended preference language due to Wilson [7, 6] (denoted $\mathcal{L}_{Ext}$) allows arbitrary preference statements of the form $y : x \succ_i x'[\mathcal{Z}]$ where $X \in \mathcal{X}$, $x, x' \in D_X$, $y \in \mathcal{Y} \subseteq \mathcal{X} \backslash \{X\}$, $\mathcal{Z} \subseteq \mathcal{X} \backslash \mathcal{Y} \backslash \{X\}$.

We make the following observations:

- $\mathcal{L}$ is neither more expressive nor less expressive compared to $\mathcal{L}_{CP}$. $\mathcal{L}$ allows the expression of relative importance while $\mathcal{L}_{CP}$ does not; and $\mathcal{L}_{CP}$ allows the expression of conditional intra-variable preferences while $\mathcal{L}$ does not.

- $\mathcal{L}$ is less expressive than $\mathcal{L}_{TCP}$ because it does not allow the expression of conditional intra-variable preferences and relative importance.

- When $\mathcal{L}_{TCP}$ is restricted to unconditional intra-variable and unconditional relative importance preferences, its expressiveness is the same as that of $\mathcal{L}$.

- $\mathcal{L}_{Ext}$ is more expressive than $\mathcal{L}_{CP}$ and $\mathcal{L}_{TCP}$ [7, 6], and therefore is more expressive than $\mathcal{L}$ as well.

We next consider several alternative semantics for the unconditional preference language $\mathcal{L}$ in terms of a binary preference relation $\succ$ (dominance) over outcomes, which is derived from the input preferences $\{\succ_i\}$ and $\rhd$.

## 2 Dominance under *Ceteris Paribus* Semantics

One of the first formal semantics for preference languages involving conditional intra-variable and relative importance preferences in terms of the *ceteris paribus* interpretation was given by Brafman et al. in [2]. Under this interpretation, the dominance relation $\succ^\circ$ over the set of possible outcomes is defined as *any* strict partial order that is *consistent* with the input preferences $\{\succ_i\}$ and $\rhd$ (as given in Definition 6 in [2]). Dominance testing between two outcomes is then cast as a search for an *improving flipping sequence* of outcomes from either outcome to the other. In what follows, we describe dominance testing based on the search for a flipping sequence for the restricted case of language $\mathcal{L}$.

**Definition 1** (Improving flipping sequence: adapted from [2] for the case of unconditional preferences)**.** *A sequence of outcomes* $\beta = \gamma_1, \gamma_2, \cdots \gamma_{n-1}, \gamma_n = \alpha$ *such that*

$$\alpha = \gamma_n \succ^\circ \gamma_{n-1} \succ^\circ \cdots \succ^\circ \gamma_2 \succ^\circ \gamma_1 = \beta$$

*is an **improving flipping sequence** with respect to a set of preference statements if and only if, for $1 \leq i < n$, either*

1. *(V-flip) outcome $\gamma_i$ is different from the outcome $\gamma_{i+1}$ in the value of exactly one variable $X_j$, and $\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$, or*

2. *(I-flip) outcome $\gamma_i$ is different from the outcome $\gamma_{i+1}$ in the value of exactly **two** variables $X_j$ **and** $X_k$, $\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$, and $X_j \rhd X_k$.*

Note that the notion of an I-flip in this definition revises the one presented in [2] in order to accurately reflect the semantics of $\succ^\circ$ [1]. Furthermore, this definition adapts the original definition to the unconditional case.

The following theorem due to Brafman et al. [2] establishes the equivalence between the existence of a flipping sequence between two outcomes and the dominance relationship with respect to $\succ^\circ$ between the same outcomes.

**Theorem 1.** *[2] Given a set of preference statements $N$ and a pair of outcomes $\alpha$ and $\beta$, we have that $N \models \alpha \succ^\circ \beta$ iff there is an improving flipping sequence with respect to $N$ from $\beta$ to $\alpha$.*

The following definition captures the notion of swapping sequence based dominance presented in [7, 6].

**Definition 2** (Worsening swapping sequence : adapted from [7, 6] for the case of unconditional preferences)**.** *A sequence of outcomes $\alpha = \gamma_1, \gamma_2, \cdots \gamma_{n-1}, \gamma_n = \beta$ such that*

$$\alpha = \gamma_1 \succ^{\blacksquare} \gamma_2 \succ^{\blacksquare} \cdots \succ^{\blacksquare} \gamma_{n-1} \succ^{\blacksquare} \gamma_n = \beta$$

*is an **worsening swapping sequence** with respect to a set of preference statements if and only if, for $1 \leq i < n$, either*

1. *(V-flip) outcome $\gamma_i$ is different from the outcome $\gamma_{i+1}$ in the value of exactly one variable $X_j$, and $\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$, or*

2. *(I-flip) outcome $\gamma_i$ is different from the outcome $\gamma_{i+1}$ in the value of variables $X_j$ **and** $X_{k_1}, X_{k_2}, \cdots X_{k_n}$, $\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$, and $X_j \rhd X_{k_1}, X_j \rhd X_{k_2}, \cdots, X_j \rhd X_{k_n}$.*

The corresponding theorem relating the existence of a worsening swapping sequence to dominance is as follows.

