ACOUSTOELASTICITY USING SURFACE WAVES IN SLIGHTLY
ANISOTROPIC MATERIALS

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INTRODUCTION

Acoustoelasticity is a nondestructive technique for determining applied and residual stresses in structural materials. It is based on the fact that the velocity of an ultrasonic wave varies as stress is applied to a material. In laboratory conditions, it has been demonstrated that acoustoelasticity can be used successfully [1-3], however, there are some difficulties which have delayed its practical applications. One of these difficulties is the fact that the change in velocity due to texture is of the same order as that which is due to applied stress. This means that the convenient acoustoelastic relations for an isotropic material become more complicated when anisotropic material properties are used.

In considering the acoustoelastic effect using surface waves the complications are worse than in using bulk waves because in addition to solving an eigenvalue problem one must also satisfy the stress free boundary condition. When a material has arbitrarily severe anisotropy a solution has been given for a surface wave propagating in any direction on a plate [4]. This method considers a six dimensional eigenvalue problem and reduces to an iteration procedure in which the vanishing of a real 2 x 2 determinant determines the surface wave speed. However, numerical techniques for finding the surface wave speed often give less insight to understanding the role of various parameters and also they preclude the inverse problem in which the elastic constants are determined or, given the elastic constants, the applied stress may be determined from the wave speeds. One way of avoiding the complicated surface wave solutions is to consider slightly anisotropic materials. This was done [5] by considering the effect of initial anisotropy and stress to be perturbations from the surface wave solution of an isotropic material. In making the assumption that the anisotropy and initial stresses are small an expression for the change in velocity due to both of these effects was obtained. This perturbation formalism can be applied toward the solution of the inverse problem. With such convenient expressions for the acoustoelastic response it is of interest to
determine how much the perturbative theory differs from the six dimensional approach.

The purpose of the present study is to numerically compare the perturbation technique [5] to the exact numerical solution [4] as a function of anisotropy and applied stress. In the perturbative approach the change in velocity due to anisotropy and applied stress are linearly related to the anisotropy and the principal stresses, respectively. This linearity will be checked as the anisotropy and applied stress are varied. Initially, the basic equations of surface waves will be reviewed. Equations from the six dimensional approach and the perturbative approach for determining the surface wave speed will be given and discussed. Surface wave speeds will be calculated using the six dimensional approach which allows for arbitrarily severe anisotropy. The change in velocity as a function of second- and third-order elastic constants and applied stress is studied to determine the linear range.

BASIC EQUATIONS OF SURFACE WAVES IN DEFORMED MEDIA

In the theory of acoustoelasticity an infinitesimal ultrasonic wave is superposed on a finite deformation. The position of a material point in the current configuration, $x_i$, is written as the sum of the position in the deformed configuration, $\tilde{x}_i$, plus the displacement of the infinitesimal ultrasonic wave, $u_i$. Assuming that the material and the initial deformation are homogeneous, the equations of motion can be written as

$$B_{ijk}u_{k,si} = \rho \ddot{u}_i,$$

where $B_{ijk}$ is given as

$$B_{ijk} = \rho F_i A_j B_k C_s D_{AB} \frac{\partial^2 W}{\partial E_{AB} \partial E_{CD}} + T_{ij} \delta_{jk},$$

with $F_i$ being the deformation gradient associated with $x_i$, $E_{AB}$ the Lagrange strain, $\rho$ the mass density in the deformed state, $W$ the strain energy density and $T_{ij}$ the Cauchy stress.

Consider an elastic half space with the plane defined by $x_3=0$ representing the free surface. Define unit vectors $n_i=(0,0,1)$ and $m_i=(\cos\phi, \sin\phi, 0)$ to be the inner normal to the surface and propagation direction, respectively. Assume the displacements are given by

$$u_i = A_i \exp[\mu x_1 + \mu_2 x_2 + \mu_3 x_3 - \nu t],$$

where $A_i$ is amplitude vector and $\nu$ is a complex number called the decay constant which insures that the motion remain near the surface. For nontrivial amplitude vectors the equations of motion give us the condition

$$\det\{B_{ijk}m_1 m_2 + p[B_{ijk}(m_1 n_i + n_1 m_i)]
+ p^2 B_{ijk} n_1 n_2 - \rho \nu^2 \delta_{jk}\} = 0.$$  \hspace{1cm} (4)

Eq. (4) is a bicubic equation in the decay constants, $p$, which yields three pairs of conjugate roots. Three of these roots are discarded because they do not restrict the motion to be near the surface. Using the three admissible decay constants, amplitude vectors $A_{k\alpha}$ can be found.
where the subscript $a$ denotes that this amplitude vector is associated with the $a$th admissible decay constant.

