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A formalization of logic in diagonal-free cylindric algebras

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CHAPTER 1. BACKGROUND MATERIAL

1.1 Introduction

In a 1915 paper, Leopold Löwenheim first conjectured that "all important questions of mathematics" may be reducible to equations of the Peirce-Schröder calculus of binary relations [Löw15]. Taking up this task, in 1942-1943 Alfred Tarski proved that almost all axiomatized set theories can be formalized as sets of relation algebraic (RA) equations [Tar43, Theorem 5.22]. He developed the Translation Mapping from first-order logic (FOL) to the equational theory of RA which preserves meaning and provability for sufficiently strong theories (including most axiomatized set theories). The Translation Mapping theorem was first announced in the abstract [Tar53a], and eventually published in the monograph [TG87, 4.4(xxxiv)] (the original manuscript [Tar43] was never published in its original form).

Tarski also showed that the equational theory of RA is, in fact, rich enough to capture the whole of first-order logic, and not just set theories formalized therein [Tar43, Theorem 5.31]. This theorem was not included in [TG87] as its primary consequence (the undecidability of RA) is obtained there by other means. Consequently, it was published independently by Maddux as [Mad89a, Theorem 23]. For every first-order sentence φ (in symbols, every $\varphi \in \Sigma[\mathcal{L}]$), Tarski and Maddux give a recursive construction of an RA equation $E(\varphi)$ such that φ is logically valid iff $\mathsf{RA} \models E(\varphi)$.

Tarski's proofs in [Tar43, TG87] are syntactical and metamathematical. From an algebraic point of view, the proof of the Translation Mapping theorem can roughly be described as showing that every RA with quasi-projections is representable (more concisely stated $\mathsf{QRA} \subset \mathsf{RRA}$). Indeed, this purely algebraic result (known as the QRA theorem) can be derived from the Translation Mapping theorem, and vice versa [TG87, p. 242]. Tarski thus posed the problem of finding a purely algebraic proof of the QRA theorem which does not rely upon the Translation Mapping theorem. Roger Maddux subsequently accomplished this in [Mad78b].

Tarski's formalization of set theory has some remarkable properties, effectively eliminating many of the idiosyncrasies associated with traditional first-order logic. First and foremost, it uses no variables (thus the title of the monograph, *A formalization of set theory without variables*). This is remarkable in and of itself, especially considering that all of the variable-free formalizations (also called propositional logics) which had been previously known were proven to be decidable (therefore insufficient for the study of set theory and arithmetic). It also circumvents the need to distinguish between free and bound variables in a formula, and the (somewhat tediously defined) process of respelling bound variables. Equally remarkable from our point of view, is that the only rule of inference employed in Tarski's formalization is the "high school algebra" rule of replacing equals with equals (as opposed to the typical first-order rules, modus ponens and generalization). This second property is common to the target formalization of the present work. Indeed this is true of all algebraizations of logical formalisms, owing to the replacement of logical connectives and quantifiers with algebraic operations, which allows us to speak of equality between terms in place of the equivalence of sentences.

Relation algebras are closely related to cylindric algebras of finite dimension (CA_α for $\alpha \in \omega$) and thus to the corresponding finite-variable logics. Of the ten equational axioms which define RA, all but two can be stated and proved in restricted 3-variable first-order logic (L_3). The problematic axioms (R_1 and R_4 from Definition 12 below) assert that the relative product of binary relations is associative and that the inverse of the inverse of a relation is the same relation, both requiring 4 variables to prove. The necessity of a 4[th] variable for proof is algebraically reflected in the fact that when $\alpha \geq 4$, the natural RA reduct (in symbols, $\mathfrak{Ra}-$ [HMT85, `5.3.7`]) of any CA_α is indeed a relation algebra [HMT85, `5.3.8`], however there are algebras $\mathfrak{A} \in \mathsf{CA}_3$ for which $\mathfrak{Ra}\mathfrak{A} \not\models \mathrm{R}_1$ [HT61, p. 103], and for which $\mathfrak{Ra}\mathfrak{A} \not\models \mathrm{R}_4$. Such algebras are illustrated in [HMT85] as Constructions `3.2.71` and `3.2.69` respectively. We summarize these results by saying that CA_3 is weaker than RA, which is in turn weaker than CA_4.

In 1985 István Németi constructed an RA reduct, $\mathfrak{Ora}$ (which we call the Németi reduct) of the formula algebra of L_3, by making use of projection functions which satisfy a finite set of assumptions (similar to, but stronger than Tarski's quasi-projections) [Ném85, Proposition 2.10]. Utilizing the fact that CA_3 is an algebraization of L_3, he also showed the existence of an RA reduct of any $\mathfrak{A} \in \mathsf{CA}_3$ having sufficiently strong projection parameters [Ném85, Remark 3.13]. Németi's idea was to use projections to store information from the first two dimensions (where binary relations of the natural RA reduct are stored) in the first dimension only. Effectively, he coded binary relations as a sets (of "ordered pairs"). In this manner, he gave himself two spare variables for quantification instead of one, which turned out to be sufficient to recover both R_1 and R_4 (as it is in $\mathfrak{Ra}\mathfrak{A}$ for $\mathfrak{A} \in \mathsf{CA}_4$). Németi's technique did create a new issue with the identity law (R_3), however he was able to get around this by adding the assumption that the "ordered pairs" induced by the projections were unique. Later, in [Ném86, Theorem 9], he abandoned this assumption in favor of another technique for recovering the identity law (c.f. Remark 33).

Using his reduct, Németi constructed a recursively defined translation function from L_ω to L_3 which preserved meaning and provability (relative to theories proving the existence of such projection parameters) [Ném85, Proposition 3.3], thus solving a problem posed in [TG87, p. 143]. He also conjectured that these results could be extended to restricted 3-variable first-order logic without equality ($L_3^{\neq}$), a problem which also appears in [SA04, p. 12], [SA05, p. 476], and [Sim07, p. 300]. Just as CA_3 is an algebraization of L_3, diagonal-free cylindric algebras of dimension 3 (Df_3) constitute an algebraization of $L_3^{\neq}$. In Theorem 34, we show that every Df_3 with an equality parameter and projection parameter satisfying a finite set of assumptions has an RA reduct, thus generalizing Németi's construction from CA_3 to the weaker class Df_3, and confirming his conjecture.

In Theorem 56 we employ some algebraic properties of the variety Df_3, along with the Németi reduct, to construct an analog of the Tarski-Maddux function translating the sentences of first-order logic to the equational theory of Df_3 which: (1) preserves validity, and (2) is not relative to any particular theory. This strengthens aforementioned Tarski-Maddux result since, as we have previously mentioned and shall illustrate quite thoroughly, Df_3 is a strictly weaker variety than RA.

Remark 1 Hajnal Andréka and Németi have independently described modifications to [Ném86, p. 41-64] (in Hungarian) which yield an RA reduct of the formula algebra of $L_3^{\neq}$ [AN11, Theorem 4.3] and a formalization of set theory in $L_3^{\neq}$ [AN11, Theorem 2.1]. Somewhat similar to our own, their construction uses 4 parameters (2 for equality and 2 for projection functions), and is carried out entirely in $L_3^{\neq}$ as opposed to Df_3. $\square$

Remark 2 (Notational conventions) We use Frakture typeface (such as $\mathfrak{A}$) to denote an algebra or an operator which outputs an algebra, and sans-serif typeface (such as BA) to denote a class of algebras or an operator which outputs a class of algebras. The Greek letters $\alpha, \beta, \kappa, \lambda, \mu, \omega$ denote ordinals (with ω being the first infinite ordinal); and ρ is a rank function or sequence of ranks. We frequently use the fact that each ordinal may be defined as the set of all lesser ordinals (0 being identified with the empty set in this manner). When A and B are sets, we write ${}^{B}A$ to denote the set of all functions from B to A. In particular for any ordinal α and set A, ${}^{\alpha}A$ denotes the set of A-sequences of length α. If $x \in {}^{\alpha}A$ and $\beta \in \alpha$, then we understand x_β to be the image of β under the function x. $\square$

1.2 First-order logic

Let $\mathcal{L}$ be a first-order language with a countable set of relation symbols $\{\mathsf{R}_\kappa : \kappa \in \beta \in \omega + 1\}$, but no function symbols or constants[1]. Let $\rho \in {}^{\beta}(\omega+1)$ such that for every $\kappa \in \beta$, the relation symbol R_κ has rank ρ_κ. The variables of $\mathcal{L}$ are the countable set $\{\mathsf{v}_\kappa : \kappa \in \alpha \in \omega + 1\}$. The connectives of $\mathcal{L}$ are disjunction ($\vee$) and negation ($\neg$). The only quantifier of $\mathcal{L}$ is existential ($\exists$), and there is an equality symbol ($=$). All other logical connectives, and the universal quantifier are introduced as abbreviations (for example, $\forall_{\mathsf{v}_\kappa}\varphi := \neg\exists_{\mathsf{v}_\kappa}\neg\varphi$ and $\varphi \to \psi := \neg\varphi \vee \psi$). If $\alpha \geq \omega$ then we say $\mathcal{L}$ is an **ordinary language**. When $\rho_\kappa = \alpha$ for each $\kappa \in \beta$ we say $\mathcal{L}$ is a **full language**.

The **atomic formulas** of $\mathcal{L}$ are those of the form $\mathsf{v}_\kappa = \mathsf{v}_\lambda$ for $\kappa, \lambda \in \alpha$ (an **equation**) and $\mathsf{R}_\mu(\mathsf{v}_{\kappa_0}, \dots, \mathsf{v}_{\kappa_\eta}, \dots)_{\eta \in \rho_\mu}$ (called a **relational atomic formula**), where $\mu \in \beta$ and $\kappa \in {}^{(\rho_\mu)}\alpha$. We denote by $\Phi[\mathcal{L}]$ the set of all **formulas** of $\mathcal{L}$—the closure of the set of atomic formulas under the connectives and quantifiers of $\mathcal{L}$. For $\varphi \in \Phi[\mathcal{L}]$ let $\mathrm{Var}(\varphi)$, $\mathrm{In}(\varphi)$, and $\mathrm{Rel}(\varphi)$ be the set of indices of variables, free variables, and relation symbols occurring in φ respectively—recursively defined according to Table 1.1.