**Theorem 2.** *[7] Given a set of preference statements $N$ and a pair of outcomes $\alpha$ and $\beta$, we have that $N \models \alpha \succ^{\blacksquare} \beta$ iff there is a worsening swapping sequence with respect to $N$ from $\alpha$ to $\beta$.*

[1]Specifically, Definition 1 relaxes the stronger requirement (see Definition 13 in [2]) that "$\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$ and $\gamma_i(X_k) \succ_k \gamma_{i+1}(X_k)$" to a weaker requirement that "$\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$" – based on a personal communication exchanged by the authors with Ronen Brafman .

# 3 Our Approach to Dominance Testing

We now present our approach to dominance testing for unconditional intra-variable preferences and relative importance preferences. We define a first order logic formula parameterized by the outcomes $\alpha$ and $\beta$, and preferences $\succ_i$ and $\rhd$ such that the satisfiability of the formula determines whether or not $\alpha$ dominates $\beta$. We denote by $\succ^\bullet$ the dominance relation induced by the satisfiability of the formula over outcomes.

We proceed by defining a relation $\succeq_i$ (for each variable $X_i$) that is derived from $\succ_i$.

**Definition 3** ( $\succeq_i$). $\forall u, v \in D_i : u \succeq_i v \Leftrightarrow u = v \vee u \succ_i v$

Since $\succ_i$ is a strict partial order, i.e., irreflexive and transitive, the following property holds for $\succeq_i$.

**Proposition 1.** *$\succeq_i$ is reflexive and transitive, i.e., a preorder.*

We next define the dominance between any pair of outcomes using a logic formula, for unconditional intra-variable ($\succ_i$, $\succeq_i$) and relative importance ($\rhd$) preferences.

**Definition 4** (Dominance with Unconditional Preferences)**.** *Given input preferences $\{\succ_i\}$ and $\rhd$, and a pair of outcomes $\alpha$ and $\beta$, we say that $\alpha$* ***dominates*** *$\beta$, denoted $\alpha \succ^\bullet \beta$ whenever the following holds.*

$$\begin{aligned} \alpha \succ^\bullet \beta \Leftrightarrow \quad & \exists X_i \ : \alpha(X_i) \succ_i \beta(X_i) \wedge \\ & \forall X_k \ : \ (X_k \rhd X_i \vee X_k \sim_\rhd X_i) \\ & \qquad \qquad \Rightarrow \ \alpha(X_k) \ \succeq_k \ \beta(X_k) \end{aligned}$$

*$X_i$ is called the witness of the relation.*

Intuitively, this definition of dominance of $\alpha$ over $\beta$ (i.e., $\alpha \succ^\bullet \beta$) requires that with respect to at least one attribute, namely the witness, $\alpha$ is *preferred* to $\beta$. Further, it requires that for all attributes that are relatively more important or indifferent with respect to importance to the witness, $\alpha$ *either equals or is preferred to* $\beta$. In Example 2, $\alpha \succ^\bullet \beta$, with $X_1$ serving as the witness.

## 3.1 Properties of Dominance

We now proceed to analyze some properties of $\succ^\bullet$. Specifically, we would like to ensure that $\succ^\bullet$ has two desirable properties of preference relations: irreflexivity and transitivity, which make it a strict partial order. First, it is easy to see that $\succ^\bullet$ is irreflexive, due to the irreflexivity of $\succ_i$ (since it is a partial order).

**Proposition 2** (Irreflexivity of $\succ^\bullet$)**.** $\forall \alpha : \alpha \not\succ^\bullet \ \alpha$.

The above proposition ensures that the dominance relation $\succ^\bullet$ is strict over compositions. In other words, no composition is preferred over itself. Regarding transitivity, we observe that $\succ^\bullet$ is not transitive when $\succ_i$ and $\rhd$ are both arbitrary strict partial orders, as illustrated by the following example.

