MICROSTRUCTURAL RAYLEIGH WAVE DISPERSION ON A FLUID-COUPLED ANISOTROPIC SURFACE WITH VERTICAL LAMINATION

Adnan H. Nayfeh
Aerospace Engineering and Engineering Mechanics
University of Cincinnati
Cincinnati, OH 45221

D.E. Chimenti
Materials Laboratory
Wright-Patterson Air Force Base, OH 45433

INTRODUCTION

In a recent paper Nayfeh et al. [1] presented theoretical and experimental results for the propagation of longitudinal waves in a composite whose microstructure was large enough to cause observable velocity dispersion. Only wave propagation along the fiber axis of a uniaxial laminate was considered. A reflection coefficient was also derived for the case of normal incidence and parallel to the fibers. For ultrasonic inspection applications, what is required is the ability to analyze situations in which the wave is incident at arbitrary angles. Analysis of such general situations are, however, difficult to treat. A relatively simpler two-dimensional composite, which has been analyzed for an off-normal incident angle [2], consists of a bilaminated model with layers bonded and stacked normal to the \( x_3 \)-direction. The structure occupies the half-space \( x_2 \geq 0 \) as illustrated in Fig. 1. The composite is immersed in water such that the \( x_2 \)-direction is normal to the fluid-composite interface and the wave is incident from the fluid in the \( x_1-x_2 \) plane. For this model the reflection coefficient and the characteristic equation for the propagation of fluid-composite interfacial waves was calculated. The results reported in [2] are also restricted such that the individual composite components are isotropic.

In this paper we generalize the analysis of [2] to the case where the composite components are allowed to be anisotropic and possess as low as monoclinic symmetry. Experimental verification of the model is also conducted and included in the form of dispersion curve comparisons.

Analysis

Due to the symmetry of layering and loading we isolate from Fig. 1 the smallest repeating unit cell, which clearly consists of two half-laminates with the thicknesses \( h_1 \) and \( h_2 \) bonded at their interface. For
the convenience analysis we choose a local coordinate $x_3^{(k)}$ for each laminate. (Here, $k = 1, 2$ designates the two). In terms of the global coordinates $x_i$, $i = 1, 2$, and the local micro-coordinate $x_3$, we summarize the relevant field equations and associated interface and symmetry conditions for each material laminate $k$ as

$$\frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} - \rho \frac{\partial^2 u_\alpha}{\partial t^2} = -\frac{\partial \sigma_3}{\partial x_3}$$

(1)

The constitutive relations are:

$$\sigma_{\alpha\beta} = c_{\alpha\beta\gamma\delta} e_\gamma e_\delta + c_{\alpha\beta3} e_3$$

(2)

$$\sigma_3 = c_{33\gamma\delta} e_\gamma e_\delta + c_{3333} e_3$$

(3)

$$\sigma_3 = c_{3333} e_3$$

(4)

with

$$u_\alpha^{(1)} = u_\alpha^{(2)}, \quad \sigma_\alpha^{(1)} = \sigma_\alpha^{(2)}, \quad u_3^{(1)} = u_3^{(2)}, \quad \sigma_3^{(1)} = \sigma_3^{(2)}$$

(5)
at the interface $x^{(1)}_3 = h_1$ and $x^{(2)}_3 = -h_2$, and

$$\sigma_{a3}^{(k)} = 0, \quad u_{3}^{(k)} = 0,$$  \hspace{1cm}  (6)

on the central planes $x^{(k)}_3$. In the above equations the usual tensorial summation convention holds.