φ	$\mathrm{Var}(\varphi)$	$\mathrm{In}(\varphi)$	$\mathrm{Rel}(\varphi)$
$\mathsf{R}_\mu(\mathsf{v}_{\kappa_0}, \dots, \mathsf{v}_{\kappa_\eta}, \dots)_{\eta \in \rho_\mu}$	$\{\kappa_\eta : \eta \in \rho_\mu\}$	$\{\kappa_\eta : \eta \in \rho_\mu\}$	$\{\mu\}$
$\mathsf{v}_\kappa = \mathsf{v}_\lambda$	$\{\kappa, \lambda\}$	$\{\kappa, \lambda\}$	0
$\neg\varphi$	$\mathrm{Var}(\varphi)$	$\mathrm{In}(\varphi)$	$\mathrm{Rel}(\varphi)$
$\exists_{\mathsf{v}_\kappa}\varphi$	$\mathrm{Var}(\varphi)$	$\mathrm{In}(\varphi) \setminus \{\kappa\}$	$\mathrm{Rel}(\varphi)$
$\varphi \vee \psi$	$\mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi)$	$\mathrm{In}(\varphi) \cup \mathrm{In}(\psi)$	$\mathrm{Rel}(\varphi) \cup \mathrm{Rel}(\psi)$

Table 1.1 Definitions of $\mathrm{Var}(\varphi)$, $\mathrm{In}(\varphi)$, and $\mathrm{Rel}(\varphi)$

The set of **sentences** of $\mathcal{L}$ is $\Sigma[\mathcal{L}] = \{\varphi \in \Phi[\mathcal{L}] : \mathrm{In}(\varphi) = \emptyset\}$. The set of **subformulas** of $\varphi \in \Phi[\mathcal{L}]$ is defined to be the smallest set $\Gamma \subset \Phi[\mathcal{L}]$ such that $\varphi \in \Gamma$, and $\psi \in \Gamma$ whenever: $\exists_{\mathsf{v}_\kappa}\psi \in \Gamma$, or $\neg\psi \in \Gamma$, or $\psi \vee \chi \in \Gamma$ for some $\chi \in \Phi[\mathcal{L}]$. We say φ is a **restricted formula** when all of the relational atomic subformulas have the form $\mathsf{R}_\mu(\mathsf{v}_0, \dots, \mathsf{v}_\eta, \dots)_{\eta \in \rho_\mu}$, and set $\Phi_r[\mathcal{L}]$ to be the set of all such formulas. We say $\varphi \in \Phi[\mathcal{L}]$ is an **equality-free formula** when all of the atomic subformulas of φ are relational and set $\Phi_e[\mathcal{L}]$ to be the set of all such formulas. The sets $\Phi_{r,e}[\mathcal{L}]$, $\Sigma_r[\mathcal{L}]$, $\Sigma_e[\mathcal{L}]$, and $\Sigma_{r,e}[\mathcal{L}]$ are named and defined analogously.

[1] The elimination of function symbols and constants does not really restrict our choice of language since every n-ary function can be equivalently looked upon as an $(n+1)$-ary relation, and similarly every constant as a unary relation containing only a single element.

Definition 3 A **realization** for $\mathcal{L}$ is a structure of the form $\mathfrak{U} = \langle U, \mathbf{R} \rangle$ where U is a nonempty set and $\mathbf{R} = \langle R_\mu : \mu \in \beta \rangle$ is a sequence of relations on U with $R_\mu \subseteq {}^{(\rho_\mu)}U$ for each $\mu \in \beta$. $\mathsf{RE}[\mathcal{L}]$ is the class of all realizations for $\mathcal{L}$. Given $\mathbf{a} \in {}^\omega U$, and $\varphi, \psi \in \Phi[\mathcal{L}]$, we say:

1. $\mathfrak{U} \models \mathsf{v}_\kappa = \mathsf{v}_\lambda[\mathbf{a}]$ if and only if $a_\kappa = a_\lambda$,

2. $\mathfrak{U} \models \mathsf{R}_\mu \mathsf{v}_{\kappa_0} \ldots \mathsf{v}_{\kappa_{\rho_\mu - 1}}[\mathbf{a}]$ if and only if $\langle a_{\kappa_0}, \ldots, a_{\kappa_{\rho_\mu - 1}} \rangle \in R_\mu$,

3. $\mathfrak{U} \models \varphi \vee \psi[\mathbf{a}]$ if and only if $\mathfrak{U} \models \varphi[\mathbf{a}]$ or $\mathfrak{U} \models \psi[\mathbf{a}]$

4. $\mathfrak{U} \models \neg\varphi[\mathbf{a}]$ if and only if it is not the case that $\mathfrak{U} \models \varphi[\mathbf{a}]$,

5. $\mathfrak{U} \models \exists_{\mathsf{v}_\kappa}\varphi[\mathbf{a}]$ if and only if there is some $\mathbf{b} \in {}^\omega U$ such that $b_\lambda = a_\lambda$ for each $\lambda \in \mathrm{In}(\exists_{\mathsf{v}_\kappa}\varphi)$ and $\mathfrak{U} \models \varphi[\mathbf{b}]$.

This recursively defines $\mathfrak{U} \models \varphi[\mathbf{a}]$ for every formula φ. We say $\mathfrak{U} \models \varphi$ when $\mathfrak{U} \models \varphi[\mathbf{a}]$ for every $\mathbf{a} \in {}^\omega U$. In particular, for every $\varphi \in \Sigma[\mathcal{L}]$ and $\mathbf{a} \in {}^\omega U$, $\mathfrak{U} \models \varphi[\mathbf{a}]$ iff $\mathfrak{U} \models \varphi$. When $\mathfrak{U} \models \varphi$ for every $\mathfrak{U} \in \mathsf{RE}[\mathcal{L}]$ we write $\models \varphi$, and say that φ is **logically valid**. For a set of formulas $\Gamma \subset \Phi[\mathcal{L}]$ we say $\Gamma \models \varphi$ if and only if $\mathfrak{U} \models \varphi$ whenever $\mathfrak{U} \models \psi$ for each $\psi \in \Gamma$. When $\models \varphi \leftrightarrow \psi$, we say that φ and ψ are **logically equivalent**. □

In an ordinary language, every formula φ is logically equivalent to some restricted formula φ_r [HMT85, `4.3.6`]. Similarly, if $\Gamma \subseteq \Phi[\mathcal{L}]$ then there is some $\Gamma_r \in \Phi_r[\mathcal{L}]$ such that $\Gamma \models \varphi$ if and only if $\Gamma_r \models \varphi_r$ [HMT85, `4.3.15`].

Next we introduce a proof system suitable for $\mathcal{L}$ and several of its subformalisms. A set of formulas Γ is said to be closed under **detachment** (also called modus ponens) iff $\psi \in \Gamma$ whenever $\{\varphi \to \psi, \varphi\} \subset \Gamma$. The set Γ is said to be closed under **generalization** if $\forall_{\mathsf{v}_\kappa}\varphi \in \Gamma$ whenever $\varphi \in \Gamma$.

Definition 4 The set of **logical axioms** for $\mathcal{L}$ is denoted by $\Lambda[\mathcal{L}]$. The sets $\Lambda_r[\mathcal{L}]$, $\Lambda_e[\mathcal{L}]$, and $\Lambda_{r,e}[\mathcal{L}]$ are the corresponding restricted, equality-free, and restricted equality-free logical axioms. $\Lambda[\mathcal{L}]$ consists of all formulas of the following kind, where $\varphi, \psi \in \Phi[\mathcal{L}]$ and $\kappa, \lambda, \mu \in \alpha$ are arbitrary:

(1) φ, a propositional tautology,

(2) $\forall_{\mathsf{v}_\kappa}(\varphi \to \psi) \to (\forall_{\mathsf{v}_\kappa}\varphi \to \forall_{\mathsf{v}_\kappa}\psi)$,

(3) $\forall_{\mathsf{v}_\kappa}\varphi \to \varphi$,

(4) $\varphi \to \forall_{\mathsf{v}_\kappa}\varphi$, if v_κ does not occur free in φ,

(5) $\mathsf{v}_\kappa = \mathsf{v}_\kappa$,

(6) $\exists_{\mathsf{v}_\kappa}(\mathsf{v}_\kappa = \mathsf{v}_\lambda)$,

(7) $\mathsf{v}_\kappa = \mathsf{v}_\lambda \to (\mathsf{v}_\kappa = \mathsf{v}_\mu \to \mathsf{v}_\lambda = \mathsf{v}_\mu)$,

(8) $\mathsf{v}_\kappa = \mathsf{v}_\lambda \to [\varphi \to \forall_{\mathsf{v}_\kappa}(\mathsf{v}_\kappa = \mathsf{v}_\lambda \to \varphi)]$ if $\kappa \neq \lambda$.

For $\Gamma \cup \{\varphi\} \subseteq \Phi[\mathcal{L}]$, define $\Gamma \vdash \varphi$ if $\varphi \in \Omega$ whenever $\Omega \supset \Gamma \cup \Lambda$ is closed under detachment and generalization. Analogously define $\vdash_r$, $\vdash_e$, and $\vdash_{r,e}$ for $\Gamma \cup \{\varphi\}$ contained in Φ_r, Φ_e, and $\Phi_{r,e}$ respectively. Restrictions of $\mathcal{L}$ which use these proof systems are summarized in Table 1.2. □

Language	Variables	Formulas	Proof system
L_α	$\{\mathsf{v}_\kappa : \kappa \in \alpha\}$	$\Phi[L_\alpha] = \Phi_r[\mathcal{L}]$	$\vdash_r$
$L_\alpha^{\neq}$	$\{\mathsf{v}_\kappa : \kappa \in \alpha\}$	$\Phi[L_\alpha^{\neq}] = \Phi_{r,e}[\mathcal{L}]$	$\vdash_{r,e}$

Table 1.2 Restrictions of $\mathcal{L}$

By Gödel's completeness theorem, we have $\models \varphi$ if and only if $\vdash \varphi$ when $\alpha \geq \omega$. As we shall see, this is not the case for $\alpha < \omega$.

Remark 5 As a historical note, we mention that the formalism which might best be described in our notation as $\mathcal{L}_3$ is also of interest in the field of algebraic logic. It uses only 3 variables, but admits relational atomic formulas which are not restricted. As a consequence of this, the identity law (R_4) becomes $\mathcal{L}_3$-provable, leaving only R_1 without proof. It was this that motivated Maddux's invention of so-called semi-associative relation algebras (SA). In this broader class, the axiom R_1 asserting the associativity of relative product is replaced with the special case:

$$(x;1);1 = x;(1;1),$$

which *is* provable using only 3 variables [Mad78b]. Since L_3 is strictly weaker than $\mathcal{L}_3$, we are justified in concluding that the corresponding variety of algebras, CA_3 is strictly weaker than SA. As with CA_3, however, the QRA theorem does not extend to SA. While representable SA's coincide with representable RA's, Németi has shown in [Ném85] that there are quasi-projectional SA's which are not even relation algebras. □

1.3 General algebra and Boolean algebras with operators

We begin our investigation by defining some general algebra (also called universal algebra) notions. These provide the framework upon which our target formalism shall be built. For a more detailed and comprehensive exposition of results and definitions concerning general algebra, we refer the reader to e.g. [HMT71, §0].

Definition 6 A sequence $\mathfrak{A} = \langle A, \sigma_i \rangle_{i \in I}$ is an **algebra** if and only if A is a set and for each $i \in I$, σ_i is an operation on A to A of rank ρ_i. In case $\rho_i = 0$ we call σ_i a **distinguished element** of $\mathfrak{A}$. We call the sequence $\langle \rho_i : i \in I \rangle$ the **similarity type** of $\mathfrak{A}$, and the set A the **universe** of $\mathfrak{A}$. Now let K be a class of similar algebras and $\mathfrak{A} = \langle A, \sigma_i \rangle_{i \in I} \in \mathsf{K}$.

