GREEN'S FUNCTION FOR LAMB'S PROBLEM AND RAYLEIGH WAVE PROPAGATION IN GENERAL TRANSVERSELY ISOTROPIC MATERIALS

Martin Spies and Michael Kröning
Fraunhofer-Institute for Nondestructive Testing (IZFP)
66123 Saarbrücken, Germany

INTRODUCTION

Composite materials have gained considerable industrial importance, being widely applied e.g. in aerospace industries. The need for their proper testing in view of delaminations, inclusions and other defects has correspondingly stimulated the interest in describing wave propagation in such anisotropic media. In this study, Lamb's problem of determining the disturbance resulting from a point source in a half-space [1] is investigated for the case of transversely isotropic (TI) symmetry, which is characteristic for unidirectional fiber composites and extruded metal-matrix composites, but also for fiber-textured columnar-grained steels. Using the dyadic and triadic full-space Green's functions obtained previously in their 2d-space-time spectral representations [2], a corresponding representation of Green's dyad for the half-space has been derived exploiting the boundary condition of the stress-free surface. The resulting dyadic function is the solution of the elastic wave equation with point forces applied at the surface or within the uniform half-space, the fiber orientation being variable. First numerical evaluations have been performed with respect to Rayleigh-surface wave propagation by determining the zeroes of the corresponding Rayleigh function, which is included in the analytical expressions. Resulting slowness and wave curves are presented for several materials. The work presented can be further applied, e.g., to determine Rayleigh wave directivity patterns for point sources on the half-space as well as to model laser-generated wave propagation in composites. Application in the field of seismic wave propagation is also possible.

PLANE WAVES IN GENERAL TI-MEDIA

The equation of motion for the displacement vector \( \mathbf{u} \) in a homogeneous anisotropic solid reads [3]

\[
(\nabla \cdot \underline{C} \cdot \nabla) \cdot \mathbf{u}(R, \omega) + \varrho \omega^2 \mathbf{u}(R, \omega) = - \mathbf{f}(R, \omega),
\]

\[ \text{(1)} \]

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This contribution is dedicated to Professor Paul Hölter, former director of IZFP, on the occasion of his 70th birthday on March 22nd, 1995.
where \( \nabla \) is the gradient vector, \( \varrho \) is the material density, \( f \) accounts for the volume force density and \( \omega \) denotes the circular frequency, if time dependence \( \sim e^{-j\omega t} \) is assumed. The elastic stiffness tensor \( G \) for the general transversely isotropic case (arbitrary fiber direction) has been given e.g. in Ref. [2].

The plane harmonic wave solutions of Eq. (1) \((f=0)\) are in the form
\[
\hat{u}_\alpha(R, \omega) = U \hat{u}_\alpha \exp \left[j K_\alpha \hat{K} \cdot R \right],
\]
and have been obtained previously [4]. With \( \hat{K} \) being the propagation direction, the wave numbers \( K_\alpha \) are given by the dispersion relations as:
\[
K_{SH}^2 = \frac{\varrho \omega^2}{[\mu_0 + (\mu_\parallel - \mu_\perp)(\hat{a} \cdot \hat{K})^2]},
\]
\[
K_{SV, qP}^2 = \frac{\varrho \omega^2 (B \pm (B^2 - 4A)^{1/2})/(2A)},
\]
where \( A \) and \( B \) depend on the elastic constants, the fiber direction \( \hat{a} \) and the propagation direction \( \hat{K} \). The decomposition into a pure transverse wave \((SH: \text{shear horizontal})^2 \) and two quasi-waves \((qSV: \text{quasi-shear vertical}, qP: \text{quasi-pressure})\) is apparent from the polarization vectors, which are
\[
\hat{u}_{SH} = N_{SH} (\hat{a} \times \hat{K}),
\]
\[
\hat{u}_{SV, qP} = N_{qSV, qP} \left( F_1^{qSV, qP} \hat{K} + F_2^{qSV, qP} \hat{a} \right).
\]
The abbreviated quantities can be found in Ref. [4]. Since the \( SH\) polarization is always perpendicular to the \( \hat{a} - \hat{K} \)-plane, i.e. in general not horizontally polarized, whereas the quasi-wave polarizations lie within this plane, Helbig’s designations as \( \text{cross-plane} S\)-wave and \( \text{in-plane} \) waves, respectively, seem to be less misleading [5].