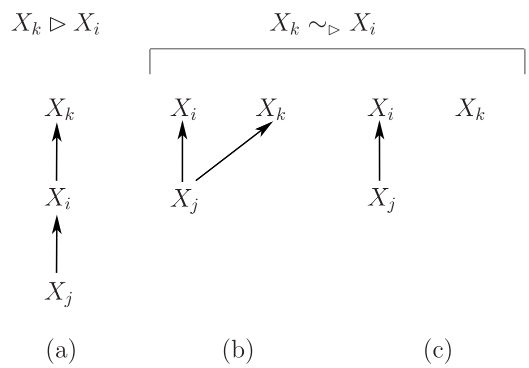


Figure 1: $X_i \rhd X_j \wedge (X_k \rhd X_i \vee X_k \sim_\rhd X_i)$

**Example 1.** *Let $\mathcal{X} = \{X_1, X_2, X_3, X_4\}$, and for each $X_i \in \mathcal{X} : D_i = \{a_i, b_i\}$ with $a_i \succ_i b_i$. Suppose that $X_1 \rhd X_3$ and $X_2 \rhd X_4$. Let $\alpha = \langle a_1, a_2, b_3, b_4\rangle$, $\beta = \langle b_1, a_2, a_3, b_4\rangle$ and $\gamma = \langle b_1, b_2, a_3, a_4\rangle$. Clearly, we have $\alpha \succ^\bullet \beta$ (with $X_1$ as witness), $\beta \succ^\bullet \gamma$ (with $X_2$ as witness), but there is no witness for $\alpha \succ^\bullet \gamma$, i.e., $\alpha \not\succ^\bullet \gamma$ according to Definition 4.*

Because transitivity of preference is a necessary condition for rational choice [5, 4], we proceed to investigate the possibility of obtaining such a dominance relation by restricting $\rhd$. In particular, we find that $\succ^\bullet$ is transitive when $\rhd$ is restricted to a special family of strict partial orders, namely *interval orders* as defined below. We prove that such a restriction is necessary and sufficient for the transitivity of $\succ^\bullet$.

**Definition 5** (Interval Order)**.** *A binary relation $\boldsymbol{R} \subseteq \mathcal{X} \times \mathcal{X}$ is an interval order iff it is irreflexive and satisfies the* ferrers *axiom [3]: for all $X_i, X_j, X_k, X_l \in \mathcal{X}$, we have:*
$(X_i \,\boldsymbol{R}\, X_j \wedge X_k \,\boldsymbol{R}\, X_l) \Rightarrow (X_i \,\boldsymbol{R}\, X_l \vee X_k \,\boldsymbol{R}\, X_j)$

We now proceed to establish the transitivity of $\succ^\bullet$ when $\rhd$ is an interval order. We make use of two intermediate propositions 3 and 4 that are needed for the task.

In Proposition 3, we prove that if an attribute $X_i$ is relatively more important than $X_j$, then $X_i$ is not more important than a third attribute $X_k$ implies that $X_j$ is also not more important than $X_k$. This will help us prove the transitivity of the dominance relation. Figure 1 illustrates the cases that arise.

**Proposition 3.** $\forall X_i, X_j, X_k : X_i \rhd X_j \Rightarrow$
$\Big((X_k \rhd X_i \vee X_k \sim_\rhd X_i) \Rightarrow (X_k \rhd X_j \vee X_k \sim_\rhd X_j)\Big)$

The proof follows from the fact that $\rhd$ is a partial order.

*Proof.*

1. $X_i \rhd X_j \quad (Hyp.)$
2. $X_k \rhd X_i \vee X_k \sim_\rhd X_i \quad (Hyp.)$ Show $X_k \rhd X_j \vee X_k \sim_\rhd X_j$
   2.1. $X_k \rhd X_i \Rightarrow X_k \rhd X_j$ By transitivity of $\rhd$ and (1.); see Figure 1(a)

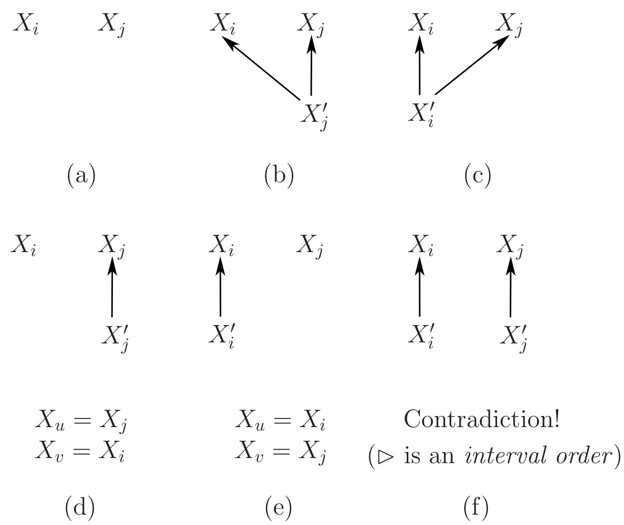


Figure 2: $X_i \sim_\triangleright X_j$

2.2. $X_k \sim_\triangleright X_i \Rightarrow X_k \triangleright X_j \vee X_k \sim_\triangleright X_j$

i. $X_k \sim_\triangleright X_i \quad (Hyp.)$

ii. $(X_k \triangleright X_j) \vee (X_j \triangleright X_k) \vee \; (X_k \sim_\triangleright X_j)$ Always; see Figure 1(b,c)

iii. $X_j \triangleright X_k \Rightarrow X_i \triangleright X_k \quad (1.)$ Contradiction!