1. A **subalgebra** of $\mathfrak{A}$ is an algebra $\mathfrak{B} = \langle B, \sigma_i \rangle_{i \in I}$ such that $B \subset A$. Symbolically, we write $\mathfrak{B} \leq \mathfrak{A}$. That $\mathfrak{B}$ is an algebra with the same operations as $\mathfrak{A}$ entails that each distinguished element of $\mathfrak{A}$ is also an element of B, and that B is closed under all of the operations of $\mathfrak{A}$. The class of all subalgebras of algebras in K is $\mathbf{S}\mathsf{K}$.

2. A **homomorphic image** of $\mathfrak{A}$ is an algebra $\mathfrak{B} = \langle B, \tau_i \rangle_{i \in I}$ of the same similarity type as $\mathfrak{A}$ such that there is a function f from A onto B which respects the operations of $\mathfrak{A}$ (such an f is called a **homomorphism**). That is to say, for each $i \in I$ and $\mathbf{x} \in {}^{\rho_i}A$, $f\sigma_i\mathbf{x} = \tau_i f\mathbf{x}$ (where $f\mathbf{x}$ denotes the sequence $\langle fx_0, \ldots, fx_{\rho_i - 1} \rangle \in {}^{\rho_i}B$). The class of all homomorphic images of algebras in K is $\mathbf{H}\mathsf{K}$. The operator $\mathbf{I}$ (the class of all **isomorphic images**) is defined just as $\mathbf{H}$, but with the additional requirement that f be bijective.

3. A **direct product** of an indexed system of algebras $\boldsymbol{\mathfrak{A}} \in {}^J\mathsf{K}$ is the algebra $\mathfrak{B} = \prod \boldsymbol{\mathfrak{A}} = \prod_{j \in J} \mathfrak{A}_j$ (with, say $\mathfrak{A}_j = \langle A_j, \sigma_i^{\mathfrak{A}_j} \rangle_{i \in I}$ for each $j \in J$) whose universe is the (not necessarily finite) Cartesian product $B = \prod_{j \in J} A_j$ and whose operations $\sigma_i^{\mathfrak{B}}$ are defined component-wise. That is to say, if $\mathbf{x} \in {}^{\rho_i}B$ (so that for $\lambda \in \rho_i$ and $\kappa \in \alpha$, we have $x_\lambda \in B$ and $(x_\lambda)_\kappa$ or simply $x_{\lambda,\kappa}$ is an element of A_κ), then $\sigma_i^{\mathfrak{B}}\mathbf{x} := \langle \sigma_i^{\mathfrak{A}_\kappa} x_{0,\kappa} \ldots x_{\rho_i - 1,\kappa} : j \in J \rangle \in B$. We define $\mathbf{P}\mathsf{K}$ to be the class $\mathbf{I}\{\prod_{j \in J} \mathfrak{A}_j : J \text{ is a set and } \boldsymbol{\mathfrak{A}} \in {}^J\mathsf{K}\}$.

We say $\mathfrak{A}$ is a **subdirect product** of algebras in K when $\mathfrak{A} \in \mathbf{SP}\mathsf{K}$, i.e. $\mathfrak{A}$ is a subalgebra of a direct product of algebras in K (up to isomorphism). We say that K is a **variety** just in case $\mathbf{HSP}\mathsf{K} = \mathsf{K}$. $\mathbf{HSP}\mathsf{K}$ is called the **variety generated by** K. □

The reason for closing the set of direct products under $\mathbf{I}$ in the definition of $\mathbf{P}$ is that this ensures that if K is closed under isomorphisms, then $\mathbf{P}\mathsf{K}$ will be as well. It is easy to see that $\mathbf{S}\mathsf{K}$ inherits this closure from K and that $\mathbf{H}\mathsf{K}$ is such for every K. If K is a variety then it is not difficult to show that $\mathsf{K} = \mathbf{H}\mathsf{K} = \mathbf{S}\mathsf{K} = \mathbf{P}\mathsf{K}$ [HMT71, `0.3.13`]. Garret Birkhoff's famous theorem [Bir35, Theorem 10] (sometimes called Birkhoff's **HSP** theorem) says that a class of similar algebras K is a variety iff K is an equational class, i.e. K is precisely the class of all algebras satisfying a certain set of equations.

Definition 7 An algebra $\mathfrak{A}$ is said to be **simple** if and only if the only homomorphic images of $\mathfrak{A}$ are itself and the trivial 1-element algebra. The simplicity of $\mathfrak{A}$ can be symbolically expressed as: $\mathfrak{B} \in \mathbf{H}\{\mathfrak{A}\}$ implies either $\mathfrak{A} \cong \mathfrak{B}$ or $|B| = 1$. We say $\mathfrak{A}$ is **semisimple** if it is isomorphic to a subdirect product of simple algebras. □

Simple algebras often have more convenient properties than the varieties that they generate. Moreover they form a sufficiently large class to generate the varieties we are concerned with. These facts will be stated more precisely after we introduce CA and RA (c.f. Theorems 22-24 and 29-31). We do note here, however, that the algebraic systems first used by Tarski (beginning with [Tar41]) coincide with what we now call simple relation algebras [Mad06, p. ix]. A definition equivalent to the modern variety RA was first published in the abstract [JT48].

Definition 8 Fix any similarity type $\rho = \langle \rho_i : i \in I \rangle$ and let $\langle \sigma_i : i \in I \rangle$ be an arbitrary sequence of operation symbols. Let $\mathfrak{A} = \left\langle A^{\mathfrak{A}}, \sigma_i^{\mathfrak{A}} \right\rangle_{i \in I}$ be an algebra and K be a class of similar algebras, both having similarity type ρ.

1. The set of $\mathfrak{A}$ **terms** (in symbols $\mathrm{Tm}[\mathfrak{A}]$), is the closure of the variables $\{\mathsf{w}_\kappa : \kappa \in \omega\}$ under the operation symbols $\{\sigma_i\}_{i \in I}$. Thus $\{\mathsf{w}_\kappa : \kappa \in \omega\} \subset \mathrm{Tm}[\mathfrak{A}]$ and $\sigma_i t_0 \cdots t_{\rho_i - 1} \in \mathrm{Tm}[\mathfrak{A}]$ whenever $i \in I$ and $t_0, \ldots, t_{\rho_i - 1} \in \mathrm{Tm}[\mathfrak{A}]$. For each $t_0, t_1 \in \mathrm{Tm}[\mathfrak{A}]$, we call $t_0 \doteq t_1$ an $\mathfrak{A}$ **equation** and let $\mathrm{Eq}[\mathfrak{A}]$ be the set of all $\mathfrak{A}$ equations. Neither $\mathrm{Tm}[\mathfrak{A}]$ nor $\mathrm{Eq}[\mathfrak{A}]$ depend on anything about $\mathfrak{A}$ except its similarity type, ρ. Thus we analogously define $\mathrm{Tm}[\mathsf{K}]$ and $\mathrm{Eq}[\mathsf{K}]$, or even $\mathrm{Tm}[\rho]$ and $\mathrm{Eq}[\rho]$.

2. By an **assignment** to $A^{\mathfrak{A}}$ we mean a sequence $\mathbf{s} \in {}^{\omega}(A^{\mathfrak{A}})$. If $t \in \mathrm{Tm}[\mathfrak{A}]$ and $\mathbf{s}$ is an assignment to $A^{\mathfrak{A}}$, then the **interpretation of** t **under** $\mathbf{s}$ is defined to be the element $t[\mathbf{s}] \in A^{\mathfrak{A}}$ obtained by replacing each variable w_κ with $s_\kappa \in A^{\mathfrak{A}}$ and each operation symbol σ_i with the function $\sigma_i^{\mathfrak{A}} : {}^{(\rho_i)}(A^{\mathfrak{A}}) \to A^{\mathfrak{A}}$.

3. Let $E = t_0 \doteq t_1 \in \mathrm{Eq}[\rho]$ and $\mathbf{s}$ be an assignment to $A^{\mathfrak{A}}$.

   (a) $\mathfrak{A} \models E[\mathbf{s}]$ (in words $\mathfrak{A}$ **models** E **under** $\mathbf{s}$) if and only if $t_0[\mathbf{s}] = t_1[\mathbf{s}]$.

   (b) $\mathfrak{A} \models E$ if and only if $\mathfrak{A} \models E[\mathbf{s}]$ for every $\mathbf{s} \in {}^{\omega}A$.

   (c) $\mathsf{K} \models E$ if and only if $\mathfrak{A} \models E$ for every $\mathfrak{A} \in \mathsf{K}$. □

It is easy to see that whenever $\mathsf{K} \models E$, $\mathbf{HSP}\mathsf{K} \models E$ as well.

Next we turn our attention to a special class of algebras which contains all three of RA, CA, and Df; namely Boolean algebras with operators. Named for George Boole, who pioneered their study in the mid-nineteenth century, Boolean algebras are an algebraization of propositional (truth table) logic.

As suggested by the name, Boolean algebras with operators extend Boolean algebras. Thus we can extend all Boolean definitions, notations, and abbreviations to this much larger class. For more results concerning Boolean algebras with operators, we refer the reader to [JT51, JT52].

Definition 9 The algebra $\mathfrak{A} = \langle A, +, ^- \rangle$ of similarity type $\langle 2, 1 \rangle$ is a **Boolean algebra** whenever it satisfies all of the identities in Table 1.3 for each $x, y, z \in A$.

1. If $\mathfrak{A}$ is a Boolean algebra and $\sigma : {}^{\rho}A \to A$, then we say that σ is **additive** if and only if σ distributes over Boolean addition. Precisely, when $\mathbf{x}, \mathbf{y} \in {}^{\rho}A$ and $\mathbf{x} + \mathbf{y} := \langle x_\kappa + y_\kappa : \kappa \in \rho \rangle \in {}^{\rho}A$, we have $\sigma(\mathbf{x} + \mathbf{y}) = \sigma\mathbf{x} + \sigma\mathbf{y}$.

2. $\mathfrak{A} = \langle A, +, ^-, \sigma_i \rangle_{i \in I}$ is called a **Boolean algebra with operators** when $\mathfrak{A}$ is an algebra, the **Boolean reduct** $\mathfrak{Bl}\mathfrak{A} := \langle A, +, ^- \rangle$ is a Boolean algebra, and each σ_i is additive.

We let BA and Bo be the classes of all Boolean algebras and algebras with operators, respectively. $\square$

B_1	$x + y = y + x$	$+$ - commutativity
B_2	$x + (y + z) = (x + y) + z$	$+$ - associativity
B_3	$(\overline{x} + \overline{y})^- + (\overline{x} + y)^- = x$	Huntington's axiom

Table 1.3 BA axioms

We use any of $-x$, $\overline{x}$, and x^- interchangeably for the Boolean complement of x. Since $x + \overline{x} = y + \overline{y}$ holds for every $\mathfrak{A} \in \mathsf{BA}$ and $x, y \in A$, we may unambiguously define $1 := x + \overline{x}$ and $0 := \overline{1}$. We also make use of the defined operations $x \cdot y := (\overline{x} + \overline{y})^-$ and $x \oplus y := x \cdot \overline{y} + y \cdot \overline{x}$. As one last abbreviation we let $x \leq y$ indicate $x + y = y$.

