Ride and handling models of a vehicle with active suspensions

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Ride and handling models of a vehicle with active suspensions

by

Jay E. Shannan

A Thesis Submitted to the Graduate Faculty in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

Major: Mechanical Engineering

Signatures have been redacted for privacy

Iowa State University
Ames, Iowa

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LIST OF NOMENCLATURE

a  longitudinal distance from sprung mass center of gravity to front axle centerline

A  system matrix, defined in text

b  longitudinal distance from sprung mass center of gravity to rear axle centerline

B₁  matrix in feedback control loop, defined in text

B₂  input matrix, defined in text

C₇  damping coefficient of one front shock absorber

Cᵣ  damping coefficient of one rear shock absorber

Fₐ  total left front suspension force

Fₐ  total left rear suspension force

Fₐ  total right front suspension force

Fₐ  total right rear suspension force

Fₐ  left front actuator force

Fₐ  left rear actuator force

Fₐ  right front actuator force

Fₐ  right rear actuator force

I₇  mass moment of inertia of one front unsprung mass about vertical axis through its mass center of gravity

Iᵣ  mass moment of inertia of one rear unsprung mass about vertical axis through its mass center of gravity

Iₓ  sprung mass moment of inertia about longitudinal axis through sprung mass center of gravity

Iᵧ  sprung mass moment of inertia about lateral axis through sprung mass center of gravity

Iᵦ  sprung mass moment of inertia about vertical axis through sprung mass center of gravity
K  feedback gain matrix
k_f  spring constant of one front suspension spring
k_R  spring constant of one rear suspension spring
k_t  vertical spring constant of one tire
M_s  sprung mass
M_1  left front unsprung mass
M_2  left rear unsprung mass
M_3  right front unsprung mass
M_4  right rear unsprung mass
P  vertical distance from pitch center of sprung mass to sprung mass center of gravity
R  vertical distance from roll center of sprung mass to sprung mass center of gravity
R_f  vertical distance from front roll center to sprung mass center of gravity
R_r  vertical distance from rear roll center to sprung mass center of gravity
X_1  longitudinal force on sprung mass due to left front tire
X_2  longitudinal force on sprung mass due to left rear tire
X_3  longitudinal force on sprung mass due to right front tire
X_4  longitudinal force on sprung mass due to right rear tire
Y_1  lateral force on sprung mass due to left front tire
Y_2  lateral force on sprung mass due to left rear tire
Y_3  lateral force on sprung mass due to right front tire
Y_4  lateral force on sprung mass due to right rear tire
t_f  distance between centerlines of the front tires
\( t_r \) distance between centerlines of the rear tires
\( u \) vector of actuator forces
\( V_x \) longitudinal velocity of sprung mass
\( V_y \) lateral velocity of sprung mass
\( w \) vector of road inputs
\( Z \) vertical position of sprung mass center of gravity
\( Z_a \) vertical position of left front corner of sprung mass
\( Z_b \) vertical position of left rear corner of sprung mass
\( Z_c \) vertical position of right front corner of sprung mass
\( Z_d \) vertical position of right rear corner of sprung mass
\( Z_{lf} \) height of road under left front unsprung mass
\( Z_{lr} \) height of road under left rear unsprung mass
\( Z_{rf} \) height of road under right front unsprung mass
\( Z_{rr} \) height of road under right rear unsprung mass
\( Z_1 \) vertical position of left front unsprung mass
\( Z_2 \) vertical position of left rear unsprung mass
\( Z_3 \) vertical position of right front unsprung mass
\( Z_4 \) vertical position of right rear unsprung mass
\( \phi \) roll angle of sprung mass
\( \theta \) pitch angle of sprung mass
\( \Pi \) cost function used to determine feedback gain matrix
\( \omega_z \) angular velocity of sprung mass about the vertical axis
\( \cdot \) denotes differentiation with respect to time
\( \cdot\cdot \) denotes differentiation twice with respect to time
\( T \) denotes transposition
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1. INTRODUCTION

In recent years, the increasing cost of fuel has resulted in smaller and lighter vehicles. This has resulted in degradation of ride performance. One solution to this problem may be active suspension systems. Active suspension systems consist of force elements in addition to the normal spring and damper assembly. The force elements continuously vary the force between the wheel and the chassis of the automobile. The forces are computed using a control law which is a function of the state of the vehicle. These force elements may input, store, or dissipate energy. By comparison, a passive suspension with ordinary springs and dampers may only store and dissipate energy. The drawbacks to active suspension systems are increased complexity, weight, power consumption, and cost.

A related class of suspension systems is known as semi-active. These systems contain force elements that may only store or dissipate energy. The difference between semi-active and passive is that the amount of storage or dissipation can be changed continuously. These suspensions often use the same control laws as active systems, the difference being when the force elements are called upon to input energy into the system they cannot. Examples of such force elements are continuously variable rate springs and dampers. These systems are not as complex or as costly as
active suspensions and require little additional energy. The additional energy is used to change damping and spring rates.

The objective of this thesis is to formulate several vehicle models which include active suspension elements and to use these models to study the effects of active suspensions on vehicle ride and handling. The major contribution of this thesis is to study the effects of active suspensions on vehicle handling.

The first study will use seven degrees of freedom (DOF) to model an automobile with an active suspension system. This model is used to obtain the control law and to study the effect of the active suspension on ride. The control algorithm for the active suspension will be developed using linear optimal control techniques.