TI-FULL-SPACE GREEN’S FUNCTIONS

Green’s dyadic function \( G \) is defined by the differential equation [3]
\[
(\nabla \cdot G \cdot \nabla) \cdot G(R, \omega) + \varrho \omega^2 G(R, \omega) = - \delta(R),
\]
where \( \delta \) is Dirac’s three-dimensional delta function. By application of spatial Fourier-Transform (FT) with respect to \( R \) in terms of
\[
\hat{G}(K, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(R, \omega) e^{-jK \cdot R} d^3R,
\]
a 3d-space-time spectral representation (STSR) of \( G \) can be derived; the 2d-STSR -transformed with respect to \( x \) and \( y \) - is determined by one-dimensional inverse FT according to [2]
\[
\hat{G}(K_x, K_y, z, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(K_z, \omega) e^{jzK_x} dz.
\]
Defining \((\alpha = SH, qSV, qP)\)
\[
K_\alpha \equiv K_x \hat{e}_x + K_y \hat{e}_y + K_{\alpha z} \hat{e}_z,
\]
\footnote{Note that in Eq. (3) only the shear-moduli parallel and perpendicular to the fiber direction - \( \mu_\parallel \) and \( \mu_\perp \) - enter.}
where the sign accounts for up- and down-going wave components, provides

\[
\hat{G}(K_x, K_y, z, \omega) = \hat{G}_0^{SH} e^{izK_z^zSH} + \hat{G}_0^{qSV} e^{izK_z^zSV} + \hat{G}_0^{qP} e^{izK_z^zP}.
\]  

(11)

Here, the positive sign applies for \( z > 0 \), the negative sign for \( z < 0 \). The \( K_{z0}^{\pm} \) are functions of the elastic constants, the fiber direction and - for the case of horizontal fiber direction (THI-media) - of the vector \( K_0 \equiv (K_x, K_y, 0) \). The explicit representations have also been given in Ref. [2], the corresponding results for a variable in 3d-space can be obtained following Ref. [6].

In analogy to Hooke's Law, Green's triadic function \( \Xi \) is defined as [2]

\[
\Xi(R, \omega) = \frac{\partial}{\partial z} G(R, \omega) = \Xi^{213}(R, \omega).
\]  

(12)

Thus \( \Xi(R, \omega) \) is a third rank tensor, symmetrical in the first two indices. Using the \( K \)-vectors defined in (10), its 2d-STSR has been derived as

\[
\hat{\Xi}(K_x, K_y, z, \omega) = \hat{\Xi}_0^{SH} e^{izK_z^zSH} + \hat{\Xi}_0^{qSV} e^{izK_z^zSV} + \hat{\Xi}_0^{qP} e^{izK_z^zP},
\]  

(13)

where again the positive sign is valid for \( z > 0 \), the negative sign for \( z < 0 \). The explicit expressions for \( \hat{\Xi}_0^{SH} \) and \( \hat{\Xi}_0^{qSV} \) (\( \alpha = SH, qSV, qP \)) have been given in Ref. [2].

DERIVATION OF GREEN'S DYADIC FUNCTION FOR LAMB'S PROBLEM

In the following, a THI-half-space with fiber direction parallel to the surface is considered, the free surface being identical with the \( x-y \)-plane (Fig. 1). The problem is to determine the disturbance due to a point source in the half-space at position \( z' \). As indicated in Fig. 1, the displacement at an observation point (position \( z \)) results from the 3 direct waves \( \alpha \) with spectral components \( K_{z0}^{\pm} \) and 9 reflected waves \( \beta \rightarrow \alpha \), the respective spectral components being accordingly \( \hat{K}_\beta \) and \( \hat{K}_\alpha^+ \). While the direct contributions are described by full-space Green's function, the reflected contributions are to be determined. Mathematically, the basic problem is to solve Eq. (7) under the condition that the stresses vanish at the surface \( z = 0 \).