Remark 10 The axiomatization given here is due to Edward V. Huntington [Hun33b, Hun33a][2]. A beautiful feature of Huntington's axiomatization is that axioms B_1-B_2 together with the Boolean dual of B_3:

$$((x + \overline{y})^- + (x + y)^-)^- = x,$$

also axiomatize BA—a surprisingly non-trivial fact. Soon after Huntington's axiomatization was published, Herbert Robbins conjectured that this was the case. The dual of B_3 became known as the Robbins axiom, and the class of algebras satisfying B_1-B_2 plus Robbins axiom were called Robbins algebras. The problem gained much notoriety over the years, and remained open until 1996 when W.

[2]The idempotence axiom $x + x = x$ was included in the axiomatization appearing in [Hun33b], along with an incorrect proof of its independence of the remaining three. This was corrected in [Hun33a].

McCune was able to construct a proof of B_3 from B_1, B_2, and Robbins's axiom using his automated theorem prover, EQP [McC97] (for a simplified derivation, see [Dah98]). It turns out that it is even possible to axiomatize BA using only one operation (the Sheffer stroke, $x \uparrow y = \overline{x} + \overline{y}$, from which $x + y$ and $\overline{x}$ can both be defined) and a single axiom (also found with computer assistance, this is known to be the shortest possible axiomatization) [Wol02]. $\square$

In addition to the three operators described in Definition 6, Boolean algebras with operators yield a fourth way of creating new algebras from old ones (which may or may not lead out of the variety generated by the parent algebra).

Definition 11 Let $\mathfrak{A} = \langle A, +, ^-, \sigma_i \rangle_{i \in I}$ be a Boolean algebra with operators and $a \in A$. We define $\mathrm{Rl}_a A := \{x \cdot a : x \in A\}$, $\overline{x}^a := a \cdot \overline{x}$, and for each $i \in I$, $x \in {}^{\rho_i}\mathrm{Rl}_a A$ we let $\sigma_i^a(\mathbf{x}) := a \cdot \sigma_i(\mathbf{x})$. Then $\mathfrak{Rl}_a\mathfrak{A} := \langle \mathrm{Rl}_a A, +, ^{-a}, \sigma_i^a \rangle_{i \in I}$ is the **relativization to a of $\mathfrak{A}$.** $\square$

It is well known and easy to show that $\mathfrak{Rl}_a\mathfrak{A} \in \mathsf{BA}$ when $\mathfrak{A} \in \mathsf{BA}$ and $a \in A$. This does not generally extend to other varieties contained in the class Bo. If $\mathsf{K} \subset \mathsf{Bo}$ is a variety and $\mathfrak{A} \in \mathsf{K}$ with $a \in A$, then it is possible that $\mathfrak{Rl}_a\mathfrak{A} \notin \mathsf{K}$ (though the structure is, by definition, guaranteed to be an algebra with the same similarity type as K). Examples of such algebras (for $\mathsf{K} = \mathsf{RA}$) can be found in [Mad06, p. 379] (along with some sufficient conditions for $\mathfrak{Rl}_a\mathfrak{A} \in \mathsf{RA}$). Relativizations of CA_α's are discussed in [HMT71, §2.2].

When studying the algebraization of a particular logic, those algebras which correspond to models of the logic are especially interesting. We call these algebras **representable**. The class of representable Boolean algebras consists of those which are isomorphic to an algebra $\mathfrak{A} = \langle A, +, ^- \rangle$ is such that A is a family of subsets of some largest set $1^{\mathfrak{A}}$, $+$ is union, and $^-$ is complementation with respect to $1^{\mathfrak{A}}$. Since propositional logic is complete, every BA is representable. Conversely, by Stone's representation theorem for Boolean algebras [Sto36], every BA is representable, and thus propositional logic is seen to be complete. Just as Bo extends BA, the notion of representability for the Boolean operations extends, for example, to RA. Thus a representable relation algebra will be isomorphic to some algebra where the interpretations of $+$ and $^-$ are union and complementation. Of course, we shall impose requirements on the additional operators of RA as well, making the problem of classifying representable algebras much more difficult.

1.4 Relation algebras

Next we investigate relation algebras. There is a natural correspondence between the equations which hold in every RA and the formulas of Tarski's $\mathcal{L}^{\times}$ which are provable. We shall not discuss $\mathcal{L}^{\times}$ in any detail here, but mention in passing that: (1) the works [Tar43, Tar53a, Tar53b, TG87] are primarily formulated in $\mathcal{L}^{\times}$ rather than what we would call relation algebras, (2) the formulas of $\mathcal{L}^{\times}$ involve no variables and are generated from a single binary predicate symbol, and (2) the provability relation in $\mathcal{L}^{\times}$ falls somewhere between that of L_3 and L_4 (which is reflected in our discussion of $\mathfrak{RaCA}_\alpha$ for $3 \leq \alpha \leq 4$ on page 2).

Definition 12 A **relation algebra** is any algebra

$$\mathfrak{A} = \langle A, +, ^{-}, ;, ^{\smile}, 1' \rangle$$

with similarity type $\langle 2, 1, 2, 1, 0 \rangle$ which satisfies axioms R_0-R_7 in Table 1.4 (due to Tarski). The binary operation ; is called **relative product** and the unary $^{\smile}$ is called **converse**. The variety consisting of all relation algebras is denoted RA. [Mad06, 6.0.1] □

R_0	$\langle A, +, ^{-} \rangle \in \mathsf{BA}$	
R_1	$x;(y;z) = (x;y);z$	; - associativity
R_2	$(x+y);z = x;z + y;z$	; - distributivity
R_3	$x;1' = x$	identity law
R_4	$\breve{\breve{x}} = x$	$^{\smile}$ - involution
R_5	$(x+y)^{\smile} = \breve{x} + \breve{y}$	$^{\smile}$ - distributivity
R_6	$(x;y)^{\smile} = \breve{y};\breve{x}$	$^{\smile}$ - involutive distributivity
R_7	$\breve{x};(x;y)^{-} \leq \overline{y}$	Tarski's formulation of De Morgan's Theorem K

Table 1.4 RA axioms

We also define the additional relational constant $0' := \overline{1'}$, called the **diversity** relation. Similarly to the unary $^{-}$, we use $\breve{x}$ and $x^{\smile}$ interchangeably (dissimilarly, we do not use any prefix notation for the operation). In everything to follow, we let unary operations bind closer than binary operations and relational operations bind closer than Boolean operations. For example, we unambiguously read $x;y \cdot z$ as $(x;y) \cdot z$. When $\mathfrak{A} \in \mathsf{RA}$ and $x, y, z \in A$, all of the theorems in Table 1.5 hold. For reference, we have included theorem numbers from [Mad06] where proofs of each can be found.

$x \leq y \Leftrightarrow \breve{x} \leq \breve{y}$	*240*	$x;(y+z) = x;y + x;z$	*261*
$\breve{0} = 0$	*242*	$x \leq y \Rightarrow z;x \leq z;y$	*262*
$\breve{1} = 1$	*243*	$\breve{1}\text{'} = 1\text{'}$	*269*
$\breve{\overline{x}} = \overline{\breve{x}}$	*246*	$1\text{'};x = x = x;1\text{'}$	*270*
$(x \cdot y)^{\smile} = \breve{x} \cdot \breve{y}$	*247*	$\breve{0}\text{'} = 0\text{'}$	*272*
$(x+y)^{\smile} = \breve{x} + \breve{y}$	*249*	$x;0 = 0 = 0;x$	*287*
$x \leq y \Rightarrow x;z \leq y;z$	*253*	$1;x = x \Rightarrow 1;\overline{x} = \overline{x}$	*305(ii)*

Table 1.5 Important RA theorems

Definition 13 When U is any set and $R, S \subseteq U \times U$, we let:

$$R \cup S \coloneqq \{\langle x, y\rangle \ :\ \langle x, y\rangle \in R \text{ or } \langle x, y\rangle \in S\}, \qquad R|S \coloneqq \{\langle x, y\rangle \ :\ \exists_z[\langle x, z\rangle \in R \text{ and } \langle z, y\rangle \in S]\},$$

$$\sim\! R \coloneqq \{\langle x, y\rangle \ :\ \langle x, y\rangle \notin R\}, \qquad R^{-1} \coloneqq \{\langle x, y\rangle \ :\ \langle y, x\rangle \in R\},$$

$$\mathrm{Re}U \coloneqq \mathrm{Sb}(U \times U), \qquad Id_U \coloneqq \{\langle x, x\rangle \ :\ \langle x, x\rangle \in 1\},$$

$$\mathfrak{Re}U \coloneqq \left\langle \mathrm{Re}(U), \cup, \sim, |, ^{-1}, Id_U \right\rangle.$$

We call $\mathfrak{Re}U$ the **full relation algebra on** U and any subalgebra of $\mathfrak{Re}U$ is called a **proper relation algebra**. We denote the class of all full relation algebras by Rs. A **representable relation algebra** is an algebra which is isomorphic to a subdirect product of algebras of the form $\mathfrak{Re}U$. The class of representable relation algebras is denoted RRA. [Mad06, 6.0.3] □

Every representable relation algebra is indeed a relation algebra [Mad06, *229*]. The containment $\mathsf{RRA} \subset \mathsf{RA}$ was first shown to be proper by Roger Lyndon [Lyn50]. Lyndon's non-representable relation algebra is discussed in [Mad06, p. 358-62] (along with several other constructions). It turns out that RRA is a variety, but is not finitely based. Tarski first showed that RRA is closed under homomorphic images in [Tar55], thereby establishing that RRA is an equational class via Birkhoff's theorem (this is also proved as [Mad06, *120*]). J. Donald Monk first showed that RRA has no finite axiomatization in [Mon64] (he also gave an infinite axiomatization in [Mon69], where he also established that RCA_α is not finitely axiomatizable for $3 \leq \alpha \in \omega$).

The provable sentences of $\mathcal{L}^\times$ correspond to those equations which are true in the free relation algebra on one generator. The existence of non-representable relation algebras which are generated by a single element establishes that $\mathcal{L}^\times$ is incomplete. Maddux has given a construction of an infinite sequence of one-generated non-representable relation algebras with a representable ultraproduct [Mad89b]. This shows that $\mathcal{L}^\times$ is essentially incomplete in the sense that the addition of finitely many logical axiom schemes will not suffice for completion of the provability relation. Despite this inherit deficiency of RA, we *can* impose some simple (though obviously not equational) conditions on a relation algebra which are sufficient to guarantee representability.

