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# Abelian qo-groups and atomic pseudo-valuation domains

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**Abelian  $q_0$ -groups and atomic pseudo-valuation domains**

by

Elijah James Stines

A dissertation submitted to the graduate faculty  
in partial fulfillment of the requirements for the degree of  
**DOCTOR OF PHILOSOPHY**

Major: Mathematics

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## DEDICATION

I would like to dedicate this thesis to my wife Katie and to my daughter Avery, without whose support I would not have been able to complete this work. I would also like to dedicate this work to my late grandmother Kay, who never stopped learning.

## TABLE OF CONTENTS

<b>LIST OF FIGURES</b> . . . . .	v
<b>ACKNOWLEDGEMENTS</b> . . . . .	vi
<b>ABSTRACT</b> . . . . .	vii
<b>CHAPTER 1. OVERVIEW</b> . . . . .	1
1.1 Introduction . . . . .	1
1.2 Ordered Algebraic Systems . . . . .	2
<b>CHAPTER 2. MONADICITY OF ABELIAN QO-GROUPS OVER SET</b>	
<b>MONOMORPHISMS</b> . . . . .	9
2.1 Introduction . . . . .	9
2.2 Adjunctions for Relevant Categories . . . . .	10
2.3 Free Abelian Qo-Groups . . . . .	18
2.4 Monadicity for Qo-Groups . . . . .	21
<b>CHAPTER 3. ATOMIC PSEUDO-VALUATION DOMAINS</b> . . . . .	26
3.1 Introduction . . . . .	26
3.2 Preliminary Facts . . . . .	27
3.3 PVDs From Lexicographic Sums . . . . .	28
3.4 Classification of Atomic PVDs . . . . .	31
<b>CHAPTER 4. GENERALIZED SERIES RINGS AND CONGRUENCE LAT-</b>	
<b>TICES</b> . . . . .	34
4.1 Introduction . . . . .	34
4.2 PVDs from Generalized Power Series . . . . .	34

4.3	A Particular Class of PVDs . . . . .	38
<b>CHAPTER 5. IMPLICATION RINGS . . . . .</b>		<b>43</b>
5.1	Introduction . . . . .	43
5.2	Relationship to Rings . . . . .	47
5.3	Finite Fields . . . . .	48
<b>CHAPTER 6. FUTURE DIRECTIONS . . . . .</b>		<b>51</b>
6.1	Introduction . . . . .	51
6.2	Monadicity Theorems . . . . .	51
6.3	Complete Classification of PVDs . . . . .	52
6.4	Implication Rings . . . . .	54
<b>BIBLIOGRAPHY . . . . .</b>		<b>56</b>

## LIST OF FIGURES

2.1	A morphism of $\mathbf{QAb}^i$ . . . . .	10
2.2	A morphism of $\mathbf{Set}^i$ . . . . .	11
2.3	A morphism of $\mathbf{QAb}^m$ . . . . .	12
2.4	Forgetful Functors . . . . .	15
2.5	Identity and Associative Laws of a Monad . . . . .	21
2.6	Identity and associative laws of a $\mathbf{C}^T$ -algebra . . . . .	22
2.7	A $\mathbf{C}^T$ -Morphism $f$ . . . . .	22
2.8	The Eilenberg-Moore comparison . . . . .	22
4.1	A portion of the divisibility relation on $\mathbb{F}_2 + X\mathbb{F}_4[[X]]$ . . . . .	39
4.2	Hasse diagram of principal ideals of $\mathbb{F}_2 + X\mathbb{F}_4[[X]]$ . . . . .	39
4.3	Congruence lattice of $R$ . . . . .	40
4.4	Basic structure of congruence lattice . . . . .	42
4.5	Congruence lattice of $\mathbb{Q} + X(\mathbb{Q}[\sqrt[3]{2}])([X])$ . . . . .	42

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## ABSTRACT

The notion of imposing a partial ordering on algebraic structures is one of the most fundamental concepts in abstract algebra. Even further, the notion of a quasi-ordering is also used at the most basic algebraic levels. These orderings, however, are not very useful or well behaved when studying universal algebra. The first topic of this thesis is to examine the concept of a quasi-ordered abelian group and create a category, over which, the category of quasi-ordered abelian groups is monadic. This monadicity theorem allows one to examine the category of quasi-ordered abelian groups in a more algebraic setting.

The second focus is on the partial ordering of divisibility on a class of integral domains, known as pseudo-valuation domains. It has been known for some time that pseudo-valuation domains have a fairly predictable divisibility structure. Here, it is shown that this divisibility structure can be used to find sufficient criteria to ensure a domain is a pseudo-valuation domain. This criteria is then used, along with a classification of a related class of domains, to classify all atomic pseudo-valuation domains. This classification is done solely in terms of the divisibility structure of the domains.

Within the discussion of pseudo-valuation domains there is a classification of the lattice of ideals of a certain class of pseudo-valuation domains, so called restricted power series. Additionally, there is a classification of the groups of divisibility of generalized restricted power series which provides further evidence for the conjecture that every pseudo-valuation domain can be classified in terms of its group of divisibility alone.

The thesis concludes with a discussion of the variety generated by the collection of all fields considered as algebras with two binary operations, division and subtraction. We develop an axiomatic approach to obtain an idea on what would be included in this variety as well as a discussion of some of the properties of subvarieties generated by individual finite fields.



## CHAPTER 1. OVERVIEW

### 1.1 Introduction

In any early study of algebra, the concept of ordering plays a significant role. In fact, most of the applications of algebra are consequences of the order structure of the real numbers  $\mathbb{R}$ , the integers  $\mathbb{Z}$ , and the positive natural numbers  $\mathbb{N}^*$  by divisibility. It is natural, then, to extend this concept of an ordered algebraic system to more a more abstract setting.

One of the more troublesome aspects of working with ordered algebraic systems is that the order structure is, often, independent of the algebra structure. In fact, most often the order structure cannot even be described in algebraic terms. In Chapter 2, we discuss how to examine the category of abelian quasi-ordered groups in a more algebraic setting by defining a new base category (the category of set monomorphisms) that is more suited to this task than the more standard category of sets.

Admittedly, partial orderings are slightly more natural to think about than quasi-orderings. One major application of partial orderings is in the arena of ordering ring elements by divisibility and examining the ring structure using a partially ordered group, called the group of divisibility. In Chapter 3, we use the divisibility structure of a class of integral domains, called pseudo-valuation domains, to find several equivalent conditions for a given integral domain to be an atomic pseudo-valuation domain. We also find a sufficient condition on the group of divisibility of an integral domain to guarantee that it is a (not necessarily atomic) pseudo-valuation domain.

Applying some of the results in Chapter 3, we are able to find many more examples of pseudo-valuation domains and classify their groups of divisibility and congruence (ideal) lattices. Chapter 4 builds on Ribenboim's (27) definition of a generalized power series to do so.

Deviating from the major theme of ordered algebraic systems, in Chapter 5 we discuss an approach to a solution to one of the most vexing problems in algebra, how to define division in a field axiomatically as a binary operation. There are many different ways to accomplish this feat, including an approach described by Carlström in (6) utilizing unary operations which simulate division by ring elements. While this approach has great merit, the resulting structure has some unexpected behavior that rings do not have, such as the additive identity being a sink. The approach taken here is to define the variety of algebras modeling fields in terms of two binary operations and axioms which give a ring structure and several other desirable properties of fields.

## 1.2 Ordered Algebraic Systems

The seminal work in the exploration of ordered algebraic systems was (12), where the majority of the preliminary definitions are derived. Other major definitions and conventions come from (2) as well.

**Definition 1.2.1** *Given a set  $X$ , a subset  $\leq$  of the Cartesian product (a relation) on  $X^2$  is said to be a quasi-ordering if the following conditions hold:*

- *If  $a \in X$ , then  $(a, a) \in \leq$  ( $\leq$  is reflexive)*
- *If  $(a, b) \in \leq$  and  $(b, c) \in \leq$ , implies  $(a, c) \in \leq$  ( $\leq$  is transitive)*

*The most common notation for  $(a, b) \in \leq$  is the infix notation  $a \leq b$ . Also, if  $(a, b)$  and  $(b, a) \in \leq$ , then  $a = b$  ( $\leq$  is symmetric), the relation  $\leq$  is a partial-ordering.*

This paper uses the universal algebraic notions of algebra and variety as follows. Much of the notation and terminology is in keeping with (5).

**Definition 1.2.2** *Given a set  $A$  and a set of functions  $\mathcal{F}$ , where each  $f \in \mathcal{F}$  is a function  $f : A^n \rightarrow A$  for some  $n \in \mathbb{N}$ , we say that  $A$  is an algebra with basic operations  $\mathcal{F}$ . The power  $n$  on  $A$  for each basic operation  $f$  is known as the arity of the operation and all of the arities together is known as the type.*

The collection of all algebras of the same type that satisfy a fixed set of equations (identities) is known as a variety.

**Example 1.2.3** A group  $G$  is an algebra with a binary operation  $\cdot$ , a unary operation  $-1$  (inversion), and a nullary operation  $1$  (identity element). The collection of all groups is a variety, the identities are:

- $a(bc) = (ab)c$  (the associative identity)
- $aa^{-1} = 1$  (the inversion identity)
- $a1 = a$  and  $a = 1a$  (the unital identities)

Other common examples of collections of algebras that form varieties are; rings, modules over a fixed ring, lattices, and Heyting algebras. One of the most fundamental theorems in universal algebra is Birkhoff's HSP theorem, showing the connection between the satisfaction of a fixed set of identities and the constructions of taking quotients, subalgebras, and products of algebras from a given collection.

**Definition 1.2.4** Given algebras  $A$  and  $B$  of the same type and a function  $f : A \rightarrow B$ , we say that  $f$  is an homomorphism if, for every operation  $g_A$  on  $A$  and the corresponding operation  $g_B$  on  $B$ , and every  $(a_1, a_2, \dots, a_n) \in A^n$ , we have that  $f(g_A(a_1, a_2, \dots, a_n)) = g_B(f(a_1), f(a_2), \dots, f(a_n))$ .

For every homomorphism  $f$  of algebras, the relation  $\theta_f = \{(a_1, a_2) \in A^2 \mid f(a_1) = f(a_2)\}$ , is called the congruence associated with  $f$ .

If  $A$  is an algebra and  $f$  is a homomorphism whose domain is  $A$ , the image of  $A$  under  $f$  is also an algebra (of the same type), and is known as either the homomorphic image of  $A$  under  $f$  or the quotient of  $A$  by  $\theta_f$ .

If  $A$  is an algebra and  $B$  is a subset of  $A$  such that, for every basic operation  $f$  of  $A$ , and any  $b_1, b_2, \dots, b_n \in B$  (where  $n$  is the arity of  $f$ ) implies  $f(b_1, b_2, \dots, b_n) \in B$ , we say that  $B$  is a subalgebra of  $A$ . Note that if  $A$  has no nullary operations, then  $B = \emptyset$  is a subalgebra of  $A$ .

**Example 1.2.5** *Common examples of congruences include normal subgroups  $N$  of a group  $G$ , where  $a\theta b$  if and only if  $ab^{-1} \in N$ . Similarly the (two sided) ideals of rings represent the ring congruences. Generally speaking, understanding the congruence structure of a class of algebras will tell one a great deal about the properties of any algebra in the class.*

**Theorem 1.2.6** (2) *A collection of algebras forms a variety if and only if every homomorphic image of a subalgebra of a product of algebras from the collection  $\mathcal{A}$  is a member of  $\mathcal{A}$ . More succinctly,  $HSP(\mathcal{A}) = \mathcal{A}$ .*

We now introduce the notion of a partially-ordered algebra, and exhibit some of the differences between them and algebras which do not necessarily have an order structure.

**Definition 1.2.7** *An algebra  $A$  is said to be partially (or respectively quasi-) ordered if there is a partial (or quasi-) ordering  $\leq$  ( $\preceq$ ) on the set  $A$  such that each basic operation  $\rho$  is order-preserving or order-reversing in each of its arguments. The homomorphisms between such algebras are the algebra homomorphisms which are order-preserving, that is,  $a \leq b$  implies  $f(a) \leq f(b)$  for the orderings in the respective algebras.*

Since the order preserving homomorphisms can be composed and the identity function is order preserving, we may regard collections of partially-ordered algebras and quasi-ordered algebras and their order preserving functions as categories. From this point forward the notation po-group will be used for a partially-ordered group and qo-group will be used for a quasi-ordered group. The categories of abelian po-groups and abelian qo-groups with order preserving homomorphisms will be denoted as **PAb** and **QAb** respectively.

There are many accessible examples of partially-ordered groups, but examples of quasi-ordered groups may not be readily accessible to many. Here are a few examples of each to keep in mind. There is a definition from ring theory that is relevant and will be important for the rest of the paper.

**Definition 1.2.8** *For any ring  $S$ , denote the nonzero elements of  $S$  by  $S^\#$ . For an integral domain  $R$ , define the ring  $QF(R)$ , the quotient field (fraction field) of  $R$ , to be the smallest*

field containing  $R$  as a subring. (This construction can also be obtained by creating formal quotients of  $R$  and imposing an equivalence relation on them,  $\frac{a}{b} = \frac{c}{d}$  if and only if  $ad = cb$ .)

**Example 1.2.9** *The following are examples of po-groups:*

- $\mathbb{Q}^\# / \{\pm 1\}$  where  $p \leq q$  if and only if  $qp^{-1} \in \mathbb{Z}^\#$
- $\mathbb{R}$  (under the usual ordering)
- Any group  $G$  where  $a \leq b$  if and only if  $a = b$

*The following are examples of qo-groups:*

- $\mathbb{Z}$  where  $a \preceq b$  if and only if  $a \leq b$  in the usual sense and  $b - a \equiv 0 \pmod{2}$
- The group  $QF(R)^\#$  ordered by,  $a \preceq b$  if and only if there exists  $r \in R$  such that  $ar = b$

While it may be fairly obvious that the first two examples are indeed po-groups, close inspection will verify that the third is a po-group. To verify that the last two examples are qo-groups we use a feature of ordered groups called the positive cone. We will generally use the positive cone to describe the order structure on the ordered group. One can also see how the definition of an abelian qo-group would not be expected to be entirely algebraic in nature, that is, the structure is not solely defined in terms of operations. Also, requiring algebra homomorphisms to satisfy an additional property, such as order preservation, is not part of traditional universal algebra. In the course of this discussion we shall need to reinterpret **PAb** and **QAb** into more algebraic terms. We are in need of an efficient way to discern the partial order structure on one of these objects via algebraic conditions. We begin by defining the positive cone of an abelian qo-group, which is the key concept in understanding the overall procedure.

**Definition 1.2.10** *For an abelian qo-group  $(G, \preceq)$ , we define*

$$G_0^+ := \{g \in G : 0 \preceq g\}$$

*to be the positive cone of  $(G, \preceq)$ .*

We now characterize those subsets of abelian groups which may be considered as positive cones with respect to some quasi-ordering or partial ordering. For a monoid  $M$ , we denote the invertible elements of  $M$  by  $M^*$ .