The model will then be extended to ten DOF to include yaw plane dynamics. This results in a nonlinear model which cannot be used to compute a linear optimal control law. The control algorithm developed for the seven DOF model will be used in the ten DOF model to determine the effects of the optimal ride active suspension on directional response.

Finally, the optimal control law developed using the linear model will be slightly altered and the responses compared to those previously obtained. This will show some possible affects of an active suspension on the handling of
an automobile.

This thesis will not deal with the implementation of these suspension systems, but will study the effects that idealized active suspension systems have on the dynamics of the vehicle.
2. LITERATURE REVIEW

An overview of active suspensions in automobiles and railway vehicles was given by Goodall and Kortum [1] in 1983. This review will be limited to automobile suspensions.

Researchers have studied active and semi-active suspensions using a number of different vehicle models [1-22]. Figure 2.1 presents several of the more common vehicle models used in these studies. Many investigators have utilized quarter car models consisting of one DOF. Margolis [2] compared the responses of one DOF active and semi-active suspensions. He assumed that the response of a so-called "sky-hook" damper, a fictitious damper between the sprung mass and a fixed-height point, was the ideal case. This "ideal response" was then used to find the feedback gains. Based on position response, both suspensions exhibited improved ride when compared to passive systems. The active suspension was found to be only marginally better than the semi-active system. Karnopp and Margolis [3] also utilized a one DOF model in their work on semi-active suspensions. The results were similar to those presented in [2]. Karnopp and Margolis [3] present some advantages of preparing the suspension for a maneuver. For example, the brake pressure could be measured as a leading indicator for dive, and steering wheel angular velocity could be measured
as a leading indicator for roll.

Thompson presented two papers [4,5] utilizing a two DOF quarter car model with an active suspension. Both papers used optimal control techniques to determine the feedback gains of the system. Optimal control is based on the assignment of costs to the state variables and the control forces. The cost functions and dynamics of the system yield a Riccati equation [23] which is solved to determine the optimal feedback gains. Based on these gains and the state of the automobile, the control forces are calculated. When compared to passive suspensions, Thompson has shown substantial increases in ride quality by assigning costs to the dynamic tire deflection and suspension travel as well as the control forces.

Wilson, Sharp, and Hassan [6] developed several different control algorithms for use on two DOF quarter car models. Some of their algorithms used full state information and others used partial state information. The main thrust behind their work was an attempt to eliminate the road to sprung mass height measurement necessary in most full state feedback systems.

Sutton [7,8] also presented a two DOF quarter car model with an active suspension. To verify the theoretical results, an experimental suspension was constructed. Problems were encountered involving nonlinearities of the
FIGURE 2.1. Schematics of various suspension models
actuators. However, the work did show increased isolation of the sprung mass from input disturbances.

Hrovat and Hubbard [9] utilized a one DOF quarter car model with an active suspension. In this case, the cost function included jerk, the time rate of change of acceleration, of the sprung mass. The active control significantly improved the isolating characteristics of the suspension.

Karnopp [10,11] utilized both one and two DOF bounce models to investigate the effects of active damping. Active damping is a scheme in which the active force element is controlled by velocity feedback only. He found that active damping improved the frequency response characteristics of the one DOF system but only marginally improved the response of the two DOF model. An active suspension which contained position and velocity feedback was also investigated. This resulted in marked improvements over the passive and the active damping systems.

Sachs [12] presented a two DOF quarter car model utilizing adaptive control concepts. Sample road profiles were measured and used with a vehicle simulation to compute the response of the vehicle. Using this information and minimizing sprung mass acceleration subject to suspension travel constraints, optimal spring constants and damping coefficients were calculated for the vehicle at various
speeds. The test data and related optimal constants could be stored in a computer on board the vehicle. While the vehicle is operating on the road, the accelerations of the vehicle could be measured and compared to the stored response data. Once a match is found, the spring constant and damping coefficient are set to the previously calculated values. The parameters are updated as the measured motions change. Computer simulations showed that this system resulted in significant decreases in sprung mass accelerations.

Research has also been conducted in the area of preview control. This work is based on the idea that sensing the changes in road profile height ahead of the wheel will enable the suspension to be prepared for the upcoming disturbances and result in a smoother ride. Results of Bender [13], Masayoshi [14], and Thompson, Davis, and Pearce [15] showed the potential for marked improvements based on preview control. These papers did not take into account any of the practical difficulties of the measurement of the road profile.

Li, Meiry, and Roeseler [16] presented a model which used roll as the only degree of freedom to be controlled. The paper presented a suspension design that decoupled roll from the vertical motion of the sprung mass. This scheme used one suspension system to isolate the roll of the
vehicle. A second suspension system was utilized to isolate the sprung mass from road disturbances. It was found that roll could be significantly controlled by the proposed scheme.

Karnopp, Crosby, and Harwood [17] introduced a pitch plane two DOF one-half car model. Frequency response results indicated an improvement in ride using a semi-active system to control pitch and bounce.

Margolis [18] utilized a similar pitch plane two DOF model to investigate active and semi-active control schemes. He first used velocity feedback to obtain the response given by front and rear sky-hook dampers. He then developed a model using both position and velocity feedback. Both models exhibited improvement over a model using a passive suspension.

Thompson and Pearce [19] developed a four DOF pitch plane one-half car model which included unsprung masses and an active suspension. The active suspension, utilizing optimal control cost functions similar to those by Thompson [4], exhibited improved performance compared to passive systems.