The ansatz for the general solution of this problem is chosen as

\[
\hat{G}_0^{half} = \sum_\alpha \hat{G}_0^{\alpha} e^{i(x-z')K_\alpha^z} + \sum_\alpha \hat{\nu}_\alpha \hat{\nu}_\alpha^* e^{i(x+z')K_\alpha^z},
\]  

(14)

where \( \hat{\nu}_\alpha \) designates the plane wave polarization vectors. In Eq. (14), the first term represents the solution to the inhomogeneous problem (full-space Green's function), while the second term involving the arbitrary constant vectors \( \hat{\nu}_\alpha \) is the solution to the homogeneous problem that remains bounded as \( z \) goes to infinity.

The next step is to apply the stress boundary conditions at the free surface according to

\[
\varepsilon_2 \cdot T \bigg|_{z=0} = \varepsilon_2 \cdot \left( \frac{\partial}{\partial z} \right) \bigg|_{z=0} = 0.
\]  

(15)

Transforming the stress tensor into the 3d-spectral representation yields

\[
\hat{T} = j \hat{C} \hat{\nu}.
\]  

(16)
Performing one-dimensional inverse Fourier-transformation according to Eq. (9) and replacing \( \hat{\mathbf{u}} \) by \( \hat{\mathbf{G}}_{\text{half}} \) allows to formulate

\[
\hat{x}_z \equiv \mathbf{e}_z \cdot \hat{\mathbf{F}} = \sum \alpha \mathbf{e}_z \cdot \left( \mathbf{C} : \mathbf{K}_\alpha \hat{\mathbf{G}}_{\text{half}, \alpha} \hat{\mathbf{i}} \right) = \mathbf{e}_z \cdot \left( \hat{\mathbf{x}}_{\text{half}} \cdot \hat{\mathbf{f}} \right) = \hat{\mathbf{S}}_{\text{half}} \cdot \hat{\mathbf{f}} \, , \tag{17}
\]

where \( \hat{\mathbf{x}}_{\text{half}} \equiv \mathbf{e}_z \cdot \hat{\mathbf{G}}_{\text{half}} \) has been introduced. It follows that

\[
\hat{\mathbf{x}}_{\text{half}} = \sum \alpha \hat{\mathbf{x}}_\alpha e^{i(z-z')K_{z\alpha}^z} + \sum \alpha \left[ j \mathbf{e}_z \cdot \left( \mathbf{C} : \mathbf{K}_\alpha \hat{\mathbf{u}} \hat{\mathbf{a}}_{\alpha} \right) e^{i(z+z')K_{z\alpha}^z} \right] , \tag{18}
\]

and

\[
\hat{\mathbf{G}}_{\text{half}} = \sum \alpha \hat{\mathbf{x}}_\alpha e^{i(z-z')K_{z\alpha}^z} + \sum \alpha \hat{x}_z \hat{\mathbf{a}}_{\alpha} e^{i(z+z')K_{z\alpha}^z} . \tag{19}
\]

For arbitrary force density \( \hat{\mathbf{f}} \), Eq. (17) is fulfilled if \( \hat{\mathbf{G}}_{\text{half}} \) vanishes at \( z = 0 \). Thus the arbitrary vectors \( \hat{\mathbf{x}}_{\alpha} \) can be determined from

\[
\sum_{\beta} \hat{x}_\beta + \hat{\mathbf{x}}_{\text{SH}} \hat{\mathbf{b}}_{\text{SH}} + \hat{\mathbf{x}}_{\text{SV}} \hat{\mathbf{b}}_{\text{SV}} + \hat{\mathbf{x}}_{\text{P}} \hat{\mathbf{b}}_{\text{P}} = 0 \, , \tag{20}
\]

where

\[
\hat{\mathbf{x}}_\alpha = \hat{x}_z e^{jz'K_{z\alpha}^z} , \quad \hat{\mathbf{x}}_{\alpha} = \hat{x}_z e^{-jz'K_{z\alpha}^z} \, . \tag{21}
\]

