ACOUSTIC TOMOGRAPHIC RECONSTRUCTION OF ANOMALIES 
IN THREE-DIMENSIONAL BODIES

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SUMMARY

We present a theory of acoustic tomography based on data processing shear wave scattered field over an observational plane, including frequency and polarization diversities. The theory is based on the Gubernatis formulation of scattering and does not require solution of the Fredholm equation for material displacement $u$. An essential feature of the theory is an expansion of $v_{jk}u_k$ in even powers of frequency to obtain an "equivalent frequency insensitive" source in the anomaly.

We treat data inversion both in cartesian coordinates via the two-dimensional fast-Fourier transform (FFT) and in cylindrical coordinates via the one-dimensional Hankel transform. We note the advantages of polarization diversity. Sampling formulae are quoted. AR (auto regressive) and ARMA (auto regressive moving-average) modeling are mentioned as means of improving anomaly resolution. Both frequency and polarization diversities tend to reduce speckle noise.

THEORETICAL FORMULATION

In the Gubernatis formulation of scattering,\textsuperscript{1} the $i$th cartesian component of scattered (superscript $s$) displacement $u^s_i(\vec{r})$ at observational point $\vec{r}_o$ due to a shear wave with propagation vector $\vec{k}$ is

$$
u^s_i(\vec{r}) = (\delta_{ij} - \hat{r}_i \cdot \hat{r}_j) \frac{k^2}{4\pi\rho\omega} \int_{\text{an}} d\vec{r} e^{-i\vec{k}\cdot(\vec{r}_o-\vec{r})} \sum_k v_{jk}(\vec{r}) u_k(\vec{r}), \quad (1)$$

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where $\hat{r}_i, \hat{r}_j$ are unit vectors, $\rho = \text{ambient density in the body}$, $\omega = \text{circular frequency (exp (i\omega t) dependence)}$, $\vec{r} = \text{"local" position vector in the anomaly}$, $\nu_{jk}$ is the material parameter with terms proportional to the changes in density, $\delta\rho(\vec{r})$, and stiffness $\delta C(\vec{r})$ in the anomaly, and $u_k$ is the $k$th cartesian component of strain.

Displacement $u_k$ in the anomaly obeys the second-type Fredholm equation.

$$u_k(\vec{r}) = u_k^0(\vec{r}) + \int d\vec{r} g_{kl}(\vec{r}, \vec{r}') v_{lm}(\vec{r}') u_m(\vec{r}) \quad ,$$

(2)

$u_k^0$ being the incident wave and $g_{kl}$ the Green function with a shear term proportional to $\exp(-i\beta R)/R$ and a mixed term proportional to $[\exp(-i\beta R)/R - \exp(-i\alpha R)/R]$, $R = |\vec{r} - \vec{r}'|$. $g$ must be appropriately modified for the boundaries of a real body.

It seems reasonable to expand $v_{jk}u_k$ of (1) in even powers of frequency in the anomaly as

$$\sum_k v_{jk}(\vec{r}) u_k(\vec{r}) = \omega^2 J_j^{(1)}(\vec{r}) + \omega^4 J_j^{(2)}(\vec{r}) + \ldots \quad ,$$

(3)

in which the $J_j^{(n)}$ are considered frequency insensitive. Then, the scattered field $u^S$ without the directional factor in parentheses is proportional to

$$f_j(\vec{r}) = \frac{1}{4\pi \rho c_S^2} e^{-i\vec{K} \cdot \vec{r}_0} \int d\vec{r} \omega^2 J_j^{(1)}(\vec{r}) + \omega^4 J_j^{(2)}(\vec{r}) + \ldots$$

(4)

$c_s = \omega/k$, shear wave velocity.

The $\omega^2n$ -integral in (4) can be rewritten

$$\int d\vec{r} e^{i\vec{K} \cdot \vec{r}} \omega^2 J_j^{(n)}(\vec{r}) = (-)^n c_s^2 \omega^2 \int d\vec{r} e^{i\vec{K} \cdot \vec{r}} \omega^{2n} J_j^{(n)}(\vec{r}) \quad ,$$

(5)

with which we put (4) into the form

$$f_j(\vec{r})e^{-i\vec{K} \cdot \vec{r}_0} = \int_{\text{an}} d\vec{r} e^{i\vec{K} \cdot \vec{r}} K_j(\vec{r}) \quad ,$$

(6)

where the equivalent frequency insensitive source term $K_j$ is strong.
inside and on the anomaly and zero outside,

\[ K_j(r) = \frac{1}{4\pi p C_s^2} \sum_{n=1}^{N} (-1)^n c_s^{2n} v_n^2 j_i(n)(F) \]  