References 20-22 are summaries of the work of Malek and Hedrick [20], Fruhauf, Kasper, and Luckel [21], and Barak and Sachs [22] who used seven DOF models. These references are summaries of papers that were presented at the 1985
Symposium on Dynamics of Vehicles on Roads and Tracks, Stockholm, Sweden.

For the most part, the previous results dealt with mathematical models. A few papers have been published presenting results from test vehicles equipped with active and semi-active suspensions. For example, Mizuguchi et al. [24] dealt with a system which switches between a soft and a hard suspension when certain conditions are met. A soft suspension effectively isolates road disturbances from the sprung mass. Hard suspensions allow better road following capability but do not isolate the sprung mass from road disturbances as well as a soft suspension. This suspension is not continually variable but simply switches between two modes.

Dominy and Bulman [25] presented a semi-active suspension developed by the Lotus racing team for its Formula One racing car. The suspension was developed during 1982 to facilitate the use of "ground effects" aerodynamics. The undersides of the cars were designed to develop an area of low pressure air underneath the car. This generated downward forces which facilitated increased cornering accelerations and improved racing performance. However, for the ground effects to work, the car had to be maintained at a controlled height above the roadway. If this was accomplished with passive suspensions, the driver was
subjected to an extremely harsh ride. The semi-active suspension was successful in controlling the height while giving the driver a much better ride. (Regulations eliminating ground effects were adopted for the 1983 racing season. Thus, the added complexity of the semi-active suspension was not warranted for continued use on subsequent Formula One cars.)

Baker [26] described Lotus' continuing efforts in the active suspension area. An active suspension has been installed in a test vehicle with promising results. It was reported that speeds through corners increased as much as 10% while providing a ride that was better than the ride provided by a similar car with a passive suspension. Baker [27] briefly discussed Lotus' most recent active suspension system which focuses on ride rather than handling.

The analytical models presented in this review have indicated that active suspensions may provide better ride characteristics. The two documented cases of actual implementation of these suspensions tends to confirm the expectations.
3. SEVEN DEGREE OF FREEDOM LINEAR RIDE MODEL

This chapter presents a linear seven degree of freedom ride model. The Riccati equation is used to obtain feedback gains with the goal of isolating the sprung mass from road input disturbances. The model is used to simulate a vehicle on several different road profiles. The response of the vehicle with and without active suspensions is presented and compared.

3.1 Equations of Motion Including Feedback Control

The seven DOF linear ride model is developed assuming the automobile is composed of five discrete rigid bodies, the chassis or sprung mass, and four wheels or unsprung masses. The sprung mass is assumed to have three DOF: bounce, pitch, and roll. Each unsprung mass is assumed to have only a bounce degree of freedom. This model is shown in Figure 3.1.

The equations of motion of the model are developed by summing moments about the x and y axes of the sprung mass and summing vertical forces on the sprung and unsprung masses. The roll and pitch angles are assumed to be about the sprung mass center of gravity. Assuming small roll and pitch angles and independence of the roll and pitch of the sprung mass allows the equations of motion to be linearized. The equations for the sprung mass are:
The equations for the unsprung masses are:

\[ \Sigma F_{Z_1} = M_1 \ddot{Z}_1 \]  
\[ \Sigma F_{Z_2} = M_2 \ddot{Z}_2 \]  
\[ \Sigma F_{Z_3} = M_3 \ddot{Z}_3 \]  
\[ \Sigma F_{Z_4} = M_4 \ddot{Z}_4 \]  

The forces on the sprung mass result from the springs, dampers, and actuators between the sprung mass and the unsprung masses. The tires are modeled as linear springs.
between the unsprung masses and the ground. The damping due to the tires is ignored. The inputs to the model are the vertical height velocities of the road under each tire. This model does not have a longitudinal DOF. The road is assumed to move underneath the automobile. The fully developed equations are given in Appendix A. The vehicle model for the simulations presented in this thesis has roll, pitch, and bounce natural frequencies of 1.38 Hz, 0.9 Hz, and 1.31 Hz, respectively. The damping ratios for the roll, pitch, and bounce modes are 0.2, 0.08, and 0.15, respectively. The vehicle parameters obtained from measurements and test data are given in Appendix B.

To help develop the control algorithm for the active suspension, the model is represented by the block diagram as shown in Figure 3.2.

The equations for the model may be written in matrix form:

\[ \dot{x} = A \, x + B_1 \, u + B_2 \, w \]  

where \( x \) is the vector of the 18 state variables. The equations of motion for a seven DOF model are seven second order differential equations. They may be reduced to 14 first order differential equations. The additional four equations are due to using road height velocities for the inputs. These velocities are integrated to determine the road heights at each of the tires. Using road velocities
FIGURE 3.2. Block diagram of seven DOF model

instead of road positions increases the amount of information available for use in the control law.

\[ x = \begin{bmatrix} \dot{\phi} & \dot{\theta} & \phi & \theta & Z & \dot{Z} & Z_1 & \dot{Z}_1 & Z_2 & \dot{Z}_2 & Z_3 & \dot{Z}_3 & Z_4 & \dot{Z}_4 \\ Z_{1f} & Z_{1r} & Z_{rf} & Z_{rr} \end{bmatrix}^T \]  
(3.9)

The vector \( u \) gives the four control forces that act between the sprung and unsprung masses.

\[
u = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \quad (3.10)\]

The vector \( w \) gives the inputs which consist of road height
velocities at the four tires.

$$
\begin{align*}
\mathbf{w} &= \begin{bmatrix}
\dot{Z}_{1f} \\
\dot{Z}_{1r} \\
\dot{Z}_{rf} \\
\dot{Z}_{rr}
\end{bmatrix}
\end{align*}
$$

The system matrices, $A$, $B_1$, and $B_2$ are given in Appendix A. Converting the model into state-space form facilitated the use of modern linear control techniques.