**Definition 1.2.11** *A cancellative monoid  $(M, +, 0)$  is said to be conical if  $M^* = \{0\}$ .*

**Proposition 1.2.12** *Let  $G$  be an abelian group and  $M \subseteq G$ . Then:*

- (a)  *$M$  is the positive cone of a qo-group  $(G, \preceq)$  if and only if  $M$  is a monoid.*
- (b)  *$M$  is the positive cone of a po-group  $(G, \leq)$  if and only if  $M$  is a conical monoid.*

**Proof 1.2.13** *Suppose that  $M$  is a submonoid of  $G$ . Define the quasi-order as  $a \preceq b$  if and only if  $b - a \in M$ . We have that  $a - a \in M$  for  $a \in G$ , thus  $\preceq$  is reflexive. Furthermore if  $a \preceq b$  and  $b \preceq c$ , then  $c - a = (c - b) + (b - a) \in M$ . Finally if  $a \preceq b$  and  $c \in G$ , we have  $(b - c) + (c - a) = b - a \in M$ . Now suppose that  $M$  is the positive cone for an abelian qo-group  $(G, \preceq)$ . We must have  $0 \in M$  by reflexivity, and if  $a, b \in M$  so  $0 \preceq a$  and  $0 \preceq b$ . Now  $b \preceq a + b$  by translation invariance, so  $0 \preceq b \preceq a + b$  implies  $0 \preceq a + b$  by transitivity.*

*In the special case where  $M$  is a conical submonoid of  $G$ , we must have that if  $b - a, a - b \in M$  then  $(b - a) + (a - b) = 0$ , so  $a - b = 0$  and  $a = b$ . Therefore the quasi-ordering created is, in fact, a partial ordering. On the other hand, if  $(G, \leq)$  is an abelian po-group, then if  $0 \leq a$  and  $0 \leq -a$ , we must have  $a \leq 0 \leq a$ , so  $a = 0$ . Finally, since  $M$  is a submonoid of an abelian group  $G$ , it is necessarily cancellative.*

**Proposition 1.2.14** *Given an abelian group  $G$  with a submonoid  $M$  of  $G$ , we have that  $M$  is the positive cone of an abelian po-group  $(G, \leq)$  if and only if  $M \setminus \{0\}$  is a subsemigroup of  $G$ .*

**Proof 1.2.15** *Suppose that  $M$  is the positive cone of an abelian po-group  $(G, \leq)$ . From Proposition 1.2.12 we know that  $M$  is a conical monoid. Since  $M$  is a monoid, it is closed under the binary operation, and since  $M$  is conical, the set of nonzero elements is also closed under the binary operation. Thus  $M \setminus \{0\}$  is a semigroup.*

*On the other hand, if  $M \setminus \{0\}$  is a subsemigroup of  $G$ , we have that  $M$  is a conical monoid. Thus, there is a partial order  $\leq$  such that  $M$  is the positive cone of  $(G, \leq)$ .*

We are now in a position to construct many more examples of abelian qo-groups, which should aid in the understanding of their nature. In view of Proposition 1.2.12, we may construct many abelian qo-groups as follows.

**Example 1.2.16** *Given a finite abelian group  $G$  and a nonzero submonoid  $M$  of  $G$ , the monoid  $M$  is the positive cone of a qo-group  $(G, \preceq)$  which is not an abelian po-group. This is because each element of  $M$  must have torsion, as  $M$  is finite.*

**Example 1.2.17** *Given an integral domain  $R$  and its quotient field  $QF(R)$ , we may see that  $QF(R)^\sharp$  is a group with submonoid  $R^\sharp$ . Therefore  $R^\sharp$  defines a quasi-order on the group  $QF(R)^\sharp$ , the divisibility quasi-order. This quasi-order is a partial-order if and only if  $U(R)$  is trivial, which is rarely the case.*

In the same way that one may define an equivalence relation on a quasi-ordered set identifying all element pairs  $a, b$  with  $a \preceq b$  and  $b \preceq a$ , one may do the same for abelian qo-groups. The process of taking the quotient by this equivalence relation (which is, in fact, a congruence) is called the *antisymmetrization* of the abelian qo-group.

**Lemma 1.2.18** *Consider an abelian qo-group  $(G, \preceq)$  with positive cone  $M$ . Then for elements  $a, b$  of  $G$ , the following are equivalent:*

- (1)  $a \preceq b$  and  $b \preceq a$ ;
- (2)  $a + M^* = b + M^*$ .

**Proposition 1.2.19** *Given an abelian qo-group  $(G, \preceq)$  with positive cone  $M$ , the quotient  $(G/M^*, \preceq')$ , is an abelian po-group with partial ordering  $a + M^* \preceq' b + M^*$  if and only if  $b - a \in M^*$ .*

**Proof 1.2.20** *First note that  $M^*$  is a (normal) subgroup of  $G$ , so the quotient  $G/M^*$  is an abelian group. It is clear that  $M/M^*$  is a submonoid of  $G/M^*$ . We now verify that  $M/M^*$  is conical. Suppose that  $a + b + M^* = M^*$  for  $a, b \in M$ . That is,  $a + b \in M^*$ . Then  $-b - a \in M^* \subset M$ , so  $-b - a + a = -b \in M$ , whence  $b \in M^*$  and  $(G/M^*, \preceq')$  is an abelian po-group.*

In light of Example 1.2.16, the antisymmetrization of a finite abelian qo-group is trivially ordered. We also have that antisymmetrization is idempotent.

The discussion of the positive cone for an abelian po-group reveals an intrinsic connection between the two. Since the morphisms  $f$  of  $\mathbf{QAb}$  are order-preserving, we have that  $0 \preceq f(a)$  for any element  $a$  of the positive cone. So we see that the order-preserving morphisms are precisely the group homomorphisms mapping positive cones into positive cones.



## CHAPTER 2. MONADICITY OF ABELIAN QO-GROUPS OVER SET MONOMORPHISMS

In this chapter we examine the relationship between  $\mathbf{QAb}$  and the categories of set injections and set monomorphisms. We begin by defining the relevant categories and notation to the discussion and use it to examine some adjunctions between the categories. After the initial discussion of adjunctions we examine the notion of free abelian qo-groups and discuss some of their aspects. We conclude the chapter with a monadic adjunction for the category of abelian qo-groups over the category of set monomorphisms.

### 2.1 Introduction

We begin with the definitions that will reexamine abelian qo-groups in a way that will be more beneficial to understanding the algebraic structure of the object.

**Definition 2.1.1** *Let*

$$C : \mathbf{QAb} \rightarrow \mathbf{Mon}$$

*be the functor from abelian qo-groups to monoids which assigns to each abelian qo-group its positive cone, and to each order-preserving morphism, the corresponding monoid homomorphism between the positive cones. Let*

$$U' : \mathbf{QAb} \rightarrow \mathbf{Mon}$$

*be the forgetful functor which forgets both the inverses and quasi-ordering. Finally, let*

$$\tau : C \rightarrow U'$$

*be the natural transformation between these functors with components as in Figure 2.1.*

$$\begin{array}{ccc}
G & G_0^+ & \xrightarrow{\tau_G} & U'G \\
\downarrow f & \downarrow Cf & & \downarrow U'f \\
H & H_0^+ & \xrightarrow{\tau_H} & U'H
\end{array}$$

Figure 2.1 A morphism of  $\mathbf{QAb}^i$ 

The components of  $\tau$  at an abelian qo-group will be the objects of a category  $\mathbf{QAb}^i$  with vertical composition of the maps as described in (20), where each map is induced by an order-preserving homomorphism  $f : G \rightarrow H$  as in Figure 2.1. The most important observation is that  $\mathbf{QAb}^i$  is isomorphic to  $\mathbf{QAb}$  itself.

**Theorem 2.1.2** *There is an isomorphism of categories between  $\mathbf{QAb}$  and  $\mathbf{QAb}^i$ .*

**Proof 2.1.3** *Consider the functor  $A : \mathbf{QAb} \rightarrow \mathbf{QAb}^i$  which assigns, to each order-preserving morphism on the left of Figure 2.1, the commuting diagram on the right. Further, consider the functor  $B : \mathbf{QAb}^i \rightarrow \mathbf{QAb}$ , which assigns, to any component of the natural transformation  $\tau_G$ , an abelian qo-group  $G$  with positive cone  $C(G)$ . The functor  $B$  sends the commuting diagram on the right of Figure 2.1 to the morphism on the left. It is readily observed that  $AB = 1_{\mathbf{QAb}^i}$  and  $BA = 1_{\mathbf{QAb}}$ . Therefore the two categories are isomorphic.*

A component  $\tau_G$  of the natural transformation  $\tau$  can be considered as the function that inserts the positive cone of the abelian qo-group  $G$  into the group. We see that instead of taking the group as the main object of interest in our study of abelian qo-groups, we may instead concentrate on the monoid insertion  $\tau_G$ . This technique will be used to establish a monadic relationship between abelian qo-groups and a base category in Section 2.3.

## 2.2 Adjunctions for Relevant Categories

In Section 2.1 we examined some of the fundamental properties of ordered abelian groups, and reexamined the relationship between the positive cone of an abelian qo-group and the group itself. Specifically, we saw that the category  $\mathbf{QAb}$  is isomorphic to the category  $\mathbf{QAb}^i$ , the category of components of the natural transformation  $\tau : C \rightarrow U'$  with vertical composition

of the induced maps. It is now possible to relate the category  $\mathbf{QAb}^i$  to suitable base categories, the categories of set insertions and set monomorphisms. Further, we develop a more general category  $\mathbf{QAb}^m$  which will be seen to be equivalent to  $\mathbf{QAb}^i$ .

**Definition 2.2.1** *The objects of the category  $\mathbf{Set}^i$  of set insertions are the insertions  $i : X' \hookrightarrow X$  of subsets into supersets. The morphisms of the category are pairs of set maps*

$$(f_1, f_2) : (i : X' \hookrightarrow X) \rightarrow (j : Y' \hookrightarrow Y)$$

where  $f_1 : X \rightarrow Y$ ,  $f_2 : X' \rightarrow Y'$ , and  $j \circ f_2 = f_1 \circ i$ . The composition is clearly associative by the inspection and verification of the commuting squares of Figure 2 and regarding  $f \circ g = (f_1 \circ g_1, f_2 \circ g_2)$ .

$$\begin{array}{ccc} X' & \xhookrightarrow{i} & X \\ f_2 \downarrow & & \downarrow f_1 \\ Y' & \xhookrightarrow{j} & Y \end{array}$$

Figure 2.2 A morphism of  $\mathbf{Set}^i$

In Definition 2.2.1 we have set insertions as the objects of the category. In the current form it would be acceptable to view each of the objects as a pair, say  $(X', X)$  with no real confusion as to how  $X'$  is identified inside of the superset  $X$ . It is possible to generalize the situation by defining an equivalent related category.

**Definition 2.2.2** *The objects of the category  $\mathbf{Set}^m$  of set monomorphisms are injective set maps  $i : X' \rightarrow X$ . The morphisms of the category are pairs of set maps*

$$(f_1, f_2) : (i : X' \rightarrow X) \rightarrow (j : Y' \rightarrow Y)$$

where  $f_1 : X \rightarrow Y$ ,  $f_2 : X' \rightarrow Y'$ , and  $j \circ f_2 = f_1 \circ i$ . Composition of morphisms in  $\mathbf{Set}^m$  is given by the same “componentwise” composition as defined for  $\mathbf{Set}^i$ .

**Theorem 2.2.3** *There is an equivalence of categories between  $\mathbf{Set}^i$  and  $\mathbf{Set}^m$ .*

**Proof 2.2.4** Define a functor

$$A : \mathbf{Set}^i \rightarrow \mathbf{Set}^m$$

on the morphisms as  $A(i : X' \hookrightarrow X) = Ai : X' \rightarrow X$ , which forgets that set insertions are insertions and simply considers them as set monomorphisms. Define  $B : \mathbf{Set}^m \rightarrow \mathbf{Set}^i$  as  $B(i : X' \rightarrow X) = Bi : i(X') \hookrightarrow X$ , which is the insertion of the image of  $i$  into the set  $X$ .

We must show that  $A$  and  $B$  constitute an equivalence of categories. We show that  $A$  is full, faithful, and dense in  $\mathbf{Set}^m$ . Suppose  $f, g \in \mathbf{Set}^i(i, j)$  with  $f \neq g$ . We have that  $f_1$  and  $g_1$  must differ on at least one element of  $X$ , the codomain of  $i$ . This is because  $f$  and  $g$  are uniquely determined by their definition on  $X$ , as described in (17). Then, since  $Af$  and  $Ag$  return the same maps  $f_1$  and  $g_1$  on  $X$ , we can be sure that they are not the same.

Now consider  $f \in \mathbf{Set}^m(Ai, Aj)$ . The morphism  $f$  is defined by its two components  $f_1$  on  $X$  and  $f_2$  on  $X'$ . Since  $i$  and  $j$  are monomorphisms, they are injective. Thus the definition of  $f_1$  uniquely determines the definition of  $f_2$  via  $j \circ f_2 = f_1 \circ i$ . Therefore  $f = Af'$  for some  $f' \in \mathbf{Set}^m(i, j)$ .

Finally, we have that  $A$  is dense since, given any object  $i : X' \rightarrow X$ , we have that  $i \cong ABi$ , by virtue of the morphism induced from the identity map on  $X$ .

Recall that in  $\mathbf{QAb}^i$  we may think of each of the components of  $\tau$  as the insertion map of the positive cone  $G_0^+$  into the group  $G$ . While this does provide us with quite a succinct way to view  $\mathbf{QAb}$  through an isomorphic category, we may use the principle outlined in Theorem 2.2.3 to define a category  $\mathbf{QAb}^m$  equivalent to  $\mathbf{QAb}^i$ .

$$\begin{array}{ccc} M & \xrightarrow{i} & U'G \\ f_2 \downarrow & & \downarrow U'f_1 \\ N & \xrightarrow{j} & U'H \end{array}$$

Figure 2.3 A morphism of  $\mathbf{QAb}^m$

**Definition 2.2.5** Let the objects of  $\mathbf{QAb}^m$  be monoid monomorphisms  $i : M \rightarrow U'G$ , where  $U'G$  is the monoid reduct of an abelian group  $G$ . Let the morphisms of  $\mathbf{QAb}^m$  be commuting

squares as in Figure 2.2, induced by a group homomorphism  $f : G \rightarrow H$  with componentwise composition similar to that of  $\mathbf{Set}^m$ .

**Theorem 2.2.6** *There is an equivalence of categories between  $\mathbf{QAb}^i$  and  $\mathbf{QAb}^m$ .*

**Proof 2.2.7** *Define  $A : \mathbf{QAb}^i \rightarrow \mathbf{QAb}^m$  to be the functor that forgets that Figure 2.1 came from a natural transformation, and simply considers it as a diagram in  $\mathbf{Mon}$ . Define  $B : \mathbf{QAb}^m \rightarrow \mathbf{QAb}^i$  to be the functor that assigns, to each monoid monomorphism  $i : M \rightarrow U'G$ , the component  $\tau_G$  where  $G$  is the abelian qo-group with positive cone  $i(M) \subseteq G$ . The morphism part of  $B$  simply assigns the morphism of  $\mathbf{QAb}^i$  corresponding to the abelian qo-group morphism induced by  $f : G \rightarrow H$ . The fact that  $A$  and  $B$  form an equivalence is simply a special case of Theorem 2.2.3.*

In defining  $\mathbf{QAb}^m$  as we have, we have moved the emphasis from the actual group  $G$  and the submonoid  $M$  of the positive cone to the way in which the elements of some monoid  $M$  are identified inside of  $G$ . In particular, the existence of a monoid monomorphism  $i : M \rightarrow U'G$  implies that  $M$  is commutative and cancellative. The focus on the identification provides a sufficiently abstracted framework for us to discuss a monadic adjunction between  $\mathbf{QAb}^m$  and  $\mathbf{Set}^m$  in Section 2.3. Before we describe such an adjunction, let us examine some specific objects of  $\mathbf{QAb}^m$ .

**Example 2.2.8** *In each of the following cases, we consider a qo-group structure on  $\mathbb{Z}$ .*

1. *We may identify the abelian l-group of  $\mathbb{Z}$  under the usual order as the object  $i : \mathbb{N} \rightarrow U'\mathbb{Z}$ , where  $i$  is the usual inclusion.*
2. *Let  $2 : \mathbb{N} \rightarrow U'\mathbb{Z}$  be the map that doubles every element and inserts it into the group  $\mathbb{Z}$ . We obtain the abelian po-group  $(\mathbb{Z}, \leq)$  with  $m \leq n$  if and only if  $n - m = 2k$  for  $k \in \mathbb{N}$ .*
3. *Letting  $2 : \mathbb{Z} \rightarrow U'\mathbb{Z}$  be a doubling map again, the resulting abelian qo-group is  $(\mathbb{Z}, \preceq)$ , where  $m \preceq n$  if and only if  $n - m = 2k$  for  $k \in \mathbb{Z}$ .*

Example 2.2.8 shows that it is not the positive cone monoid itself that is important for the structure of an abelian qo-group, but rather the way in which that cone is identified inside the group. This is why we take monomorphisms as the objects, and not just the pairs  $(M, G)$ .

In the special case of abelian po-groups, we define categories corresponding to  $\mathbf{QAb}^i$  and  $\mathbf{QAb}^m$ .