(7)

Our tomographic inversion method reconstructs the "object" \( K_j(F) \) from spatially limited wavenumber data on an observation plane which yields \( f_j(F) \). Frequency \( \omega \) indexes \( k \), the wavenumber magnitude, in the range \((\omega_1, N\omega_1)\). Since \( K_j \) is frequency insensitive, we interpret (6) to be one member of a three-dimensional Fourier-transform pair between \( f_j \) in wavenumber \(-K\) space and \( K_i \) in real \(-R\) space. The problem is to invert (6) and resolve \( K_j \) sensibly from \( f_j \)-data sampled throughout a hemisphere of \( K\)-space bounded by radii \( k_1 = \omega_1/C_s \) and \( k_2 = N\omega_1/C_s \). (Note the \( K\)-space range is compatible with an observation plane which subtends a solid angle considerably less than \( 2\pi^2 \) steradians).

In the subsequent discussion, we ignore the subscript \( j \) in (6) for convenience.

Let the origin of a cartesian coordinate system be located in or near the anomaly with the observation plane perpendicular to the \( z\)-axis at \( z_0 \) \((< 0 \) on one side) and extending over the range \( \pm x_{\text{max}}/2 \), \( \pm y_{\text{max}}/2 \). The process of inverting (6) starts by writing the left side on the \( z_0\)-plane as

\[ f(x,y,\omega/C_s)e^{ik_z(x+y+k_z z_0)}, \quad k_z = \sqrt{(\omega/C_s)^2 - k_x^2 - k_y^2} \]  

(8)

Equation (6) says this is independent of \( x \) and \( y \) on the observation plane, so we can integrate it over \( x,y \) and divide by \( x_{\text{max}} y_{\text{max}} \) without changing its value:

\[ (x_{\text{max}} y_{\text{max}})^{-1} \int_{\text{obs}} dx \int dy f(x,y,\omega/C_s)e^{ik_x x + ik_y y} e^{ik_z z_0} = F(k_x,k_y) e^{ik_z z_0} \]  

(9)

We rewrite the right side of (6) by first defining the weighted projection (Fourier transform) of \( K(R) \) as

\[ \tilde{K}(x,y,k_z) = \int_{\text{an}} dz e^{ik_z z} K(x,y,z) \]  

(10)

whereupon the right side of (6) is
\[ i(k_x x + k_y y) \sim \int \int \rho d \rho \sin \beta e^{i(k_x x + k_y y)} \quad (11) \]

From (9) equal to (11) we have a two-dimensional Fourier transform relation between the \( \mathbf{k} \)-space data \( F(k_x, k_y) \) on the \( \omega/C_s \)-shell and \( K(x, y, k_z) \). By projecting the data from the various shells onto the \( k_z \)-axis we can obtain (9) over discrete \( k_x, k_y, k_z \) space. Equating (9) and (11) over this space we have

\[ F(k_x, k_y, k_z) e^{ik_z z_0} = \int \int \rho d \rho \sin \beta e^{i(k_x x + k_y y)} K(x, y, k_z) \quad (12) \]

Formally, the solution for \( K(x, y, k_z) \) is

\[ K(x, y, k_z) = \frac{1}{4\pi^2} \int dk_x \int dk_y F(k_x, k_y, k_z) e^{ik_z z_0} e^{i(k_x x + k_y y)} \quad (13) \]

and \( K(x, y, z) \) follows from the transform inverse to (10).

**Inversion in Rectangular Coordinates**

The 2D FFT can be used to solve (9) with the data at discrete intervals of \( \Delta x, \Delta y \), and also (13) for given \( k_z \) at intervals of \( \Delta k_x, \Delta k_y \). \( K(x, y, z) \) then follows from the 1D FFT of \( K \) at intervals of \( \Delta k_z \) by (10).