The algebraic Riccati equation is used to determine the optimal feedback gain matrix $K$. The Riccati equation requires a cost be applied to the various state variables, combinations of state variables, and control forces. In this thesis, the following cost function was used:

$$
\Pi = 10 \{ \phi^2 + \theta^2 + z^2 + \dot{\phi}^2 + \dot{\theta}^2 + \dot{z}^2 \} +
10 \{ (Z_{1f} - Z_1)^2 + (Z_{1r} - Z_2)^2 +
(Z_{rf} - Z_3)^2 + (Z_{rr} - Z_4)^2 \} +
( (Z_1 - Z_a)^2 + (Z_2 - Z_b)^2 +
(Z_3 - Z_c)^2 + (Z_4 - Z_d)^2 \} +
10^{-8} \{ F_1^2 + F_2^2 + F_3^2 + F_4^2 \} 
$$

This function assigns cost coefficients to roll, pitch, and vertical positions and velocities of the sprung mass. Nonzero cost coefficients are used for the tire deflections. This tends to even out the force on each tire. The control forces are assigned nonzero cost coefficients. This limits the sprung mass accelerations. The relative suspension
travels are also weighted in an effort to limit the travel of the unsprung masses.

The relative magnitudes of these cost coefficients are assigned such that the contribution from each term in the cost function is approximately the same order of magnitude. For example, quantities such as suspension travel and dynamic tire deflection have units of meters and experience a range of only a small portion of a meter. Other quantities, such as the control forces, have units of Newtons and undergo changes of several hundred Newtons. Therefore, the coefficients on the suspension travel and dynamic tire deflections are much larger than the weighting on the control forces. The resulting feedback gain matrix is given in Appendix C.

Since the cost function is derived in an intuitive way, it may not optimize ride in the traditional sense. However, these cost coefficients give good results and are used throughout this work. Thompson [4] used similar coefficients for the relative suspension travel and dynamic tire deflection.

3.2 Simulation Results

Systems with multiple inputs do not easily lend themselves to analysis in the frequency domain. Therefore, selective time responses will be used to evaluate the active
suspension system. This section presents several time plots showing the effects of the active and passive suspensions on the sprung mass.

Using the feedback gain matrix computed with the Riccati equation and given in Appendix C, the vehicle is simulated over various terrains. One of the terrains chosen is a slanted bump as shown in Figure 3.3. This terrain excites the roll mode of the sprung mass. The vehicle is assumed to be traveling forward at 24.59 m/s (55 mph). The simulation is then run for ten seconds. Figure 3.4 presents the time histories of the bumps that the four tires experience.

The vehicle is simulated over this terrain with and without the active suspension. Figure 3.5 presents the roll angle as a function of time. Figure 3.6 presents the pitch angle as a function of time. Figure 3.7 presents the vertical displacement of the mass center of the sprung mass as a function of time. These figures indicate that the response of the sprung mass with the active suspension is reduced compared to the passive model. In particular, peak roll is reduced by 95%, peak pitch is reduced by 50%, and peak bounce is reduced by 80%. In addition, the model with active suspension damps out quicker than the vehicle with the passive suspension. To determine the required bandwidth of the active suspension system, the actuator forces are
FIGURE 3.3. Slanted bump terrain

converted from the time domain into the frequency domain by
the use of a Fast Fourier Transform algorithm. Figure 3.8
presents the left front and left rear actuator forces as
functions of frequency. Very little force is required above
12 Hz. This indicates that for this bump, responses above
12 Hz could be ignored in the control law. Figure 3.9
presents the total passive suspension force versus frequency
for the same bump. The amplitudes of the forces due to the
spring and damper in the passive suspension are generally
higher than the amplitudes of the forces due to the
actuators in the active suspension.
FIGURE 3.4. Time histories of the tires over the slanted bump.
FIGURE 3.5. Roll angle during the slanted bump simulation
FIGURE 3.6. Pitch angle during the slanted bump simulation

WITH CONTROL

WITHOUT CONTROL
FIGURE 3.7. Vertical displacement during the slanted bump simulation
FIGURE 3.8. Actuator forces during the slanted bump simulation.
FIGURE 3.9. Passive suspension forces during the slanted bump simulation
The second terrain used is the double bump. It excites the pitch mode of the sprung mass at the pitch natural frequency. The amplitude of the bumps on the right side of the automobile are twice as large as the bumps on the left side. Thus, the roll mode is also excited. Figure 3.10 presents an overhead view of the road while Figure 3.11 presents the time histories of the bumps the four tires experience. The vehicle is again assumed to be traveling forward at 24.59 m/s (55 mph). This simulation is also run for ten seconds.
FIGURE 3.11. Time histories of the tires over the double bump
Figures 3.12 through 3.14 present the roll angle, pitch angle, and the vertical displacement of the sprung mass, respectively. These plots show that the active suspension reduces the response of the sprung mass substantially. In particular, peak roll is reduced by 93%, peak pitch is reduced by 91%, and peak bounce is reduced by 93%. As in the previous case, the response damps out quicker than the response from the vehicle with a passive suspension. Figure 3.15 presents the actuator forces in the frequency domain. In this case, the control forces are small above 10 Hz. This would indicate, for this bump, inputs above 10 Hz could be ignored in the control law. Figure 3.16 presents the total passive suspension force in the frequency domain for the double bump road profile. The amplitudes of the forces due to the actuators in the active suspension are generally much lower than the forces in the passive suspension.
FIGURE 3.12. Roll angle during the double bump simulation.
FIGURE 3.13. Pitch angle during the double bump simulation
FIGURE 3.14. Vertical displacement during the double bump simulation
FIGURE 3.15. Actuator forces during the double bump simulation.
FIGURE 3.16. Passive suspension forces during the double bump simulation.
4. TEN DEGREE OF FREEDOM MODEL

