ACOUSTIC SCATTERING BY SUBMERGED ELASTIC BODIES:
A BOUNDARY ELEMENT APPROACH

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INTRODUCTION

The scattering of time-harmonic acoustic waves from an elastic solid immersed in a fluid with the transmission of elastic waves into the solid is a generic problem of interest to various disciplines. A solution strategy for this class of problems is of direct significance to the NDE community, where such knowledge can contribute to a simulation scheme for an ultrasonic immersion scanning system with randomly distributed, subsurface flaws. Use of the boundary element method (BEM) is known to be an effective tool in handling such scattering problems, especially in the mid range frequencies where asymptotic approximations fail. The strength of the method lies in exact modelling of the interaction; the drawback being its loss of efficiency in the high frequency regime.

This paper highlights the strength of BEM to capture this exact fluid-solid coupling and presents a numerical solution strategy. Our earlier work [1] concentrated on developing the formalism and treating only a few examples involving spherical scatterers. Here, we present verification of results for spherical and non-spherical elastic scatterers of a wide range of impedances (aluminum, brass, lucite, solder etc.) Non-spherical shapes include axisymmetric bodies like capped cylinders and spheroids, for which comparison solutions are available from other existing numerical schemes. The method, however, is not limited by the axial symmetry of the scatterer and is capable of handling more general flaw shapes.

FLUID-SOLID INTERACTION

The complexity involved in numerical modelling of problems of this class is two fold. First, a full vector model of the wave field in
solid is required. Second, the scalar field in fluid must be coupled with the vector field of the solid. The BEM involves a numerical solution of the boundary integral equations for both the scattering solid (assumed isotropic) and the transmitting acoustic medium coupled through appropriate boundary conditions at the fluid-solid interface. The harmonic term $\exp(-i\omega t)$ is assumed for all the field variables.

The boundary variables (pressure $p$, displacement $\bar{u}$, traction $\bar{t}$ and $\frac{\partial p}{\partial n}$) are a priori unknown and satisfy the interface continuity conditions:

$$t_i = -pn_i \quad (1)$$

$$\frac{\partial p}{\partial n} = \rho_f \omega^2 \bar{u} \cdot \bar{n} \quad (2)$$

where $\bar{n}$ is the geometric normal pointing inwards into the solid, $\omega$ is the circular frequency and $\rho_f$ is the inviscid fluid density.

The determination of the boundary variables demands a simultaneous solution of the familiar vector and scalar integral equations pertaining to the solid and fluid domains, i.e.,

$$\bar{C}_s^T(x)\bar{u}(x) = \int_{\partial B} \left[ \bar{U}^T(x,y)\bar{t}(y) - \bar{T}^T(x,y)\bar{u}(y) \right] ds(y) \quad (3)$$

$$C_f(x)p(x) = \int_{\partial B} \left[ \frac{\partial p}{\partial n}(y)G(x,y) - p(y)\frac{\partial G}{\partial n}(x,y) \right] ds(y) + p^I(x) \quad (4)$$

where, the subscripts $f$ and $s$ refer to the fluid and the solid respectively and $G$, $\frac{\partial G}{\partial n}$, $\bar{U}$ and $\bar{T}$ are the kernels or Green's functions characterizing fundamental point disturbances in the fluid and solid. The superscript $T$ denotes tensor transposition and $\partial B$ refers to the surface of the scatterer. The examples presented here are for plane incident wave $(p^I(x) = \rho_0 e^{ikx})$, although other more sophisticated incident wave models can be easily incorporated into the code.