iv. $X_k \triangleright X_j \vee X_k \sim_\triangleright X_j \quad (2.2.ii., iii.)$

3. $X_i \triangleright X_j \Rightarrow \Big( (X_k \triangleright X_i \vee X_k \sim_\triangleright X_i) \Rightarrow (X_k \triangleright X_j \vee X_k \sim_\triangleright X_j) \Big) \quad (1., 2.1, 2.2)$

□

Proposition 4 states that if attributes $X_i, X_j$ are such that $X_i \sim_\triangleright X_j$ then at least one of them, $X_u$ is such that with respect to the other, $X_v$, there is no attribute $X_k$ that is less important while at the same time $X_k \sim_\triangleright X_u$. This result is needed to establish the transitivity of the dominance relation.

**Proposition 4.** *If $\triangleright$ is an interval order, then*
$\forall X_i, X_j, u \neq v, X_i \sim_\triangleright X_j$
$\Rightarrow \exists X_u, X_v \in \{X_i, X_j\}, \nexists X_k : (X_u \sim_\triangleright X_k \wedge X_v \triangleright X_k)$.

*Proof.* Let $X_i \sim_\triangleright X_j$, and $X'_i$ and $X'_j$ be variables that are less important than $X_i$ and $X_j$ respectively (if any). Figure 2 illustrates all the possible cases that arise. Figure 2(a, b, c, d, e) illustrates the cases when *at most* one of $X'_i$ and $X'_j$ exists, and in each case the claim holds trivially. For example, in the cases of Figure 2(a, b, c), both $X_u = X_i; X_v = X_j$ and $X_u = X_j; X_v = X_i$ satisfy the implication, and in the cases of Figure 2(d, e), the corresponding satisfactory assignments to $X_u$ and $X_v$ are shown in the figure. The final case (Figure 2(f)) corresponds to $\triangleright$ not being an interval order (see Definition 5). Hence, the proposition holds in all cases. □

The above proposition reflects the interval order property of the $\triangleright$ relation, and relates to Example 1 in which $\succ^\bullet$ was shown to be intransitive when $\triangleright$ is not an interval order. In fact, if relative importance was defined as a strict partial order instead, it is easy to see that the above proof does not hold. Given that $\alpha \succ^\bullet \beta$ with witness $X_i$ and $\beta \succ^\bullet \gamma$ with witness $X_j$, the above proposition guarantees that one among $X_i$ and $X_j$ can be chosen as a potential witness for $\alpha \succ^\bullet \gamma$ so that the conditions demonstrated in Example 1 are avoided. Using the propositions 3 and 4, we are now in a position to prove the transitivity of $\succ^\bullet$ in Proposition 5.

**Proposition 5** (Transitivity of $\succ^\bullet$)**.** *$\forall \alpha, \beta, \gamma$,*
*$\alpha \succ^\bullet \beta \wedge \beta \succ^\bullet \gamma \Rightarrow \alpha \succ^\bullet \gamma$ when $\rhd$ is an interval order.*

The proof proceeds by considering all possible relationships between $X_i, X_j$, the respective attributes that are *witnesses* of the dominance of $\alpha$ over $\beta$ and $\beta$ over $\gamma$. Lines $5, 6, 7$ in the proof establish the dominance of $\alpha$ over $\gamma$ in the cases $X_i \rhd X_j$, $X_j \rhd X_i$ and $X_i \sim_\rhd X_j$ respectively. In the first two cases, the more important attribute among $X_i$ and $X_j$ is shown to be the witness for $\alpha \succ^\bullet \gamma$ with the help of Proposition 3; and in the last case we make use of Proposition 4 to show that at least one of $X_i, X_j$ is a witness for $\alpha \succ^\bullet \gamma$.