This chapter presents a model used to study the effects of an active suspension system on handling. Three additional degrees of freedom are added to the seven DOF model: longitudinal, lateral, and yaw motions. The vehicle model is used to simulate several handling and braking maneuvers. The response of the vehicle with and without active suspension is presented and compared.

4.1 Equations of Motion

Adding three degrees of freedom to the model introduced three additional differential equations. The sprung mass has six DOF: longitudinal, lateral, vertical, pitch, roll, and yaw. Each unsprung mass has one bounce DOF. The equations for this model are given in Appendix D. The model was assumed to have negligible roll steer.

Because the yaw angle is not small, the equations for this model are nonlinear. Therefore, the model cannot be converted into state-space form and a new feedback gain matrix cannot be computed by using the Riccati equation. However, it seems reasonable to use the parameters for the active suspension derived for the seven DOF model. This will illustrate the effect of an active suspension designed only for ride comfort on directional response.
A nonlinear lateral tire force model developed by Dugoff [28] is used for the ten DOF model. The lateral forces are nonlinear functions of longitudinal and lateral tire velocities, steering angles, and normal loads. The longitudinal braking forces are applied at the tire-road interface. Due to the load transfer during deceleration, the braking forces are proportioned such that 70% of the total force is on the front tires and 30% is on the rear tires.

4.2 Simulation Results

This section presents the results of the ten DOF handling model. Three different maneuvers are presented here. The first is a step steer of .04 radians. This input will lead to a so-called J-turn. The second is a double sinusoidal steer. This steering input gives a response similar to a double lane change maneuver. The third is a 0.4 g straight line braking maneuver. The model is simulated with both active and passive suspensions for each steering and braking input. The road profile is flat and the vehicle is initially travelling forward at 24.59 m/s (55 mph).

Roll steer is not included in this model. However, the previous results lead to the expectation that the active system will significantly reduce sprung mass motion during
handling maneuvers. This would indicate that if roll steer is present in a vehicle, an active suspension system would significantly alter the handling of the vehicle. This effect is not studied here. Rather, the concern is with directional changes which may result when the active suspension influences the normal load at the tire-road interface.

Figure 4.1 presents the lateral acceleration of the sprung mass for the J-turn maneuver. Note that even though the acceleration level is about 0.5 g, the active and passive suspensions give almost identical lateral acceleration results. However, Figure 4.2 indicates that the roll angle of the sprung mass is significantly reduced by the active suspension.

Figure 4.3 presents the steering input for a double sine wave steer maneuver. Figure 4.4 presents the lateral acceleration of the sprung mass. This figure shows only a slight difference in the directional response of the passive and active suspensions. This again indicates that the handling is unaffected by a ride optimized active suspension. However, Figure 4.5 indicates that the roll is significantly reduced during the handling maneuver.

Figure 4.6 presents the longitudinal acceleration for the braking maneuver. Again, the figure indicates little difference between the active and passive suspensions. This
FIGURE 4.1. Lateral acceleration during step steer maneuver
FIGURE 4.2. Roll angle during step steer maneuver
FIGURE 4.3. Steering angle versus time for sinusoidal steer maneuver
FIGURE 4.4. Lateral acceleration during sinusoidal steer maneuver
FIGURE 4.5. Roll angle during sinusoidal steer maneuver
suggests that braking is unaffected by this active suspension. However, Figure 4.7 shows that the pitch angle is considerably reduced by the active suspension.

This active suspension system, whose gains are calculated based on ride considerations only, exhibits essentially the same handling and braking response as the passive suspension system. However, as expected, the ride quality is greatly improved by the active suspension system.
FIGURE 4.6. Longitudinal acceleration during braking maneuver
FIGURE 4.7. Pitch angle during braking maneuver
5. THE EFFECTS OF ACTIVE SUSPENSIONS ON HANDLING

This chapter presents an active suspension used to alter the handling characteristics of the vehicle. Several researchers have shown the relative roll stiffnesses of the front and rear suspensions affect the handling of the vehicle. This effect depends on tire nonlinearities, and thus will not affect handling in the linear range. The linear range for this model is up to about 0.3 g. The active suspension system can be used to change the relative roll stiffness of the front and rear suspensions. To do this, the entries in the gain matrix corresponding to the roll angle are modified, thus changing the handling characteristics. It is important to note that the control system is no longer an optimal system in relation to the cost function defined in section 3.1.

