BIE ANALYSIS OF ELASTIC WAVE SCATTERING
BY A NONLINEAR INTERFACE

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INTRODUCTION

Material interfaces play an important role in determining the failure strength of composite materials and bonded materials. In order to evaluate mechanical properties of interfaces quantitatively by using an ultrasonic technique, it is necessary to make sure how elastic waves are scattered by interfaces. There are, however, many possibilities for mechanical properties of interfaces. One of the most important and interesting behaviors may be nonlinear characteristics such as contact-separation interaction and plastic large deformation at interfaces.

In this paper, scattering problems of elastic waves by a three-dimensional inclusion with nonlinear interface conditions are solved by a boundary integral equation (BIE) method. Since the problem is time-dependent nonlinear, the BIE is formulated in the time domain. The nonlinear conditions are expressed by using a distribution of nonlinear springs located at the matrix-inclusion interface. The springs have perfect elastoplastic characteristics, which implies that the springs can not transmit forces over a threshold value. The contact condition, which implies that no overlap between matrix and inclusion is admitted, is also considered. Contrary to these nonlinear behaviors at interfaces, it is postulated that the matrix and inclusion themselves are linear at all times. The problem being considered here is, thus, a boundary-type nonlinear problem, in which nonlinearities are included in the boundary conditions only. The usual time-domain BIE method can still be applied to the problem, and nonlinear dynamic behaviors at the interface are pursued by solving the BIE by an iterative time marching scheme.

SCATTERING PROBLEMS WITH NONLINEAR INTERFACE CONDITIONS

Let $\bar{D}$ be an inclusion in a three-dimensional infinite body $D$ as shown in Fig. 1. Both $D$ and $\bar{D}$ are assumed to be homogeneous, isotropic, linear elastic solid, for which material constants are given by $(c_L, c_T, \rho)$ and $(\bar{c}_L, \bar{c}_T, \bar{\rho})$, respectively. Here, $c_L$ and $c_T$ are...
velocities of longitudinal and transverse waves, respectively, and \( \rho \) is the mass density. The bar \( \bar{\cdot} \) is used to represent the quantity on the inclusion side. When the incident wave \( \mathbf{u}^{in} \) strikes a point on the boundary \( \partial D \), the scattered wave \( \mathbf{u}^{sc} \) is generated by the interaction with the inclusion. The total wave field \( \mathbf{u} \) is represented as the sum of the incident wave \( \mathbf{u}^{in} \) and the scattered wave \( \mathbf{u}^{sc} \), i.e., \( \mathbf{u} = \mathbf{u}^{in} + \mathbf{u}^{sc} \).

In an inclusion analysis, it is usually assumed that the inclusion is in a perfect bond with the matrix, and that both displacement and traction are continuous across the interface \( \partial D \) [1]:

\[
\begin{align*}
\mathbf{u} &= \bar{\mathbf{u}}, \quad \mathbf{t} = -\bar{\mathbf{t}} \quad \text{on} \quad \partial D
\end{align*}
\]  

(1)

where \( \mathbf{t} \) represents the traction defined by

\[
\mathbf{t}(\mathbf{x},t) = C_{ijpq} n_j(\mathbf{x}) \partial u_p(\mathbf{x},t)/\partial x_q.
\]  

(2)

Here \( \mathbf{n} \) denotes the unit normal vector on the boundary, and \( C_{ijpq} \) are elastic constants given by

\[
C_{ijpq} = \lambda \delta_{ij}\delta_{pq} + \mu (\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) \quad (\lambda, \mu: \text{Lamé constants}).
\]  

(3)

If elastic and linear springs are distributed at the interface, then the interface conditions may be written as

\[
\begin{align*}
\mathbf{t} &= -\mathbf{t} = \mathbf{k} \cdot (\bar{\mathbf{u}} - \mathbf{u}) \quad \text{on} \quad \partial D
\end{align*}
\]  

(4)

where \( \mathbf{k} \) are spring constants. In the linear spring model, the traction are always continuous, but the displacements can be discontinuous across the interface \( \partial D \). Kitahara et al. [2] used the linear spring model in the BIE analysis in the frequency domain to solve the elastic scattering problem by an interface crack.

