CRACK-TIP DIFFRACTION IN A TRANSVERSELY ISOTROPIC SOLID

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ABSTRACT

Crack diffraction in a transversely isotropic material is analyzed. The solution is given for the diffracted field generated by incidence of a plane time-harmonic wave on a semi-infinite crack located in a plane normal to the axis of symmetry of the material. The exact solution is obtained by Fourier integral methods and the Wiener-Hopf technique. The diffraction coefficients have been used in the context of the geometrical theory of diffraction to compute high-frequency scattering by a crack of finite length. Applications to scattering by delaminations in a medium of periodic layering have been considered for the case that the wavelength and the crack length are of the same order of magnitude, but both are much larger than the larger layer thickness.

INTRODUCTION

In this paper we consider the problem of diffraction from a stress-free crack in an anisotropic solid. The same problem in an isotropic solid is now well understood, and has been discussed by many authors, see Ref.[1] for a review. However, in practice, most materials exhibit some form of anisotropy. Examples of transversely isotropic solids are laminated media with periodic layering (on a length scale much larger than the characteristic thicknesses of the layers), and metals that have been subjected to certain manufacturing processes (e.g., rolling). Anisotropy may also be induced in an isotropic solid by large pre-stressing or straining [2].
We first consider a semi-infinite crack located in a plane of symmetry of a transversely isotropic material. The incident wave motion is confined to the plane which is normal to the crack edge. Therefore, the scattering geometry is two dimensional. A related problem of wave motion generated by symmetric concentrated forces on the crack faces in a material of cubic isotropy was considered in Ref.[3]. The current problem can be viewed as a canonical problem whose solution is necessary for the development of an anisotropic geometrical theory of diffraction (GTD).

The number of known solutions for scattering problems in anisotropic solids is much smaller than for isotropic elasticity. The simplest problems, reflection and refraction of plane waves, have been discussed in detail in Ref.[4]. The present paper, which is based on Ref.[5], is concerned only with the direct scattering from cracks in anisotropy. The inverse problem of locating and sizing a crack in anisotropy has been discussed in Ref.[6].

THE INCIDENT PLANE WAVES

As a preliminary to the discussion of diffraction by a semi-infinite crack, we briefly review the propagation of plane waves in uncracked transversely isotropic solids. The geometry is shown in Fig. 1. The y-axis is parallel to the axis of symmetry of the solid. The term exp(-iωt) will be omitted.

For plane strain the stress-strain relations may be expressed in the form
\[ \sigma_x = \rho [au_x + (c-d)v_y] \]  
\[ \sigma_y = \rho [(c-d)u_x + bv_y] \]  
\[ \sigma_{xy} = \rho d(u_y + v_x), \]  

where \( u_x = \partial u/\partial x, \) etc., and

\[ a = \frac{C_{11}}{\rho}, \quad b = \frac{C_{33}}{\rho}, \quad c = \frac{(C_{13} + C_{44})}{\rho}, \quad d = \frac{C_{44}}{\rho}. \]  

Here \( C_{11}, C_{13}, C_{33} \) and \( C_{44} \) are four of the five elastic constants that enter in the 3-D stress-strain relations of a transversely isotropic solid. It is noted that for cubic isotropy we have \( a = b, \) while the equations reduce to those for an isotropic solid when \( a = b = c = a - d \) (then \( \rho a = \lambda + 2\mu, \rho c = \lambda + \mu \) and \( \rho d = \mu \)). By the use of (2.1) - (2.3) the displacement equations of motion are obtained as

\[ a u_{xx} + d u_{yy} + c v_{xy} = -\omega^2 u \]  
\[ c u_{xy} + d v_{xx} + b v_{yy} = -\omega^2 v. \]  

Two types of plane waves are possible. These are of the form

\[ u = \tilde{d}^n(\lambda, \gamma) \exp(i\omega(\lambda x + \gamma y)), \]  

where \( n = 1 \) or 2. The unit vectors \( \tilde{d}^1 \) and \( \tilde{d}^2 \) are the displacement vectors given by

\[ \frac{\tilde{d}^n_2}{\tilde{d}^n_1} = M_n(\lambda)/c \beta_n. \]  

The quantities \( \beta_n \) and \( M_n \) are

\[ \beta^2_n = \frac{b+d-L\lambda^2}{2bd} + (-1)^n \left[ \frac{(b+d-L\lambda^2)^2}{2bd} - \frac{a^2}{b^2} \mu_n^2 \mu_2^2 \lambda^2 \right], \]  

and

\[ M_n(\lambda) = -a\lambda^2 - d\beta^2_n + 1, \]  

where

\[ L = ab + d^2 - c^2, \mu^2_n = p_n^2 - \lambda^2; \quad p_1^2 = 1/a, \quad p_2^2 = 1/d. \]
In the isotropic limit, type 1 and 2 waves are the usual longitudinal and transverse waves, respectively. Therefore, we refer to waves of type 1 and 2 as quasi-longitudinal (qL) quasi-transverse (qT) waves, respectively.