To determine the steady state effects of the various active suspensions on the handling characteristics of the automobile, several steady turn simulations were run. The vehicles travel at a constant forward speed of 25 m/s. Different steering angles are input to determine the steady state lateral accelerations. The first active suspension has a larger roll stiffness in the front and a smaller roll stiffness in the rear than the ride optimized active suspension. The second case uses an active suspension in which the roll stiffness is increased in the rear and
decreased in the front. The ride optimized active suspension and passive suspension cases are also included. Figure 5.1 presents a summary of the steady state portion of the results. All of the vehicles exhibit an understeer of approximately three degrees per g in the linear range. The vehicle with the roll stiffness shifted forward has more understeer in the nonlinear range than the ride optimized active suspension. The vehicle with the roll stiffness shifted rearward has less understeer in the nonlinear range. However, all of the vehicles with active suspensions, as well as the vehicle with the passive suspension, exhibit limit understeer with a plow out response at about 0.6 g.

As a specific example, the vehicle with each of the four suspensions was simulated in the nonlinear range in a 0.5 g J-turn. Figure 5.2 presents the trajectories of a vehicle using four different suspension systems: three active and one passive. The steering angle was identical for all of the maneuvers. The suspension with the roll stiffness shifted rearward caused the car to steer a tighter arc when compared to the car with the ride optimized active suspension. The vehicle with the suspension in which the roll stiffness is shifted forward exhibits a larger arc. Figure 5.3 shows that the roll angle is affected by altering the gain matrix. The car with the roll stiffness shifted rearward has a larger roll angle than the standard active
FIGURE 5.1. Steady state lateral acceleration versus steering angle
suspension. The converse is true for the car with the roll stiffness shifted forward. The roll angles for all of the active suspensions are smaller than the passive suspension.
FIGURE 5.2. Trajectories of the vehicles during steering
FIGURE 5.3. Roll angle of the sprung mass during steering.
6. CONCLUSIONS

This thesis developed two different vehicle models to investigate the effects of an active suspension. One was a linear seven degree of freedom model. The second was a nonlinear ten degree of freedom model.

The seven degree of freedom model was used to study the ride effects. The active suspension reduced the motions of the sprung mass by a substantial amount.

The ten degree of freedom model was used to study the effects of the active suspension on the directional response characteristics of the vehicle. The handling characteristics exhibited by the active suspension are very similar to those of the passive suspension. However, the active suspension did significantly reduce sprung mass motions during the handling maneuvers. It was then illustrated that by altering various feedback gains, active suspensions can be made to change the handling characteristics in the nonlinear range.

Of course, the goal of all work in this area is to design better suspensions. Future work should include analysis which considers the penalty for giving up hard to sense state variables and experiments which elucidate the potential and limitations of active suspensions. It seems that an important link between active suspensions and braking and turning would be to analyze and implement
suspension actions which use braking and steering as state variables.
7. BIBLIOGRAPHY


8. APPENDIX A - EQUATIONS OF MOTION FOR SEVEN DOF MODEL

Equations of motion:

\[ M_s \ddot{Z} = F_A + F_B + F_C + F_D \]

\[ I_x \dot{\phi} = \frac{t_f F_A}{2} - \frac{t_f F_C}{2} + \frac{t_r F_B}{2} - \frac{t_r F_D}{2} \]

\[ I_y \ddot{\theta} = -a F_A + b F_B - a F_C + b F_D \]

\[ M_1 \ddot{Z}_1 = -F_A + k_t (Z_{1f} - Z_1) \]

\[ M_2 \ddot{Z}_2 = -F_B + k_t (Z_{1r} - Z_2) \]

\[ M_3 \ddot{Z}_3 = -F_C + k_t (Z_{rf} - Z_3) \]

\[ M_4 \ddot{Z}_4 = -F_D + k_t (Z_{rr} - Z_4) \]
where:

\[
\begin{align*}
F_A &= k_f (Z_1 - Z_a) + C_f (\dot{Z}_1 - \dot{Z}_a) + F_1 \\
F_B &= k_r (Z_2 - Z_b) + C_r (\dot{Z}_2 - \dot{Z}_b) + F_2 \\
F_C &= k_f (Z_3 - Z_c) + C_f (\dot{Z}_3 - \dot{Z}_c) + F_3 \\
F_D &= k_r (Z_4 - Z_d) + C_r (\dot{Z}_4 - \dot{Z}_d) + F_4 \\
Z_a &= Z + \frac{t_f}{2} \phi - a \theta \\
\dot{Z}_a &= \dot{Z} + \frac{t_f}{2} \dot{\phi} - a \dot{\theta} \\
Z_b &= Z + \frac{t_r}{2} \phi + b \theta \\
\dot{Z}_b &= \dot{Z} + \frac{t_r}{2} \dot{\phi} + b \dot{\theta} \\
Z_c &= Z - \frac{t_f}{2} \phi - a \theta \\
\dot{Z}_c &= \dot{Z} - \frac{t_f}{2} \dot{\phi} - a \dot{\theta} \\
Z_d &= Z - \frac{t_r}{2} \phi + b \theta \\
\dot{Z}_d &= \dot{Z} - \frac{t_r}{2} \dot{\phi} + b \dot{\theta}
\end{align*}
\]
Columns 1 and 2 of the A matrix:

\[
A = \begin{pmatrix}
\frac{-C_f t_f^2 - C_r t_r^2}{2 I_x} & 0 \\
0 & \frac{2}{I_x} (-a^2 C_f - b^2 C_r) \\
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & \frac{2}{M_2} (a C_f - b C_r) \\
0 & 0 \\
\frac{t_f C_f}{2 M_1} & -a C_f \\
0 & -a C_f \\
\frac{t_r C_r}{2 M_1} & -b C_r \\
0 & -b C_r \\
\frac{t_f C_f}{2 M_2} & -a C_f \\
0 & -a C_f \\
\frac{t_r C_r}{2 M_2} & -b C_r \\
0 & -b C_r \\
\frac{t_f C_f}{2 M_3} & -a C_f \\
0 & -a C_f \\
\frac{t_r C_r}{2 M_3} & -b C_r \\
0 & -b C_r \\
\frac{t_f C_f}{2 M_4} & -a C_f \\
0 & -a C_f \\
\frac{t_r C_r}{2 M_4} & -b C_r \\
0 & -b C_r \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]
Columns 3 and 4 of the A matrix:

\[
\begin{array}{cccc}
-k_f t_f^2 - k_r t_r^2 & 0 \\
\frac{2}{I_x} & \frac{2}{I_y} \left(-a^2 k_f - b^2 k_r\right) \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\frac{t_f k_f}{2 M_1} & \frac{-a k_f}{2 M_1} \\
0 & 0 \\
\frac{t_r k_r}{2 M_2} & \frac{b k_r}{2 M_2} \\
0 & 0 \\
\frac{t_f k_f}{2 M_3} & \frac{-a k_f}{2 M_3} \\
0 & 0 \\
\frac{t_r C_r}{2 M_4} & \frac{b C_r}{2 M_4} \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{array}
\]
Columns 5 and 6 of the A matrix:

\[
\begin{array}{cccc}
0 & 0 & \frac{2}{I_y} \left( a k_f - b k_r \right) & \frac{2}{I_y} \left( a c_f - b c_r \right) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & \frac{2}{M_s} \left( -k_f - k_r \right) & \frac{2}{M_s} \left( -c_f - c_r \right) \\
0 & \frac{k_f}{M_1} & \frac{c_f}{M_1} & 0 \\
0 & \frac{k_r}{M_2} & \frac{c_r}{M_2} & 0 \\
0 & \frac{k_f}{M_3} & \frac{c_f}{M_3} & 0 \\
0 & \frac{k_r}{M_4} & \frac{c_r}{M_4} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}
\]
Columns 7 through 10 of the A matrix:

\[
\begin{array}{cccc}
{t_f} & {k_f} & {t_f} & {C_f} \\
\frac{2}{I_x} & \frac{2}{I_x} & \frac{2}{I_x} & \frac{2}{I_x} \\
-a & k_f & -a & C_f \\
\frac{I_y}{I_y} & \frac{I_y}{I_y} & \frac{I_y}{I_y} & \frac{I_y}{I_y} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{k_f}{M_s} & \frac{C_f}{M_s} & \frac{k_r}{M_s} & \frac{C_r}{M_s} \\
0 & 1 & 0 & 0 \\
\frac{-k_f - k_t}{M_1} & \frac{-C_f}{M_1} & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & \frac{-k_r - k_t}{M_2} & \frac{-C_r}{M_2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}
\]
Columns 11 through 14 of the A matrix:

<table>
<thead>
<tr>
<th>-t_f k_f</th>
<th>-t_f C_f</th>
<th>-t_r k_r</th>
<th>-t_r C_r</th>
</tr>
</thead>
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<td>( \frac{2}{I_x} )</td>
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<tr>
<td>-a k_f</td>
<td>-a C_f</td>
<td>b k_r</td>
<td>b C_r</td>
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<td>( \frac{C_f}{M_s} )</td>
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<td>( \frac{C_r}{M_s} )</td>
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<td>( \frac{-k_f - k_t}{M_3} )</td>
<td>( \frac{-C_f}{M_3} )</td>
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<td>( \frac{-k_r - k_t}{M_4} )</td>
<td>( \frac{-C_r}{M_4} )</td>
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</tbody>
</table>
Columns 15 through 18 of the A matrix:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{k_t}{M_1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \frac{k_t}{M_2} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{k_t}{M_3} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{k_t}{M_4} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]
The $B_1$ matrix:

$$
B_1 = \begin{bmatrix}
\frac{t_f}{2} & \frac{t_f}{2} & -\frac{t_r}{2} & -\frac{t_r}{2} \\
2I_x & 2I_x & 2I_x & 2I_x \\
-a & b & -a & b \\
\overline{I}_y & \overline{I}_y & \overline{I}_y & \overline{I}_y \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
\overline{M}_s & \overline{M}_s & \overline{M}_s & \overline{M}_s \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & \overline{M}_3 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$
The $B_2$ matrix:

$$
B_2 = 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
$$
9. APPENDIX B - PARAMETERS FOR VEHICLE MODELS.