In the present work, the linear spring model is extended into nonlinear spring models in the time domain analysis. Two types of nonlinearities are considered. One is the nonlinear spring model without contact conditions. The relation between the traction \( t_{N(S)} \) and the displacement discontinuity \( \bar{u}_{N(S)} - u_{N(S)} \) is shown in Fig. 2 (a). Here, the subscripts \( N \) and \( S \) indicate the inward normal and tangential components at the interface \( \partial D \), respectively. In the normal direction, the spring yields for a tensile force beyond the threshold value \( t_{N}^{\gamma} \), but remains linear for any compressive force. In the tangential direction, the spring yields for a shear force beyond the critical value \( |t_{S}^{\gamma}| \). The interface condition for the nonlinear spring model may be represented in the following form:
Fig. 2. Nonlinear relation between traction and displacement discontinuity at the interface \( \partial D \); (a) Nonlinear spring without contact condition and (b) Nonlinear spring with contact condition.

\[
\begin{align*}
\{ & t_N = -\tilde{t}_N = k_N(\bar{u}_N - u_N) \quad \text{if} \quad \bar{u}_N - u_N \leq k_N^{-1}t_N^\sigma \\
& t_N = -\tilde{t}_N = t_N^\sigma \quad \text{if} \quad \bar{u}_N - u_N > k_N^{-1}t_N^\sigma \\
\{ & t_S = -\tilde{t}_S = k_S(\bar{u}_S - u_S) \quad \text{if} \quad |\bar{u}_S - u_S| \leq k_S^{-1}t_S^\sigma \\
& t_S = -\tilde{t}_S = \text{sgn}(\bar{u}_S - u_S)t_S^\sigma \quad \text{if} \quad |\bar{u}_S - u_S| > k_S^{-1}t_S^\sigma 
\end{align*}
\]

As seen in eqs. (5) and (6), the transition from the linear state to the yielding and vice versa depend on the displacement discontinuities \( \bar{u}_{N(S)} - u_{N(S)} \). So the interface conditions will show nonlinear variations with time.

The other interface condition considered here is the nonlinear spring model with contact conditions. The relation between the traction and the displacement discontinuity is shown in Fig. 2 (b). The difference from Fig. 2 (a) is only the contact condition in the normal component. The interface conditions for the normal component may be expressed as follows.

\[
\begin{align*}
\{ & t_N = -\tilde{t}_N = k_N(\bar{u}_N - u_N) \quad \text{if} \quad 0 \leq \bar{u}_N - u_N \leq k_N^{-1}t_N^\sigma \\
& t_N = -\tilde{t}_N = t_N^\sigma \quad \text{if} \quad \bar{u}_N - u_N > k_N^{-1}t_N^\sigma \\
& \bar{u}_N = u_N \quad \text{if} \quad t_N = -\tilde{t}_N < 0 
\end{align*}
\]

As seen in eq. (7), the displacement discontinuity in the normal direction can not be negative so that no overlap between matrix and inclusion occurs. It is supposed that there is no displacement gap between the matrix and the inclusion prior to the incidence of an elastic wave.

BOUNDARY INTEGRAL EQUATIONS

Since \( D \) and \( \bar{D} \) are always linearly elastic solids, the integral
representations for wave fields in $D$ and $D$ can be obtained in the same way as in a linear case. The boundary integral equations for the total wave field in $D$ may be written as [3]
\[
c_{ik}(x)u_i(x,t) + \text{p.v.} \int_{\partial D} T_{ik}(y,t;x) \ast u_i(y,t) dS_y - \int_{\partial D} U_{ik}(y,t;x) \ast t_i(y,t) dS_y = u^n_k(x,t) \quad x \in \partial D
\] (8)

where $c_{ik}(x)$ is the free term of the double layer potential, the asterisk $\ast$ indicates the convolution integral with respect to time and $\text{p.v.}$ denotes the principal value of the integral. Also, $U_{ik}$ and $T_{ik}$ are fundamental solutions in elastodynamics given by
\[
U_{ik}(y,t;x) = \frac{1}{4\pi \rho} \left\{ \left( \frac{3r_{ik}}{r} - \frac{\delta_{ik}}{r} \right) \int_{c_L^{-1}}^{c_L} \lambda \delta(t - \lambda r) d\lambda + \frac{r_{ik}}{c_T} \left[ \frac{1}{c_L^2} \delta(t - \frac{r}{c_L}) - \frac{1}{c_T^2} \delta(t - \frac{r}{c_T}) \right] + \frac{\delta_{ik}}{rc_T^2} \delta(t - \frac{r}{c_T}) \right\}
\] (9)
\[
T_{ik}(y,t;x) = C_{ijk} \frac{\partial}{\partial y_j} U_{ik}(y,t;x) n_j(y)
\] (10)

where $r = |x - y|$, $r_{ik} = \partial r/\partial y_i$.

