AN EDDY-CURRENT MODEL AND INVERSION ALGORITHMS
FOR THREE-DIMENSIONAL FLAW RECONSTRUCTION

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INTRODUCTION

We have developed a reconstruction algorithm based on a rigorous electromagnetic model. This model and its associated algorithm assume that the flaw or 'anomalous region' lies within a cylinder whose properties are known, and for which we can easily compute a Green's function. Figure 1 shows a cross-section of the physical system. We have normalized the system equations so that if a voxel is completely covered by a flaw, its conductivity is -1, if a voxel is completely without a flaw, its conductivity is 0, and if a voxel is partially covered by a flaw, its conductivity is between -1 and 0. Our objective is to produce a conductivity map for the mathematical mesh. This approach shows excellent promise as a basis for quantitative nondestructive evaluation.

Fig. 1. The Physical Model Problem
The model equations give the components of the perturbed magnetic induction field in terms of the two dimensional Fourier transform of the conductivity distribution of the anomalous region at each of the \( N_r \) layers within the cylinder wall. The prototype model that we have analyzed most, and for which we shall present results, uses the \( z \)-component of the perturbed field; a combination of all three components could also be used.

The field component is sensed at a single radius, \( r \), but at many different \( z \) and \( \phi \) locations. Then, when the two-dimensional Fourier transform of the field is taken with respect to the \( \phi \) and \( z \) variables, the result is the following transform equation (see [1] for details):

\[
\hat{B}(m,h,w) = \sum_{i=1}^{N_r} \hat{H}_k(m,h,\omega) \hat{\sigma}_k(m,h)
\]

where \( \hat{B}(m,h,\omega) \) is the two-dimensional transform of the \( z \)-component of the perturbed induction field, and \( \hat{\sigma}_k(m,h) \) is the two-dimensional transform of the conductivity distribution at the \( k \)-th layer of the anomalous region. \( \hat{H}_k(m,h,\omega) \) is the transfer function from the \( k \)-th layer to the sensor. \( m \) is the spatial frequency in the \( \phi \) direction, and is an integer, being a simple Fourier series harmonic, and \( h \) is the spatial frequency in the \( z \)-direction. In practice the two-dimensional Fourier transform is approximated by a two-dimensional discrete Fourier transform, which is then computed using a fast Fourier transform (FFT) algorithm. The availability of the FFT is a significant advantage for casting all models in terms of Fourier transforms. We used the mixed radix FFT algorithm of Singleton [4] in performing all numerical experiments; this algorithm allows a radix other than two and is easily used in multidimensional problems.

We have shown an explicit dependence of \( \hat{B} \) and \( \hat{H}_k \) on the temporal frequency, \( \omega \), whereas \( \hat{\sigma}_k \) is independent of \( \omega \). This allows us to acquire data to solve (1) for \( \hat{\sigma}_k \) as a function of \( (m,h) \). We do this by measuring the perturbed induction field at several frequencies, and then writing down an equation such as (1) for each frequency. If there are \( N_f \) frequencies, then, clearly, (1) becomes a system of \( N_f \) equations in \( N_r \) unknowns, in each couple \( (m,h) \):
Fig. 2. Reconstruction of Inverted Pyramid Flaw (Flaw 1).
\[
\tilde{B}(m,h,\omega_k) = \sum_{k=1}^{N_r} \tilde{H}_k(m,h,\omega_k) \tilde{\sigma}_k(m,h)
\]

When the system (2) is nonsquare, as it usually will be, least-squares methods are used to solve it. This model, and the corresponding algorithm, can also be called a 'multifrequency model' because of the manner of acquiring data. We solve (2) using least squares techniques with a Levenberg-Marquardt (LM) smoothing parameter.

The system we solve, for each \((m,h)\), is:

\[
\begin{bmatrix}
\tilde{H}_1(m,h,\omega_1) & \cdots & \tilde{H}_N(m,h,\omega_1) \\
\vdots & \ddots & \vdots \\
\tilde{H}_1(m,h,\omega_{N_f}) & \cdots & \tilde{H}_N(m,h,\omega_{N_f}) \\
\end{bmatrix}
\begin{bmatrix}
\tilde{\sigma}_1(m,h) \\
\vdots \\
\tilde{\sigma}_N(m,h) \\
\end{bmatrix}
= \begin{bmatrix}
\tilde{B}(m,h,\omega_1) \\
\vdots \\
\tilde{B}(m,h,\omega_{N_f}) \\
\end{bmatrix}
\]

where \(\lambda\) is a Levenberg-Marquardt parameter.