Definition 14 A **quasi-projectional relation algebra** is an algebra $\mathfrak{A} \in \mathsf{RA}$ which has a pair of **(conjugated) quasi-projections** $p_0, p_1 \in A$ satisfying the following equations.

$$\breve{p}_0 ; p_0 + \breve{p}_1 ; p_1 \leq 1' \tag{1.1}$$

$$\breve{p}_0 ; p_1 = 1 \tag{1.2}$$

The class of all quasi-projectional relation algebras is denoted by QRA. □

Theorem 15 *If* $\mathfrak{A} \in \mathsf{QRA}$ *then* $\mathfrak{A}$ *is representable. Equivalently,* $\mathsf{QRA} \subseteq \mathsf{RRA}$. □

The above is famously known as Tarski's QRA theorem. It was first announced in the abstract [Tar53b, VII], and it appears as Theorem 8.4(iii) in [TG87] with a metamathematical proof. In [Mad78b] Maddux generalized the theorem to the broader class of relation algebras satisfying:

$$\sum_{p,q \in \mathrm{Fn}\, A} \breve{p} ; q = 1,$$

where $\mathrm{Fn}\, A$ is defined to be the set of all $p \in A$ such that $\breve{p};p \leq 1'$ (the **functional** elements). Algebras satisfying this condition are called **tabular** relation algebras. It is quite easy to see that every QRA is tabular since the quasi-projections p_0 and p_1 are functional by (1.1). Maddux's purely algebraic proof of this generalization (and respectively the QRA corollary) can also be found as [Mad78a, Theorem 7, Corollary 8] and [Mad06, *423*, *427*].

Condition (1.2) of Definition 14 asserts that for each x, y there is some z such that $p_0 z = x$ and $p_1 z = y$, and thus p_0 and p_1 act as projection functions on an abstract pairing. In case $\mathfrak{A} = \mathfrak{Re} U$, this "pairing" consists of a one-to-one correspondence between $U \times U$ and U. The existence of such a correspondence implies that the cardinality of U is either 0 or infinite. Therein lies the reason that the formalization of [TG87] is restricted to set theory. Since RRA corresponds to the models of $\mathcal{L}^\times$ and every nontrivial $\mathfrak{A} \in \mathsf{QRA} \subset \mathsf{RRA}$ is infinite, the class QRA is naturally suited for those theories whose models are infinite. However, as Tarski was aware, Maddux later rediscovered, and we shall soon see; QRA is not *exclusively* suited for such theories.

A slight idiosyncrasy associated with QRA is that the quasi-projections are not included in the similarity type of the algebra, i.e. any given QRA has multiple choices of conjugated quasi-projections. It is easy to see, for example, that interchanging the roles of p_0 and p_1 yields another pair of quasi-projections. The reason for this convention is that the inclusion of the quasi-projections would mean that the natural representation of a QRA would be a TPA (see Definition 19 below) instead of an RRA. This would be especially disadvantageous from a general algebra perspective in light of the fact that **HSP**TPA is not a axiomatizable by any decidable system of equations [MSS92, Theorem 2].

We can actually use this minor quirk to our advantage by strengthening the conditions on quasi-projections in various ways. It is not generally necessary to do this (c.f. Remark 33), but we find it more convenient and as long as we keep the number of assumptions finite, we find that it creates no difficulties (c.f. commentary following Theorem 31). Our first such assumption (which can be made without loss of generality) is that the domains of p_0 and p_1 coincide (see e.g. [Sim07, Lemma 2.6] for proof).

Proposition 16 *Let $\mathfrak{A} \in$ QRA with conjugated quasi-projections p_0 and p_1. Then $p_0' = p_0 \cdot p_1 ; 1$ and $p_1' = p_1 \cdot p_0 ; 1$ also satisfy (1.1)-(1.2), and additionally satisfy:*

$$p_0' ; 1 = p_1' ; 1. \tag{1.3}$$

In other words, p_0' and p_1' are conjugated quasi-projections which have a common domain. □

Next we introduce the assumption that "pairs" formed by p_0 and p_1 are unique. This becomes particularly useful in proving that the Németi reduct of a CA_α or Df_α obeys the identity law. In contrast to (1.3), we cannot assume that *every* QRA has such quasi-projections, and so are forced to define a proper subclass.

Definition 17 $\mathsf{Q^+RA}$ is the subclass of QRA where the conjugated quasi-projections p_0 and p_1 additionally satisfy the following equation.

$$(p_0 ; \breve{p}_0) \cdot (p_1 ; \breve{p}_1) \leq 1' \tag{1.4}$$

In words, the pairs formed by p_0 and p_1 are unique. □

If p_0 and p_1 satisfy (1.4), then p_0' and p_1' from Proposition 16 must as well. This follows immediately from the monotonicity of conversion and relative product [Mad06, *240*, *253*, *262*]. In view of this we can equivalently define $\mathsf{Q^+RA}$ as the class of all relation algebras having elements p_0 and p_1 satisfying (1.1)-(1.4); and similarly QRA satisfying (1.1)-(1.3). We note that it can also be assumed without loss of generality that the functions p_0 and p_1 are entire by simply defining $p_i(x) = x$ for each x outside the domain of p_i. However, from our point of view this would be quite disadvantageous (c.f. Proposition 20 and the commentary following).

We have not yet addressed the question of whether or not QRA and $\mathsf{Q^+RA}$ are varieties. It turns out that neither are closed under subalgebras, for whenever U is infinite, $\mathfrak{Re}U \in \mathsf{Q^+RA}$, but the subalgebra of $\mathfrak{Re}U$ with universe $\{0, 1, 0\text{'}, 1\text{'}\}$ is clearly not a QRA. It is straightforward to see that QRA and $\mathsf{Q^+RA}$ (indeed, every class defined by extending the conditions on quasi-projections by equations without variables) are closed under $\mathbf{H}$ and $\mathbf{P}$. By a result of Maddux the operators $\mathbf{H}$ and $\mathbf{S}$ commute on RA and classes contained in RA, so $\mathbf{S}\mathsf{QRA}$ and $\mathbf{S}\mathsf{Q^+RA}$ can be seen to be varieties [Mad78a, Lemma 10] (also found as [Mad06, *112*]). Partially because of this observation, but more so because we wish to be able to discuss $\mathsf{Q^+RA}$ equations which involve the projection functions, we have the following definition.

Definition 18 An algebra $\mathfrak{A} = \langle A, +, ^-, ;, ^\smile, 1\text{'}, p_0, p_1\rangle$ is called a **pairing relation algebra** whenever $\langle A, +, ^-, ;, ^\smile, 1\text{'}\rangle \in \mathsf{RA}$ and p_0, p_1 satisfy Equations (1.1)-(1.3). We call the class of all such algebras PRA. The class $\mathsf{P^+RA}$ consists of those PRA's which also satisfy (1.4). $\square$

Clearly every QRA can be extended to a PRA and every PRA has a QRA reduct; and similarly for $\mathsf{Q^+RA}$ and $\mathsf{P^+RA}$. Since PRA and $\mathsf{P^+RA}$ are equationally defined, they form varieties by Birkhoff's theorem. The difference here is that since the quasi-projections are included in the similarity type, they must be preserved by subalgebras. Thus in the counterexample we gave previously, $\{0, 1, 0\text{'}, 1\text{'}\}$ fails to be the universe of a subalgebra.

Definition 19 Let U be any set.

1. $\ddot{U} = \bigcap\{W : U \cup (W \times W) \subset W\}$
2. $\mathsf{p}_0^U = \{\langle\langle x, y\rangle, x\rangle \ : x, y \in \ddot{U}\}$
3. $\mathsf{p}_1^U = \{\langle\langle x, y\rangle, y\rangle \ : x, y \in \ddot{U}\}$
4. $\mathfrak{B}(U) = \langle \mathrm{Re}\ddot{U}, \cup, \sim, \mid, ^{-1}, Id_{\ddot{U}}, \mathsf{p}_0^U, \mathsf{p}_1^U\rangle$

$\mathfrak{B}(U)$ is the **true pairing algebra** on U. The class of true pairing algebras is denoted $\mathsf{TPA} := \{\mathfrak{B}(U) : \ddot{U} \times \ddot{U} = \ddot{U} \setminus U \neq \emptyset\}$. $\square$

In the previous definition (taken from [Mad89a, Definitions 18, 20]), $\ddot{U}$ is seen to be the closure of U under ordered pairs. The requirement $\ddot{U} \times \ddot{U} = \ddot{U} \setminus U$ for membership of $\mathfrak{B}(U)$ in TPA ensures that U is free of ordered pairs of its own elements. Whenever U is nonempty, $\ddot{U}$ is necessarily infinite. Next we shall see that U (Id_U to be precise) can be expressed in terms of the projection functions in TPA. This is the key to expressing theories with finite models. Even though $\ddot{U}$ is infinite (as it must be, since every TPA has a natural $\mathsf{Q^+RA}$ reduct), we can refer back to the original set U which does not have such a restriction. For more results concerning TPA, see [Sai00, MSS92].

Proposition 20 *If* $\mathfrak{B}(U) \in \mathsf{TPA}$ *then* $\mathsf{u} := 1\text{'} \cdot (p_0 ; 1 \cdot p_1 ; 1)^-$ *denotes* Id_U. $\square$

Proof By definition, $\mathsf{p}_k^U;1 = \left\{\langle\langle x,y\rangle, z\rangle : x,y,z \in \ddot{U}\right\}$. Since $\mathfrak{B}(U) \in \mathsf{TPA}$ we have $\ddot{U} \times \ddot{U} = \ddot{U} \setminus U$, so we may write $\mathsf{p}_k^U;1 = \left\{\langle x,y\rangle \in \ddot{U} \times \ddot{U} : x \notin U\right\}$. Then the interpretation of u is seen to be $Id_{\ddot{U}} \cap \left\{\langle x,y\rangle \in \ddot{U} \times \ddot{U} : x \in U\right\} = Id_U$. ■

Remark 21 In [Mad89a], Maddux originally used the simpler condition $U \cap (U \times U) = \emptyset$ to define TPA. This does guarantee that U is free of ordered pairs of its own elements, however it was pointed out in [Mad93] that this condition is insufficient for our purposes. In particular, Proposition 20 fails when $U = \{x, \langle\langle x,x\rangle, x\rangle\}$ because while $U \cap (U \times U) = \emptyset$, the interpretation of u becomes $\{\langle x,x\rangle\} \neq Id_U$. □

We conclude this section with some useful general algebraic results concerning RA. That each of the next three theorems has a counterpart for (diagonal-free) cylindric algebras (Theorems 29-31) further illustrates the connections between these varieties. Theorems 22-23 are found as Theorems 4.10,11,15 of [JT52]. Theorem 24 is a corollary which was originally observed by Ernst Schröder in 1895 [Mad91, Theorem 24].