The boundary integral equations for the wave field in $D$ may also be obtained in the following form.
\[
c_{ik}(x)\ddot{u}_i(x,t) + \text{p.v.} \int_{\partial D} T_{ik}(y,t;x) \ast \ddot{u}_i(y,t) dS_y - \int_{\partial D} \ddot{U}_{ik}(y,t;x) \ast \dot{t}_i(y,t) dS_y = 0 \quad x \in \partial D.
\] (11)

In numerical calculations, the integral equations (8) and (11) are discretized by using appropriate interpolation functions for unknown terms. In the present paper, the unknown displacement $u$ and traction $t$ are represented by a piecewise linear function and a piecewise step function in time, respectively. For the spatial distributions, constant elements are used for both $u$ and $t$. For the time $t = M \Delta t$, equations (8) and (11) may be discretized into the following matrix forms.
\[
\sum_{m=1}^{M} \left[ [T]^{M-m} \{u\}^m - [U]^{M-m} \{t\}^m \right] = \{u^n\}^M
\] (12)
and
\[
\sum_{m=1}^{M} \left[ [T]^{M-m} \{\ddot{u}\}^m - [U]^{M-m} \{\dot{t}\}^m \right] = \{0\}
\] (13)

where $\{\}$ stands for the quantities related to the time step $m$ and the free terms $c_{ik}$ and $\ddot{c}_{ik}$ in equations (8) and (11) are involved in the matrices $[T]$ and $[T]$, respectively. For details on the discretization technique of the time-domain BIE, see Hirose [3].

SYSTEM OF EQUATIONS

Equations (12) and (13) are combined by using the interface conditions given by eqs. (1), (4), (5)-(7).

In the case of a perfect bonding interface, the combination of eqs. (12) and (13) by using eq. (1) yields the system of equations as follows.
Fig. 3. A spherical inclusion subjected to an incident plane wave.

\[ \begin{bmatrix} T & -U \\ T & U \end{bmatrix} \begin{bmatrix} u \\ t \end{bmatrix}^0 = \begin{bmatrix} u^m \\ 0 \end{bmatrix}^M \]

\[ - \sum_{m=1}^{M-1} \begin{bmatrix} T & -U \\ T & U \end{bmatrix}^{M-m} \begin{bmatrix} u \\ t \end{bmatrix}^m \]

(14)

Since all terms on the right-hand side are known, the unknowns \( \{u, t\}^M \) \((M = 1, 2, \ldots)\) are obtained by a step-by-step time marching calculation.

In the case of a linear spring interface, the system of equations becomes

\[ \begin{bmatrix} T + U \cdot k & -U \cdot k \\ -U \cdot k & T + U \cdot k \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \end{bmatrix}^0 = \begin{bmatrix} u^m \\ \dot{u}^m \end{bmatrix}^M \]

\[ - \sum_{m=1}^{M-1} \begin{bmatrix} T + U \cdot k & -U \cdot k \\ -U \cdot k & T + U \cdot k \end{bmatrix}^{M-m} \begin{bmatrix} u \\ \dot{u} \end{bmatrix}^m \]

(15)

For a nonlinear spring interface without contact conditions given by eqs. (5) and (6), the system of equations has almost the same structure as eq. (15), except that the spring constants \( k \) can vary with the displacement discontinuity \( \ddot{U}_N(s) - u_N(s) \). In the case of a nonlinear spring interface with contact conditions, it is difficult to write down a system of equations in an explicit form as in linear cases. Because there is a variety of the structure for the system of equations, depending on the contact-noncontact condition in each component on each element. For noncontact components, the equation is similar to eq. (15), while for contact components, the equation has the same structure as eq. (14). In either nonlinear analysis, some iterative calculations are necessary to converge the solution at each time step.