It is well known that the energy of a wave in an anisotropic solid does not propagate in the same direction as the phase [4]. The phase velocity is the inverse of the slowness \((\lambda^2 + \beta^2)^{1/2}\). The corresponding ray velocity is always greater than or equal to the phase velocity. A simple algorithm relating the velocities and directions of propagation is given in Ref.[6]. The ray velocity and direction are of most importance physically since they determine the speed and direction of ultrasonic signals.

**DIFFRACTION BY A SEMI-INFINITE CRACK**

The total field is expressed as the sum of the incident and the diffracted fields

\[
\begin{align*}
\mathbf{u}^{\text{tot}} &= \mathbf{u}^{\text{in}} + \mathbf{u}^{\text{d}},
\end{align*}
\]

where \(\mathbf{u}^{\text{in}}\) is given by (2.18) and \(\mathbf{u}^{\text{d}}\) represents the diffracted field. In the following we denote by \(\lambda_1\) the fixed value of \(\lambda\) for the incident wave.

We will seek a solution as a superposition of plane waves of types 1 and 2:

\[
\begin{align*}
u &= \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} A_m(\lambda) e^{i\omega(\lambda x + \beta_m y)} d\lambda \\
v &= \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{M_m(\lambda)}{c \lambda \beta_m} A_m(\lambda) e^{i\omega(\lambda x + \beta_m y)} d\lambda.
\end{align*}
\]

The stress components of the diffracted field are denoted by \(\sigma_x\), \(\sigma_y\), and \(\sigma_{yx}\). Since the total tractions must vanish on the faces of the crack, we have for \(y=0, x \geq 0\):

\[
\begin{align*}
\sigma_y &= -\sigma_y^0 \exp(i\omega \lambda_1 x), \\
\sigma_{yx} &= -\sigma_{yx}^0 \exp(i\omega \lambda_1 x),
\end{align*}
\]

where \(\sigma_y^0(\lambda_1)\) and \(\sigma_{yx}^0(\lambda_1)\) are constants by

\[
\begin{align*}
\sigma_y^0 = i\omega \beta_n G_n, \\
\sigma_{yx}^0 = i\omega \beta_n F_n,
\end{align*}
\]
where
\[ G_n = c(c-d)\lambda^2 + bM_n \quad F_n = \left( d\lambda/\beta_n \right) \left( c\beta_n^2 + M_n \right). \] (3.6)

The plane of the crack is perpendicular to the axis of the transversely isotropic material, and hence (3.4) and (3.5) will give rise to displacement fields that are symmetric and antisymmetric, respectively, with respect to the plane \( y=0 \). It is then convenient to separately formulate symmetric and antisymmetric problems for the half-plane \( y \geq 0 \). These are

**Symmetric Problem:**
\[ y = 0, \quad -\infty < x < \infty : \quad \sigma_{yx} = 0 \] (3.7)
\[ -\infty < x \leq 0 : \quad v = 0 \] (3.8)
\[ 0 < x < \infty : \quad \sigma_y = -\sigma_y \exp(i\omega \lambda x) \] (3.9)

**Antisymmetric Problem:**
\[ y = 0, \quad -\infty < x < \infty : \quad \sigma_y = 0 \] (3.10)
\[ -\infty < x \leq 0 : \quad u = 0 \] (3.11)
\[ 0 < x < \infty : \quad \sigma_{yx} = -\sigma_{yx} \exp(i\omega \lambda x) \] (3.12)

The symmetric and antisymmetric problems are formulated in Ref.[5] as Wiener-Hopf problems for the functions \( U(\lambda) \) and \( V(\lambda) \), respectively, where
\[ (U(\lambda),V(\lambda)) \equiv (2\pi\omega)^{-1} \int_{-\infty}^{\infty} (u(x,0),v(x,0))e^{-i\omega \lambda x} \, dx. \] (3.13)

The solution of the Wiener-Hopf problem is discussed in detail in Ref.[5]. We will omit all details here.

The solution for \( u \) to the full problem may be expressed as in eqns.(3.2) and (3.3) where \( A_1(\lambda) \) and \( A_2(\lambda) \) are now
\[ A_1(\lambda) = \frac{a\mu_1 G_1 U - b\mu_2 F_2 V}{abd\mu_1(\beta_1^2-\beta_2^2)} \] (3.14)
\[ A_2(\lambda) = \frac{-a\mu_1 F_1 U + b\mu_2 F_1 V}{abd\mu_1(\beta_1^2-\beta_2^2)}. \] (3.15)
Having found the exact solution as an integral transform, the far-field can be computed by the method of stationary phase. Let $\theta_n$, $n = 1$ or 2 be the angle of incidence, defined in Fig. 1 and let $\theta$ be the angle of observation. The observation point is at a radial distance $r$ from the crack tip. From Ref.[5] we state the result that the far-field of type $m$ due to an incident wave of type $n$ is:

$$u_m^m \sim \frac{1}{(p_2 \omega r)^{1/2}} \frac{d_m^m(\lambda_0)D_n^m(\theta; \theta_n)}{\omega r} \exp[i\omega r(\lambda_o \cos \theta + \beta_m^m(\lambda_o) \sin \theta)].$$  \hspace{1cm} (3.16)