The stress free boundary condition is written as

$$T_{ij}n_i = B_{3jk}u_{ks} = 0, \text{ when } x_3=0. \quad (5)$$

Let the displacement field be written as a linear combination of straight crested waves having phase velocity $V$, but with $p$ equal to one of the admissible roots $p_a$. Then the stress free boundary condition can be written as [4]

$$\det(B_{3jk}A_{\alpha m_{\alpha}}) = 0, \alpha=1,2,3 \text{ not summed,} \quad (6)$$

where $m_{1a}=m_1$, $m_{2a}=m_2$, and $m_{3a}=p_a$.

To obtain the velocity for a surface wave using an iterative approach, a velocity is chosen and decay parameters $p_a$ are calculated from Eq. (4). The amplitude vectors $A_{i\alpha}$ are computed and then Eq. (6) is evaluated to determine if the boundary condition is satisfied. If Eq. (6) is not zero then another velocity is chosen and the process is repeated.

One can now see how much more complicated an explicit surface wave solution is compared to the bulk wave solution. Eq. (4) is essentially an eigenvalue problem whose eigenvalues, $p$, must be determined. After this, the corresponding eigenvectors must be found and then all of these expressions substituted into the stress free boundary condition. For waves propagating in a direction which does not correspond to an axis of material symmetry the final expression for satisfying the stress free boundary can become very complicated. In addition to that, the decay constants and amplitude vectors are, in general, complex which makes the surface wave speed a real root of a complex expression. Contrasting this to finding longitudinal and shear wave speeds where one only has to find the eigenvalues of the characteristic equation, since there is no stress free boundary condition which has to be satisfied, it is clear that an explicit surface wave solution will be much more involved if not intractable.

SLIGHTLY ANISOTROPIC MATERIALS

One way of obtaining a more tractable solution for the surface wave speed is to consider materials in which the anisotropy is small. In doing this an expression for the change in velocity due to slight anisotropy and applied stress can be found which lends itself to the inverse problem. In this section, a brief description of a perturbation technique from Ref. 5 for surface waves in deformed, slightly orthotropic materials will be given followed by the resulting expression for the change in surface wave speed due to slight anisotropy and applied stress. In this expression, the change in velocity varies linearly in anisotropy and applied stress. To study the range of linearity an expression will be given from Ref. 4 from which the surface wave speed can be determined for a material whose anisotropy is not necessarily slight. For a complete discussion on the derivation of either of these expressions, the reader should refer to Refs. 4 and 5. In the next section, numerical results from the latter technique will be given.

A perturbation method for determining the change in surface wave speeds in slightly anisotropic, deformed materials starts with the unperturbed case of an unstressed, isotropic material [5]. After finding
the surface wave velocity in the unperturbed state, the effect of a small perturbation on the surface wave speed is considered and an expression for the change in velocity due to this perturbation is established. This perturbation could be due to slight anisotropy, applied stress, small temperature change or any other cause which affects the surface wave speed. Let the second-order elastic constants (SOEC) be given by

\[
C_{kk} = \lambda + 2\mu + C'_{kk}, \quad k=1,2,3, \text{ not summed,}
\]
\[
C_{kk} = \mu + C'_{kk}, \quad k=4,5,6, \text{ not summed}
\]
\[
C_{km} = \lambda + C'_{km}, \quad k\neq m=1,2,3,
\]

where \(\lambda\) and \(\mu\) represent some average values of the Lame constants for an isotropic material and \(C'_{km}\) are the perturbations of the elastic constants from isotropy. Consider a wave propagating on the surface \(x_3=0\) at an angle \(\phi\) from a material symmetry axis. With small perturbations, the change in velocity, \(V'\), due to applied stress and anisotropy can be added to give

\[
V'/V = \left\{ \alpha_{2222} \left[ \sin^4 \phi C'_{11} + \cos^4 \phi C'_{22} + 2 \sin^2 \phi \cos^2 \phi (C'_{12} + 2C'_{66}) \right] \right. \\
+ \alpha_{2233} \left[ \sin^2 \phi C'_{13} + \cos^2 \phi C'_{23} + \alpha_{23} (\sin^2 \phi C'_{55} + \cos^2 \phi C'_{44}) \right] \\
+ \alpha_{3333} C'_{33} \right\}/2\mu
\]

where \(V\) is the velocity of a surface wave in an unstressed, isotropic media, \(A_{2222}, A_{2233}, A_{23}, A_{3333}, a_0, a_0, a_{k}, a_{k}\) and \(\sigma_{\mu}\) are all functions of \(\lambda\) and \(\mu\), \(\phi\) are the three isotropic third-order elastic constants (TOEC), \(\sigma\) and \(\mu\) are the sum and difference of the applied principal stresses and \(\phi\) is the angle between the material symmetry direction and one principal stress direction. Eq. (8) shows the linear dependence of \(V'/V\) on \(C'_{km}\), the third-order elastic constants, and the principal stresses (or their sum or difference) for this theory.