Theorem 22 *Every* RA *is semisimple.* □

Theorem 23 *For every* $\mathfrak{A} \in \mathsf{RA}$*, the following are equivalent:*

(i) $\mathfrak{A}$ *is simple,*

(ii) every subalgebra of $\mathfrak{A}$ *is simple,*

(iii) for every $x \in A$*,* $x \neq 0$ *iff* $1;x;1 = 1$. □

Theorem 24 *For every Boolean combination* φ *of relation algebraic equations, there is a correlated term* φ^* *such that* φ *and* $\varphi^* \doteq 0$ *are equivalent in every simple* RA*. The mapping* * *is defined thusly:*

(i) $(t_0 \doteq t_1)^* = t_0 \oplus t_1$,

(ii) $(\varphi \wedge \psi)^* = \varphi^* + \psi^*$,

(iii) $(\neg\varphi)^* = (1;\varphi^*;1)^-$. □

In order to apply Theorems 23 and 24 to PRA and $\mathsf{P^+RA}$, we require the following lemma.

Lemma 25 *Every* PRA *is semisimple.* □

Proof Let $\mathfrak{A} \in \mathsf{PRA}$ and $\mathfrak{A}'$ be the RA reduct of $\mathfrak{A}$. Clearly since the universes of $\mathfrak{A}$ and $\mathfrak{A}'$ coincide, any homomorphism on $\mathfrak{A}$ is also a homomorphism on $\mathfrak{A}'$ and vice versa. Thus $\mathfrak{A}$ is simple iff $\mathfrak{A}'$ is simple. We have that $\mathfrak{A}'$ is simple iff $\mathfrak{A}'$ is subdirectly indecomposable iff $\mathfrak{A}'$ is directly indecomposable by [TG87, Theorem 8.2(v)]. We show that if $\mathfrak{A}'$ is directly decomposable, then so is $\mathfrak{A}$. Then if $\mathfrak{A}$ is subdirectly indecomposable (hence also directly), $\mathfrak{A}'$ must be as well so that $\mathfrak{A}'$ is simple and we may conclude that $\mathfrak{A}$ is simple. This is sufficient because every general algebra can be written as a subdirect product of subdirectly indecomposable algebras by [Bir44].

We use p and q instead of p_0 and p_1 to avoid confusion with subscripts. Suppose $\mathfrak{A}' \cong \mathfrak{B} \times \mathfrak{C}$ by a homomorphism h. Let $h(p) = \langle p_B, p_C \rangle$ and $h(q) = \langle q_B, q_C \rangle$. We claim that $\mathfrak{A} \cong \langle \mathfrak{B}, p_B, q_B \rangle \times \langle \mathfrak{C}, p_C, q_C \rangle$ and that $\langle \mathfrak{B}, p_B, q_B \rangle, \langle \mathfrak{C}, p_C, q_C \rangle \in \mathsf{PRA}$. Again, since the universes of $\mathfrak{A}$ and $\mathfrak{A}'$ are identical, h is an isomorphism satisfying the first condition. By symmetry we only need show that $\langle \mathfrak{B}, p_B, q_B \rangle \in \mathsf{PRA}$. First, note that since $\breve{p} ; q = 1$ in $\mathfrak{A}'$ we have $h(p)^{\smile} ; h(q) = \langle 1_B, 1_C \rangle$, so:

$$\begin{aligned} \langle 1_B, 1_C \rangle &= \langle p_B, p_C \rangle^{\smile} ; \langle q_B, q_C \rangle \\ &= \langle \breve{p}_B, \breve{p}_C \rangle ; \langle q_B, q_C \rangle \\ &= \langle (\breve{p}_B ; q_B), (\breve{p}_C ; q_C) \rangle . \end{aligned}$$

Thus $\breve{p}_B ; q_B = 1_B$ as required. Similarly $h(p)^{\smile} ; h(p) = \langle 1\text{'}_B, 1\text{'}_C \rangle$, so:

$$\begin{aligned} \langle 1\text{'}_B, 1\text{'}_C \rangle &= \langle p_B, p_C \rangle^{\smile} ; \langle p_B, p_C \rangle \\ &= \langle \breve{p}_B, \breve{p}_C \rangle ; \langle p_B, p_C \rangle \\ &= \langle (\breve{p}_B ; p_B), (\breve{p}_C ; p_C) \rangle . \end{aligned}$$

Thus $\breve{p}_B ; p_B = 1\text{'}_B$ as required. A symmetric argument establishes the same for q_B so $\mathfrak{B} \in \mathsf{QRA}$ implying $\langle \mathfrak{B}, p_B, q_B \rangle \in \mathsf{PRA}$. In fact, if p and q satisfy the uniqueness of pairs (1.4), then p_B and q_B must as well. We conclude that every $\mathfrak{A} \in \mathsf{PRA}$ is isomorphic to a subdirect product of subdirectly indecomposable (hence simple) algebras in PRA, as desired. ■

1.5 Cylindric algebras

We now turn our attention to the classes of cylindric and diagonal-free cylindric algebras. Included throughout this section in `typewriter font`, are references to [HMT71, HMT85][3] where definitions and proofs of theorems can be found. We maintain this practice for annotating proofs in the chapters to come.

Definition 26 (`1.1.1, 1.1.2`) Let α be any ordinal.

[3] The chapters in [HMT71] and [HMT85] are respectively numbered 0-2 and 3-5, so we do not risk ambiguity with this referencing system.

1. A **cylindric algebra of dimension** α is any algebra,

$$\mathfrak{A} = \left\langle A, +, ^{-}, \mathsf{c}_\kappa, \mathsf{d}_{\kappa\lambda} \right\rangle_{\kappa,\lambda\in\alpha}$$

with similarity type $\langle 2, 1, \mathbf{1}, \mathbf{0} \rangle$ which satisfies axioms C_0-C_7 in Table 1.6.

2. A **diagonal-free cylindric algebra of dimension** α is any algebra,

$$\mathfrak{A} = \left\langle A, +, ^{-}, \mathsf{c}_\kappa \right\rangle_{\kappa\in\alpha}$$

with similarity type $\langle 2, 1, \mathbf{1} \rangle$ which satisfies axioms C_0-C_4 in Table 1.6.

CA_α and Df_α respectively denote the classes of all such algebras. CA and Df are the classes of cylindric and diagonal-free cylindric algebras of arbitrary dimension. The unary operations c_κ are called **cylindrifications** and the distinguished elements $\mathsf{d}_{\kappa\lambda}$ are called **diagonal elements**. □

C_0	$\langle A, +, ^{-} \rangle \in \mathsf{BA}$	
C_1	$\mathsf{c}_\kappa 0 = 0$	
C_2	$x \leq \mathsf{c}_\kappa x$	
C_3	$\mathsf{c}_\kappa (x \cdot \mathsf{c}_\kappa y) = \mathsf{c}_\kappa x \cdot \mathsf{c}_\kappa y$	
C_4	$\mathsf{c}_\kappa \mathsf{c}_\lambda x = \mathsf{c}_\lambda \mathsf{c}_\kappa x$	
C_5	$\mathsf{d}_{\kappa\kappa} = 1$	
C_6	$\mathsf{d}_{\lambda\mu} = \mathsf{c}_\kappa (\mathsf{d}_{\lambda\kappa} \cdot \mathsf{d}_{\kappa\mu})$	$\kappa \neq \lambda, \mu$
C_7	$\mathsf{c}_\kappa (\mathsf{d}_{\kappa\lambda} \cdot x) \cdot \mathsf{c}_\kappa (\mathsf{d}_{\kappa\lambda} \cdot \overline{x}) = 0$	$\kappa \neq \lambda$

Table 1.6 CA_α and Df_α axioms

Although C_0 and C_2 are not equations, C_0 could be replaced by B_1-B_3 and C_2 abbreviates $x + \mathsf{c}_\kappa x = \mathsf{c}_\kappa x$ by definition. Thus Birkhoff's theorem applies to CA_α and Df_α, so that both can be seen to be varieties. If $\mathfrak{A} = \langle A, +, ^{-}, \mathsf{c}_\kappa, \mathsf{d}_{\kappa\lambda} \rangle_{\kappa,\lambda\in\alpha} \in \mathsf{CA}_\alpha$ then clearly

$$\mathfrak{Df}\mathfrak{A} := \left\langle A, +, ^{-}, \mathsf{c}_\kappa \right\rangle_{\kappa\in\alpha} \in \mathsf{Df}_\alpha$$

[HMT71, `1.1.3`]. One might ask whether *every* $\mathfrak{A} \in \mathsf{Df}_\alpha$ is equal to (or at least isomorphic to) $\mathfrak{Df}\mathfrak{B}$ for some $\mathfrak{B} \in \mathsf{CA}_\alpha$, and whether $\mathfrak{Df}\mathfrak{B} \cong \mathfrak{Df}\mathfrak{C}$ implies $\mathfrak{B} \cong \mathfrak{C}$ (i.e. whether the function $\mathfrak{Df}$ is surjective and injective, respectively). Except in the somewhat trivial case $\alpha \leq 1$, the answer to both is negative [HMT85, `5.1.4`, `5.1.9`]. Moreover, when $\alpha \geq 4$ there is an equation not involving diagonal elements which hold in every CA_α but not in every Df_α [HMT85, `5.1.6`] (originally appearing with a metalogical formulation in [Hen67]). Whether or not there is such an equation for $\alpha = 3$ is an open question. Next we turn to defining the notion of representability for cylindric algebras.

Definition 27 (`1.1.5, 5.1.33`) The **full cylindric set algebra of dimension** α **with base** U, in symbols $\mathfrak{Cs}_\alpha U$, is the algebra $\langle \mathrm{Sb}\,{}^\alpha U, \cup, \sim, \mathbf{c}_\kappa, \mathbf{d}_{\kappa\lambda}\rangle_{\kappa\in\alpha}$ where U is any set and:

$$\mathrm{Sb}\,{}^\alpha U := \{X : X \subset {}^\alpha U\},$$

$$\mathbf{c}_\kappa X := \{\mathbf{y} \in {}^\alpha U : \exists \mathbf{x} \in X\, [x_\lambda = y_\lambda \text{ for each } \lambda \neq \kappa]\},$$

$$\mathbf{d}_{\kappa\lambda} := \{\mathbf{x} \in {}^\alpha U : x_\kappa = x_\lambda\},$$

and $\cup$ and $\sim$ are union and complementation with respect to ${}^\alpha U$. The class of all such algebras is Cs. An algebra $\mathfrak{A}$ is called a **representable cylindric algebra of dimension** α if it is isomorphic to a subdirect product of algebras in Cs. RCA_α denotes the class of all representable cylindric algebras of dimension α. A **representable diagonal-free cylindric algebra** (and the corresponding class RDf_α) are defined analogously by omitting any reference to $\mathbf{d}_{\kappa\lambda}$.[4] □

As we would expect, for any set U we have that $\mathfrak{Cs}_\alpha U \in \mathsf{CA}_\alpha$ [HMT71, `1.1.6`]. Since CA_α is a variety, this also implies that $\mathsf{RCA}_\alpha \subseteq \mathsf{CA}_\alpha$; and similarly $\mathsf{RDf}_\alpha \subseteq \mathsf{Df}_\alpha$. The proper containment of representable algebras greatly on the value of α. We briefly summarize results concerning the different cases—what follows reflects comments from [HMT71, p. 171-3] as well as items `3.1.108`, `3.2.65`, `4.1.3-7`, and `5.1.57,62,63` from [HMT71, HMT85].[5] Firstly, for every α all four classes are axiomatizable. When $\alpha \geq \omega$, none of the four are finitely axiomatizable. When $\alpha \leq 1$ all four varieties coincide, and when $\alpha = 0$ they coincide with BA. When $\alpha = 2$, $\mathsf{RDf}_\alpha = \mathsf{Df}_\alpha \neq \mathsf{CA}_\alpha \neq \mathsf{RCA}_\alpha$, and RCA_α is finitely axiomatizable. When $3 \leq \alpha < \omega$, neither of RCA_α and RDf_α are finitely axiomatizable, while CA_α and Df_α are so by definition.