**Definition 2.2.9** *The category  $\mathbf{PAb}^i$  is the full subcategory of  $\mathbf{QAb}^i$  where all the positive cones are positive cones of abelian po-groups. The category  $\mathbf{PAb}^m$  is the full subcategory of  $\mathbf{QAb}^m$  where all the monoids are conical.*

**Proposition 2.2.10**  *$\mathbf{PAb}^i$  is equivalent to  $\mathbf{PAb}^m$  and isomorphic to  $\mathbf{PAb}$ .*

**Proof 2.2.11** *Recall the functors  $A$  and  $B$  in Theorem 2.2.6. Take the restrictions of the functors to the respective subcategories  $\mathbf{PAb}^i$  and  $\mathbf{PAb}^m$ . For the second statement, take the restrictions of the functors of Theorem 2.1.2 to the respective subcategories  $\mathbf{PAb}^i$  and  $\mathbf{PAb}^m$ .*

In defining  $\mathbf{QAb}^i$  and  $\mathbf{PAb}^i$ , we identified the properties that a monoid must have in order for it to be the positive cone of an abelian po-group or an abelian qo-group. Namely, we saw that an arbitrary submonoid of a group could be the positive cone of an abelian qo-group, and an arbitrary conical monoid could be the positive cone of an abelian po-group. Example 2.2.8 shows that there is no comparable characterization of the monoids which may be the positive cone of an abelian l-group, since Example 2.2.8(1) and Example 2.2.8(2) have isomorphic monoids as their positive cones, while the first example is an l-group and the second is not. In order to define categories  $\mathbf{LAb}^i$  and  $\mathbf{LAb}^m$  corresponding to  $\mathbf{QAb}^i$  and  $\mathbf{QAb}^m$  for lattice-ordered abelian groups, we thus use extrinsic properties rather than an internal definition relying on positive cones.

**Definition 2.2.12** *Let  $\mathbf{LAb}^i$  be the category whose objects are insertions  $i : G_0^+ \hookrightarrow G$  of the positive cone of an abelian l-group into the abelian l-group, and whose morphisms are pairs  $f = (f_1, f_2)$ , where  $f_1$  is an abelian l-group morphism and  $f_2$  is the restriction of  $f_1$  to the positive cone.*

$$\begin{array}{ccccccc}
\mathbf{PAb} & \xrightarrow{U_{PQ}} & \mathbf{QAb} & \xrightarrow{U_Q} & \mathbf{Set} & \xleftarrow{U_L} & \mathbf{LAb} \\
\uparrow U_P^i & & \uparrow U_{QQ}^i & & \uparrow U^i & & \uparrow U_{LL}^i \\
\mathbf{PAb}^i & \xrightarrow{U_{PQ}^i} & \mathbf{QAb}^i & \xrightarrow{U_Q^i} & \mathbf{Set}^i & \xleftarrow{U_L^i} & \mathbf{LAb}^i \\
\downarrow U_P^{im} & & \downarrow U_Q^{im} & & \downarrow U^{im} & & \downarrow U_L^{im} \\
\mathbf{PAb}^m & \xrightarrow{U_{PQ}^m} & \mathbf{QAb}^m & \xrightarrow{U_Q^m} & \mathbf{Set}^m & \xleftarrow{U_L^m} & \mathbf{LAb}^m
\end{array}$$

Figure 2.4 Forgetful Functors

Similarly, define  $\mathbf{LAb}^m$  be the category whose objects are monoid monomorphisms  $i : M \rightarrow G$  of  $M$  onto the positive cone of an abelian  $l$ -group  $G$ . The morphisms of  $\mathbf{LAb}^m$  are pairs  $f = (f_1, f_2)$ , where  $f_1$  is an abelian  $l$ -group morphism and  $f_2$  is a monoid morphism, with  $j \circ f_2 = f_1 \circ i$ .

**Remark 2.2.13 (Notational conventions)** In the rest of this section, a functor denoted by a decorated  $U$  will be a forgetful functor. A left adjoint to one of these will be denoted by  $A$  with the same subscripts and superscripts. The subscripts  $P$ ,  $L$ , and  $Q$  denote the order structure on the domain and codomain of the forgetful functor. If there is only one letter in the subscript, either the codomain is related to  $\mathbf{Set}$  or the order type is not changing. The superscripts  $i$  and  $m$  denote insertions or monomorphisms for the domain and codomain. If there is only one superscript, the functor disregards the insertion.

There is a collection of evident forgetful functors between all of the categories discussed so far, as recorded in Figure 2.4. We have already seen some of the left adjoints to these functors, and some are particularly well known. We shall conclude this section by indicating all of the adjoints to these forgetful functors. The forgetful functors  $U_L^i$  and  $U_L^m$  have not yet been defined, but will also be seen as right adjoints by the end of this section.

**Corollary 2.2.14** Consider the forgetful functors of Figure 2.4. Then  $U_{LL}^i, U_P^i$ , and  $U_{QQ}^i$  are all isomorphisms of categories. Furthermore, the forgetful functors  $U_L^{im}, U_P^{im}, U_Q^{im}$ , and  $U^{im}$  provide equivalences of categories.

**Proof 2.2.15** The first assertion is a summary of the statements of Proposition 2.2.10 and Theorem 2.1.2, and the special case of the restriction of the isomorphisms to  $\mathbf{LAb}$  and  $\mathbf{LAb}^i$ .

We have that  $U_Q^{im}$  is an equivalence by Theorem 2.2.6, the functor  $U_P^{im}$  is an equivalence by the first portion of Proposition 2.2.10, and the fact that  $U^{im}$  is an equivalence results from Theorem 2.2.3. Finally, the fact that  $U_L^{im}$  is an equivalence follows by restriction from **PAb** to **LAAb**.

**Proposition 2.2.16** *Each of the forgetful functors  $U_Q, U^i, U_Q^i$ , and  $U_Q^m$  of Figure 2.4 is a right adjoint.*

**Proof 2.2.17** *Let us begin with  $U^i$ . Consider the functor  $A^i : \mathbf{Set} \rightarrow \mathbf{Set}^i$  with morphism part which sends  $f : X \rightarrow Y$  to  $A^i f = (f, 0)$ , where  $0 : \emptyset \hookrightarrow Y'$ . We may see that  $U^i A^i$  is the identity functor, so the unit of the adjunction is the identity natural transformation. Similarly  $F^i A^i$  sends the morphism  $(f_1, f_2)$  to the morphism  $(f_1, 0)$ , so the component of the counit of the adjunction at an object  $X' \rightarrow X$  is the pair consisting of the identity map on  $X$  and the empty map  $\emptyset \hookrightarrow X'$ .*

We now show that  $U_Q$  is the right adjoint of an adjunction. The fact that  $U_Q^i$  and  $U_Q^m$  are right adjoints then follows from a restatement of the definition of the left adjoint to  $U_Q$ . Define  $A_Q : \mathbf{Set} \rightarrow \mathbf{QAb}$  to have morphism part sending  $f : X \rightarrow Y$  to  $\bar{f} : F_1 X \rightarrow F_1 Y$ , where  $\bar{f}$  is the induced map from the free abelian group  $F_1 X$  to the free abelian group  $F_1 Y$ , in which both  $F_1 X$  and  $F_1 Y$  are trivially ordered. Then, by definition, all group homomorphisms from  $F_1 X$  to an abelian *qo*-group  $G$  are necessarily order-preserving, which means we may use the restrictions of the usual unit and counit components from the adjunction between **Set** and **Ab**.

**Proposition 2.2.18** *Each of the forgetful functors  $U_{PQ}, U_{PQ}^i$ , and  $U_{PQ}^m$  of Figure 2.4 is a right adjoint.*

**Proof 2.2.19** *Define the functor  $A_{PQ} : \mathbf{QAb} \rightarrow \mathbf{PAb}$  to have a morphism part that takes  $f : G \rightarrow H$  to*

$$f/G_0^{+*} : G/G_0^{+*} \rightarrow H/H_0^{+*}; g + G_0^{+*} \mapsto f(g) + H_0^{+*}.$$

*This is a well defined group homomorphism, since if  $a \in G_0^{+*}$ , then  $-a \in G_0^{+*}$  by definition. Further, since  $f$  is order-preserving, we must have that  $f(a) \in H_0^+$  and  $f(-a) \in H_0^+$ , so  $f(a) \in H_0^{+*}$ .*



The functor  $A_{PQ}U_{PQ}$  is the identity, so  $\varepsilon_G : A_{PQ}F_2G = G \rightarrow G$  is the identity map. Also,  $\eta_G : G \rightarrow U_{PQ}A_{PQ}G = U_{PQ}G/G_0^{+*}$  is the quotient map. It is clear that

$$A_{PQ} \xrightarrow{A_{PQ}\eta} A_{PQ}U_{PQ}A_{PQ} \xrightarrow{\varepsilon_{A_{PQ}}} A_{PQ}$$

and

$$U_{PQ} \xrightarrow{\eta_{U_{PQ}}} U_{PQ}A_{PQ}U_{PQ} \xrightarrow{U_{PQ}\varepsilon} U_{PQ}$$

are identity morphisms.

If we define  $A_{PQ}^i(i : G_0^+ \hookrightarrow G) = A_{PQ}^i i : G_0^+/G_0^{+*} \hookrightarrow G/G_0^{+*}$  and extend it to a morphism part in a similar fashion to  $A_{PQ}$ , we obtain  $A_{PQ}^i$  as the left adjoint to  $U_{PQ}^i$ .

Finally, the specification of

$$A_{PQ}^m : (i : M \rightarrow G) \mapsto (A_{PQ}^m i : M/M^* \rightarrow G/i(M^*))$$

will provide a left adjoint to  $U_{PQ}^m$ .

The adjoint situations for  $U_L : \mathbf{LAb} \rightarrow \mathbf{Set}$ ,  $U_L^i : \mathbf{LAb}^i \rightarrow \mathbf{Set}^i$ , and  $U_L^m : \mathbf{LAb}^m \rightarrow \mathbf{Set}^m$  are well known and, in fact, the first adjunction is even monadic. What is notable about the adjunction between  $\mathbf{LAb}$  and  $\mathbf{Set}$  are the free objects. Consider the following result due to Birkhoff (2).

**Example 2.2.20** *Let  $X = \{x\}$  be a set. Then the free lattice-ordered group  $A_L(X)$  on  $X$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$  with the componentwise ordering. The generator for this lattice-ordered group is obtained by mapping  $x$  to  $(1, -1)$ .*

It was also shown in (14) that, given any finitely generated (not necessarily free) abelian l-group, the underlying group is a free abelian group. At this point, it is important to note that there is no left adjoint to the forgetful functor from  $\mathbf{PAb}$  and  $\mathbf{LAb}$ . This issue was discussed by Weinberg (32), and later refined by Conrad (10). The main result from (10) is as follows. We begin with a definition.

**Definition 2.2.21** *Given a group  $G$  we say that a partial order  $\leq$  on  $G$  is a right order if  $\leq$  is a linear order and  $ab \leq ac$  whenever  $b \leq c$ .*

For an abelian po-group, the notions of a right order and a linear order are equivalent.

**Theorem 2.2.22** (10) *For a po-group  $G$  the following are equivalent:*

1. *There exists a free l-group over  $G$ ;*
2. *There exists a **PAb** isomorphism between  $G$  and the underlying po-group of an l-group;*
3. *The positive cone  $G_0^+$  of  $G$  is an intersection of right orders.*

We now have the following.

**Corollary 2.2.23** *The forgetful functor  $U : \mathbf{LAb} \rightarrow \mathbf{QAb}$  has no left adjoint.*

**Proof 2.2.24** *Let  $T$  be a finitely generated abelian qo-group with non-trivial torsion elements and antichain ordering. Since  $A_{PQ}$  takes the antisymmetrization of the qo-group, we have that there is no free abelian l-group over  $A_{PQ}T$ , since condition (2) of Theorem 2.2.22 is violated (finitely generated l-groups cannot have nontrivial torsion elements). With this observation, and the fact that adjoints are unique, we have that there is no left adjoint to  $U$ , as desired.*

### 2.3 Free Abelian Qo-Groups

In the preceding section we discussed the relationships between several categories related to **QAb** and summarized the relevant adjunctions between them in Figure 2.4. Of particular interest at this point is the left adjoint to the functor  $U_Q^m : \mathbf{QAb}^m \rightarrow \mathbf{Set}^m$  (hereafter abbreviated to  $U$ ), since the images of **Set** <sup>$m$</sup> -objects under the left adjoint will be regarded as free abelian qo-groups. (The images of objects under the left adjoint from **Set** to **QAb** have trivial order structure, and thus fail to encompass the full gamut expected of free abelian qo-groups.) We restate the situation as follows.

**Definition 2.3.1** *Let  $f : (i : X' \rightarrow X) \rightarrow (j : Y' \rightarrow Y)$  be a morphism in **Set** <sup>$m$</sup> , where  $f$  is induced by the set map  $f_1 : X \rightarrow Y$ . Define  $Ff : (Fi : F_2X' \rightarrow U'F_1X) \rightarrow (Fj : F_2Y' \rightarrow U'F_1Y)$  to be the **QAb** <sup>$m$</sup> -morphism determined by the group homomorphism  $F_1f : F_1X \rightarrow F_1Y$ , where  $F_1X$  is the free abelian group on  $X$  and  $F_2X'$  is the free commutative monoid on  $X'$ .*

**Definition 2.3.2** Let  $U_1 : \mathbf{Ab} \rightarrow \mathbf{Set}$  be the forgetful functor, part of the adjunction  $(F_1, U_1, \eta_1, \varepsilon_1)$ .

**Theorem 2.3.3** Let  $U$  be the forgetful functor from  $\mathbf{QAb}^m$  to  $\mathbf{Set}^m$ . Consider  $F, F_1, F_2$ , and  $U'$  as in Definitions 2.3.1 and 2.3.2. Then  $F$  is left adjoint to  $U$ .

**Proof 2.3.4** We prove the theorem by constructing the unit and counit of the claimed adjunction, and verifying the triangular identities. Define

$$\eta_i : (i : X' \rightarrow X) \rightarrow (UFi : U_2F_2X' \rightarrow U_2U'F_2X),$$

the map induced by the set homomorphism  $\eta_{1X}$ . Define

$$\varepsilon_j : (FUj : F_2U_2M \rightarrow F_1U_2U'G) \rightarrow (j : M \rightarrow U'G),$$

the map induced by the abelian group homomorphism  $\varepsilon_{1G}$ .

We have  $F_1 \xrightarrow{F_1\eta_1} F_1U_1F_1 \xrightarrow{\varepsilon_1F_1} F_1$  as the identity at each set and  $U_1 \xrightarrow{\eta_1U_1} U_1F_1U_1 \xrightarrow{U_1\varepsilon_1} U_1$  as the identity at each abelian group. Furthermore, if  $f : (i : X' \rightarrow X) \rightarrow (i : X' \rightarrow X)$  is induced by the identity on  $X$ , the commuting diagram and the monic nature of  $i$  forces  $f$  to be the identity morphism. The same holds for a morphism in  $\mathbf{QAb}^m$ .

**Definition 2.3.5** The image of an object of  $\mathbf{Set}^m$  under  $F$  is described as a free object of  $\mathbf{QAb}^m$ , or as a free abelian qo-group.

Recall the characterization of free abelian groups and free commutative monoids.

**Proposition 2.3.6** An abelian group  $G$  is a free abelian group if and only if there is an indexing set  $X$  so that  $G \cong \bigoplus_{x \in X} \mathbb{Z}$ . A commutative monoid  $M$  is a free commutative monoid if and only if there is an indexing set  $X'$  so that  $M \cong \bigoplus_{x \in X'} \mathbb{N}$ .

**Definition 2.3.7** In the event that the indexing set for a free abelian group (or commutative monoid) is finite, the cardinality of  $X$  is commonly known as the rank of the free abelian group (or commutative monoid). The rank determines a free abelian group (or commutative monoid) up to isomorphism. The function  $\text{rk}(G)$  assigns, to each free abelian group (or commutative monoid)  $G$ , its rank.

In light of Proposition 2.3.6, we may now characterize free abelian qo-groups in the context of the base category  $\mathbf{Set}^m$  as follows.

**Theorem 2.3.8** *An abelian qo-group associated with  $i : M \rightarrow G$  is a free abelian qo-group if and only if  $M$  and  $G$  are free.*

**Proposition 2.3.9** *If  $i : M \rightarrow G$  is free, then  $\text{rk}(M) \leq \text{rk}(G)$ .*

Since the positive cone monoid determines the order structure on an abelian qo-group, we may observe the following.