In order to examine the range of linearity for the change in velocity as a function of \(C'_{km}\) and applied stresses a six dimensional approach will be used in which the surface wave speed is found from the vanishing of a real 2 x 2 determinant [4]. This formulation does not restrict the material to be slightly anisotropic which will allow for studying the linearity of the change in velocity as a function of \(C'_{km}\) and applied stress. The surface wave speed is determined from

\[
det B_{mg} = 0,
\]

where

\[
2\pi B_{mg} = (m n - n m) \left\{ B_{wmis} - \rho V^2 m s \right\} \left\{ B_{ng} - \rho V^2 n p \right\} \right. \\
x(n n - m m) M_{ir} + m n N_{ir} - n m O_{ir}
\]

and where \(M_{ir}\), \(N_{ir}\), and \(O_{ir}\) are 3 x 3 matrices whose components are various combinations of second- and third-order elastic constants averaged over the plane containing the propagation direction and the normal to the free surface. For a complete derivation of Eq. (10) the reader should see Ref. 4.
In this section the six dimensional surface wave theory will be
used to calculate the change in surface wave speed as a function of
anisotropy of the second- and third-order elastic constants and stress.
Questions to be addressed here are: i) To what extent is the
acoustoelastic response linear in $C'_{km}$ and stress? and ii) To what extent
can the effects of anisotropy and stress be treated separately?

$V'/V$ as a function of stress and $C'_{km}$

In studying the change in velocity as a function of $C'_{km}$, a range of
stiffnesses must be chosen which is large enough so that any nonlinearity
will show up. Second-order elastic constants have been measured for
7039-T6 aluminum alloy [6] and are given in Table I. To calculate
averaged isotropic values the average of $C_{12}$, $C_{13}$, and $C_{23}$ is taken as $\lambda$
and the average of $C_{11}$, $C_{22}$, and $C_{33}$ is taken as $\lambda+2\mu$. We see that most
of the constants differ from isotropy by less than 1%. The maximum that
any constant varies from isotropy is 2.8% for $C_{55}$. Since we are
interested in looking at a range in which the change in velocity is
linear in $C'_{km}$, it will be allowed to vary 12% from its isotropic value
which is about how much the SOEC of copper differs from isotropy.

Figure 1 shows the change in velocity normalized by the velocity of
a wave in an unstressed, isotropic material as a function of $C'_{11}$
normalized by its isotropic value. All other elastic constants are kept
at their isotropic value as $C'_{11}$ is varied. In this plot three
propagation directions are considered: 0 deg corresponds to propagation
in the $x_1$ direction, 90 deg corresponds to propagation in the $x_2$
direction, and 45 deg corresponds to propagation midway between the $x_1$
and $x_2$ axis. The symbols plotted are the values calculated using the six
dimensional approach which is not restricted to slightly anisotropic
materials. For each propagation direction, a line is drawn which is
tangent to the curve at $C'_{11}$=0. We see that for small values of $C'_{11}$
that the normalized change in velocity is well approximated as linear.
For a material like aluminum, whose SOEC differ from isotropy by about
1%, a theory which is linear in $C'_{11}$ would be a good approximation.
However, a material such as copper which has SOEC that differ from
isotropy by about ten times that of aluminum, the linear approximation
may not be well suited. Also note in Fig. 1 that the velocity of a wave
propagating in the $x_2$ direction does not change as $C'_{11}$ changes. This
does not mean that as the material becomes less isotropic there are
certain directions which the phase velocity is not affected, it only
means that as $C_{11}$ becomes less isotropic the phase velocity in the $x_2$
direction is unaffected. As other elastic constants change, for instance
$C_{55}$, the phase velocity of the surface wave travelling in this direction
will change.