By the usual discretization of $\partial B$ (six and eight node bi-quadratic elements are used here) and collocation at the nodes [1], Eqs. (3) and (4) may be written in discrete form as

$$[A]\{\bar{u}\} + [B]\{\bar{t}\} = \{\bar{0}\} \quad (5)$$

$$[C]\left\{\frac{\partial p}{\partial n}\right\} + [D]\{p\} = \{p^I\} \quad (6)$$

Using discrete form of the boundary conditions (1) and (2), $\{t\}$ and $\{\frac{\partial p}{\partial n}\}$ are eliminated from the Eqs. (5) and (6) and they are coupled to give

$$[A]\{\bar{u}\} + [B]\{ -p\bar{n}\} = \{\bar{0}\} \quad (7)$$

$$[(\rho\omega^2 n_i)C]\{u_i\} + [D]\{p\} = \{p^I\} \quad (8)$$

or,

$$[A]\{\bar{u}\} + [B^*]\{p\} = \{\bar{0}\} \quad (9)$$

$$[C^*]\{\bar{u}\} - [D]\{p\} = \{p^I\} \quad (10)$$

where, from Eqs. (1) and (2)

$$[B]\{\bar{t}\} = [B]\{ -p\bar{n}\} = [B^*]\{p\}$$
\[
[C'] \left\{ \frac{\partial p}{\partial n} \right\} = [C'] \left\{ \rho \omega^2 u_{n_1} \right\} = [C''] \{ \bar{u} \}
\]

If there are \( N \) nodes in the surface discretization, then \([A]\) is a \( 3N \times 3N \) matrix, \([B']\) is \( 3N \times N \), \([C'']\) is \( N \times 3N \), and \([D]\) is \( N \times N \).

**SOLUTION STRATEGIES**

In principle, one could evaluate \( \{u\} \) and \( \{p\} \) by solving the combined system

\[
\begin{bmatrix}
    A & B' \\
    C' & D
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    p
\end{bmatrix} =
\begin{bmatrix}
    0 \\
    0 \\
    0 \\
    p'
\end{bmatrix}
\]

However, the combined matrix is normally ill-conditioned due to the weak coupling between the two field equations whenever realistic material parameters (such as steel in water) are used. The solution of the combined system can therefore be numerically demanding. We can avoid a simultaneous solution of the combined matrix by using one of the following two approaches.

The first approach keeps the individual systems separate and substitutes equivalent expression for scalar unknown \( p \) in the solid equation as follows.

From Eqn. (10),

\[
D \{p\} = \{p'\} - [C''] \{u\}
\]

or,

\[
\{p\} = [D]^{-1} \left[ \{p'\} - [C''] \{u\} \right]
\]

Substituting this expression for \( \{p\} \) in Eqn. (9), one has

\[
A \{u\} = -[B'] \{p\} = -[B'] [D]^{-1} \{p'\} + [B'] [D]^{-1} [C''] \{u\}
\]

i.e.,

\[
[A - B' D^{-1} C''] \{u\} = -[B' D^{-1}] \{p'\}
\]

From Eqn. (13) we can solve for \( \{\bar{u}\} \) and once \( \{\bar{u}\} \) is known, \( \{p\} \) is easily evaluated from (12).

The second approach builds the elastic solution in an iterative way. The fluid equations are first solved by assuming a rigid scatterer (i.e., \( \frac{\partial p}{\partial n} = 0 \)) and then the solid equations are solved as a Neumann problem, where the applied tractions are obtained from the solved pressure \( p \) using the appropriate boundary condition (\( t_i = -p n_i \)). The fluid equations are then solved again with the now known pressure gradients as obtained from the solved displacement values (\( \frac{\partial p}{\partial n} = \rho_f \omega^2 \bar{u} \bar{n} \)). The process is repeated until some preset convergence criterion is reached.

The first approach appears to be numerically less elegant as it involves actual matrix inversion and many matrix multiplications. Our experience with the second (iterative) approach is that it generates reliable solutions at low to moderate frequencies. A combined solution, if attempted, should use at least double precision computations. The choice of an appropriate solution procedure should depend on the given frequency, the surface discretization being used and proximity to certain eigen frequencies [2], the details of which are beyond the scope of this discussion.
NUMERICAL EXAMPLES

The results presented in this paper are obtained either by the iterative or the combined solution. The available comparison data are for far field scattered pressure in fluid which we calculate from the surface solution through an integral representation [1]. The results are normalized with respect to the far field distance [1].