**Proposition 2.3.10** *Any free abelian qo-group  $i : M \rightarrow G$  can be viewed, in terms of  $\mathbf{QAb}$ , as  $(\bigoplus_{x \in X'} (\mathbb{Z}, \leq)) \oplus (\bigoplus_{x \notin X'} (\mathbb{Z}, =))$ , where the first term is a sum of copies of  $\mathbb{Z}$  with the usual order, while the second term is a sum of  $\mathbb{Z}$  with the discrete order.*

**Proof 2.3.11** *We translate the expression of the group in terms of direct sums into an expression of the insertion of a positive cone in  $\mathbf{QAb}^i$ . We recognize that in terms of the group structure we have  $\bigoplus_{x \in X} \mathbb{Z}$ , and that an element of this group is in the positive cone if and only if  $a_x \geq 0$  for each term in the sum. This means that  $a_x = 0$  for each  $x \notin X'$ . Thus we may recognize the submonoid of the positive cone as  $\bigoplus_{x \in X'} \mathbb{Z}$  and take the monoid monomorphism as the natural set insertion.*

**Proposition 2.3.12** *Suppose that  $i : M \rightarrow G$  is a free abelian qo-group over  $i' : X' \rightarrow X$ . Then:*

1.  $M$  is a conical monoid;
2.  $i$  is associated with an abelian qo-group;
3.  $i$  is associated with a lattice if and only if  $i'$  is an isomorphism;
4.  $i$  is associated with a chain if and only if  $i'$  is an isomorphism and  $X = \{x\}$ ;
5.  $i$  is associated with an antichain ordering if and only if  $X' = \emptyset$ .

**Proposition 2.3.13** *There are  $n + 1$  isomorphism classes for free abelian qo-groups whose abelian group reducts are free groups of rank  $n$ .*

**Proof 2.3.14** *Since every free abelian qo-group is a direct sum of copies of  $\mathbb{Z}$ , we may arrange the sums so that the terms corresponding to the image of  $i'$  come first. Thus two free abelian qo-groups are isomorphic if the sizes of the generating sets of the group part and the monoid part are of the same cardinality.*

**Definition 2.3.15** *For a free object  $i : M \rightarrow G$ , we define the defect of  $i$  to be  $\text{defect}(i) := \text{rk}(G) - \text{rk}(M)$ . We say that a free object  $i$  is defective if  $\text{defect}(i) = \text{rk}(G)$ .*

**Proposition 2.3.16** *For a free object  $i : M \rightarrow G$ , the pair*

$$(\text{defect}(i), \text{rk}(G))$$

*determines the abelian qo-group up to isomorphism.*

The difference in our approach to free abelian qo-groups from that adopted in (25) is that we are focusing on the adjunction between  $\mathbf{Set}^m$  and  $\mathbf{QAb}^m$ , so we obtain free abelian qo-groups with nontrivial order structure.

## 2.4 Monadicity for Qo-Groups

**Definition 2.4.1** *A monad over a category  $\mathbf{C}$  is a triple  $(T, \eta, \mu)$  consisting of an endofunctor  $T : \mathbf{C} \rightarrow \mathbf{C}$ , and two natural transformations  $\eta : 1 \rightarrow T$  and  $\mu : T^2 \rightarrow T$ , called the unit and the multiplication respectively. The triple must satisfy the identity and associative laws in Figure 2.5.*

$$\begin{array}{ccc}
 T & \xrightarrow{T\eta} & T^2 \\
 \eta T \downarrow & \searrow & \downarrow \mu \\
 T^2 & \xrightarrow{\mu} & T
 \end{array}
 \qquad
 \begin{array}{ccc}
 T^3 & \xrightarrow{T\mu} & T^2 \\
 \mu T \downarrow & & \downarrow \mu \\
 T^2 & \xrightarrow{\mu} & T
 \end{array}$$

Figure 2.5 Identity and Associative Laws of a Monad

**Definition 2.4.2** Each monad  $(T : \mathbf{C} \rightarrow \mathbf{C}, \eta, \mu)$  yields the category  $\mathbf{C}^T$  of Eilenberg-Moore algebras over that monad. The objects of  $\mathbf{C}^T$  are pairs  $(x, h)$ , where  $x$  is a  $\mathbf{C}$ -object and  $h : Tx \rightarrow x$  is a  $\mathbf{C}$ -morphism satisfying the associative and identity laws of Figure 2.6. The morphisms of  $\mathbf{C}^T$  are  $\mathbf{C}$ -morphisms  $f$  that make Figure 2.7 commute.

$$\begin{array}{ccc}
 x & \xrightarrow{\eta_x} & Tx \\
 \parallel & & \downarrow h \\
 & & x
 \end{array}
 \qquad
 \begin{array}{ccc}
 T^2x & \xrightarrow{\mu_x} & Tx \\
 Th \downarrow & & \downarrow h \\
 Tx & \xrightarrow{h} & x
 \end{array}$$

Figure 2.6 Identity and associative laws of a  $\mathbf{C}^T$ -algebra

$$\begin{array}{ccc}
 Tx & \xrightarrow{Tf} & Ty \\
 h \downarrow & & \downarrow h' \\
 x & \xrightarrow{f} & y
 \end{array}$$

Figure 2.7 A  $\mathbf{C}^T$ -Morphism  $f$

For any adjunction  $(F, U, \eta, \varepsilon)$  with  $F : \mathbf{C} \rightarrow \mathbf{D}$ , there is a monad  $(UF, \eta, U\varepsilon F)$ . Furthermore, for this monad, there is an Eilenberg-Moore category  $\mathbf{C}^{U\varepsilon F}$ . The Eilenberg-Moore category is a terminal object in that there is a unique functor  $\bar{G}$  making Figure 2.8 commute. If  $\bar{G}$  is an equivalence, the adjunction is said to be *monadic*. (As noted in (20), some authors require an isomorphism between  $\mathbf{C}^T$  and  $\mathbf{A}$  for an adjunction to qualify as monadic.) If there exists a monadic adjunction with  $\mathbf{C} = \mathbf{Set}$ , the category  $\mathbf{D}$  is said to be *algebraic*. In particular, any variety of algebras is algebraic, motivating the terminology (29, Cor. IV.4.2.8).

$$\begin{array}{ccccc}
 \mathbf{Set}^m & \longrightarrow & \mathbf{Set}^{mT} & \longrightarrow & \mathbf{Set}^m \\
 \parallel & & \uparrow \bar{G} & & \parallel \\
 \mathbf{Set}^m & \xrightarrow{F} & \mathbf{QAb}^m & \xrightarrow{U} & \mathbf{Set}^m
 \end{array}$$

Figure 2.8 The Eilenberg-Moore comparison

We shall show that the adjunction of Theorem 2.3.3 is monadic. When writing elements of a free group or monoid, we adopt the convention of juxtaposition to represent a word, using 1 for the empty word. Since the initial operation in an abelian qo-group will always be denoted

additively, there should be no confusion as to which elements are considered as words and which are considered as elements of the corresponding monoids or groups.

For an object  $(i, h)$  with  $i : X' \rightarrow X$  of  $\mathbf{Set}^{mUF}$ , define

$$\overline{F}(i : X' \rightarrow X) = (i : X' \rightarrow X).$$

Define an operation

$$+_1 : X^2 \rightarrow X; (x, y) \mapsto h_1(xy)$$

on  $X$  and an operation

$$+_2 : X'^2 \rightarrow X'; (x, y) \mapsto h_2(xy)$$

on  $X'$ . Define

$$- : X \rightarrow X; x \mapsto h(x^{-1}).$$

Finally, define  $0_1 = h_1(1)$  and  $0_2 = h_2(1)$ .

**Lemma 2.4.3** *As defined,  $\overline{F}$  maps from the object class of  $\mathbf{Set}^{mUF}$  to the object class of  $\mathbf{QAb}^m$ .*

**Proof 2.4.4** *We must show, by using the associative and identity laws of the structure map  $h$  as expressed in Figure 2.6, that  $i : X' \rightarrow X$  is indeed a monoid monomorphism from the monoid  $X'$  to the underlying monoid of the group  $X$ . Note that in Figure 2.6,  $\eta$  is the identity of the monad and the unit of adjunction and  $\mu = U\varepsilon F$  is the multiplication, and  $T = UF$  using the adjunction  $(F, U, \eta, \varepsilon)$  from Definition 2.3.1.*

*We see that  $+_1$  is associative since  $(x +_1 y) +_1 z = h_1(h_1(xy)z)$ , and the associative law provides that  $h_1(U_1 F_1 h_1) = h_1(U_1 \varepsilon_1 F_1)$ , that is to say,  $h_1(h_1(xy)h_1(z)) = h_1(xyz) = h_1(h_1(x)h_1(yz))$ , and the identity law gives that  $h_1(z) = z$  for every  $z \in X$ . Thus  $(x +_1 y) +_1 z = x +_1 (y +_1 z)$ . The associative law for  $+_2$  follows similarly from the fact that  $h_2(U_2 F_2 h_2) = h_2(U_2 \varepsilon_2 F_2)$ .*

*The equality  $0_1 +_1 x = x = x +_1 0_1$  follows directly from the identity law and the fact that since  $F_1$  generates a free abelian group, the letters commute. The identity for  $X'$  is similar. We also have that  $i(0_2) = 0_1$ , since  $i(h_2(1)) = h_1(UF_i(1))$  and  $Fi(1) = 1$ . Furthermore, we have*

that  $h_1(UFi(xy)) = i(h_2(x, y))$ , so  $i$  is a monoid homomorphism which is a monomorphism since  $i$  is a set monomorphism.

**Lemma 2.4.5** *The object map as defined in Lemma 2.4.3 has an extension to a morphism part.*

**Proof 2.4.6** *For  $f \in \mathbf{Set}^{mUF}(i, j)$ , define  $\overline{F}f \in \mathbf{QAb}^m(\overline{F}i, \overline{F}j)$  as follows. Define the individual maps  $\overline{F}_1f_1$  and  $\overline{F}_2f_2$  from  $X$  to  $Y$  and from  $X'$  to  $Y'$  to be the same set maps as in  $\mathbf{Set}^{mUF}$ . We need only verify that  $f_1$  and  $f_2$  preserve  $+_1$  and  $+_2$  respectively. Since  $f_1 \circ h_1 = g_1 \circ U_1F_1f_1$  we must have  $f_1(h_1(x, y)) = g_1(f_1(x), f_1(y))$  where  $g_1$  is the part of the structure map  $j : Y' \rightarrow Y$  which acts on  $Y$ . Consequently  $\overline{F}_2f_2$  is a monoid homomorphism as well.*

**Lemma 2.4.7** *The morphisms*

$$\overline{F} : \mathbf{Set}^{mUF} \rightarrow \mathbf{QAb}^m \text{ and } \overline{G} : \mathbf{QAb}^m \rightarrow \mathbf{Set}^m$$

*are mutually inverse.*

**Proof 2.4.8** *Let  $f : i \rightarrow j$  be a morphism in  $\mathbf{QAb}^m$ . If we apply  $\overline{G}$ , we obtain the morphism  $\overline{G}f : Ui \rightarrow Uj$ , where  $Ui$  has structure map  $h = (h_1, h_2)$  with  $h_1(xy) = x +_1 y$ ,  $h_2(xy) = x +_2 y$ , and  $Uj$  has structure map  $g = (g_1, g_2)$  with  $g_1(xy) = x \cdot_1 y$ ,  $g_2(xy) = x \cdot_2 y$ . Then applying  $\overline{F}$  to this morphism we obtain  $\overline{F}\overline{G}f : Ui \rightarrow Uj$  with binary operations on  $Ui$  defined as  $x +_1 y = h_1(xy)$ ,  $x +_2 y = h_2(xy)$ , and on  $Uj$  as  $x \cdot_1 y = g_1(xy)$ ,  $x \cdot_2 y = g_2(xy)$ , which was the morphism that was initially present.*

*Let  $f : i' \rightarrow j'$  be a morphism in  $\mathbf{Set}^m$ . As before, we see that the underlying sets of the morphism are unchanged under application of  $\overline{F}$  and  $\overline{G}$ . The operations  $+_1$ ,  $+_2$ ,  $\cdot_1$ , and  $\cdot_2$  are all determined by the action of the structure map on words of length two. Likewise, the structure maps are determined by the values of each of the binary operations previously listed.*

**Theorem 2.4.9** *The category  $\mathbf{QAb}$  of abelian qo-groups is monadic over the category  $\mathbf{Set}^m$ .*

**Proof 2.4.10** *We must show that  $\mathbf{QAb}$  is equivalent to  $\mathbf{Set}^{mT}$  for some endofunctor  $T$ . Using Lemmas 2.4.3 to 2.4.7 we see that  $\mathbf{Set}^{mUF}$  is isomorphic to  $\mathbf{QAb}^m$ . In turn, Theorem 2.1.2*



shows that  $\mathbf{QAb}^i$  is isomorphic to  $\mathbf{QAb}$ . Furthermore Theorem 2.2.6 shows that  $\mathbf{QAb}^i$  is equivalent to  $\mathbf{QAb}^m$ . Therefore we have  $\mathbf{Set}^{mUF} \cong \mathbf{QAb}^m \simeq \mathbf{QAb}^i \cong \mathbf{QAb}$  as desired.

It is readily verified that the adjunction between  $\mathbf{Set}^i$  and  $\mathbf{QAb}^i$  is also a monadic adjunction. One might hope that there would be a monadic adjunction between  $\mathbf{Set}^m$  and  $\mathbf{PAb}^m$  defined in a similar way as in Definition 2.3.1, since all of the objects in the image of  $F$  are isomorphic to objects of  $\mathbf{PAb}^m$ . That this is not the case, however, is shown by the following example.

**Example 2.4.11** Let  $X' = X = \{x, y\}$  and  $i : X' \rightarrow X$  be the identity; also define  $h : UFi \rightarrow i$  have  $h_1 : U_1F_1X \rightarrow X$  where  $x, xx,$  and  $yy$  are sent to  $x$  and both  $y$  and  $xy$  are sent to  $y$ . Then the  $UF$ -algebra  $(i, h)$  becomes isomorphic to  $\mathbb{Z}_2$  with nontrivial order, which by Example 1.2.16 is not isomorphic to an object of  $\mathbf{PAb}^m$ . Therefore  $\mathbf{PAb}^m$  is not monadic over  $\mathbf{Set}^m$  with the adjunction described.

We end by returning to quasi-ordered groups of divisibility under the partial order of divisibility.

**Example 2.4.12** Let  $\mathbf{Dom}_i$  be the category of integral domains with injective ring homomorphisms. Let  $\mathbf{Fld}$  be the category of fields. Then there is a well known functor  $Q : \mathbf{Dom}_i \rightarrow \mathbf{Fld}$  which assigns to each integral domain its quotient field. There is another functor  $G : \mathbf{Fld} \rightarrow \mathbf{Ab}$  which assigns to each field its multiplicative group of nonzero elements. The quasi-ordered group of divisibility functor is the functor  $D : \mathbf{Dom}_i \rightarrow \mathbf{QAb}^m$  taking a domain  $R$  to  $i : R^* \rightarrow U'GQR$ .

One may also construct the quasi-ordered group of divisibility using the multiplication operation on the quotient field as  $+_1$  and the original multiplication of the ring as  $+_2$ , which is seen to be much more compact in many respects.

The antisymmetrization of this object will produce the classical group of divisibility.

## CHAPTER 3. ATOMIC PSEUDO-VALUATION DOMAINS

In this chapter we apply the theory of abelian po-groups to answer some questions about the structure of integral domains. Specifically, we wish to classify whether or not a given integral domain falls into a certain class, the class of pseudo-valuation domains, given only its divisibility structure.

We begin by introducing the relevant concepts associated with pseudo-valuation domains. Afterwards, we discuss a sufficient condition on the divisibility structure of an integral domain to guarantee that it is a pseudo-valuation domain. This chapter concludes with several equivalent characterizations of atomic pseudo-valuation domains.

### 3.1 Introduction

Given an integral domain  $R$ , a prime ideal  $I$  of  $R$  is said to be strongly prime if, for every  $a, b \in QF(R)$ ,  $ab \in I$  implies either  $a \in I$  or  $b \in I$ . In their 1978 paper (16), Hedstrom and Houston investigated the class of domains where every prime ideal is strongly prime. They named these domains pseudo-valuation domains.

Recall that a domain  $V$  is a valuation domain if, for every  $a \in QF(V)$ , either  $a \in V$  or  $a^{-1} \in V$ . An equivalent definition of PVDs is that they are the domains  $R$  which have unique valuation overrings having the same po-set of prime ideals.