<table>
<thead>
<tr>
<th>Constant</th>
<th>Measured [6] (GPa)</th>
<th>Isotropic (GPa)</th>
<th>$C'<em>{km}$ / $C</em>{km}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{11}$</td>
<td>109.9</td>
<td>109.73</td>
<td>0.2</td>
</tr>
<tr>
<td>$C_{12}$</td>
<td>56.6</td>
<td>56.7</td>
<td>-0.2</td>
</tr>
<tr>
<td>$C_{13}$</td>
<td>56.9</td>
<td></td>
<td>0.4</td>
</tr>
<tr>
<td>$C_{22}$</td>
<td>108.9</td>
<td></td>
<td>-0.8</td>
</tr>
<tr>
<td>$C_{23}$</td>
<td>56.6</td>
<td></td>
<td>-0.2</td>
</tr>
<tr>
<td>$C_{33}$</td>
<td>110.4</td>
<td></td>
<td>0.6</td>
</tr>
<tr>
<td>$C_{44}$</td>
<td>26.2</td>
<td>26.52</td>
<td>-1.2</td>
</tr>
<tr>
<td>$C_{55}$</td>
<td>25.8</td>
<td></td>
<td>2.8</td>
</tr>
<tr>
<td>$C_{66}$</td>
<td>26.8</td>
<td></td>
<td>1.1</td>
</tr>
</tbody>
</table>
As is the case with any numerical procedure, it is advisable to verify the results with another method when possible. The results presented in Fig. 1 for a wave propagating in the 0 deg direction were verified using a different method for calculating surface wave speeds in an unstressed, orthorhombic media [7, Eq. 21].

Now consider the change in velocity as a function of applied stress. Starting with an isotropic material, it will be loaded in the rolling, or x₁, direction and the change in velocity will be calculated for propagation in the same three directions as before. Since the theory we are using is for elastic deformations the stress applied to the material was limited to 280 MPa. Fig. 2 shows the change in velocity normalized by the unstressed, isotropic velocity as a function of applied stress and for this range of applied stress the response was very well approximated as linear.

Treating the effects of anisotropy and stress separately

In the perturbative treatment the effects of anisotropy and applied stress are treated separately and then added together to give the response of a slightly anisotropic material which is initially deformed. This is demonstrated in Eq. (8) where the first three lines describe the changes in velocity due to the anisotropy of an unstressed material through the SOEC and the fourth line of that equation reflects the change in velocity of a wave propagating in an isotropic material which is initially deformed. The essence of this assumption is that we are considering a material whose SOEC are slightly anisotropic while the TOEC are isotropic. This means that the acoustoelastic constants, that is the change in velocity per unit stress, will be the same for a slightly anisotropic material as for an isotropic material.

As a way of illustrating this consider the acoustoelastic constants for surface waves propagating on an isotropic material which is loaded in
Figure 2. Relative change in velocity in an isotropic material as a function of applied stress in the 0 degree or rolling direction.

the $x_1$, or rolling, direction. The slope of the lines in Fig. 2 represent the acoustoelastic constants for propagation directions of 0 deg, 45 deg, and 90 deg. In the perturbative approach, the slope of these lines will remain the same while the initial anisotropy will cause a nonzero change in velocity in the unstressed state.

Using the six dimensional approach the independence of initial anisotropy and applied stress can be examined concomitantly since no assumption on the degree of anisotropy is made. Fig. 3 shows the change in velocity as a function of applied stress for an isotropic material with $C_{11}$ perturbed by 1% and 8%. Again, the symbols are the values calculated from the six dimensional approach by applying stress to a material whose elastic constant $C_{11}$ is 1 and 8 percent below its isotropic value. The straight lines the same slope as those in Fig. 2, the isotropic case, but in the unstressed state there is a nonzero change in velocity due to the initial anisotropy. That is, the straight lines are what is predicted if you assume that slight anisotropy and applied stress can be treated separately.

Considering a surface wave propagating in the 0 deg or $x_1$ direction in an isotropic material the acoustoelastic constant, or the slope of the solid line in Fig. 2, is $-0.1005 \text{ TPa}^{-1}$. In the perturbative treatment, the anisotropy and the applied stress are independent so this slope stays the same and the solid lines in both plots of Fig. 3 have this same slope. When anisotropy is introduced in the six dimensional approach, that is, when $C_{11}$ is 1% below its isotropic value, the acoustoelastic constant changes 8.2% from the isotropic case to $-0.1087 \text{ TPa}^{-1}$. More anisotropy is introduced when $C_{11}$ is set 8% below its isotropic value and the acoustoelastic constant becomes $-0.1736 \text{ TPa}^{-1}$ which is 73% less than the isotropic constant. Clearly, the change in velocity due to anisotropy and applied stress are not independent, but if the anisotropy is small enough the error in treating them separately may be small. In addition to the coupling of the stress and the SOEC, we also know that a material which is nearly isotropic in its SOEC is not necessarily nearly isotropic in its TOEC [6]. This would add additional errors to a theory that considers the TOEC to be isotropic.
Figure 3. Acoustoelastic response for a material with $C_{11}$ 1% and 8% below its isotropic value.

REFERENCES