Theorem 28 (`4.3.57, 4.3.59`) *Let L_α be a full language and $f : \mathrm{Tm}[\mathsf{CA}_\alpha] \to \Phi[L_\alpha]$ be recursively defined as follows, where $s, t \in \mathrm{Tm}[\mathsf{CA}_\alpha]$ are arbitrary:*

$$f(\mathsf{w}_\kappa) := \mathsf{R}_\kappa \mathsf{v}_0 \ldots \mathsf{v}_{\alpha-1}, \qquad f(\mathsf{d}_{\kappa\lambda}) := \mathsf{v}_\kappa = \mathsf{v}_\lambda, \qquad f(s+t) := f(s) \vee f(t),$$

$$f(\mathsf{c}_\kappa s) := \exists_{\mathsf{v}_\kappa} f(s), \qquad f(\overline{s}) := \neg f(s).$$

Then for any $s, t \in \mathrm{Tm}[\mathsf{CA}_\alpha]$ we have:

$$\models f(s) \leftrightarrow f(t) \quad \Leftrightarrow \quad \mathsf{RCA}_\alpha \models s \doteq t,$$

$$\vdash_{\overline{r}} f(s) \leftrightarrow f(t) \quad \Leftrightarrow \quad \mathsf{CA}_\alpha \models s \doteq t.$$

[4] Note: The classes RCA_α and RDf_α are called IGs_α and IGsdf_α respectively in [HMT71, HMT85]. Our definition is closer in character to the description on p. 172 of [HMT71]. Definition `5.1.33` is more precise (and consequently more complex) than necessary for our discussion.

[5] Especially important here, is Theorem `4.1.3` which establishes the nonfinite axiomatizability of RCA_α for $3 \leq \alpha \in \omega$. This originally appeared in [Mon69] and the corresponding proof for RDf_α (`5.1.57`) is analogous.

If $s, t \in \mathrm{Tm}[\mathsf{Df}_\alpha] \subset \mathrm{Tm}[\mathsf{CA}_\alpha]$ *then* $f(s), f(t) \in \Phi[L_\alpha^{\neq}]$ *and:*

$$\models f(s) \leftrightarrow f(t) \qquad \Leftrightarrow \qquad \mathsf{RDf}_\alpha \models s \doteq t,$$

$$\models_{r,e} f(s) \leftrightarrow f(t) \qquad \Leftrightarrow \qquad \mathsf{Df}_\alpha \models s \doteq t. \qquad \square$$

Theorem 28 provides a good indication[6] of the relationship between the algebras CA_α and Df_α, and the languages L_α and $L_\alpha^{\neq}$. In light of the preceding paragraph, it also establishes the incompleteness of the latter because for $\alpha \geq 3$ there are equations holding in the representable algebras which don't hold in their respective abstractions. From now on we shall concern ourselves primarily with Df_3. We conclude this section with the promised cylindric counterparts to Theorems 22-24. The first of these is due to Don Pigozzi.

Theorem 29 (2.4.53) *Every* $\mathfrak{A} \in \mathsf{CA}_\alpha \cup \mathsf{Df}_\alpha$ *is semisimple.* $\square$

Theorem 30 (2.3.14,16) *For every* $\mathfrak{A} \in \mathsf{CA}_\alpha \cup \mathsf{Df}_\alpha$*, the following are equivalent:*

(i) $\mathfrak{A}$ *is simple,*

(ii) every subalgebra of $\mathfrak{A}$ *is simple,*

(iii) for every $x \in A$, $x \neq 0$ *iff* $\mathsf{c}_{(\alpha)}x = 1$. $\square$

As they appear in [HMT71], the above are actually a theorems about CA_α only, but as noted in [HMT85, p. 187-8] the proofs carry through for Df_α without changes. As a corollary of Theorem 30, we obtain the following marvelous result.

Theorem 31 *For every Boolean combination* φ *of (diagonal-free) cylindric algebraic equations, there is a correlated term* φ^* *such that* φ *and* $\varphi^* \doteq 0$ *are equivalent in every simple* CA_α *(or* Df_α*). The mapping* * *is defined thusly:*

(i) $(t_0 \doteq t_1)^* = t_0 \oplus t_1$,

(ii) $(\varphi \wedge \psi)^* = \varphi^* + \psi^*$,

(iii) $(\neg\varphi)^* = -\mathsf{c}_{(\alpha)}\varphi^*$. $\square$

Theorem 29 allows us to work exclusively with simple algebras (since they generate their parent varieties, and **HSP** preserves equational satisfaction). Theorems 30-31 then allow us to code finitely many assumptions into a single equation. This is exemplified in Proposition 55 below.

[6] For a thorough and formal exposition of the connections between these formalisms and their corresponding algebras, we refer the reader to [HMT85, §4.3 and p. 203]

CHAPTER 2. RELATION ALGEBRAIC REDUCTS

2.1 Definitions

Throughout this chapter we fix $\mathfrak{A} = \langle A, +, ^-, \mathsf{c}_\kappa \rangle_{\kappa \in 3} \in \mathsf{Df}_3$ and $e, q \in A$. The parameter e simulates $\{\mathbf{x} : x_0 = x_1 = x_2\}$ and q simulates $\{\mathbf{x} : P_0(x_0) = x_1, P_1(x_0) = x_2\}$ where P_0 and P_1 are some conjugated quasi-projections. Occasionally, we shall informally refer to these "functions" in order to clarify certain concepts and definitions, but we emphasize that they *need not actually exist*. We frequently use finite sequences of 0's and 1's and let S_n be the set of such sequences having length less than $n+1$ and $2^* = \cup_{n \in \omega} S_n$. Concatenation of sequences shall be denoted by juxtaposition. All definitions pertinent to this chapter follow immediately. We recommend that the reader bookmark this page.

Definition 32 For $\{\kappa, \lambda, \mu\} = 3$ define the **diagonal elements**:

$$\mathsf{d}_{\kappa\lambda} := \mathsf{c}_\mu e$$

$$\mathsf{d}_{\kappa\kappa} := \mathsf{c}_\lambda \mathsf{c}_\mu e.$$

For $\kappa, \lambda \in 3$ and $x \in A$ define the **λ-for-κ substitution** [HMT71, `1.5.1`]:

$$\mathsf{s}^\kappa_\lambda x := \begin{cases} x & \kappa = \lambda \\ \mathsf{c}_\kappa \left(\mathsf{d}_{\kappa\lambda} \cdot x\right) & \kappa \neq \lambda. \end{cases}$$

Define the **quasi-projections** to be $\mathsf{p}_0 = \mathsf{c}_2 q$ and $\mathsf{p}_1 = \mathsf{s}^2_1 \mathsf{c}_1 q$, and for $n \in 2$, let

$$\mathsf{p}_n(\kappa, \lambda) := \begin{cases} \mathsf{s}^0_\kappa \mathsf{s}^1_\lambda \mathsf{p}_n & \kappa \leq \lambda \\ \mathsf{s}^2_\kappa \mathsf{s}^1_\lambda \mathsf{s}^0_2 \mathsf{p}_n & \kappa > \lambda. \end{cases}$$

Let $\{\kappa, \lambda, \mu\} = 3$; $i, j \in 2^*$; $n \in 2$; and $\kappa' = \kappa +_3 1$ (where $+_3$ denotes addition modulo 3). Then the **projectional diagonal elements** are defined as follows.

$$\mathsf{d}_{\kappa\langle\rangle\lambda\langle\rangle} := \mathsf{d}_{\kappa\lambda} \tag{2.1}$$

$$\mathsf{d}_{\kappa\langle n\rangle\lambda\langle\rangle} := \mathsf{d}_{\lambda\langle\rangle\kappa\langle n\rangle} := \mathsf{p}_n(\kappa, \lambda) \tag{2.2}$$

$$\mathsf{d}_{\kappa\langle in\rangle\lambda\langle\rangle} := \mathsf{d}_{\lambda\langle\rangle\kappa\langle in\rangle} := \mathsf{c}_\mu(\mathsf{d}_{\kappa\langle i\rangle\mu\langle\rangle} \cdot \mathsf{d}_{\mu\langle n\rangle\lambda\langle\rangle}) \qquad |i| \neq 0 \tag{2.3}$$

$$\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle} := \mathsf{c}_\mu(\mathsf{d}_{\mu\langle\rangle\kappa\langle i\rangle} \cdot \mathsf{d}_{\mu\langle\rangle\lambda\langle j\rangle}) \qquad |i| \neq 0 \neq |j| \tag{2.4}$$

$$\mathsf{d}_{\kappa\langle i\rangle\kappa\langle j\rangle} := \mathsf{c}_{\kappa'}(\mathsf{d}_{\kappa'\langle\rangle\kappa\langle i\rangle} \cdot \mathsf{d}_{\kappa'\langle\rangle\kappa\langle j\rangle}). \tag{2.5}$$

Let Ax to be the union of the following sets of equations.

$$\mathrm{Ax}_0 := \{\mathsf{d}_{\kappa\kappa} = 1 : \kappa \in 3\}$$

$$\mathrm{Ax}_1 := \left\{\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle} \cdot \mathsf{d}_{\lambda\langle jl\rangle\mu\langle k\rangle} \leq \mathsf{d}_{\kappa\langle il\rangle\mu\langle k\rangle} : \kappa, \lambda, \mu \in 3; i, jl, kl \in S_3\right\}$$

$$\mathrm{Ax}_2 := \left\{\mathsf{d}_{\kappa\langle i\rangle\kappa\langle i\rangle} \cdot \mathsf{d}_{\lambda\langle j\rangle\lambda\langle j\rangle} \leq \mathsf{c}_\mu\left(\mathsf{d}_{\mu\langle 0\rangle\kappa\langle i\rangle} \cdot \mathsf{d}_{\mu\langle 1\rangle\lambda\langle j\rangle}\right) : \mu \neq \kappa, \lambda \in 3; i, j \in S_3\right\}$$

$$\mathrm{Ax}_3 := \left\{\mathsf{d}_{\kappa\langle i0\rangle\lambda\langle j0\rangle} \cdot \mathsf{d}_{\kappa\langle i1\rangle\lambda\langle j1\rangle} \leq \mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle} : \kappa, \lambda \in 3; i, j \in S_2\right\}$$