**Example 3.1.1** *The following is a list of some common examples of PVDs:*

- *valuation domains*
- *rings of the form  $K + XF[[X]]$ , where  $K \subset F$  is a field extension and  $K[[X]]$  is the ring of formal power series*

- $\mathbb{Z}[\sqrt{m}]_{(2,1+\sqrt{m})}$ , where  $m$  is square free and  $m \equiv 5 \pmod{8}$

**Definition 3.1.2** *Given a domain  $R$  in which every nonzero element can be factored into unique lengths of irreducibles (in other words,  $R$  is a half factorial domain or HFD),  $R$  is a boundary valuation domain or BVD if every element of the quotient field of  $R$  with more irreducibles on the numerator is in  $R$  itself. That is to say, for every  $\frac{a}{b} \in QF(R)$  with irreducible factorization  $\frac{\pi_1 \dots \pi_n}{\eta_1 \dots \eta_m}$ , with  $n > m$ ,  $\frac{a}{b} \in R$ .*

The interplay between PVDs and their valuation overrings was examined implicitly by Maney in (21). In that paper, the class of all BVDs was characterized solely in terms of necessary and sufficient divisibility properties. The main result at present is a complete characterization of all atomic PVDs in terms of their divisibility properties. It is shown that the class of atomic PVDs is precisely the class of BVDs. This characterization is an important tool for the study of the structure of domains by using their divisibility properties.

In Chapter 4, we use the investigation of the divisibility structure of PVDs to construct the lattice of all congruences for domains  $R$  of a certain subclass of PVDs, those arising from restricting coefficients of power series rings. Taking this notion a step further, it is seen that many more examples of PVDs can be constructed in a similar fashion by generalizing the exponents from a power series ring.

### 3.2 Preliminary Facts

It is essential to recall some basic facts about groups of divisibility and ordered groups from (2) and (22). It is also necessary to record some facts about PVDs proved in (16) that are relevant to this investigation.

**Definition 3.2.1** *Expanding on Definition 1.2.8, let  $R$  be a domain. We denote the subset of nonzero elements  $R^\#$  and group of units  $U(R)$ . The group of divisibility of  $R$  is the quotient group  $G(R) := QF(R)^\# / U(R)$ . For any integral domain,  $G(R)$  is partially ordered by divisibility  $|$ , meaning  $\alpha U(R) \leq \beta U(R)$  if and only if  $\alpha | \beta$ , that is  $\exists r \in R$  such that  $\alpha r = \beta$ .*

**Proposition 3.2.2** (16) *The set of prime ideals of a PVD  $R$  is linearly ordered. As a consequence of, all PVDs are local.*

The following theorem shows the importance of the unique maximal ideal  $M$  of a PVD.

**Theorem 3.2.3** (16) *For a given local domain  $R$  with maximal ideal  $M$ , the following are equivalent:*

1.  $R$  is a PVD;
2.  $R$  has a unique valuation overring  $V$  with maximal ideal  $M$ ;
3.  $R$  has a unique maximal ideal  $M$  and  $M$  is strongly prime;
4. There exists a valuation overring  $V$  in which every prime ideal of  $R$  is a prime ideal of  $V$ .

### 3.3 PVDs From Lexicographic Sums

In this section, conditions are placed on the group of divisibility of a domain  $R$ , sufficient for  $R$  to be a PVD. First, recall the following definitions from ring theory and the theory of po-sets.

**Definition 3.3.1** *A subset  $S$  of a ring  $R$  is said to be saturated multiplicative if it is a wall under multiplication. That is,  $xy \in S$  if and only if  $x \in S$  and  $y \in S$ .*

**Definition 3.3.2** *For a po-set  $X$ , a subset  $C$  is convex if, whenever  $a, b \in C$  with  $a \leq b$ , then  $c \in C$  whenever  $a \leq c \leq b$ . Furthermore, if  $X$  is a po-group a subset  $C$  is directed if every element of  $C$  can be written as a difference of positive elements.*

**Definition 3.3.3** *For abelian po-groups  $A$  and  $B$ , there is a po-group  $A \circ B$ , called the lexicographic sum of  $A$  and  $B$ . The group structure on  $A \circ B$  is that of  $A \oplus B$ , the direct sum of  $A$  and  $B$ , with order relation  $(a_1, b_1) \leq (a_2, b_2)$  if and only if  $a_1 < a_2$  or  $a_1 = a_2$  and  $b_1 \leq b_2$ . The other common partial order on the group  $A \oplus B$  is the product order, where  $(a_1, b_1) \leq (a_2, b_2)$  if and only if  $a_1 \leq a_2$  and  $b_1 \leq b_2$ . The po-group with this partial order will be denoted  $A \oplus B$ .*

The following theorem, proved in (23), provides insight into the interplay between the structure of the group of divisibility of a domain and the structure of the domain itself.

**Theorem 3.3.4** (23) *Let  $R$  be a domain and  $G(R)$  its group of divisibility. Then there is a one to one order reversing correspondence between the saturated multiplicative subsets of  $R$  and the convex directed subgroups of  $G(R)$ .*

The results that follow show that if the group of divisibility of a domain is a lexicographic sum of a linearly ordered group and a trivially ordered group (an antichain group), then  $R$  is a PVD. It is not clear, however, if this condition on the group of divisibility is necessary for  $R$  to be a PVD.

**Lemma 3.3.5** *If  $R$  is a domain with  $G(R) \cong_o L \circ A$ , where  $L$  is linearly ordered and  $A$  is trivially ordered, then the set of convex directed subgroups is linearly ordered.*

**Proof 3.3.6** *Suppose  $M$  and  $N$  are two convex directed subgroups of  $G(R)$ . Since the order isomorphism holds, we know that  $M$  and  $N$  correspond to two convex directed subgroups in  $L \circ A$ . We would like to prove that the groups  $M$  and  $N$  are related by inclusion. Since  $M$  and  $N$  are convex directed, they are generated by their positive elements. Suppose that  $m \in M^+$ . If  $m \leq n$  for some  $n \in N^+$ , then  $m \in N^+$ , because  $N^+$  is convex directed and  $M \subseteq N$ . Alternatively, if for every  $n \in N^+$  we have  $n \leq m$ , then  $N \subseteq M$ . The only other case to consider is that there exists an  $n \in N^+$  such that  $n$  is incomparable to  $m$ , which means if  $m \mapsto (l_1, a_1)$  in  $L \circ A$  and  $n \mapsto (l_2, a_2)$ , then  $l_1 = l_2$  and  $a_1 \neq a_2$ . So we have that  $(l_1, a_1) \leq (2l_1, 2a_2)$  since  $l_1 \in L$  and  $l_1 \geq 0$ . Thus  $(l_1, a_1) \leq (2l_1, 2a_2) = (2l_2, 2a_2) = 2(l_2, a_2)$  which corresponds to  $2n \in N^+$ , hence  $m \leq 2n$ ,  $m \in N^+$ , so  $M \subseteq N$  and the set of convex directed subgroups is linearly ordered.*

**Lemma 3.3.7** *Let  $R$  be as in Lemma 3.3.5. Then:*

1. *The set of prime ideals of  $R$  is linearly ordered.*
2. *The domain  $R$  is local.*

**Proof 3.3.8** *From Lemma 3.3.5, the set of convex directed subgroups of  $R$  is linearly ordered. By Theorem 3.3.4, there is a one-to-one order correspondence between the convex directed subgroups of  $G(R)$  and the saturated multiplicative subsets of  $R$ . Thus, the saturated multiplicative*

subsets of  $R$  are linearly ordered by set inclusion as well. If  $P_1, P_2$  are prime ideals of  $R$  and each prime ideal is the complement of a saturated multiplicative subset in  $R$ , we may say that  $S_1 = R \setminus P_1$  and  $S_2 = R \setminus P_2$  where  $S_1$  and  $S_2$  are saturated multiplicative subsets of  $R$ . Without loss of generality suppose that  $S_1 \subseteq S_2$ , so we have that  $P_2 \subseteq P_1$  and the prime ideals of  $R$  are linearly ordered. Thus, since  $\text{Max}(R)$  is nonempty,  $\text{Max}(R) = \{M\}$  for  $M$  some maximal ideal of  $R$ . Further, we know that  $M = R \setminus U(R)$  because  $U(R)$  is a saturated multiplicative subset. Therefore,  $R \setminus U(R)$  is a prime ideal of  $R$  and  $M \subseteq R \setminus U(R)$ , so from  $M$  being maximal and the set of prime ideals being linearly ordered we have  $M = R \setminus U(R)$ .

Since the set of prime ideals of  $R$  is linearly ordered by set inclusion, if there were two maximal ideals, they would be comparable under set inclusion. This clearly shows that  $R$  is a local domain.

**Lemma 3.3.9** *Let  $R$  be as in Lemma 3.3.5. Then the maximal ideal of  $R$  is strongly prime.*

**Proof 3.3.10** *From Lemma 3.3.7 we know that  $R$  has a unique maximal ideal  $M$ . Suppose that  $\alpha, \beta \in QF(R)$  the quotient field of  $R$ , such that  $\alpha\beta \in M$ . We want to show that  $\alpha \in M$  or  $\beta \in M$ . First, note that  $\alpha\beta = 0$  if and only if  $\alpha = 0$  or  $\beta = 0$  since  $R$  is a domain, hence  $\alpha \in M$  or  $\beta \in M$ . Alternatively, if  $\alpha\beta \neq 0$  then since  $\alpha\beta \in M$  then  $\alpha\beta$  corresponds to  $\alpha\beta U(R)$  in  $G(R)$  and that corresponds to  $(l_1 + l_2, a_1 + a_2) > (0, 0)$  in  $L \circ A$ . This is because all elements in  $G(R)$  map to positive elements in  $L \circ A$ , thus  $l_1 + l_2 > 0$  so  $l_1 > -l_2$  so if  $l_1 > 0$  then  $\alpha \in M$  otherwise  $l_1 \leq 0$  means that  $l_2 > 0$  and  $\beta \in M$  and thus  $\alpha \in M$  or  $\beta \in M$  and  $M$  is strongly prime and  $R$  is a PVD.*

**Theorem 3.3.11** *If  $R$  is a domain with  $G(R) \cong_o L \circ A$ , where  $L$  is linearly ordered and  $A$  is trivially ordered, then  $R$  is a PVD.*

**Proof 3.3.12** *From Lemma 3.3.7 we have that  $R$  must have a unique maximal ideal. From Lemma 3.3.9 we see that this maximal ideal must be strongly prime. From Theorem 3.2.3, this is enough to show that  $R$  is a PVD.*

Integral domains  $R$  where  $G(R)$  satisfies the conditions of Theorem 3.3.11 are quite common. Section 4.3 contains many different kinds of examples. Further, all valuation domains are

included in this collection, as is the class of all BVDs.

### 3.4 Classification of Atomic PVDs

In this section, Theorem 3.3.11 is used along with Maney's classification of BVDs in (21) to give several equivalent conditions for a domain  $R$  to be an atomic PVD. There are many nonatomic PVDs, for example as exhibited in the previous section. It is also seen that the assumption of atomicity on a PVD  $R$  implies  $R$  is an HFD.

**Definition 3.4.1** *Given an HFD  $R$  with quotient field  $QF(R)$ , an overring of  $R$  is any ring  $T$  such that  $R \subseteq T \subset QF(R)$ . An overring  $T$  is boundary positive if every element  $x \in T^\#$  has at least as many irreducibles of  $R$  on the numerator as the denominator. We say that  $T$  is boundary complete if, for every  $x \in T^\#$  with an equal amount of irreducibles on the numerator and denominator, we have  $x \in R^\#$ , which is equivalent to  $x \in U(R)$ .*

Of particular interest at this point is the the classification of BVDs in terms of their groups of divisibility, proven in (21).

**Theorem 3.4.2** *Let  $R$  be a domain with complete integral closure  $R'$ . Then  $R$  is a BVD if and only if  $G(R) \cong_o \mathbb{Z} \circ U(R')/U(R)$ .*

The final result in this chapter is the characterization of the class of atomic PVDs as domains whose group of divisibility is an element of a certain isomorphism class of po-groups. This kind of result has precedents in the literature. As previously mentioned, it was proven by Hahn and referred to in (12) that  $V$  is a valuation domain if and only if  $G(V) \cong_o L$  for some linearly ordered group  $L$ . Two other classic results referred to in (22) are that  $R$  is a UFD if and only if  $G(R) \cong_o \bigoplus_{p \in \mathcal{P}} \mathbb{Z}$  with  $\mathcal{P}$  the set of prime elements and sum ordered under the cardinal order. The other result is that  $R$  is a GCD domain if and only if  $G(R)$  is a lattice-ordered group.

**Theorem 3.4.3** *For an integral domain  $R$ , the following are equivalent:*

1.  $R$  is a BVD;

2.  $R$  is an HFD with boundary positive, boundary complete, valuation overring  $V$  with  $G(V) \cong_o \mathbb{Z}$ ;
3.  $G(R) \cong_o \mathbb{Z} \circ U(V)/U(R)$  for some overring  $V$  of  $R$ ;
4.  $R$  is an atomic PVD.

**Proof 3.4.4** (1)  $\Rightarrow$  (2) For this implication we refer to Theorem 3.4.2 with the overring  $V = R'$ , the complete integral closure of  $R$ .

(2)  $\Rightarrow$  (3) It must be shown that the group of divisibility  $G(R) \cong_o \mathbb{Z} \circ U(V)/U(R)$  for an overring  $V$  of  $R$ . The overring used is the one provided by assuming (2). Since  $V$  is an overring of  $R$ , it is evident that the quotient field of  $V$  is the same as that of  $R$ . So, any element  $\frac{\alpha}{\beta} \in QF(R)$  may be written as  $\frac{z^m u}{z^n v}$  where  $z$  is the unique prime of the rank-1 DVR overring  $V$ ,  $n, m \in \mathbb{N}$ , and  $u, v \in U(V)$ .

Construct set map

$$\phi : G(R) \rightarrow \mathbb{Z} \circ U(V)/U(R); \frac{z^m u}{z^n v} U(R) \mapsto (m - n, uv^{-1}U(R))$$

This is a homomorphism of abelian groups. First, suppose that there are two coset representatives for an element, that is,

$$\frac{z^m u}{z^n v} U(R) = \frac{z^m u u'}{z^n v} U(R). \text{ These elements map to the same pair}$$

$(m - n, uu'v^{-1}U(R))$  since  $U(R) \subseteq U(V)$ . This homomorphism also preserves the operation of multiplication of quotient field elements as  $\frac{z^{m_1} u_1}{z^{n_1} v_1} \frac{z^{m_2} u_2}{z^{n_2} v_2} = \frac{z^{m_1+m_2} u_1 u_2}{z^{n_1+n_2} v_1 v_2}$ . The kernel of  $\phi$  is the set of all  $\frac{z^m u}{z^n v} \in QF(R)$  such that  $m = n$ , and  $uv^{-1} \in U(R)$ , which is simply the set  $U(R)$ . Finally,  $\phi$  is surjective because, for every  $(m, vU(R)) \in \mathbb{Z} \circ U(V)/U(R)$  we may write this as  $\phi(z^m vU(R))$  if  $m \geq 0$ , or  $\phi(\frac{v}{z^{-m}}U(R))$  if  $m < 0$ .

To conclude this implication it must be shown that the isomorphism is, in fact, an order isomorphism of abelian groups. It is sufficient to show the preservation of positive cones. Suppose that  $\frac{z^m u}{z^n v} U(R)$  is positive. that means  $\frac{z^m u}{z^n v} \in R$ . That means  $m \geq n$  since, if not,  $\frac{z^m u}{z^n v} \notin V$  and  $R \subseteq V$ . If  $m > n$ , then  $z^{m-n} uv^{-1} \in V$  is a non zero non unit and is thus in  $R$ . If, on the other hand,  $m = n$ , to be in  $R$  means  $uv^{-1}$  is in  $R$  and its inverse  $u^{-1}v$  is also in  $R$ , which shows that  $\phi$  is order preserving. If  $(n, uU(R))$  is a positive element of  $\mathbb{Z} \circ U(V)/U(R)$ ,



$n > 0$  in which case  $(n, uU(R)) = \phi(z^n uU(R))$  and  $z^n u \in R$  or  $n = 0$  and  $u \in U(R)$ , this means  $(n, uU(R)) = \phi(uU(R))$  and  $u \in R$ . This finishes the verification that  $\phi$  is, in fact, an isomorphism of po-groups.