With \( k = x, y, z \), it is

\[
\sum_{\beta} \hat{x}_\beta \cdot \mathbf{e}_k + \hat{\mathbf{x}}_{\text{SH}} \hat{\mathbf{b}}_{\text{SH}} + \hat{\mathbf{x}}_{\text{SV}} \hat{\mathbf{b}}_{\text{SV}} + \hat{\mathbf{x}}_{\text{P}} \hat{\mathbf{b}}_{\text{P}} = 0 \, , \tag{22}
\]

which can be rewritten as

\[
\sum_{\beta} \hat{x}_\beta \cdot \mathbf{e}_k + \left( \begin{array}{ccc} \hat{\mathbf{x}}_{\text{SH}} & \hat{\mathbf{x}}_{\text{SV}} & \hat{\mathbf{x}}_{\text{P}} \\ \hat{\mathbf{b}}_{\text{SH}} & \hat{\mathbf{b}}_{\text{SV}} & \hat{\mathbf{b}}_{\text{P}} \end{array} \right) \cdot \left( \begin{array}{c} \hat{\mathbf{b}}_{\text{SH}} \\ \hat{\mathbf{b}}_{\text{SV}} \\ \hat{\mathbf{b}}_{\text{P}} \end{array} \right) = 0 \, . \tag{23}
\]
Defining the matrix

$$\mathbf{T}_x = \begin{pmatrix} I^{SH} & I^{SV} & I^{SP} \end{pmatrix}, \quad (24)$$

Eq. (23) yields the components $\hat{b}_k^\alpha$ according to

$$\begin{pmatrix} \hat{b}_k^{SH} \\ \hat{b}_k^{SV} \\ \hat{b}_k^{SP} \end{pmatrix} = - \mathbf{T}_x^{-1} \cdot \sum_\beta \hat{e}_\beta \cdot \mathbf{e}_k \quad (25)$$

or in vector notation:

$$\begin{align*}
\hat{b}_k^{SH} &= - \mathbf{e}_x \cdot \mathbf{T}_x^{-1} \cdot \sum_\beta \hat{e}_\beta, \\
\hat{b}_k^{SV} &= - \mathbf{e}_y \cdot \mathbf{T}_x^{-1} \cdot \sum_\beta \hat{e}_\beta, \\
\hat{b}_k^{SP} &= - \mathbf{e}_z \cdot \mathbf{T}_x^{-1} \cdot \sum_\beta \hat{e}_\beta.
\end{align*} \quad (26-28)$$

Rewriting Eq. (24) according to

$$\mathbf{T}_x = I^{SH} \cdot \mathbf{e}_x + I^{SV} \cdot \mathbf{e}_y + I^{SP} \cdot \mathbf{e}_z \quad (29)$$

and applying dyadic algebra allows to determine the inverse $\mathbf{T}_x^{-1} = \text{adj} \mathbf{T}_x / \text{det} \mathbf{T}_x$. The adjoint and the determinant are given by

$$\begin{align*}
\text{adj} \mathbf{T}_x &= \mathbf{e}_x \mathbf{T}_x^{SH} + \mathbf{e}_y \mathbf{T}_x^{SV} + \mathbf{e}_z \mathbf{T}_x^{SP}, \\
\text{det} \mathbf{T}_x &= I^{SH} \cdot \mathbf{T}_x^{SH} (= I^{SV} \cdot \mathbf{T}_x^{SV} = I^{SP} \cdot \mathbf{T}_x^{SP}),
\end{align*} \quad (30-31)$$

where the vectors $\mathbf{T}_x^\alpha$ are defined as

$$\mathbf{T}_x^{SH} = I^{SV} \times I^{SP}, \quad \mathbf{T}_x^{SV} = I^{SP} \times I^{SH}, \quad \mathbf{T}_x^{SP} = I^{SH} \times I^{SV}. \quad (32)$$

Thus, it is

$$\hat{b}_k^{\alpha} = - \mathbf{T}_x^\alpha \cdot \sum_\beta \hat{e}_\beta / \text{det} \mathbf{T}_x, \quad (33)$$

$$\equiv \sum_\beta \hat{b}_k^{\beta} \varepsilon^{\beta\alpha} e^{-j\pi(K_\alpha - K_\beta)} / R(K_\alpha). \quad (34)$$