*Proof.*

1. $\alpha \succ^\bullet \beta \quad (Hyp.)$
2. $\beta \succ^\bullet \gamma \quad (Hyp.)$
3. $\exists X_i \ : \ \alpha(X_i) \succ'_i \beta(X_i) \quad (1.)$
4. $\exists X_j \ : \ \beta(X_j) \succ'_j \gamma(X_j) \quad (2.)$
   Three cases arise: $X_i \rhd X_j (5.)$, $X_j \rhd X_i (6.)$ and $X_i \sim_\rhd X_j (7.)$.
5. $X_i \rhd X_j \Rightarrow \alpha \succ^\bullet \gamma$
   5.1. $X_i \rhd X_j \quad (Hyp.)$
   5.2. $\beta(X_i) \succeq'_i \gamma(X_i) \quad (2., 5.1.)$
   5.3. $\alpha(X_i) \succ'_i \gamma(X_i) \quad (3., 5.2.)$
   5.4. $\forall X_k : (X_k \rhd X_i \vee X_k \sim_\rhd X_i) \Rightarrow \alpha(X_k) \succeq'_k \gamma(X_k)$
      i. Let $X_k \rhd X_i \vee X_k \sim_\rhd X_i \quad (Hyp.)$
      ii. $\alpha(X_k) \succeq'_k \beta(X_k) \quad (1., 5.4.i.)$
      iii. $X_k \rhd X_j \vee X_k \sim_\rhd X_j \quad (5.4.i., Proposition\ 3)$
      iv. $\beta(X_k) \succeq'_k \gamma(X_k) \quad (2., 5.4.iii.)$
      v. $\alpha(X_k) \succeq'_k \gamma(X_k) \quad (5.4.ii., 5.4.iv.)$
   5.5. $X_i \rhd X_j \Rightarrow \alpha \succ^\bullet \gamma \quad (5.1., 5.3., 5.4.)$
6. $X_j \rhd X_i \Rightarrow \alpha \succ^\bullet \gamma$
   6.1. This is true by symmetry of $X_i, X_j$ in the proof of $(5.)$; in this case, it can easily be shown that $\alpha(X_j) \succ'_i \gamma(X_j)$ and $\forall X_k : (X_k \rhd X_j \vee X_k \sim_\rhd X_j) \Rightarrow \alpha(X_k) \succeq'_k \gamma(X_k)$.
7. $X_i \sim_\rhd X_j \Rightarrow \alpha \succ^\bullet \gamma$
   7.1. $X_i \sim_\rhd X_j \quad (Hyp.)$

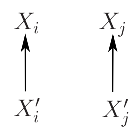


Figure 3: A $2 \oplus 2$ substructure, not an Interval Order

7.2. $\exists X_u, X_v \in \{X_i, X_j\} : X_u \neq X_v \wedge \nexists X_k : (X_u \sim_{\rhd} X_k \wedge X_v \rhd X_k) \quad (7.1., Proposition\ 4)$

7.3. Without loss of generality, suppose that $X_u = X_i, X_v = X_j \quad (Hyp.)$.

7.4. $\beta(X_i) \succeq'_i \gamma(X_i) \quad (2., 7.1.)$

7.5. $\alpha(X_i) \succ'_i \gamma(X_i) \quad (3., 7.4.)$

7.6. $\forall X_k : X_k \rhd X_i \Rightarrow \alpha(X_k) \succeq'_k \gamma(X_k)$.

   i. $X_k \rhd X_i \quad (Hyp.)$

   ii. $\alpha(X_k) \succeq'_k \beta(X_k) \quad (1., 7.6.i.)$

   iii. $X_k \rhd X_j \vee X_k \sim_{\rhd} X_j$ Because $X_j \rhd X_k$ Contradicts $(7.1., 7.6.i.)$!

   iv. $\beta(X_k) \succeq'_k \gamma(X_k) \quad (2., 7.6.iii.)$

   v. $\alpha(X_k) \succeq'_k \gamma(X_k) \quad (7.6.ii., 7.6.iv.)$

7.7. $\forall X_k : X_k \sim_{\rhd} X_i \Rightarrow \alpha(X_k) \succeq'_k \gamma(X_k)$

   i. $X_k \sim_{\rhd} X_i \quad (Hyp.)$

   ii. $\alpha(X_k) \succeq'_k \beta(X_k) \quad (1., 7.7.i.)$

   iii. $X_k \rhd X_j \vee X_k \sim_{\rhd} X_j$ Because $X_j \rhd X_k$ Contradicts $(7.2., 7.3.)$!

   iv. $\beta(X_k) \succeq'_k \gamma(X_k) \quad (2., 7.7.iii.)$

   v. $\alpha(X_k) \succeq'_k \gamma(X_k) \quad (7.7.ii., 7.7.iv.)$

7.8. $\forall X_k : X_k \rhd X_i \vee X_k \sim_{\rhd} X_i \Rightarrow \alpha(X_k) \succeq'_k \gamma(X_k) \quad (7.6., 7.7.)$

7.9. $X_i \sim_{\rhd} X_j \Rightarrow \alpha \succ^{\bullet} \gamma \quad (7.5., 7.8.)$

8. $(X_i \rhd X_j \vee X_j \rhd X_i \vee X_i \sim_{\rhd} X_j) \Rightarrow \alpha \succ^{\bullet} \gamma \quad (5., 6., 7.)$

9. $\alpha \succ^{\bullet} \beta \wedge \beta \succ^{\bullet} \gamma \Rightarrow \alpha \succ^{\bullet} \gamma \quad (1., 2., 8.)$ $\square$

From Propositions 2 and 5, we have the first main result of this paper as follows.