(3)  $\Rightarrow$  (4) Since  $G(R) \cong_o L \circ A$ , Theorem 3.3.11 states that  $R$  is a PVD. We need only show that  $R$  is atomic. Let  $r \in R^\sharp$ . It must be shown that we may write  $r = ux_1x_2\dots x_n$  where  $u \in U(R)$  and  $x_i$  are irreducibles. Since  $r \in R^\sharp$ , the element  $rU(R)$  is in the positive cone of  $G(R)$ . This means  $\phi(rU(R)) = (n, uU(R)) = \sum_{k=1}^{n-1} (1, U(R)) + (1, uU(R))$ . Furthermore, since  $\phi$  is an isomorphism of po-groups and each  $(1, U(R))$  and  $(1, uU(R))$  is a minimal positive element of  $\mathbb{Z} \circ U(V)/U(R)$ ,  $\phi^{-1}(\sum_{k=1}^{n-1} (1, U(R)) + (1, uU(R)))$  is a product of minimal positive elements of  $G(R)$ , which equal  $rU(R)$ . Thus,  $r$  may be written as a product of minimal positive elements in  $G(R)$ , which means that  $R$  is atomic.

(4)  $\Rightarrow$  (1) Suppose that  $R$  is an atomic PVD. To see that  $R$  is a BVD it is sufficient to show that, for any  $\frac{x}{y}U(R) \in G(R)$  with  $\partial_R(\frac{x}{y}) \neq 0$ , we have either  $\frac{x}{y}$  or  $\frac{y}{x} \in R$ . Since  $R$  has a unique valuation overring  $V$ , we have, without loss of generality,  $\frac{x}{y} \in V$ . The only way that  $\frac{x}{y} \in V \setminus R$  is if  $\frac{x}{y} \in U(V) \setminus U(R)$ , since  $V \setminus R = U(V) \setminus U(R)$ . Suppose that, in fact,  $\frac{x}{y} \in U(V) \setminus U(R)$ . This means  $\partial_V(\frac{x}{y}) = 0$ . But, since  $V$  and  $R$  share the same unique maximal ideal  $M$ , if  $\partial_V(\frac{x}{y}) = 0$ , then  $\partial_R(\frac{x}{y}) = 0$ , contradicting our assumption. Therefore  $R$  is a BVD.

## CHAPTER 4. GENERALIZED SERIES RINGS AND CONGRUENCE LATTICES

In this chapter we discuss two different aspects of pseudo-valuation domains. We first use the sufficient condition on the group of divisibility of a domain to prove that generalized series rings form a subclass of pseudo-valuation domains. The chapter concludes with a discussion of how to construct the congruence lattice of an atomic generalized series ring using its group of divisibility.

### 4.1 Introduction

#### 4.2 PVDs from Generalized Power Series

In the previous section, a source of PVDs was obtained from restricting the leading coefficients of power series over a field. Another source of PVDs is found by restricting the generalized power series from Ribenboim's (27). This method allows one to construct many PVDs which are not necessarily atomic, as is exhibited.

**Definition 4.2.1** *A commutative monoid  $M$  on a po-set under  $\leq$  is called a linearly ordered monoid if  $\leq$  is a total order and if, for every  $a, b, c \in M$ , with  $a \leq b$ ,  $ac \leq bc$ .*

**Definition 4.2.2** *A po-set  $X$  is said to be narrow if each induced antichain is finite. A po-set  $X$  is said to be Artinian (Noetherian) if there are no infinite decreasing (increasing) sequences in  $X$ .*

Generalized power series over rings  $R$  are given by specifying a partially ordered monoid  $M$ . The usual power series are given by considering generalized power series over rings with monoid  $\mathbb{N}$ . The definition is as follows.

**Definition 4.2.3** Given a ring  $R$  and a partially ordered monoid  $M$ , the generalized power series ring denoted  $A = R[[M]]$  is the collection of functions  $f : M \rightarrow R$  with support on a narrow, Artinian subset of  $M$ . Addition is given by  $(f+g)(m) = f(m)+g(m)$  and multiplication given by the convolution  $(f * g)(m) = \sum_{m_1 \in M} f(m_1)g(m - m_1)$ . If, in addition,  $M$  is linearly ordered, the support of each function is well ordered and we may define the minimum of the support, denoted  $\min(f)$ .

Given an element  $m \in M$ , we may define the delta function centered at  $m$  as  $\delta_m(x) := \delta_{mx}$ , the Kronecker delta.

The units of generalized power series have been characterized in a similar fashion to those of classical power series. The theorem is stated for the special case that  $R$  is a field.

**Proposition 4.2.4** (27) Let  $A = R[[M]]$  be a generalized power series where  $R$  is a field. Then  $U(A) = \{f(m) | f(0) \neq 0\}$ .

Of particular interest at present is when the partially ordered monoid  $M$  is the positive cone of a linearly ordered group  $G$ . Generalized power series rings of this type have been studied for some time and the following result, due to Hahn in (12), is the standard example for a class of rings showing that the classification in terms of a group of divisibility has at least one ring for each isomorphism class of po-groups.

**Theorem 4.2.5** (12) The generalized power series ring  $R = F[[\Gamma^+]]$  over a field  $F$  and linearly ordered group  $\Gamma$  has group of divisibility  $G(R) \cong_o \Gamma$ .

**Proof 4.2.6** Given any element  $f \in R^\#$  we have that  $f = (\delta_{\min(f)}) * u$  where  $u$  is a unit of  $R$ . So, given any element  $\frac{f}{g} \in QF(R)^\# / U(R)$ , we may write  $\frac{f}{g}$  as  $\frac{\delta_{\min(f)}}{\delta_{\min(g)}} U(R)$ .

Additionally,  $\frac{\delta_{\min(f)}}{\delta_{\min(g)}} \in R$  if and only if  $\min(g) \leq \min(f)$ . This is due to the fact that a delta function convolved with another function  $f$  simply translates the support of  $f$  by the support of the delta function. Therefore, the convolution of a delta function with another function  $f$  is again a delta function if and only if  $f$  is a delta function.

We now consider the map

$$\phi : G(R) \rightarrow \Gamma; \frac{\delta_{\min(f)}}{\delta_{\min(g)}} U(R) \mapsto \min(f) - \min(g)$$

It must be shown that  $\phi$  is an isomorphism of po-groups. We know that  $u \in U(R)$  if and only if  $\min(u) = 0$ . Therefore  $\phi$  is well defined on cosets. Further, since  $\min : R \rightarrow \Gamma^+$  is an order preserving group homomorphism, we have that  $\phi(R/U(R)) \subseteq \Gamma^+$  so  $\phi$  is order preserving and  $\phi$  is a group homomorphism when extended to  $G(R)$ . Since  $\phi(\delta_\gamma U(R)) = \gamma$  and  $\phi(\frac{1}{\delta_\gamma} U(R)) = -\gamma$ ,  $\phi$  is surjective. Finally,  $\phi^{-1}$  is order preserving, since  $\phi^{-1}(\gamma) = \delta_\gamma U(R)$  for all  $\gamma \in \Gamma^+$ .

**Definition 4.2.7** Let  $K \subseteq F$  be a field extension. Let  $\Gamma$  be a linearly ordered group with positive cone  $\Gamma^+$ . The subring  $S$  of  $R = F[[\Gamma^+]]$  consisting of functions  $f$ , for which  $f(0) \in K$  is called the series ring over  $K \subseteq F$ , with exponents in  $\Gamma^+$ .

**Lemma 4.2.8** The group of units of a series ring  $S$  over  $K \subseteq F$  is  $U(S) = U(R) \cap S$ .

**Proof 4.2.9** Suppose that  $f \in S$  is a unit of  $S$ . This means there exists an element of  $S$  so that  $f * g = 1 \in S \subseteq R$ , which means  $f \in U(R)$ . Now suppose that  $f \in U(R) \cap S$ . This means  $f(0) \in F^\# \cap K$  which means  $f$  has an inverse in  $R$  whose component at 0 is in  $K$ . Thus  $g^{-1} \in S$  as well.

**Theorem 4.2.10** Let  $S$  be a series ring over  $K \subseteq F$ , with exponents in  $\Gamma^+$ . Then  $G(S) \cong_o \Gamma \circ F^\# / K^\#$ .

**Proof 4.2.11** Observe that every element of  $S^\#$  may be written as

$$f(\min(f))\delta_{\min(f)} * u$$

where  $u \in U(S)$ . It must be noted that  $f(\min(f)) \in F^\#$  and not necessarily in  $K^\#$ . In the case where  $f(\min(f)) \in K^\#$ ,

$$f(\min(f))\delta_{\min(f)} * u = \delta_{\min(f)} * v$$

Elements of the group of divisibility  $G(S)$  are therefore of the form  $\frac{f(\min(f))\delta_{\min(f)}}{g(\min(g))\delta_{\min(g)}} U(S)$ . This notation is shortened for the rest of the proof by identifying such an element with  $\frac{\alpha\delta_{f^0}}{\beta\delta_{g^0}} U(S)$ , where  $\alpha, \beta \in F^\#$  and  $f^0 := \min(f), g^0 := \min(g) \in \Gamma^+$ .

Define

$$\psi : G(S) \rightarrow \Gamma \circ F^\# / K^\#; \frac{\alpha\delta_{f^0}}{\beta\delta_{g^0}} U(S) \mapsto (f^0 - g^0, \frac{\alpha}{\beta} K^\#)$$

It must be shown that  $\psi$  is an isomorphism of po-groups. Observe first that  $\psi$  is well defined on cosets. This is because  $\psi(U(S)) = (0, K^\sharp) = (f^0 - g^0, \frac{\alpha}{\beta} K^\sharp) = \psi(\frac{\alpha \delta_{f^0}}{\beta \delta_{g^0}})$  if  $\frac{\alpha}{\beta} \in K^\sharp$  and  $f^0 - g^0 = 0$ . To see that  $\psi$  is an abelian group homomorphism, let  $\frac{\alpha_1 \delta_{f_1^0}}{\beta_1 \delta_{g_1^0}} U(S), \frac{\alpha_2 \delta_{f_2^0}}{\beta_2 \delta_{g_2^0}} U(S) \in G(S)$ , then

$$\begin{aligned} & \psi\left(\frac{\alpha_1 \delta_{f_1^0}}{\beta_1 \delta_{g_1^0}} \frac{\alpha_2 \delta_{f_2^0}}{\beta_2 \delta_{g_2^0}} U(S)\right) \\ &= (f_1^0 + f_2^0 - g_1^0 - g_2^0, \frac{\alpha_1 \alpha_2}{\beta_1 \beta_2} K^\sharp) = (f_1^0 - g_1^0, \frac{\alpha_1}{\beta_1} K^\sharp) + (f_2^0 - g_2^0, \frac{\alpha_2}{\beta_2} K^\sharp) \end{aligned}$$

which is clearly the sum of the images of the individual factors under  $\psi$ . It is easy to see that  $\psi$  preserves inverses.

Observe that  $\psi$  is surjective by taking any  $\gamma \in \Gamma$  and any  $\alpha K^\sharp \in F^\sharp/K^\sharp$  and writing it as the image  $\psi(\alpha \delta_\gamma)U(S)$  if  $\gamma \in \Gamma^+$ , and  $\psi(\frac{\alpha \delta_\alpha}{\delta_\gamma} U(S))$  if  $-\gamma \in \Gamma^+$ . Since  $\Gamma$  is linearly ordered, these are the only possibilities.

It must be shown that  $\psi$  is an isomorphism of abelian groups. If there were an element  $\frac{\alpha \delta_{f^0}}{\beta \delta_{g^0}} U(S)$  such that  $\psi(\frac{\alpha \delta_{f^0}}{\beta \delta_{g^0}} U(S)) = (0, K^\sharp)$ , then  $f^0 = g^0$  and  $\frac{\alpha}{\beta} \in K^\sharp$  by the definition of  $\psi$ . That is to say, the minima of the supports of  $f$  and  $g$  were equal and the ratio of the values at that location was an element of  $K^\sharp$ . Thus,  $\frac{\alpha \delta_{f^0}}{\beta \delta_{g^0}} U(S) = U(S)$  by Lemma 4.2.8.

To show that  $\psi$  and  $\psi^{-1}$  preserve order, it is sufficient to show that they preserve the positive cones of the po-groups. First, considering  $\alpha \delta_{f^0} U(S) \in G(S)^+ = S^\sharp/U(R)$ , we have that  $\psi(\alpha \delta_{f^0} U(S)) = (f^0, \alpha K^\sharp)$ . Since  $\alpha \delta_{f^0} \in S^\sharp$ ,  $f^0 \neq 0$  and  $f^0 \in \Gamma^+$  or  $f^0 = 0$  and  $\alpha \in K^\sharp$ , in either case  $(f^0, \alpha K^\sharp)$  is in the positive cone of  $\Gamma \circ F^\sharp/K^\sharp$ .

Now suppose that  $(\gamma, \alpha K^\sharp)$  is in the positive cone of  $\Gamma \circ F^\sharp/K^\sharp$ . That is, either  $\gamma \neq 0$  and  $\gamma \in \Gamma^+$  or  $\gamma = 0$  and  $\alpha \in K^\sharp$ . It is easily seen that  $\phi^{-1}(\gamma, \alpha K^\sharp) = \alpha \delta_\gamma U(S) \in R^\sharp/U(S)$ .

**Corollary 4.2.12** *Let  $S$  be a series ring over  $K \subseteq F$ , with exponents in  $\Gamma^+$ . Then  $S$  is a PVD.*

**Proof 4.2.13** *This is a consequence of Theorem 4.2.10 and Theorem 3.3.11.*

This section concludes with some examples of PVDs coming from series rings over  $K \subseteq F$ .

**Example 4.2.14** *Observe the following PVDs obtained by specifying a coefficient field extension and a positive cone of an abelian po-group:*

1. *The (nonatomic) PVDs with  $F = \mathbb{R}$ ,  $K = \mathbb{Q}$ , and  $\Gamma = \mathbb{Z} \circ \mathbb{Z}$ . These domains behave similarly to Laurent series domains over  $\mathbb{R}$  with constant terms in  $\mathbb{Q}$  and variables  $X$  and  $Y$  where  $Y$  may have negative exponents.*
2. *Series rings over field extensions  $K \subseteq F$  with  $\Gamma = \mathbb{R}$  are another class of nonatomic PVDs. These domains may be thought of as formal power series with restricted leading coefficients where the powers on the indeterminates are allowed to be any nonnegative real number.*
3. *The only atomic examples of generalized restricted power series with  $\Gamma$  linearly ordered are the standard restricted power series rings with  $\Gamma = \mathbb{Z}$ . This is a consequence of the next section.*

### 4.3 A Particular Class of PVDs

In this section, the congruence lattices of rings from a class of PVDs are characterized. The goal of this section is to represent the lattice of congruences of integral domains of the form  $K + XF[[X]]$ , where  $K \subseteq F$  is a field extension.

**Example 4.3.1** *Let  $K \subseteq F$  be a field extension. Then  $R = K + XF[[X]]$  is a PVD. This is because  $V = F[[X]]$  is an overring of  $R$  and  $V$  is a valuation domain. Also,  $R$  and  $V$  are local with maximal ideal  $\langle X \rangle$ . So  $G(R) \cong_o G(V) \circ U(V)/U(R)$ . However  $G(V) \cong_o \mathbb{Z}$ ,  $U(V) \cong F^\sharp$ , and  $U(R) \cong K^\sharp$ . So, we have that  $G(R) \cong_o \mathbb{Z} \circ F^\sharp/K^\sharp$ .*

Taking Example 4.3.1, examining a specific field extension  $K \subseteq F$ , and constructing the entire lattice helps motivate a more general method to handle all field extensions.

**Example 4.3.2** *Consider  $F = \mathbb{F}_4 = \{0, 1, a, b\}$ , the four element field, and  $K = \{0, 1\}$ . Then from Example 4.3.1,  $G(R) \cong_o \mathbb{Z} \circ \{1, a, b\}/\{1\}$ , or equivalently  $G(R) \cong_o \mathbb{Z} \circ \{1, a, b\}$ , where the*

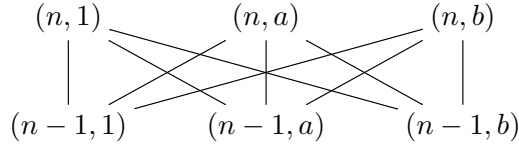


Figure 4.1 A portion of the divisibility relation on  $\mathbb{F}_2 + X\mathbb{F}_4[[X]]$

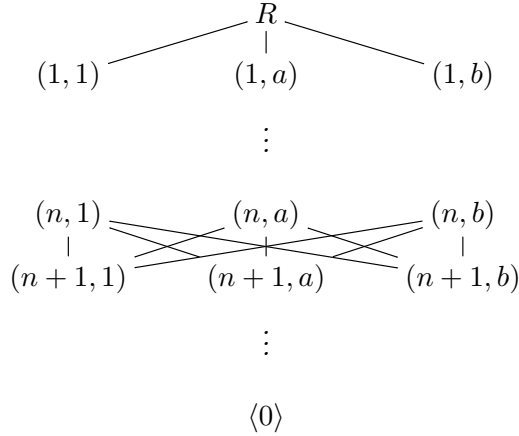


Figure 4.2 Hasse diagram of principal ideals of  $\mathbb{F}_2 + X\mathbb{F}_4[[X]]$

second factor is ordered trivially. Thus, the Hasse diagram in Figure 4.1 extended below and above for every element of  $\mathbb{Z}$  represents the dual of the entire po-set of principal ideals of  $R$ .