\[ a = 0.945 \, \text{m} \]
\[ b = 1.718 \, \text{m} \]
\[ C_f = 348 \, \text{N} \, \text{s} / \text{m} \]
\[ C_r = 782 \, \text{N} \, \text{s} / \text{m} \]
\[ I_f = 50 \, \text{kg} \, \text{m}^2 \]
\[ I_r = 40 \, \text{kg} \, \text{m}^2 \]
\[ I_x = 438 \, \text{kg} \, \text{m}^2 \]
\[ I_y = 2337 \, \text{kg} \, \text{m}^2 \]
\[ I_z = 2117 \, \text{kg} \, \text{m}^2 \]
\[ k_f = 12480 \, \text{N} / \text{m} \]
\[ k_r = 15730 \, \text{N} / \text{m} \]
\[ k_t = 240000 \, \text{N} / \text{m} \]
\[ M_s = 876 \, \text{kg} \]
\[ M_1 = 153 \, \text{kg} \]
\[ M_2 = 85 \, \text{kg} \]
\[ M_3 = 153 \, \text{kg} \]
\[ M_4 = 85 \, \text{kg} \]
\[ P = 0.178 \, \text{m} \]
\[ R = 0.178 \, \text{m} \]
\[ R_f = 0.132 \, \text{m} \]
\[ R_r = 0.263 \, \text{m} \]
\[ t_f = 1.512 \, \text{m} \]
\[ t_r = 1.470 \, \text{m} \]
10. APPENDIX C - FEEDBACK GAIN MATRIX

\[
K^T = 
\begin{bmatrix}
15962.9816 & 15037.0140 & -15962.9816 & -15037.0140 \\
-14220.3392 & 16341.7146 & -14220.3392 & 16341.7146 \\
10312.6377 & 9507.3366 & -10312.6377 & -9507.3366 \\
-10893.3714 & 8177.4130 & -10893.3714 & 8177.4130 \\
9917.9350 & 8795.8181 & 9917.9350 & 8795.8181 \\
17131.9062 & 13547.9892 & 17131.9062 & 13547.9892 \\
-10565.7290 & -3307.7182 & -8826.7066 & 5100.5252 \\
-173.9951 & 446.0606 & 72.1711 & -182.1515 \\
804.8278 & -20539.4002 & 2815.1418 & -7913.3035 \\
339.4696 & 108.3643 & -151.1296 & 75.9286 \\
-8826.7066 & 5100.5252 & -10565.7290 & -3307.7182 \\
72.1711 & -182.1515 & -173.9951 & 446.0606 \\
2815.1418 & -7913.3035 & 804.8278 & -20539.4002 \\
-151.1296 & 75.9286 & 339.4696 & 108.3643 \\
11821.8584 & 3891.8500 & 8948.8145 & -5973.8881 \\
1081.7301 & 21379.5052 & -2910.4103 & 7001.3249 \\
8948.8145 & -5973.8881 & 11821.8584 & 3891.8500 \\
-2910.4103 & 7001.3249 & 1081.7301 & 21379.5052 
\end{bmatrix}
\]
11. APPENDIX D - EQUATIONS OF MOTION FOR TEN DOF MODEL

\[
M_s \ddot{z} = F_A + F_B + F_C + F_D
\]

\[
I_x \ddot{\phi} = \frac{t_f}{2} F_A - \frac{t_f}{2} F_C + \frac{t_r}{2} F_B - \frac{t_r}{2} F_D + Y_1 R_f + Y_2 R_r + Y_3 R_f + Y_4 R_r
\]

\[
I_y \ddot{\theta} = -a F_A + b F_B - a F_C + b F_D - X_1 P - X_2 P - X_3 P - X_4 P
\]

\[
M_1 \ddot{z}_1 = -F_A + k_t (Z_{1f} - Z_1)
\]

\[
M_2 \ddot{z}_2 = -F_B + k_t (Z_{1r} - Z_2)
\]

\[
M_3 \ddot{z}_3 = -F_C + k_t (Z_{rf} - Z_3)
\]

\[
M_4 \ddot{z}_4 = -F_D + k_t (Z_{rr} - Z_4)
\]

\[
M_s (\dot{v}_x - v_y \omega_z + p \ddot{\phi}) = X_1 + X_2 + X_3 + X_4
\]

\[
M_s (\dot{v}_y + v_x \omega_z - R \ddot{\phi}) = Y_1 + Y_2 + Y_3 + Y_4
\]

\[
(I_z + 2 I_f + 2 I_r) \dot{\omega}_z = -\frac{t_f}{2} X_1 + \frac{t_f}{2} X_3 - \frac{t_r}{2} X_2 + \frac{t_r}{2} X_4 + Y_1 a + Y_2 b + Y_3 a + Y_4 b
\]
where:

\[ F_A = k_f ( z_1 - z_a ) + c_f ( \dot{z}_1 - \dot{z}_a ) + F_1 \]
\[ F_B = k_r ( z_2 - z_b ) + c_r ( \dot{z}_2 - \dot{z}_b ) + F_2 \]
\[ F_C = k_f ( z_3 - z_c ) + c_f ( \dot{z}_3 - \dot{z}_c ) + F_3 \]
\[ F_D = k_r ( z_4 - z_d ) + c_r ( \dot{z}_4 - \dot{z}_d ) + F_4 \]
\[ z_a = z + \frac{t_f}{2} \phi - a \theta \]
\[ \dot{z}_a = \dot{z} + \frac{t_f}{2} \dot{\phi} - a \dot{\theta} \]
\[ z_b = z + \frac{t_r}{2} \phi + b \theta \]
\[ \dot{z}_b = \dot{z} + \frac{t_r}{2} \dot{\phi} + b \dot{\theta} \]
\[ z_c = z - \frac{t_f}{2} \phi - a \theta \]
\[ \dot{z}_c = \dot{z} - \frac{t_f}{2} \dot{\phi} - a \dot{\theta} \]
\[ z_d = z - \frac{t_r}{2} \phi + b \theta \]
\[ \dot{z}_d = \dot{z} - \frac{t_r}{2} \dot{\phi} + b \dot{\theta} \]