**Theorem 3.** *$\succ^{\bullet}$ is a strict partial order when intra-attribute preferences $\succ_i$ are arbitrary strict partial orders and relative importance $\rhd$ is an interval order.*

The above theorem applies to all partially ordered intra-variable preferences and a wide range of relative importance preferences including total orders, weak orders and semi orders [3] which are all interval orders. Having seen in Example 1 that the transitivity of $\succ^{\bullet}$ does not necessarily hold when $\rhd$ is an arbitrary partial order, a natural question that arises here is whether there is a condition *weaker* than the interval order restriction on $\rhd$ that still makes $\succ^{\bullet}$ transitive. The answer turns out to be negative, which we show next. We make use of a characterization of interval orders by Fishburn in [3], which states that $\rhd$ is an interval order if and only if $2 \oplus 2 \nsubseteq \rhd$, where $2 \oplus 2$ is a relational structure shown in Figure 3. In other words, $\rhd$ is an interval order if and only if it has *no restriction of itself* that is isomorphic to the partial order structure shown in Figure 3.

**Theorem 4.** *For arbitrary partially ordered intra-attribute preferences $\succ^\bullet$ is transitive only if relative importance $\rhd$ is an interval order.*

*Proof.* Assume that $\rhd$ is not an interval order. This is true if and only if $2 \oplus 2 \subseteq \rhd$. However, we showed in Example 1 that in such a case $\succ^\bullet$ is not transitive. Hence, $\succ^\bullet$ is transitive only if relative importance $\rhd$ is an interval order. □

# 4 Semantics: Relationship Between $\succ^\circ$, $\succ^\blacksquare$ & $\succ^\bullet$

We now proceed to investigate the relationship between the classical semantics ($\succ^\circ$), our semantics ($\succ^\bullet$), and the revised, extended semantics proposed by Wilson ($\succ^\blacksquare$) for the language $\mathcal{L}$. The main results that we will establish are:

a) $\succ^\bullet \subseteq \succ^\blacksquare$

b) $\succ^\bullet = \succ^\blacksquare$ when $\rhd$ is an interval order

c) $(\succ^\bullet)^\star = \succ^\blacksquare$, where $(\succ^\bullet)^\star$ is the transitive closure of $\succ^\bullet$

d) $\succ^\bullet \not\subseteq \succ^\circ$ and $\succ^\circ \not\subseteq \succ^\bullet$ in general; but $\succ^\circ \subseteq \succ^\bullet$ when $\rhd$ is an interval order

**Theorem 5.** $\succ^\bullet \subseteq \succ^\blacksquare$.

*Proof.* We will show that $\alpha \succ^\bullet \beta \Rightarrow \alpha \succ^\blacksquare \beta$ for any pair of outcomes $\alpha, \beta$.

Assume $\alpha \succ^\bullet \beta$. By Definition 4, there is a witness $X_i \in \mathcal{X}$ such that:

$$\begin{aligned} \alpha \succ^\bullet \beta \Leftrightarrow \quad & \exists X_i \,:\, \alpha(X_i) \succ_i \beta(X_i) \;\wedge \\ & \forall X_k \,:\, (X_k \rhd X_i \vee X_k \sim_\rhd X_i) \\ & \qquad\qquad \Rightarrow \; \alpha(X_k) \succeq_k \beta(X_k) \end{aligned}$$

Define the sets $L = \{X_l \;:\; X_i \rhd X_l\}$, $M = \{X_l \;:\; (X_l \rhd X_i \vee X_l \sim_\rhd X_i) \wedge \alpha(X_l) \succ_l \beta(X_l)\}$, and $M' = \{X_l \,:\, (X_l \rhd X_i \vee X_l \sim_\rhd X_i) \wedge \alpha(X_l) = \beta(X_l)\}$. Clearly, the sets $\{X_i\}$, $L$, $M$, $M'$ form a partition of $\mathcal{X}$. Let $X_{t1}, X_{t2}, \ldots X_{tn}$ be an enumeration of $M$.

We now construct a sequence of outcomes $\gamma_{t1}, \gamma_{t2}, \ldots, \gamma_{tn}$ as follows. Construct outcome $\gamma_{t1} = \langle \gamma_{t1}(X_1), \gamma_{t1}(X_2), \ldots \gamma_{t1}(X_m) \rangle$ such that $\gamma_{t1}(X_{t1}) = \alpha(X_{t1})$ and $\forall X_j \in \mathcal{X} - \{X_{t1}\} : \gamma_{t1}(X_j) = \beta(X_j)$. Similarly construct outcomes $\gamma_{ti}$ corresponding to each $X_{ti}$ as follows:
$\gamma_{ti} = \langle \gamma_{ti}(X_1), \gamma_{ti}(X_2), \ldots \gamma_{ti}(X_m) \rangle$ such that $\gamma_{ti}(X_{ti}) = \alpha(X_{ti})$; and $\forall X_j \in \mathcal{X} - \{X_{ti}\} : \gamma_{ti}(X_j) = \gamma_{ti-1}(X_j)$.