NUMERICAL EXAMPLES

We consider a spherical inclusion subjected to an incident plane wave as shown in Fig. 3. The incident wave is a sinusoidal L-wave, which has stress components given by

\[ \sigma_{ij}^{in} = -\mu \frac{\lambda \delta_{ij} + 2 \mu \delta_{i3} \delta_{j3}}{\lambda + 2 \mu} \cos \left( \frac{2 \pi T}{T} \right) H(\tau), \quad T = 6 \]

(16)
Fig. 4. Time variation of displacements (-----: point A, ---: point B) and traction (----) at the bottom points A and B of the sphere. (a): Nonlinear spring without contact, (b): nonlinear spring with contact, (c): linear spring, (d): perfect bonding.
Fig. 5. Time variation of scattered far-field amplitude $\Omega_L$ for the angles $\theta = 0^\circ$ (-----), $90^\circ$ (-----) and $180^\circ$ (-----). (a): Nonlinear spring without contact, (b): nonlinear spring with contact, (c): linear spring, (d): perfect bonding.
where $T$ is a dimensionless period, $H(\cdot)$ is the Heaviside step function, $	au = (c_L t - (x_3 + a))/a$ and $a$ is the radius of the sphere. The material constants for the matrix and the inclusion are given by

$$c_L/c_L = 2, \quad c_T/c_T = 2, \quad \nu = \tilde{\nu} = 0.25, \quad \tilde{\rho}/\rho = 1$$

where $\nu$ is the Poisson's ratio. Also, the following spring constants and yielding conditions are used.

$$a_k N/\mu = a_k S/\mu = 10, \quad t_N^c/\mu = |t_S^c|/\mu = 0.7.$$ 

In numerical calculations, the spherical boundary was divided into 624 constant elements, and the time increment was chosen as $c_T \Delta t/a = 0.15$. It is noted that only 24 nodes were used as active field points by taking account of the axisymmetric condition of the problem.

Fig. 4 shows the time variation of the displacement and traction at the bottom points of the sphere (Fig. 3). Figure (a) to (d) correspond to four cases of nonlinear spring without contact, nonlinear spring with contact, linear spring, and perfect bonding, respectively. For the perfect bonding (Fig. (d)), there is no displacement discontinuity between the points A and B. For other spring models, we can see some displacement discontinuities in nonlinear cases as shown in Figs. (a) and (b), particularly, large discontinuities are observed due to the spring yielding. Figs. (a) and (b) also show that tractions are cut off at the critical value $t_N^c/\mu = 0.7$ when springs are in the yielding process. For the nonlinear spring with contact condition (Fig. (b)), the displacement discontinuity vanishes as the traction becomes negative in the contact region.

At the far-field of $|\mathbf{r}| \gg |\mathbf{y}|$ ($\mathbf{r} \in D, \mathbf{y} \in \partial D$), the scattered wave field may be written in the asymptotic form $[1]$:

$$u^c_{\beta}(\mathbf{r}, t) \approx \frac{1}{4\pi|\mathbf{r}|} \Omega_{\beta}(\mathbf{r}, t - \frac{|\mathbf{r}|}{c_{\beta}}) \quad |\mathbf{r}| \to \infty$$

where $\beta = L$ or $T$, which represents the longitudinal or transverse component, respectively, $\Omega_{\beta}$ denotes the time-domain amplitude of the scattered far-field and $\mathbf{r} = \mathbf{z}/|\mathbf{z}|$. Fig. 5 shows the longitudinal component $\Omega_L$ of the far-field amplitude as a function of $(c_L t - |\mathbf{r}|)/a$ for the angles $\theta = 0^\circ, 90^\circ$ and $180^\circ$. (See Fig. 3 for the definition of the angle $\theta$.) Figures (a) to (d) correspond to those in Fig. 4. Compared with Figs. (c) and (d), the results for nonlinear cases as shown in Figs. (a) and (b) have small ripples which may be related to the nonlinearity of the interfaces. If the Fast Fourier Transform is applied to the waveforms shown in Figs. (a) and (b), we will obtain the higher harmonics, which may be useful to determine the nonlinear parameters at the interface from the scattering data.

REFERENCES