Following the procedure of [2], Eqs. (1) and (2) are averaged across their respective thicknesses and the appropriate symmetry and continuity conditions (5) and (6) are utilized we get

$$n_k \frac{\partial^2 u_a}{\partial t^2} - n_k \frac{\partial \sigma_{a\beta}}{\partial x_\beta} = (-1)^{k+1} p_{a}, \quad a, \beta = 1, 2$$ \hspace{1cm}  (7)

$$n_k \sigma_{a\beta}^{(k)} - n_k \sigma_{a\beta}^{(k)} = (-1)^{a+1} c_{a53}^{(k)} S$$ \hspace{1cm}  (8)

where $n_k = (h_k/h)$, with $h = (h_1 + h_2)$ designates the volume fraction of material $k$; $p_a$ and $S$ are momentum and constitutive relation interaction terms given by

$$p_a h = \sigma_{a3}^{(1)}(h_1) - \sigma_{a3}^{(2)}(-h_2) = -\sigma_{a3}^{(2)} (n_2)$$ \hspace{1cm}  (9a)

$$Sh = u_3^{(1)}(h_1) - u_3^{(2)}(-h_2) = -u_3^{(2)} (n_2).$$ \hspace{1cm}  (9b)

The constitutive relations (3) and (4) will now be used to derive an approximate expression for such relations. To this end we first invert Eq. (4) and, guided by Eqs. (5) and (6) and by the definition of $p_a$ in Eq. (9), expand $\sigma_{a3}^{(a)}$ to first order

$$\sigma_{a3}^{(1)} = \frac{p_a}{n_1} x^{(1)}_3, \quad \sigma_{a3}^{(2)} = -\frac{p_a}{n_2} x^{(2)}_3.$$ \hspace{1cm}  (10)

Substituting from (10) into (4), multiplying the resulting equation by the appropriate $x^{(k)}_3$ and integrating across laminate thicknesses by parts, after neglecting $u^{(k)}_3$ and satisfying the interface continuity condition on $u_3$, yields (see [2])

$$P_1 = \frac{3}{h^2} [ G_{11} (\ddot{u}_2^{(2)} - \ddot{u}_2^{(1)}) + G_{12} (\ddot{u}_1^{(2)} - \ddot{u}_1^{(1)})]$$ \hspace{1cm}  (11)

$$P_2 = \frac{3}{h^2} [ G_{12} (\ddot{u}_2^{(2)} - \ddot{u}_2^{(1)}) + G_{22} (\ddot{u}_1^{(2)} - \ddot{u}_1^{(1)})]$$ \hspace{1cm}  (12)
where

\[ s_{44} = \frac{c_{1313}}{c_{1313}c_{2323} - c_{2313}^2}, \quad s_{55} = \frac{c_{2323}}{c_{1313}c_{2323} - c_{2313}^2} \]

\[ s_{45} = \frac{-c_{2313}}{c_{1313}c_{2323} - c_{2313}^2} \]

\[ G_{11} = \frac{(s_{55})_{(1)} n_1 + (s_{55})_{(2)} n_2)}{\Delta}, \quad G_{12} = \frac{(s_{45})_{(1)} n_1 + (s_{45})_{(2)} n_2)}{\Delta} \]

\[ G_{22} = \frac{(s_{44})_{(1)} n_1 + (s_{44})_{(2)} n_2)}{\Delta} \]

\[ \Delta = (s_{44})_{(1)} (s_{44})_{(2)} (s_{55})_{(1)} + (s_{55})_{(2)} - (s_{45})_{(1)} n_1 + (s_{45})_{(2)} n_2)^2 \]

If we average equation (3) for each material \( k \) and equate the results, we get

\[ S = \frac{1}{E} \left[ c_{33Y_6}^{(2)} e_{Y_6}^{(2)} - c_{33Y_6}^{(1)} e_{Y_6}^{(1)} \right] \]

where

\[ E = \frac{c_{3333}^{(1)}}{n_1} + \frac{c_{3333}^{(2)}}{n_2} \]

In Eqs. (12) and (13), assuming their left hand sides stay finite while letting \( h \to 0 \), dictates that \( \bar{u}_{\alpha}^{(1)} + \bar{u}_{\alpha}^{(2)} = \bar{u}_{\alpha} \). For this limiting case (15) reduces to

\[ S = \frac{1}{E} \left[ c_{33Y_6}^{(2)} - c_{33Y_6}^{(1)} \right] e_{Y_6} \]