Here the corresponding Rayleigh-function $R(K_\alpha)$ and the vectors $\hat{b}_k^{\beta}$, directly indicating the reflection process $\beta \rightarrow \alpha$, have been introduced. Inserting (34) into (14) allows to finally formulate [7]

$$\begin{align*}
\hat{G}_h = \sum_\alpha \hat{G}_0^\alpha e^{j(z-z')K_\alpha} + \sum_\alpha \sum_\beta \hat{b}_k^{\alpha \beta} e^{-j\pi(K_\alpha - K_\beta)} / R(K_\alpha). \quad (35)
\end{align*}$$

This is the complete solution of Lamb’s problem in the transform domain for any anisotropic half-space.
For TIH-media, the vectors $\hat{b}^{\beta \alpha}$ are obtained in the form

$$\hat{b}^{\beta \alpha} = - \left[ B_{\beta \alpha}^{\beta \alpha} \hat{a} + C_{\beta \alpha}^{\beta \alpha} \hat{K}_\beta + H_{\beta \alpha}^{\beta \alpha} \hat{z}_z \right], \quad (36)$$

the explicit expressions can be found in Ref. [7]. From the above equations, the Rayleigh-function can be explicitly determined as

$$R^{\text{THI}}(\hat{a}, K_r) = N \left[ \mu_1 K_{ZSH} \left( (\hat{a} \cdot K_r) T_1 + T_2 \right) \right. \left. + (\hat{a} \times K_r)^2 \left( (\mu_2 - \mu_1)(\hat{a} \cdot K_r) T_1 - \mu_1 (T_2 - 2K_{ZSH} T_3) \right) \right], \quad (37)$$

where $N, T$ depend on $N_\alpha$ and $F_{1,2}^{\text{SV},\alpha P}$ (Eqs. (5),(6)), as well as on $\hat{a}, \hat{K}$ and the elastic constants [7].

**NUMERICAL RESULTS**

The Rayleigh wave (RW) slowness curves are to be determined from

$$R^{\text{THI}}(\hat{a}, K_r) = k_{R}^{\text{THI}}(\hat{a}, \hat{K}_r) = 0. \quad (38)$$

In Fig. 2, showing the topology of $|R^{\text{THI}}(\hat{z}_r, K_r)|$ for several materials, these zeroes can be easily recognized. Beryl, cobalt and apatite have been chosen for comparison with the results on the RW-numbers given previously by Buchwald [8], the results are the same. Determination of the $k_{R}^{\text{THI}}$ according to Eq. (38) allows to further determine the corresponding group velocity curves and skewing angle diagrams. These are shown in Fig. 3 for the graphite-epoxy composite material in comparison with the respective bulk wave curves. As expected, the RW-group velocity is lower than shear (horizontal) velocity. The skewing angle diagram shows that the deviation between energy and phase directions is larger than for the $SH$-case, it amounts to almost 40° at its maximum.

**CONCLUSION**

For anisotropic half-spaces, a general solution to Lamb’s problem has been derived in form of a dyadic Green’s function. By application of a plane wave theory [4] and the 2d-space-time spectral representation of Green’s dyadic and triadic full-space functions [2] explicit expressions for general transversely isotropic media have been obtained. First numerical evaluations with respect to the Rayleigh wave numbers have been performed, which are decisive to the evaluation of the presented Green’s function, since these produce singularities in the above integral representation. Once these singularities have been determined, $\hat{G}_{\text{half}}^{\beta \alpha}$ can be evaluated as described in Refs. [9,10]. The following points are worth to be mentioned: introducing isotropic material constants to the presented expressions leads to the results given by Johnson [11] for isotropic half-spaces; the determination of the Rayleigh wave numbers from the Rayleigh function given above is less tedious than using the generalized Rayleigh function given earlier in Refs. [9,10]. As discussed in these references, the case where fiber direction $\hat{a}$ is arbitrary in 3d-space is particularly interesting in view of the existence of Rayleigh-surface waves and so-called pseudo-surface waves. Furthermore, the representation of the dyadic Green’s function described in this contribution is well
Figure 2: Topology of $| R^{TH}(g_x, K_r) |$ for a) beryl, b) graphite-epoxy, c) cobalt, d) apatite.

Figure 3: Rayleigh-wave characteristics in graphite-epoxy composite material in comparison with bulk waves: a) slowness, b) group velocity and c) skewing angles.
suited for modeling laser-generated ultrasonic wave propagation e.g. in composite materials as has been pointed out in Ref. [12].

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