SPHERICAL INCLUSION

A variety of problems are chosen for spherical scatterer for which analytical solutions are available [3]. Computations are made for the angular distribution of scattered pressure for aluminum, brass and lucite spheres. The fluid is either water or glycerine. Also presented are backscattered and bistatic (90°) echoes for a wide range of frequencies. Two different discretizations [Fig. 1] are used for
the computations. At lower frequencies (ka < 5), the 96-element model gives satisfactory results. Figure (2) compares the pressure field for an aluminum sphere in water for progressively higher frequencies. The computations at higher frequencies [ka=5 and ka=6] show a deterioration especially at the forward scattering part (0°). The error is well remedied by use of a finer mesh [Fig. 1] (144 elements on full sphere). Figure (3) shows the frequency dependence of the backscattered and bistatic echoes. The BEM data are in consistent agreement with the far field structure as calculated from the analytical (series) solution [3]. Finally Figs. (4a)-(4c) show the scattering pattern for different elastic spheres. Figure (4a) shows data for a softer material (lucite). The scattering nature of a lucite body is of particular significance as it is typically used in NDE experiments as a host material with embedded inclusions. Figures (4b) and (4c) respectively present the field for an aluminum sphere in water and a brass sphere in glycerine at a higher frequency (ka=7.0). The gradual increase of the forward scattering lobe is accurately depicted in the results. All the results presented are limited to the frequency range of ka < 8.0 because of the bad convergence of the analytical solution at higher frequencies.
CYLINDRICAL AND SPHEROIDAL SCATTERERS

The illustrations for non-spherical inclusions are divided into two categories. The frequency dependence of backscattered and bistatic (90°) far field response are shown for aluminum cylinder with hemispherical end caps and a prolate spheroid with an aspect ratio of 2:1 [Fig. 5]. All cases are for end-on incidence (i.e., angle of incidence = 0°). The plots presented are again with respect to non dimensional wave number $ka$, where $a$ is half the total length of the spheroid. For the capped cylinder, $a = h + r$, where $h$ is half length of cylinder part and $r$ is the radius of the cap.

Figure (6) compares BEM results with T-matrix [4] and a hybrid Finite Element (NASTRAN)/Boundary Element Method solutions [6] for a 2:1 aluminum spheroid in water. The hybrid FEM/BEM scheme uses a fine mesh (2000 interior nodes and 332 surface nodes) with bi-planar surface elements. Our BEM results with a 121 element (155 nodes on half spheroid) model agrees well with the FEM/BEM data at low frequencies, but shows a divergence for $ka > 2$. Use of a more uniform and finer mesh (140 elements) improves the result dramatically. The T-matrix results, although reflecting the same trend as the above two, departs significantly in their numerical values.

The illustration for the cylinder [Fig. 7] reflects a similar trend. The results are compared only with T-Matrix data due to unavailability of any other benchmark solutions. The agreement between BEM data from two different meshes (171 and 229 nodes on half cylinder) lends confidence to the BEM results for the frequency range shown in the figure. The remaining illustration [Fig. 8] compares far field BEM data for a spheroid of four different solids (aluminum, lucite, tungsten-carbide and solder) at $ka=1.25$. The BEM solution appropriately captures the relative elasticity of the different inclusions. The magnitude of the back scattered amplitude clearly indicates the relative impedance of the scatterer.

DISCUSSION

The capability of solving the problem of plane wave scattering from a finite body immersed in an infinite fluid can be extended to the

![Fig. 5. Non-spherical Scatterer Shapes](image)
case of ultrasonic immersion testing which involves transmission through a planar or curved fluid-solid interface. We can even simplify the interface model by discretizing only that part of the interface where the effect of the incident beam is likely to be significant. The theoretical limitation arising from this part discretization will be adequately offset by use of a well collimated beam, the influence of which will decay very rapidly away from the finite discretized part of the interface.

The effect of surface curvature and subsequent focussing and beam aberration will be adequately contained in such a solution as the numerical scheme actually solves the exact differential equation recast to an integral equation. Thus the formalism should be able to handle the asymmetries that the incident beam develops at interface transmission for diverging incident angle. Our future research is aimed at addressing such problems with appropriate comparison with currently developed approximate models [7]. Subsequent combination with a corresponding capability in solid regime [8] should result in a complete general scheme to provide numerical data for probability
of detection (POD) models within the context of ultrasonic immersion testing.

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