Now, we make use of Definition 2 to compare the constructed outcomes with respect to $\succ^\blacksquare$. $\gamma_{t1} \succ^\blacksquare \beta$ because $\gamma_{t1}(X_{t1}) = \alpha(X_{t1}) \succ_{t1} \beta(X_{t1})$ with $\gamma_{t1}$ and $\beta$ being equal in all variables other than $X_{t1}$. Also $\gamma_{ti+1} \succ^\blacksquare \gamma_{ti}$ because $\gamma_{ti+1}(X_{ti}) = \alpha(X_{ti}) \succ_{ti} \gamma_{ti}(X_{ti}) = \beta(X_{ti})$, with $\gamma_{ti+1}$ and $\gamma_{ti}$ being equal in variables other than $X_{ti}$. At the end of the sequence of constructed outcomes, we have $\alpha \succ^\blacksquare \gamma_{tn}$ because $\alpha(X_i) \succ_i \gamma_{tn}(X_i) = \beta(X_i)$ and $\forall X_l \in M \cup M' : \alpha(X_l) = \gamma_{tn}(X_l)$, regardless of the assignments to variables $X_j \in L$ (they are less important than $X_i$). Therefore, $\alpha \succ^\blacksquare \gamma_{tn} \succ^\blacksquare \ldots \succ^\blacksquare \gamma_1 \succ^\blacksquare \beta$.

By the transitivity of $\succ^\blacksquare$ [7, 6], we have $\alpha \succ^\blacksquare \beta$ as required. □

The above theorem establishes that $\succ^{\bullet}$ is included in $\succ^{\blacksquare}$. We now investigate whether the other side of inclusion holds.

**Example 1 (continued).** *Recall that $\alpha = \langle a_1, a_2, b_3, b_4 \rangle$, $\beta = \langle b_1, a_2, a_3, b_4 \rangle$ and $\gamma = \langle b_1, b_2, a_3, a_4 \rangle$ with $\alpha \succ^{\bullet} \beta$ (with $X_1$ as witness), $\beta \succ^{\bullet} \gamma$ (with $X_2$ as witness), but $\alpha \not\succ^{\bullet} \gamma$ according to Definition 4. However, there exists a sequence of worsening swaps from $\alpha$ to $\gamma$, namely $\alpha$, $\beta$, $\gamma$ according to Definition 2. Hence, $\alpha \succ^{\blacksquare} \gamma$.*

This example shows that $\succ^{\blacksquare} \subseteq \succ^{\bullet}$ does not hold in general. However, observe that $\succ^{\bullet}$ holds for each consecutive pair of outcomes in the swapping sequence. Hence, $\succ^{\bullet}$ is transitive, it must be possible to show that $\succ^{\blacksquare} \subseteq \succ^{\bullet}$. The following theorem proves this result using Theorem 3, which relates the interval order property of $\rhd$ to the transitivity of $\succ^{\bullet}$.

**Theorem 6.** *$\succ^{\blacksquare} \subseteq \succ^{\bullet}$ when $\rhd$ is an interval order.*

*Proof.* We show that given a set of conditional variable preferences $\succ_i$ and relative importance $\rhd$, $\alpha \succ^{\blacksquare} \beta \Rightarrow \alpha \succ^{\bullet} \beta$ when $\rhd$ is an interval order.

Let $\alpha \succ^{\blacksquare} \beta$. According to Definition 2, there exists a set of outcomes $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}, \gamma_n$ such that $\alpha = \gamma_n \succ^{\blacksquare} \gamma_{n-1} \succ^{\blacksquare} \cdots \succ^{\blacksquare} \gamma_2 \succ^{\blacksquare} \gamma_1 = \beta$ such that for all $1 \leq i < n$ there is either an improving *V-flip* or *I-flip* from $\gamma_i$ to $\gamma_{i+1}$.

*Case 1: (V-flip)* $\gamma_i$ and $\gamma_{i+1}$ differ in the value of exactly one variable $X_j$ and $\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$. With $X_j$ as the witness, the first clause in the definition of $\gamma_{i+1} \succ^{\bullet} \gamma_i$ is satisfied ($\gamma_{i+1}(X_j) \succ_i \gamma_i(X_j)$). Because $\gamma_{i+1}(X_k) = \gamma_i(X_k)$ for all $X_k \in \mathcal{X} - \{X_j\}$, we have $\forall X_k : (X_k \rhd X_j \vee X_k \sim_{\rhd} X_j) \Rightarrow \gamma_{i+1}(X_k) \succeq_k \gamma_i(X_k)$ by Definition 3. Therefore, we have $\gamma_{i+1} \succ^{\bullet} \gamma_i$ with $X_j$ as the witness.