As in any integral domain, the dual of the positive cone of  $G(R)$  represents the order relationship between the principal ideals. Therefore the principal ideals of  $R$  correspond to the Hasse diagram in Figure 4.2.

Now, all that is needed to complete the Hasse diagram of the congruences of  $R$  is the meets and joins of the principal ideals. This is trivial by looking at a specific natural number and computing the meets and joins there. So, consider the three ideals corresponding to the generators  $(n, 1), (n, a)$ , and  $(n, b)$ . These ideals are generated by power series with lowest term  $X^n, aX^n$ , and  $bX^n$  respectively. Therefore, since the joins are taken by linear combinations of the generators of these principal ideals, we have that  $(n, 1) \vee (n, a) = (n, a) \vee (n, b) = (n, 1) \vee (n, b) = \langle \mathbb{F}_4 X^n \rangle$ . This ideal may be identified by just  $\langle X^n \rangle_V$  where the subscript denotes the fact that the generation of the ideal takes place in the valuation overring.

It is also evident that  $(n, 1) \wedge (n, a) = (n, 1) \wedge (n, b) = (n, a) \wedge (n, b) = \langle X^{n+1} \rangle_V$ . So finally

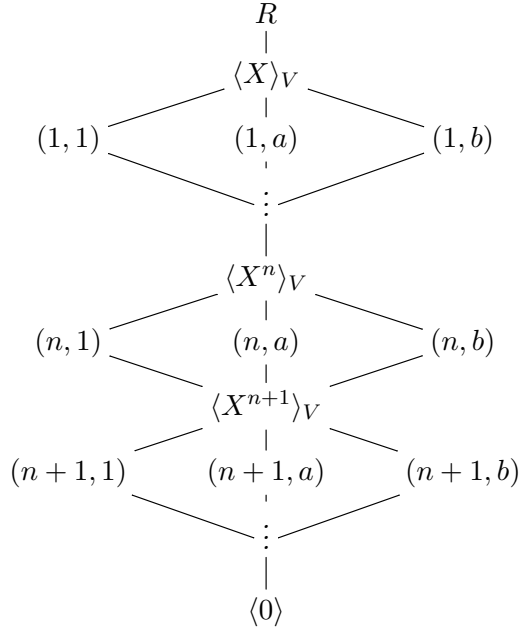


Figure 4.3 Congruence lattice of  $R$

the Hasse diagram of the congruence lattice of  $R$  is as in Figure 4.3.

As one may see, even the simplest examples of these constructions of PVDs have congruence lattices that are highly non-distributive.

The motivation for the generalization of this example to arbitrary field extensions comes from the construction of the meets and the joins of the principal ideals. Since the joins in particular can be expressed as linear combinations (over  $K$ ) of the generators of the two previous ideals, this invites us to more thoroughly investigate the vector space congruence structure of  $\text{AG}(F, K)$  (where  $K \subseteq F$ ), in order to generate the non-principal ideals of  $K + XF[[X]]$ .

**Definition 4.3.3** For a po-set  $P$  with order relation  $\leq$ , we define the dual poset  $P^\partial$  to be the poset over the set  $P$  with order relation  $\leq_\partial$ , where  $a \leq_\partial b$  if and only if  $b \leq a$ .

**Definition 4.3.4** Let  $K \subseteq F$  be a field extension. Then the poset of all nonzero subspaces of  $F$  as a vector space over  $K$  is denoted  $\text{AG}(F, K)$ .

**Theorem 4.3.5** Let  $K \subseteq F$  be a field extension, then the lattice of ideals of the power series ring  $R = K + XF[[X]]$  is lattice isomorphic to  $1 \oplus (\mathbb{N}^\partial \circ \text{AG}(F, K)) \oplus 1$ , where  $1$  is the one element lattice.



**Remark 4.3.6** *The structure of the above lattice can be decomposed into three distinct parts:*

1. *the first one-element lattice, representing the ideal  $\langle 0 \rangle$ ;*
2. *the lexicographic produce which first indexes the lowest power occurring on the indeterminate inside the ideal and then compares according to the number of generators required to create the available leading coefficients;*
3. *the last one-element lattice represents the entire ring.*

*It is certainly true that lexicographic products of po-sets do not always result in lattices. In this case, however, identifying the full vector space of  $F$  over  $K$  at a given coordinate  $n$  with the trivial subspace in the next higher coordinate  $n - 1$ , it is as if the lattice of subspaces of  $F$  over  $K$  were repeated once for each value  $n \in \mathbb{N}$ , resulting in a lattice.*

**Proof 4.3.7** *The unique valuation overring of  $R$  is  $V = F[[X]]$  which has positive cone of divisibility order isomorphic to  $\mathbb{N}$ . Also, recall, as a consequence of Example 4.3.1, that the poset of nonzero elements of  $R$  ordered under divisibility  $R^\# / U(R)$  is order isomorphic to  $V^\# / U(V) \circ U(V) / U(R)$ . Therefore, when identifying a principal ideal of  $R$  we may refer to an ordered pair  $(n, \alpha)$ , where  $n \in \mathbb{N}$  and  $\alpha \in U(V)$  is a coset representative for  $\alpha U(R)$ . Of course, the largest principal ideal is  $(0, 1)$ , which corresponds to the entire ring, and the smallest is  $\langle 0 \rangle$ , since the principal ideals are ordered dually to the lattice of divisibility.*

*Let  $L$  be the lattice of ideals of  $R$ . Then any nonzero ideal  $I \in L$  can be expressed as a join of principal ideals  $I = \bigvee_{\lambda \in \Lambda} (n_\lambda, \alpha_\lambda)$  for an indexing set  $\Lambda$ . Recall, that any principal ideal  $(n, \alpha) \subseteq (m, \beta)$  if and only if  $m > n$  or  $m = n$  and  $\alpha\beta^{-1} \in U(R)$ . So, let  $N = \inf_{\lambda \in \Lambda} n_\lambda$ , which exists since the ideal generated is not the zero ideal. Then  $I = \bigvee_{\sigma \in \Sigma} (N, \alpha_\sigma)$  for  $\Sigma = \{\lambda \in \Lambda \mid n_\lambda = N\}$ .*

*At this point, the join of the principal ideals involved can be expressed as their sum up to multiples from  $R$ . Since each principal ideal contains all series with degree lower than  $N$  we need only consider unit multiples (from  $R$ ) acting on the generating set. So, in reality*

$$I = \bigvee_{\sigma \in \Sigma} (N, \alpha_\sigma) = \left\langle x^N \alpha \mid \alpha \in \sum_{\sigma \in \Sigma} \alpha_\sigma \beta_\lambda \text{ and } \beta \in K \right\rangle$$

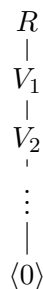
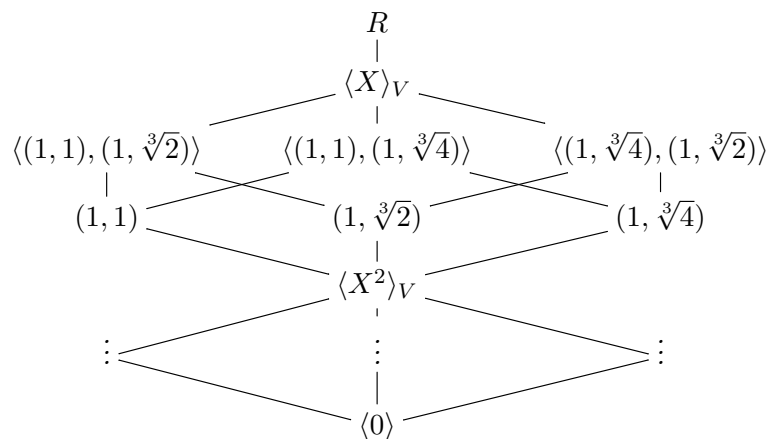


Figure 4.4 Basic structure of congruence lattice

Figure 4.5 Congruence lattice of  $\mathbb{Q} + X(\mathbb{Q}[\sqrt[3]{2}]][[X]]$ 

Thus, the lattice of ideals  $L$  is isomorphic to Figure 4.4 where  $V_k$  are copies of the poset of nontrivial subspaces of the vector space  $\text{AG}(F, K)$  where the elements generated in that subspace show up as the available coefficients on the lowest term of the series of degree  $k$ . In short  $L \cong 1 \oplus (\mathbb{N}^\partial \circ \text{AG}(F, K)) \oplus 1$  (in the category of lattices) as desired.

This chapter concludes with one final example which examines the entire lattice of ideals of a PVD of type  $K + XF[[X]]$ .

**Example 4.3.8** Let  $R = \mathbb{Q} + X(\mathbb{Q}[\sqrt[3]{2}]][[X]]$ . It is impossible to give an entire description of the lattice of ideals of  $R$  since  $\text{AG}(\mathbb{Q}[\sqrt[3]{2}], \mathbb{Q})$  is infinite, but it is possible to model what happens with selected elements. We can observe this in the lattice of Figure 4.5.

This method can be easily generalized to any field extension, but becomes very difficult to realize as the degree and the complexity of the field extension increases.

## CHAPTER 5. IMPLICATION RINGS

In (30) Smith and Romanowska describe a way to utilize the logical structure occurring in the two element field  $\mathbb{Z}_2$  to define a way to divide by zero. One of the most simple, yet perplexing problems for anyone who has learned even the most basic arithmetic is how to get around division by zero. It may seem like a triviality to try to define division by zero but, as Carlström points out in (6), there may be a situation where a computation where it may not be decidable whether a number is zero or not if it is exceedingly small. There have been many attempts to incorporate division by zero into rings, including Carlström's own attempt at doing so using an algebraic structure called a wheel.

### 5.1 Introduction

In this chapter, we develop an algebraic structure called an implication ring, which is an attempt to define a variety of rings using the binary operations of subtraction and implication (representing division) to understand how division can be defined totally on a certain rings. Unfortunately, not all rings can be considered as implication rings using the definition that we employ, meaning that division still cannot be defined as a total operation with our method on many rings. On the other hand, the approach taken in this chapter takes place wholly inside the variety of commutative rings so this construction allows the extension of division at least to some rings that are not fields, in sharp contrast to (6) where much of the basic ring structure is destroyed.

The paper (30) is not the only place where the operation of division and implication are seen to be similar. In fact, in (15), Hajek describes uses of product fuzzy logic where implication is defined as so-called goguen implication for  $x, y \in [0, 1]$ .

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{else} \end{cases}$$

The problem, of course, in generalizing goguen implication to the variety generated by all fields is that there is not necessarily a linear order on all of the generators. However, thinking of  $\rightarrow$  as we do, we may replace the condition of  $x \leq y$  with the condition of  $x = 0$  and the two definitions can be seen to be very much alike.

**Definition 5.1.1** *An implication ring is an algebra  $F$  with two binary operations  $-$  and  $\rightarrow$ . In addition to these two basic operations, there are four relevant composite operations:*

- $0 := a - a,$
- $1 := a \rightarrow a,$
- $-a := 0 - a,$
- $a + b := a - (0 - b).$

*It is important to note that, while 0 and 1 appear to be nullary operations, they are only essentially nullary and are defined in terms of a binary operation. We therefore tacitly require that  $a - a = b - b$  and  $a \rightarrow a = b \rightarrow b$ .*

*In addition to these binary operations and in the language of the composite operations we impose the following identities which guarantee that  $F$  has an abelian group reduct  $(F, +, -, 0)$  (here  $-$  is the negation composite operation):*

- $a + 0 = a,$
- $(a + b) + c = a + (b + c)$
- $a + b = b + a$
- $a + (-a) = 0$

In addition to the identities which guarantee an abelian group structure, we have identities governing a multiplicative structure. We first define the composite operation of multiplication through a series of steps:

1.  $a \triangleleft b = (a \rightarrow 1) \rightarrow b$ ,
2.  $a \triangleright b = (b \rightarrow 1) \rightarrow a$ ,
3.  $a * b = (a \rightarrow 1) \rightarrow ((b \rightarrow 1) \rightarrow 1)$ ,
4.  $a \square b = (a \triangleleft b) + (a \triangleright b) - (a * b)$ ,
5.  $ab = (((a \triangleleft b - a \square b) \rightarrow 0) \rightarrow 0) - ((b \rightarrow a) \rightarrow 0) + (a \square b)$

With multiplication defined we impose the standard commutative ring identities:

- $ab = ba$ ,
- $a(bc) = (ab)c$ ,
- $a(b + c) = (ab) + (ac)$ ,
- $0a = 0$ .

We impose identities governing the relationship between some of the composite operations and the basic operations of  $-$  and  $\rightarrow$ :

- $0 \rightarrow a = 1$
- $1 \rightarrow a = a$
- $a \rightarrow (b - c) = (a \rightarrow b) + (a \rightarrow 0) - (a \rightarrow c)$
- $a(b \rightarrow c) = b \rightarrow (ac) - (b \rightarrow 0)(1 - a)$
- $a \rightarrow 0 = (a \rightarrow 0)^2$

At present, it is an open question as to whether or not the previous list of axioms is independent or not. There is a current effort in conjunction with J.D. Phillips to verify independence or reduce the identity list to do so. This method of verification is the use of the model building software, constructing all models from a subset of axioms and testing whether any excluded axioms are satisfied by all models. Also, it may be beneficial in the future to increase the number of identities to the ones needed to construct the full variety of implication rings generated by fields if that list can be described. If that identification cannot be done, implication rings can serve as an extension of the smaller variety capturing some, but not all, of the algebraic properties of fields.

**Remark 5.1.2** *At present it is important to keep in mind that all of the identities imposed are simply in place to guarantee that all implication rings are actually commutative unital rings (with operations  $\cdot, +, 0, 1$ , and  $-$ ) in which implication is a binary operation satisfying some ‘common sense’ rules, including ‘division by 0 is 1’. The other major objective is to ensure that these definitions still make sense when the special case of fields is taken into account, the next theorem verifies this.*

**Theorem 5.1.3** *If  $F$  is a field with  $-$  defined as the subtraction on the abelian group reduct of  $F$  and  $a \rightarrow b := ba^{-1}$  if  $a \neq 0$  and 1 if  $a = 0$ , then  $F$  is an implication ring.*

**Proof 5.1.4** *We show that  $F$  satisfies all of the previously mentioned identities. The unary operators 0 and 1 of fields can also be expressed as  $a - a = 0 = b - b$  and since  $0 \rightarrow a = 1$ , we also obtain that  $a \rightarrow a = 1 = b \rightarrow b$ . It is evident, then, that the rest of the abelian group identities follow as well.*

*We show that the product  $a \cdot b$  of implication rings is precisely the ring multiplication that  $F$  was endowed with. We begin with  $a = b = 0$ . We see that  $0 \triangleleft 0 = (0 \rightarrow 1) \rightarrow 0 = 0$ , this also gives us that  $0 \triangleright 0 = 0$  and  $0 * 0 = (0 \rightarrow 1) \rightarrow ((0 \rightarrow 1) \rightarrow 1) = 1 \rightarrow 1 = 1$ . Thus,  $0 \square 0 = -1$ . Finally,  $0 \cdot 0 = (((0 \triangleleft 0 - 0 \square 0) \rightarrow 0) \rightarrow 0) - (((0 \rightarrow 0) \rightarrow 0) + 0 \square 0) = (0 \rightarrow 0) - (1 \rightarrow 0) + 0 \square 0 = 0$ .*

*Next, suppose that  $a = 0$  and  $b \neq 0$ . We see that  $0 \cdot b = (((0 \triangleleft b - 0 \square b) \rightarrow 0) \rightarrow 0) - ((b \rightarrow 0) \rightarrow 0) + 0 \square b = (0 \rightarrow 0) - ((b \rightarrow 0) \rightarrow 0) + 0 \square b = 1 - 1 + 0 = 0$ . On the other hand, if  $a \neq 0$  and  $b = 0$ ,  $a \cdot a = (((a \triangleleft 0 - a \square 0) \rightarrow 0) \rightarrow 0) - (((0 \rightarrow a) \rightarrow 0) + a \square 0) = (((0 - 0) \rightarrow 0) \rightarrow 0) - 0 + 0 = 0$ .*

Finally, suppose that  $a \neq 0$  and  $b \neq 0$ . We have that  $a \triangleleft b = ab$ ,  $a \triangleright b = ab$ ,  $a * b = ab$ , and  $a \square b = ab$ , where  $ab$  is the product endowed by the field structure. Therefore  $((a \triangleright b - a \square b) \rightarrow 0) \rightarrow 0 - ((b \rightarrow a) \rightarrow 0) + a \square b = 0 - 0 + a \square b = ab$ . Thus, the product  $a \cdot b$  corresponds to the original product of rings. The second set of identities are satisfied by fields and the result follows.