In this expression the unit displacement vector of type $m$ is divided by the square root of the non-dimensional distance $p_2 \omega r$, where $p_2$ is defined in eq.(2.11). The quantity $D_m^n$ is the diffraction coefficient which couples the incident and outgoing amplitudes. Its specific form is given in Ref.[5]. The slowness vector $(\lambda_o, \beta_m^m(\lambda_o))$ corresponds to the ray in direction $\theta$. The far-field of eq.(3.16) is singular at the physical elastodynamics reflection and shadow boundaries. This is attributable to the inadequacy of the asymptotic expansion at these boundaries. The singularities may be removed using a uniform asymptotic analysis, see for example (1).

EXAMPLES FOR A LAYERED MEDIUM

In this section we consider a laminated composite consisting of alternating plane parallel layers of two homogeneous, isotropic, elastic materials with the same density. We use an effective modulus theory [7] whereby the laminated medium is approximated by an equivalent transversely anisotropic but homogeneous elastic material. Although this is a static theory, it may be used for dynamic problems when the typical wavelength is much larger than the larger of the two layer thicknesses.

For the examples of this Section we consider the case that the elastic materials of both layers have a Poisson's ratio of 1/3, which corresponds to a bulk-wave-speed ratio of 2. Let the $x$-axis be in the direction of the layers and the $y$-axis be normal to the layers. Let $e, l$ be the thicknesses and $\mu, \mu$ the shear moduli of the two elastic layers. The parameters $a, b, c$ and $d$ follow from Ref.[7], p.33. Since it is only the relative magnitude of these constants that is of importance we need only the ratios

$$a/d = 4 + 3\epsilon(1-\epsilon)[1/(\alpha - \alpha^2) - 4]$$ \hspace{1cm} (4.1)
b/d = 4, c/d = 3,

where we have defined 0 ≤ α ≤ 1 and 0 ≤ ε ≤ 1 as

\[ \alpha = \mu / (\mu + \bar{\mu}) \]
\[ \varepsilon = l / (l + \bar{l}) . \]

We now consider a semi-infinite crack parallel to the layers in the composite medium. The solution to the diffraction problem is obtained using the effective medium hypothesis. We have taken the specific values of ε = .5 and α = (1 + \sqrt{5}/8)/2 ≈ .9 which corresponds to a/d = 9. The longitudinal wave speed parallel to the layers is then one and a half times that normal to the layers. The two slowness surfaces are shown in Fig. 2a.

Fig. 2: (a) The slowness surfaces and (b) qL-qL pulse echo diffraction coefficient for a/d = 9.

The qL to qL diffraction coefficient is plotted in Fig. 2b for θ = π + θ₁, which corresponds to pulse-echo observation. We note the singularity at θ₁ = π/2. We have also plotted in Fig. 2b the ratio of the isotropic diffraction coefficient (a/d = 4) to the anisotropic coefficient (----). The ratio differs markedly from unity over a wide range of angles.

Cracks in the direction of the layering (delaminations) frequently occur. Let a plane quasi-longitudinal wave of unit amplitude be incident on a crack of finite length D at angle θ. The diffracted quasi-longitudinal field in the back scattered direction can then be computed by using eq.(3.16) for each tip of the crack individually. Superposition of the two crack-tip diffractions gives a high-frequency approximation to the back
scattered or pulse-echo signal. It is assumed that $D$ is of the same order as the wavelength. Since the wavelength must be much larger than the layer thicknesses for the effective modulus theory to be valid, we must also require that $D$ is much larger than the larger layer thickness.

Taking the origin at the center of the crack, we have plotted the far field amplitude $A = |(r/d)^{1/2} u(r, \theta)|$ in Fig. 3 as a function of the dimensionless frequency $D\omega/\sqrt{\nu}$. We note that $\omega/\sqrt{\nu}$ is the longitudinal wave number in the direction normal to the slit ($\theta = \pi/2$). The isotropic result ($a/d = 4$) is also plotted in Fig. 3 for comparison. Because $b$ is the same for the two examples, the effect of anisotropy disappears at normal incidence.

![Fig. 3: Backscattered amplitude from a finite crack. Dashed line is the isotropic result. $k = D\omega/\sqrt{\nu}$.](image)

The isotropic results would be predicted if, for example, one assumed that the solid was isotropic with $L$-wave speed equal to the wave speed in the $y$-direction. Our results show that the anisotropy is important, especially at angles of incidence near grazing.

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