We solve this system for each \((m,h)\) by either performing a QR or a Singular Value decomposition and then constructing the minimum norm solution \((\tilde{\sigma}_1(m,h), \ldots, \tilde{\sigma}_N(m,h))\) (see [1],[2],[3]). Then for each fixed level \(i\), \(1 \leq i \leq N_r\), we have \(\tilde{\sigma}_i\) as a function of \((m,h)\), thus allowing us to solve for \(\tilde{\sigma}_i\) as a function of \((\phi,z)\) by performing an inverse Fourier transform. In this way, we can construct the conductivity map, and, hence, image the flaw.

The frequencies should be chosen so that the resulting \(N_f\) by \(N_r\) system matrix in (2) is reasonably well-conditioned in the mathematical sense, i.e., so that the columns of the matrix are relatively independent of each other. Putting it loosely, this reduces the ambiguity inherent in deciding which layer contributes to the measured induction field. In the numerical tests carried out so far we have chosen ten frequencies \((N_f = 10)\), starting at 10 kHz and ending at 5120 kHz and occurring in octave steps.
Fig. 3. Reconstruction of Upright Pyramid Flaw (Flaw 2).
RESULTS OF NUMERICAL EXPERIMENTS

The anomalous region was assumed to span the entire wall thickness of the cylinder, which was 0.049". The wall was divided into ten layers ($N_r = 10$), so that the resulting resolution in the radial direction was 0.0049". In the $z$-direction (along the axis of the tube) the anomalous region was divided into 256 intervals, each of length 0.002", and in the $\phi$-direction we divided the region into 360 intervals of 1 degree each. The inner radius of the cylinder is 0.3885", and the outer radius, 0.4375". Hence, the resolution is of the order 0.007" in the $\phi$-direction. The total number of grid points in the anomalous region is $256 \times 360 \times 10 = 921,600$, which is the total number of unknown conductivity values to be determined during reconstruction.

We simulated two flaws, an inverted pyramid and an upright pyramid. The inverted pyramid flaw is defined with respect to this grid in the following way: in the first layer the flaw occupies a square of 19 intervals in the $z$- and $\phi$-directions, centered at the origin of the grid; in the second layer the flaw occupies a square of 17 intervals, in the third layer a square of 15 intervals, and so on to the tenth layer, in which the flaw occupies a square of a single interval in $z$- and $\phi$-directions. Hence, the flaw is an inverted square pyramid; in attempts to reconstruct such a flaw using other algorithms, we were unable to accurately reconstruct more than one or two layers. In Figure 2, we see the results of applying our model and its algorithm to this flaw (note we show only a fraction of the total grid and the $r = 1, 5, 9, 10$ layers). The reconstruction is quite faithful down to the ninth layer, where dispersion in the $z$-direction reduces the resolution from three squares to five, and in the tenth layer the flaw is virtually invisible. This dispersion is due to fact that highest spatial frequencies in the $z$-direction are filtered out by use of the smoothing technique during reconstruction. Keep in mind that the 'aspect ratio' of the top to the bottom layer of the flaw is 361 to 1, so that it is not too surprising that the bottom layer would be difficult to resolve. We believe, however, that if we increase the number of frequencies, or use field components in the other two directions, we will do an even better job of reconstructing the lowest levels of this flaw. This is due to the fact that the least squares algorithms that we have used work better with over determined systems.

The second flaw that we simulated was an upright pyramid. This flaw has the same size and shape as the first flaw except that now the base
Fig. 4. Reconstruction of Upright Pyramid Flaw with Noise = 5%.
of the pyramid is at the outer surface of the cylinder and the point at the inner surface. In Figure 3, we again see a faithful reconstruction, with the upper part being nearly perfect, and some smearing at the lower levels. The improved resolution of the point of the pyramid, compared to the earlier flaw, is due to the reduced dispersion at the top surface. Thus, the higher spatial frequencies are available for accurate reconstruction of the point in flaw 2 as compared to flaw 1. The value of the LM parameter was $10^{-17}$. This value was determined after performing a singular value analysis which yields the singular values, and indicates the range of the smallest singular values. It is these singular values that contribute to the instability of the reconstruction and must be filtered by the LM parameter. In the reconstructions shown here, we kept the LM parameter fixed with respect to the couple $(m,h)$; a more sophisticated approach would allow the LM parameter to vary with $(m,h)$.

When the input data for the upright pyramid are perturbed by 5%, we get the resulting reconstruction shown in Figure 4. The size and shape of the flaw are still reasonably correct, with somewhat more smearing than before. This smearing is due to the use of a larger LM parameter, $10^{-5}$, than before, which has the effect of reducing the higher spatial frequencies. The use of the larger LM parameter is required in order to reduce the effects of noise. Note that the smearing is more pronounced in the $z$-direction. This is due to the fact that the highest spatial frequencies in the $z$-direction are filtered out by the use of a smoothing technique.

REFERENCES