## 5.2 Relationship to Rings

Since it has been verified that all fields can be thought of as implication rings and that all implication rings are, in fact, commutative unital rings, the exact nature of these inclusions should be examined more closely. In fact we show there is a large class of rings which cannot be thought of as implication rings using Definition 5.1.1. We begin with a ring theoretic observation.

**Proposition 5.2.1** *For every implication ring  $R$  and every  $r \in R$ , we have that  $r \rightarrow 0 \in \text{Ann}(\{r\})$ , the annihilator of  $\{r\}$ .*

**Proof 5.2.2** *By our assumptions in Definition 5.1.1,  $(r \rightarrow 0)r = 0 - (0(r \rightarrow 0)) = 0$ .*

Now we are able to use Proposition 5.2.1 to show a large class of rings cannot be thought of as implication rings.

**Theorem 5.2.3** *Let  $R$  be an integral domain with more than one irreducible element (up to associates). Then  $R$  cannot be an implication ring with operations defined as in Definition 5.1.1.*

**Proof 5.2.4** *Suppose that there were a binary operation on  $R$  called  $\rightarrow$  that satisfied the above axioms. It then would be true that for every  $r \in R$  we would have  $(r \rightarrow 0)r = 0 - (0(r \rightarrow 0)) = 0$ , which implies that  $r = 0$  or  $r \rightarrow 0 = 0$ . Since  $r$  was chosen arbitrarily we have that  $r \rightarrow 0 = 0$  for every  $r \in R \setminus \{0\}$ .*

*As a consequence  $(r \rightarrow s)r = s - (s(r \rightarrow 0)) = s$  for every  $s \in R$  and  $r \neq 0$ . Suppose that  $r$  and  $s$  are distinct irreducible elements of  $R$ . Then we have  $(r \rightarrow s)r = s$ . This is impossible*

because  $r$  and  $s$  are distinct irreducibles and  $r \rightarrow s$  must be an invertible element, contradicting the assumption that the irreducibles are distinct.

**Example 5.2.5** Consider  $R = \prod_{p \in \mathcal{P}} \mathbb{Z}_p$  where  $\mathcal{P}$  is the set of prime integers as an implication ring with the product implication operation. If  $UR$  is the ring reduct of  $R$ , there is a ring monomorphism  $i : \mathbb{Z} \rightarrow UR; n \mapsto (n \bmod p)$ . The image of  $\mathbb{Z}$  is a subring of  $UR$  that is not a subimplication ring. This can be seen in two ways, first we know that a ring isomorphic to  $\mathbb{Z}$  cannot be an implication ring using Theorem 5.2.3.

Another way to look at the situation is that the image of  $i$  is not closed under the implication operation on  $R$ . For example  $i(2) \rightarrow i(0) = (1, 0, 0, \dots) \notin i(\mathbb{Z})$  because coordinates in which 0 occur for a given  $n$  correspond to prime divisors of  $n$  and no integer has an infinite number of prime divisors. Therefore,  $i(\mathbb{Z})$  is a subring that is not a subimplication ring.

**Corollary 5.2.6** The ring  $\mathbb{Z}$  cannot be considered an implication ring under Definition 5.1.1.

### 5.3 Finite Fields

Since the original definition of an implication ring stems from the use of implication in the two element field (functionally equivalent to a Boolean algebra), it is worthwhile to observe the general structure of the subvariety of all implication rings generated by the single implication ring  $\mathbb{F}_2$ , which gives insight into the larger variety.

**Definition 5.3.1** Let  $\mathbb{F}_2$  the two element field considered as an implication ring. We will denote the variety of implication rings generated by  $\mathbb{F}_2$  as  $V_{IR}(\mathbb{F}_2) = B$  called Boolean implication rings and the free Boolean implication ring on a set  $S$  as  $F_B(S)$ .

**Theorem 5.3.2** The boolean implication ring  $R = F_B(\{x\})$  is isomorphic to the free Boolean ring on one generator (considered as an implication ring).

**Proof 5.3.3** Since  $R$  is in the variety  $B$ , it must satisfy all of the identities that the model  $\mathbb{F}_2$  satisfies. In particular it must satisfy the following:

- $xx = x$



- $x + x = 0$
- $x \rightarrow 0 = 1 + x$
- $x \rightarrow 1 = 1$
- $(1 + x) \rightarrow 0 = x$
- $(1 + x) \rightarrow 1 = x + 1$
- $(1 + x) \rightarrow x = x$
- $x \rightarrow (1 + x) = x + 1$

Using these identities and the standard implication ring identities from above, we determine the operation tables of  $R$  with respect to  $-$  and  $\rightarrow$ :

$-$	0	1	$x$	$1 + x$
0	0	1	$x$	$1 + x$
1	1	0	$1 + x$	$x$
$x$	$x$	$1 + x$	0	1
$1 + x$	$1 + x$	$x$	1	0

$\rightarrow$	0	1	$x$	$1 + x$
0	1	1	1	1
1	0	1	$x$	$1 + x$
$x$	$1 + x$	1	1	$1 + x$
$1 + x$	$x$	$1 + x$	$x$	1

These are the same as the corresponding tables for the Boolean ring on one generator and if considered as a Boolean algebra with  $a \wedge b = a \cdot b$ ,  $a \vee b = a + b - (ab)$ , and  $\bar{a} = a \rightarrow 0$ , we see that  $R$  is the same as the Boolean algebra on one generator.

We are now in a position to generalize the situation presented in Theorem 5.3.2 to free implication rings on finite sets, in varieties generated by any finite field.

**Theorem 5.3.4** *Let  $\mathbb{F}_q$  be a finite field and  $X = \{X_1, X_2, \dots, X_n\}$  be a finite set. The free algebra on  $X$  in the subvariety of all implication rings generated by  $\mathbb{F}_q$  can be embedded in a homomorphic image of  $\mathbb{F}_q[X_1, X_2, \dots, X_n]/I$ , where  $\mathbb{F}_q[X_1, X_2, \dots, X_n]$  is the polynomial ring with coefficients in  $\mathbb{F}_q$  and indeterminates in  $X$  and  $I$  is the ideal generated by any polynomial of the form  $p(X)^q - p(X)$ .*

**Proof 5.3.5** *Denote by  $R$ , the free implication ring on  $X$  in the subvariety generated by  $\mathbb{F}_q$ . If  $w \in R$  is an element of the free algebra, we may consider it as a function  $w : \mathbb{F}_q^m \rightarrow \mathbb{F}_q$  for some  $m \leq n$ . Since  $\mathbb{F}_q$  is simple in the variety of commutative rings, we have that every such function can be written as a polynomial in the ring operations.*

*Specifically, we observe that  $a \rightarrow b = 1 + a^{2q-3}b - a^{q-1}$  for every finite field of order  $q$ . This is because the multiplicative structure of  $\mathbb{F}_q^\#$  is a cyclic group of order  $q - 1$ . Therefore, given any word in  $R$ , any implication of polynomials  $p(X) \rightarrow q(X)$  can be rewritten as  $1 + p(X)^{2q-3}q(X) - p(X)^{q-1}$ .*

**Example 5.3.6** *As one can see, Theorem 5.3.4 provides many nontrivial examples of implication rings that are not fields due to the abundance of zero divisors.*

Since the identity  $a^2 = a$  is enough to guarantee that a ring is a Boolean ring, and can be thought of as a Boolean algebra, we know that this additional identity will create a subvariety of implication rings that are also Boolean rings. This leads to the following conjecture.

**Conjecture 5.3.7** *The variety of Boolean implication rings is the same as the subvariety of implication rings with the additional identity  $a^2 = a$  imposed.*

## CHAPTER 6. FUTURE DIRECTIONS

### 6.1 Introduction

The purpose of this chapter is to explore some possible extensions of the results discussed previously. Additionally, an attempt is made to relate the previous concepts, which may seem disparate at first glance, to one another. Some of the results are will be quite easy to obtain and should be complete within a matter of months, whereas others are not as well formed and will take longer to bring together.

### 6.2 Monadicity Theorems

One of the most obvious extensions of the work done in Chapter 2 is to find monadic adjunctions for other categories of ordered algebraic structures over  $\mathbf{Set}^m$ . The easiest extension would be for arbitrary qo-groups, which seems to be an immediate extension of the previous proof. What will be interesting to see, however, is how the positive cone of a free qo-group can be recognized within a free group. One may additionally examine the relationship between the number of generators of the positive cone and the behavior of the fractal generated within the graphical representation of the free group itself.

A second immediate generalization of the monadicity Theorem 2.4.9 is to attempt a similar result in the case of abelian l-groups. In fact, such a result should be possible using a result of (9) which classifies the positive cone of an abelian l-group as a cancellative hoop. Hoops can be defined axiomatically in the following way. The case of arbitrary l-groups may not be handled as easily but it is a possibility.

**Definition 6.2.1** *A cancellative hoop is an algebra  $A$  with two binary operations  $+$  and  $-$  and one nullary operation  $0$  such that  $A$  is a commutative monoid under  $+$  and  $0$  and the following*

*identities are satisfied:*

- $x + (y - x) = y + (x - y)$
- $(x - y) - z = x - (y + z)$
- $x - x = 0$
- $0 - x = 0$
- $(x + y) - x = y$

**Conjecture 6.2.2** *The category  $\mathbf{LAb}$  is monadic over  $\mathbf{Set}^m$ .*

If this conjecture were to be true, the next conjecture would be an immediate, and interesting, corollary.

**Conjecture 6.2.3** *The category  $\mathbf{LAb}$  is monadic over both  $\mathbf{Set}^m$  and  $\mathbf{Set}$ .*

A detailed analysis of the free objects of  $\mathbf{LAb}$  over  $\mathbf{Set}$  and  $\mathbf{Set}^m$  would reveal fairly interesting characteristics of the relationship between these categories and how closely intertwined they may be.

The full generalization of Theorem 2.4.9 may be the following conjecture.

**Conjecture 6.2.4** *Let  $\mathbf{A}$  be a category of algebras which is monadic over  $\mathbf{Set}$  and has an additional property  $P$  (such as lattice ordered, partially ordered, and quasi ordered). If  $P$  can be defined as a subset of an algebra  $A$  and the class of all subsets giving  $P$  is equationally defined, then  $\mathbf{A}$  is monadic over  $\mathbf{Set}^m$ .*

It would also be of interest to find other properties  $P$  that do not necessarily have order interpretations and investigate those categories of algebras in much the same way as  $\mathbf{QAg}^m$ .

### 6.3 Complete Classification of PVDs

To further the discussion of PVDs from Chapter 3 it would be desirable to fully classify PVDs in terms of their divisibility properties by proving the following conjecture.

**Conjecture 6.3.1** *A domain  $R$  is a PVD if and only if  $G(R) \cong_o L \circ A$  where  $L$  is a linearly ordered group and  $A$  is an antichain group. Moreover,  $L \cong_o G(V)$  and  $A \cong_o U(V)/U(R)$  where  $V$  is the unique valuation overring of  $R$ .*

The difficulty that has been encountered in attempts to prove Conjecture 6.3.1 is that  $G(V)$  is not a subgroup of  $G(R)$ . Moreover, any attempt to identify an isomorphic copy of  $G(V)$  inside  $G(R)$  or  $R$  inside  $V$  multiplicatively is hampered by an inability to select appropriate coset representatives of  $V^\sharp U(V)$  in such a way that the coset  $v_1 U(V) v_2 U(V)$  can be readily identified.

Since PVDs are examples of so-called  $D + M$  constructions, (equivalently, pullbacks of a valuation domain  $V$  and a subfield of the class field  $V/M$ ) we may use the following theorem from (22) which characterizes when the group of divisibility of a  $D + M$  construction will split lexicographically as we desire for PVDs.

**Theorem 6.3.2** (22) *If  $V$  is a valuation ring with a subring  $R = D + M$ , then the sequence  $0 \rightarrow U(V)/U(R) \rightarrow G(R) \rightarrow G(V) \rightarrow 0$  is exact. Moreover, the sequence splits if and only if there exists a valuation  $w$  on  $V$  so that, for every  $x_{v_1}, x_{v_2} \in G(V)$ ,  $(x_{v_1} x_{v_2})/x_{v_1+v_2} \in U(R)$ .*

The previous theorem shows that one way of verifying whether or not every PVD has a group of divisibility that is the lexicographic sum of a linearly ordered group and an antichain group is to find a valuation which preserves the multiplicative structure of the valuation overring  $V$  up to only unit multiples in  $R$ . For a given PVD this is generally not difficult but there does not seem to be an obvious algorithmic way to do this for every PVD. On the other hand, most of the known examples where  $0 \rightarrow G \rightarrow H \rightarrow J \rightarrow 0$  does not split for groups of divisibility  $G$ ,  $H$ , and  $J$ , require  $H$  to be lattice ordered and the only PVDs that are lattice ordered are valuation domains.

There is additional evidence for Conjecture 6.3.1 which can be obtained by examining the poset structure of  $G(R)$  without regard to any algebraic structure as follows.

**Theorem 6.3.3** *Given a PVD  $R$  with maximal ideal  $M$  and valuation overring  $V$ ,  $G(R) \cong G(V) \circ U(V)/U(R)$  as a poset.*

**Proof 6.3.4** *Any element of  $G(R)$  can be written as  $\alpha v_\gamma U(R)$ , where  $\alpha \in U(V)$  and  $\{v_\gamma | \gamma \in \Gamma\}$  is a transversal for  $G(V)$ . Then the map  $\phi : G(R) \rightarrow G(V) \circ U(V)/U(R); \alpha v_\gamma U(R) \mapsto (v_\gamma U(V), \alpha U(R))$  is clearly a poset isomorphism.*

The obvious problem at this point is to classify when the map  $\phi$  is a group homomorphism. This problem is dependent, of course, on the choice of the transversal to  $G(V)$ . If the transversal can be chosen so that the product of any two transversal elements differs by the valuation element by a unit of  $R$  only, then the result follows.

## 6.4 Implication Rings

The work done in Chapter 5 is quite preliminary. There is a tremendous amount of opportunity in this area going forward. Some of the most important aspects of the variety of implication rings have been left unexamined. The following is a list of some of the questions that may serve as a starting point for a better understanding of implication rings.

- Is the variety of implication rings generated by fields equal to the entire variety of implication rings?
- Is the variety of implication rings generated by fields finitely based?
- What is the congruence lattice structure of implication rings and which implication rings are subdirectly irreducible?
- Is there a 'normal' form for free implication rings which parallels the polynomial expression for free rings?
- Are there more precise methods of determining whether a given ring is an implication ring than the result in Theorem 5.2.3?
- What do the ultraproducts of implication rings look like?

In addition to these questions, I would like to examine a way to ensure that the variety of implication rings has a ternary discriminator term. There is a ternary discriminator in the

variety of implication rings generated by fields, it is unclear whether the full variety has one or not.

**Definition 6.4.1** *Let  $V$  be a variety. A ternary term  $t(x, y, z)$  in  $V$  is a discriminator if*

$$t(x, y, z) = \begin{cases} z & \text{if } x = y \\ x & \text{if } x \neq y \end{cases}$$

**Theorem 6.4.2** (26) *If a variety  $V$  has a ternary discriminator, then every finite algebra of  $V$  is polynomially complete, if an algebra  $A$  of  $V$  is infinite, it is locally polynomially complete.*

**Proposition 6.4.3** *The variety of implication rings generated by fields has a ternary discriminator.*

**Proof 6.4.4** *The term  $((x - y) \rightarrow 0)(z - x) + x$  is the required ternary discriminator.*

The crucial aspect of the previous discriminator was the fact that, in a field,

$$a \rightarrow 0 = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

. In the general case of an implication ring defined axiomatically we cannot necessarily be sure that  $a \rightarrow 0 = 0$  if  $x \neq 0$ .

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