*Case 2: (I-flip)* $\gamma_i$ and $\gamma_{i+1}$ differ in the value of variables $X_j$ and $X_{k_1}, X_{k_2}, \cdots X_{k_n}$, and $X_j \rhd X_{k_1}, X_j \rhd X_{k_2}, \cdots, X_j \rhd X_{k_n}$, such that $\gamma_{i+1}(X_j) \succ_j \gamma_i(X_j)$. With $X_j$ as the witness, the first clause in the definition of $\gamma_{i+1} \succ^{\bullet} \gamma_i$ is satisfied ($\gamma_{i+1}(X_j) \succ_i \gamma_i(X_j)$).

By Definition 2, $\gamma_{i+1}(X_k) = \gamma_i(X_k)$ for all $X_k \in \mathcal{X} - \{X_j, X_{k_1}, X_{k_2}, \cdots, X_{k_n}\}$. In particular, $\gamma_{i+1}(X_k) = \gamma_i(X_k)$ for all $X_k$ such that $X_k \rhd X_j \vee X_k \sim_{\rhd} X_j$, which means that $\forall X_k : (X_k \rhd X_j \vee X_k \sim_{\rhd} X_j) \Rightarrow \gamma_{i+1}(X_k) \succeq_k \gamma_i(X_k)$ by Definition 3. Therefore, we have $\gamma_{i+1} \succ^{\bullet} \gamma_i$ with $X_j$ as the witness by Definition 4[2].

From Cases 1 and 2, $\gamma_{i+1} \succ^{\bullet} \gamma_i$ for every pair of consecutive outcomes $\gamma_i$ and $\gamma_{i+1}$. Using the fact that $\succ^{\bullet}$ is transitive when $\rhd$ is an interval order (Theorem 3), we have $\alpha \succ^{\bullet} \beta$ (by Definition 4) when $\rhd$ is an interval order. Hence, $\succ^{\blacksquare} \subseteq \succ^{\bullet}$ when $\rhd$ is an interval order. □

From the fact that $\succ^{\bullet}$ holds for each pair of consecutive outcomes in a swapping sequence supporting $\alpha \succ^{\blacksquare} \beta$, we make the following observation.

**Observation 1.** *$(\succ^{\bullet})^{\star} = \succ^{\blacksquare}$, where $(\succ^{\bullet})^{\star}$ is the transitive closure of $\succ^{\bullet}$.*

Note that this observation holds even when $\rhd$ is not an interval order. However, it does not yield a computationally efficient algorithm for dominance testing in general because computing the transitive closure of $\succ^{\bullet}$ is in itself an expensive operation.

[2] Note that we do not care how $\gamma_i$ and $\gamma_{i+1}$ compare with respect to variables $\{X_{k_1}, X_{k_2}, \cdots, X_{k_n}\}$ that are less important than the witness $X_j$.

**Example 2.** *Let* $\mathcal{X} = \{X, Y, Z\}$ *and* $D_X = \{x_1, x_2\}$*;* $D_Y = \{y_1, y_2\}$*;* $D_Z = \{z_1, z_2\}$*. Suppose that the intra-variable preferences are given by* $x_1 \succ_X x_2, y_1 \succ_Y y_2$ *and* $z_1 \succ_Z z_2$*, and the relative importance among the variables is given by* $X \rhd Y$ *and* $X \rhd Z$*. Given two outcomes* $\alpha = \langle x_1, y_2, z_2 \rangle$ *and* $\beta = \langle x_2, y_1, z_1 \rangle$*, there is* ***no*** *improving flipping sequence from* $\alpha$ *to* $\beta$ *or vice versa with respect to Definition 1. Therefore,* $\alpha \not\succ^\circ \beta$ *and* $\beta \not\succ^\circ \alpha$*.*

We now investigate the relationship between $\succ^\circ$ and $\succ^\bullet$. In Example 1, $\gamma$, $\beta$, $\alpha$ forms an improving flipping sequence from $\gamma$ to $\alpha$, resulting in $\alpha \succ^\circ \gamma$ by Definition 1. However, $\alpha \not\succ^\bullet \gamma$. Since $\succ^\bullet$ holds for each pair of consecutive outcomes in a flipping sequence supporting a dominance $\alpha \succ^\circ \beta$, we have $\succ^\circ \subseteq \succ^\bullet$ when $\succ^\bullet$ is transitive. The other side of the inclusion is negated by Example 2, where $\alpha \succ^\bullet \beta$ but $\alpha \not\succ^\circ \beta$. This leads us to the following observation.

**Observation 2.** $\succ^\bullet \not\subseteq \succ^\circ$ *and* $\succ^\circ \not\subseteq \succ^\bullet$ *in general; but* $\succ^\circ \subseteq \succ^\bullet$ *when* $\rhd$ *is an interval order.*

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