**Inversion in Cylindrical Coordinates**

By converting to cylindrical coordinates,

\[ x = \rho \cos \theta, \quad y = \rho \sin \theta; \quad k_x = k_\rho \cos \beta, \quad k_y = k_\rho \sin \beta \quad (14) \]

and with

\[ e^{i(k_x x + k_y y)} = e^{i k_\rho \cos(\beta - \theta)} = \sum (-i)^n j_n(k_\rho \rho) e^{i \beta - \theta} \quad (15) \]

we write (9) with

\[ f(x, y, \omega/C_s) = \sum \sum f_\rho(\rho, \omega/C_s) e^{i \beta} \quad (16) \]

\[ F(k_x, k_y) = \sum \sum F(k_\rho) e^{i \beta} \]
and find the angular harmonics are separable. The nth equation is

$$F_n(k_\rho)e^{ik_z\zeta_0} = \frac{2\pi}{x_{\text{max}} y_{\text{max}}} (-i)^n \int \rho d\rho J_n(k_\rho \rho) f_n(\rho),$$  \hspace{1cm} (17)

with $k_z = \sqrt{(\omega/C_s)^2 - k_\rho^2}$. This transform is taken over the various $\omega/C_s$-shells and $F_n$ is projected to the $k_z$-axis to obtain $F_n(k_\rho, k_z)$ at discrete $k_z$. The inverse transform (13) for $K_n$ is, formally

$$\tilde{K}_n(\rho, k_z) = \frac{(i)^n}{2\pi} \int k_\rho dk_\rho J_n(k_\rho \rho) F_n(k_\rho) e^{ik_z\zeta_0}.$$  \hspace{1cm} (18)

Finally, $K_n(\rho, z)$ is found by the transform inverse to (10), and the entire $K$ is reconstructed as

$$K(\rho, \theta, z) = \sum_n K_n(\rho, z) e^{in\theta}.$$  \hspace{1cm} (19)

### Polarization Diversity

The data processing just described yields equivalent current $K_i(\bar{r})$ in the anomaly, and therefore its shape, for one linearly polarized incident wave. Certain polarization directions will be more favorable depending on the boundaries of the body under examination. In some cases, a weighted average of the $K$ for several polarizations will improve the overall solution and/or reveal details more clearly, such as corners or edges on the anomaly.

### Sampling Considerations

In cartesian coordinates, the 2D FFT can be utilized with the sampling intervals determined by the size of the observational plane: $\Delta x = 2\pi/x_{\text{max}}$, $\Delta y = 2\pi/y_{\text{max}}$. At any frequency $\omega$ between $\omega_1$ and $N\omega_1$ we have $k_x \text{ max} = k_y \text{ max} = \omega/C_s$, and these determine the real space sampling intervals $\Delta x = 2\pi/k_x \text{ max} = \Delta y$. $k_z \text{ max} = 2\pi N\omega_1/C_s$, and $\Delta k_z = \omega_1/C_s$. Thus, $\Delta z = 2\pi/k_z \text{ max}$ and the object is resolved over a $z$-range of $2\pi/\Delta k_z$. Outside this range, "phantoms" appear periodically.

In cylindrical coordinates, the sampled-data version of (18) is
\[ 2\pi(-i)^n k_n(p,k_z) = \frac{2}{\rho_{\max}} \sum_{i=1}^{\infty} \frac{\alpha_i F_n(\alpha_i)e^{ik_z z_0}}{J_{n+1}(x_i)} T_n(p,\alpha_i) \]  

\[ T_n(p,\alpha_i) = \int_0^{\rho_{\max}} \frac{\omega J_n(\omega p) J_n(\omega \rho_{\max})}{\alpha_i^2 - \omega^2} dw \]  

with the \( x_j \) defined by \( J_n(x_j) = 0 \) and \( \alpha_i = x_j/\rho_{\max} \). \( T_n \) is a backprojection factor from image space \( k_p = \alpha_i \) to object space \( \rho \). \( F_n(\alpha_i) \) at any \( k_z \) will usually be interpolated from the discrete \( F_n(k_p,k_z) \) data.

AR OR ARMA MODELING TO IMPROVE RESOLUTION

AR (auto regressive or all-pole) or ARMA (auto regressive moving-average, or pole-zero) modeling of the \( k_x, k_y \) data can increase the lateral x-y resolution of the tomograms. It can also be used to increase the \( k_z \)-data resolution (decrease \( \Delta k_z \)) and thus widen the \( z \)-range of reconstruction, avoiding aliasing (phantoms) in that range. AR modeling is efficiently done with the Burg maximum entropy algorithm,\(^2\) and ARMA modeling for these purposes has been described with examples in previous Quantitative NDE Proceedings.\(^3,4\)

NOISE

In microwave tomography speckle noise is caused by the coherent nature of the illumination and random scattering due to surface roughness of the target and a random distribution of scattering centers. This noise is reduced significantly by wavelength diversity,\(^5\) and polarization diversity also has a beneficial effect. We infer the same effects in the acoustic realm: reduction of material noise such as caused by pores and rough surfaces on anomalies.

REFERENCES


* This work was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under contract No. W-7405-Eng-48.