Substituting from (17) into (8) and again summing for \( \alpha = 1, 2 \) we get the effective two-dimensional constitutive relation

\[ \bar{\sigma}_{\alpha \beta} = F_{\alpha \beta Y_6} e_{Y_6} \]

where

\[ F_{\alpha \beta Y_6} = n_1 c_{\alpha \beta Y_6}^{(1)} + n_2 c_{\alpha \beta Y_6}^{(2)} + \frac{1}{E} [c_{33Y_6}^{(2)} - c_{33Y_6}^{(1)}] \{ c_{\alpha \beta Y_6}^{(1)} - c_{\alpha \beta Y_6}^{(2)} \} \]

and

\[ e_{Y_6} = \frac{1}{2} (\frac{\partial u_x}{\partial x_6} + \frac{\partial u_y}{\partial x_6}) \]

define effective mixture elastic constants and the two-dimensional strain tensor. This limit, (i.e., \( h \to 0 \)) which can also be designated as the "strong coupling" limit is equivalent to the static limit of the laminated composite. Here the composite is replaced by a homogenized and nondispersive medium whose properties are weighted functions of the individual constituents.
Now we use the representative coupled mixture equations (7) and (8) to derive the reflection coefficient from a fluid loaded half-space of the laminated composite. The fact that we are using these coupled equations will lead to a dispersive behavior of the medium which otherwise is absent for homogeneous materials. To this end we substitute from (11), (12), (13) and (15) into (8) and (7) and note that $P_a$, $S$, and $\varepsilon(k)$ are functions of the displacements $u_a(k)$. This results in the four coupled equations

$$
\rho_n \frac{\partial^2 u_a(k)}{\partial t^2} - \left[ F(k) \frac{\partial \varepsilon(k)}{\partial \gamma_a} + (-1)^{k+1} \left\{ c(k) \frac{\partial S}{\partial \gamma_a} - P_a \right\} \right] = 0, \quad (21)
$$

($k = 1, 2$ and $\alpha = 1, 2$). For harmonic waves propagating in the $x_1$ direction these four equations admit the formal solutions

$$
u_a(k) = u_a(k) \exp \left[ i \xi (x_1 + ct + nx_2) \right], \quad (22)
$$

where $u_a(k)$ are displacement amplitudes, $\xi$ is the wave number, $c = \omega/\xi$ (with $\omega$ being the circular frequency) is the phase velocity and $n_c$ is the $x_2$-direction component of the wave number. With reference to Eq. (22), for surface waves to exist, $n$ must have a positive imaginary part. If (22) is substituted into (21) we get the characteristic equation relating $n$ to $c$ and $c$ as

$$
\begin{array}{cccc}
\epsilon D_1^{-1} & 1 & \epsilon T_1 - r_1 & r_1 \\
1 & \epsilon D_2^{-1} & r_1 & \epsilon T_2 - r_1 \\
\epsilon T_1 - r_1 & r_1 & \epsilon Q_1 - r_2 & r_2 \\
r_1 & \epsilon T_2 - r_1 & r_2 & \epsilon Q_2 - r_2 \\
\end{array} = 0, \quad (23)
$$

where

$$
D_k = \rho_n k^2 c^2 - F(k) - 2F(k) \frac{n}{16} - F(k) \frac{n^2}{66} \\
Q_k = \rho_n k^2 c^2 - F(k) - 2F(k) \frac{n}{26} - F(k) \frac{n^2}{22} \\
T_k = -[F(k) - F(k) \frac{n}{16} + F(k) \frac{n}{26}] \\
r_1 = \frac{G_{12}^1}{G_{11}} \cdot r_2 = \frac{G_{22}^1}{G_{11}} \\
\xi = \frac{n^2 \omega^2}{(3G_{11})} = \frac{n^2 c^2 G_{11}}{(3e^2 G_{11})}, \quad (24)
$$

In Eq. (24) we use the contracted index notation $11-1, 22-2, 12-6$ to write out the various entries $F(k) \frac{\partial \varepsilon}{\partial \gamma_a}$. Equation (23) admits four solutions for $n^2$ as compared with two solutions for homogeneous materials or for the homogenized composite. Only two of these roots assume boundedness of the solutions. Identifying these roots by $n_1$ and $n_2$ (their actual values will be obtained numerically) we can proceed to derive the required reflection coefficient.
Using superposition, we now write (24) as

$$u_\alpha = [u_{\alpha 1} e^{i\xi n_1 x_2} + u_{\alpha 2} e^{i\xi n_2 x_2}] e^{i\xi(x_1 + ct)},$$

(25)

Here we recognize that the entire effect of the microstructure (i.e., the influence of frequency) is contained in the expressions for $n_1$ and $n_2$.

If we adopt such a passive role of the frequency then we can derive the required reflection coefficient for the homogenized medium. Invoking the standard fluid-solid boundary conditions at $x_2 = 0$ yields the reflection coefficient

$$R = (Z_1 - Z_2)/(Z_1 + Z_2),$$

(26)

$$Z_1 = Z_{11} z_{22} - Z_{12} z_{21}$$

(27a)

$$Z_2 = \frac{\rho_f c^2}{y_f^2} (W_{12} z_{22} - W_{22} z_{21})$$

(27b)

where

$$Z_{1\alpha} = F_{12} + F_{22} n_\alpha w + F_{26} (n_\alpha + W)$$

$$Z_{2\alpha} = F_{16} + F_{26} n_\alpha w + F_{66} (n_\alpha + W)$$

$$W_\alpha = \frac{[\rho_c c^2 F_{11} - 2F_{12} n_\alpha - F_{66} n_\alpha^2]}{[F_{16} + (F_{12} + F_{66}) n_\alpha + F_{26} n_\alpha^2]}$$

(28)

$$n_f^2 = \frac{c^2}{c_f^2} - 1,$$

(29)

And $\rho_f$ is the fluid density.

RESULTS AND DISCUSSION

The sample utilized in these studies is a stack of copper and stainless steel plates (elastic properties collected in ref. 3). By maintaining a large static clamping force on the stack, the welded contact boundary conditions for both stresses and displacements could be satisfied. The wave propagation surface has been carefully polished to a mirror finish to minimize any perturbing influences arising from surface roughness. The sample used for most of the measurements has a unit cell dimension of 0.79 mm.

Measurements are performed by insonifying the sample with an ultrasonic beam from a conventional piston transducer at selected angles. These angles are related to a trace velocity through Snell's law. When the trace velocity equals the phase velocity of a dispersive Rayleigh wave, strong mode coupling will deform and displace the beam energy in the reflected field. Away from this condition, the reflection will be essentially specular. To study Rayleigh wave dispersion, data are acquired in a high-resolution ultrasonic scanning system by exciting the broadband transducer.
with rf tone bursts. The reflected acoustic field is detected by a second transducer positioned at the negative incident angle. Spectra are accumulated by stepping the tone-burst frequency and observing the receiver signal. Typically, the spectra show a pronounced minimum at the frequency where strong mode conversion occurs. This fact permits us to detect the dispersive Rayleigh wave and measure its phase velocity very accurately, without performing the more difficult time-of-flight measurement.

\[ \text{FIG. 2 THEORETICAL AND EXPERIMENTAL COMPARISONS OF SURFACE VELOCITY DISPERSION.} \]

Results

Repeating such experiments at many incident angles and recording the trace velocity and minimum frequency allows us to construct a dispersion curve for Rayleigh waves on the edge-laminated plate. The results of such measurements are displayed in Fig. 2. The Rayleigh wave phase velocity is plotted on the ordinate as a function of frequency times the unit cell dimension. The solid curve is the calculation obtained from the vanishing of the denominator of the reflection coefficient (26) and the open circles are the experimental measurements. With the possible exception of the low frequency limit the agreement between the two is very good, especially considering the approximations necessary to arrive at the predicted behavior. Even in the quasistatic limit, the disparity is only 1.5%. At low frequency the Rayleigh velocity tends, as expected, to the mixture value of the two constituents. At high frequency the velocity approaches the surface wavespeed of the more compliant of the constituents. This behavior is very well rendered by the model.
REFERENCES