For $\kappa, \lambda \in 3$ and $i \in 2^*$, define the **projectional substitution**:

$$\mathsf{s}^\kappa_{\lambda\langle i\rangle} x := \begin{cases} x & \kappa = \lambda, i = \langle\rangle \\ \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\lambda\langle i\rangle} \cdot x\right) & \text{else.} \end{cases}$$

Finally, let $\kappa, \lambda, \mu \in 3$ with $\mu \neq \kappa$.

$$\delta_{\kappa\langle i\rangle\lambda\langle j\rangle} := \mathsf{d}_{\kappa\langle i0\rangle\lambda\langle j00\rangle} \cdot \mathsf{d}_{\kappa\langle i1\rangle\lambda\langle j11\rangle} \cdot \mathsf{d}_{\lambda\langle j01\rangle\lambda\langle j10\rangle} \qquad \epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle} := \mathsf{d}_{\kappa\langle i0\rangle\lambda\langle j1\rangle} \cdot \mathsf{d}_{\kappa\langle i1\rangle\lambda\langle j0\rangle}$$

$$\dot{1} := \mathsf{d}_{0\langle 0\rangle 0\langle 0\rangle} \qquad \dot{-}x := \dot{1} \cdot \overline{x}$$

$$J_\kappa := \left\{x \in \mathrm{Nr}_{\{\kappa\}}A : \mathsf{s}^\mu_\kappa \mathsf{s}^\kappa_\mu x = x\right\} \qquad J := \left\{x \cdot \dot{1} : x \in J_0\right\}$$

$$x \circ y := \mathsf{c}_1\left(\mathsf{s}^0_{1\langle 0\rangle} x \cdot \mathsf{s}^0_{1\langle 1\rangle} y \cdot \delta_{01}\right) \qquad \breve{x} := \mathsf{c}_1\left(\mathsf{s}^0_1 x \cdot \epsilon_{01}\right)$$

$$\dot{1}\text{'} := \mathsf{d}_{0\langle 0\rangle 0\langle 1\rangle}$$

We call $\mathfrak{J}\mathfrak{A} := \left\langle J, +, \dot{-}, \circ, \breve{}, \dot{1}\text{'}\right\rangle$ the **Németi reduct** of $\mathfrak{A}$. □

The diagonal element $\mathsf{d}_{\kappa\lambda}$ simulates $\{\mathbf{x} : x_\kappa = x_\lambda\}$. The λ-for-κ substitution "moves" the κ-coordinate of $\mathbf{x}$ to the λ-coordinate and cylindrifies the κ-coordinate. This mimics the process of substituting free occurrences of v_κ with v_λ in L_3.

The quasi-projection p_n simulates $\{\mathbf{x} : P_n(x_0) = x_1\}$. We note that this manner of defining p_0 and p_1 requires that our quasi-projections have the same domain. We use $\mathsf{p}_n(\kappa, \lambda)$ to simulate $\{\mathbf{x} : P_n(x_\kappa) = x_\lambda\}$. We require the rather complicated definition above because $\mathsf{CA}_\alpha \not\models \mathsf{s}^2_0\mathsf{s}^0_1\mathsf{s}^1_2 x = \mathsf{s}^2_1\mathsf{s}^1_0\mathsf{s}^0_2 x$. Thus we must make some choice regarding the order of substitution (our definition is modified from Németi's *substitution convention* [Ném85, p. 23]).

The projectional diagonal elements, $\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}$ are the algebraic counterparts to Németi's L_3 formulas $x_i \doteq y_j$ for $\{x, y\} \subset \{\mathsf{v}_0, \mathsf{v}_1, \mathsf{v}_2\}$ [Ném85, p. 38]. They are intended to simulate the set of sequences $\mathbf{x}$, such that $P_i(x_\kappa) = P_j(x_\lambda)$—where $P_{i_0 \ldots i_n} = P_{i_n} \circ \ldots \circ P_{i_0}$ and $P_{\langle\rangle}$ represents the identity function. For

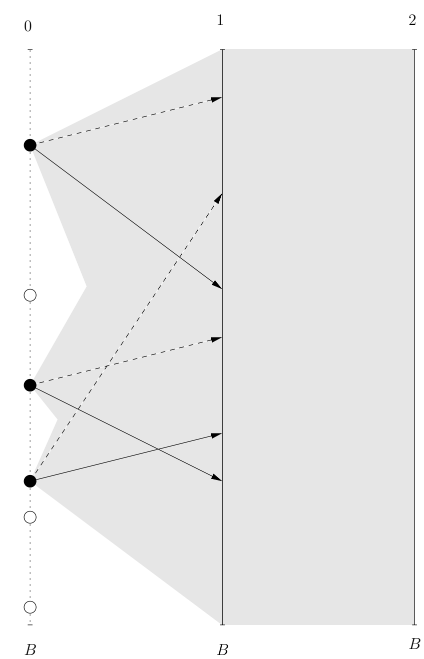

Figure 2.1 An illustration of $\dot{1}$ (more precisely, a subset of $\dot{1}$ in a full cylindric set algebra on B). Vertical lines are copies of B, dotted arrows indicate that the endpoints are the first two coordinates of an element of p_0, and solid arrows indicate that the endpoints are the first two coordinates of an element of p_1. Our $\dot{1}$ consists of all elements whose first coordinate is a closed circle, i.e. all ordered triples enclosed entirely in gray.

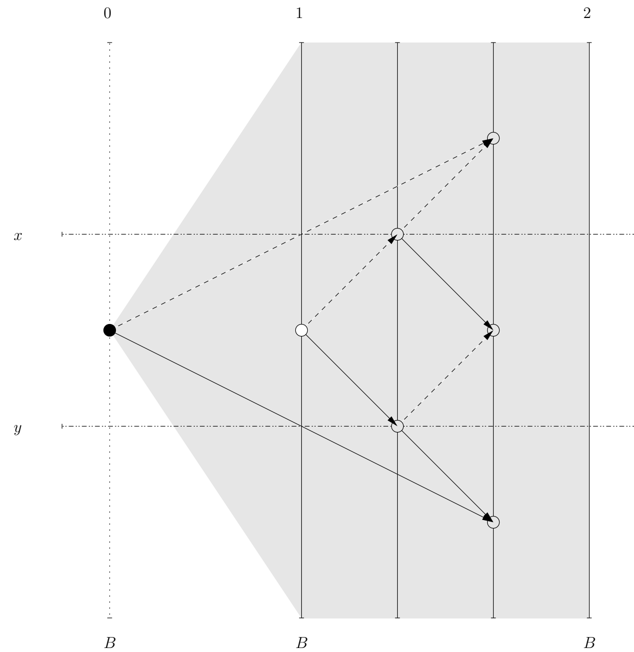

Figure 2.2 An illustration of $x \circ y$

example, the element $\mathsf{d}_{0\langle 1\rangle 1\langle 100\rangle}$ could be thought of as the set $\{\mathbf{x} : P_1(x_0) = P_0P_0P_1(x_1)\}$. We typically write e.g. $\mathsf{d}_{\kappa\langle i\rangle\lambda}$ in place of $\mathsf{d}_{\kappa\langle i\rangle\lambda\langle\rangle}$.

Ax_1 and Ax_2 are modified slightly from [Ném86, p. 42]. Ax_1 says roughly that "equality" is a congruence with respect to the "projection functions". When l is the empty sequence, Ax_1 says that "$P_i(x_\kappa) = P_j(x_\lambda)$ and $P_j(x_\lambda) = P_k(x_\mu)$ imply that $P_i(x_\kappa) = P_k(x_\mu)$." When $l \in S_1$ is nonempty, Ax_1 could be read as "$P_i(x_\kappa) = P_j(x_\lambda)$ and $P_lP_j(x_\lambda) = P_k(x_\mu)$ imply that $P_lP_i(x_\kappa) = P_k(x_\mu)$." Ax_2 expresses that any two generalized projections can be "coded" into a single element. In other words, "the existence of $P_i(x_\kappa)$ and $P_j(x_\lambda)$ imply there is some x_μ such that $P_0(x_\mu) = P_i(x_\kappa)$ and $P_1(x_\mu) = P_j(x_\lambda)$." Finally, Ax_3 expresses that "pairs" are unique—"$P_0P_i(x_\kappa) = P_0P_j(x_\lambda)$ and $P_1P_i(x_\kappa) = P_1P_j(x_\lambda)$ imply $P_i(x_\kappa) = P_j(x_\lambda)$."

The axioms Ax are more powerful than we actually need. If we so desired, it would be a straightforward but time consuming task to read through all of the proofs and record which instances of the axioms we actually use. This would substantially reduce the number of assumptions, but drastically increase the length of their enumeration. Finiteness is required so that images under our translation function from $\Sigma[\mathcal{L}]$ to Df_3 will be finite in length. That Ax be finite (which it clearly is) and true for actual projection functions and equality relations are the only requirements we must satisfy.

Remark 33 As demonstrated in [Ném86] and [AN11], Ax_3 can be eliminated from our set of assumptions. Indeed the only places it will be used are in the proofs of (2.11), (2.17), (2.18), and in showing that our projection functions satisfy the uniqueness of pairs (so that $\mathfrak{J}\mathfrak{A} \in \mathsf{Q^+RA} \subset \mathsf{QRA}$). If we eliminated Ax_3 from Ax and chose $J' := \{x : x \circ 1' = x\}$ as the universe of our reduct, then we would not require Ax_3 in the proof of Theorem 34.

In fact, the only necessary changes to the proof follow: condition I can be shown to be a consequence of Equation (2.8); condition II follows from the observation that $J' \subset J$; condition III would be satisfied trivially; and the proof of condition IV from [Ném85, p. 51-4] could be easily replicated in $\mathfrak{J}\mathfrak{A}$. We could then show that $\mathfrak{J}\mathfrak{A} \in \mathsf{QRA}$ and that for each $\mathfrak{Re}V \in \mathsf{QRA}$ there is some $\mathfrak{B} \in \mathsf{Df3}$ so that the Németi reduct $\mathfrak{J}\mathfrak{B}$ is isomorphic to $\mathfrak{Re}V$. □

Theorem 34 *If e and q satisfy the equations in* Ax, *then $\mathfrak{J}\mathfrak{A} \in \mathsf{RA}$. Moreover, $\mathfrak{J}\mathfrak{A} \in \mathsf{Q^+RA}$ with conjugated quasi-projections $P_0 = \mathsf{d}_{0\langle 00\rangle 0\langle 1\rangle}$ and $P_1 = \mathsf{d}_{0\langle 01\rangle 0\langle 1\rangle}$.* □

We reserve our proof for §2.3 as it is quite long and requires several technical lemmas.

Theorem 35 *Let $\mathfrak{A} = \mathfrak{Re}V \in \mathsf{Q^+RA}$ with quasi-projections functions p_0 and p_1. Then there is some $\mathfrak{B} \in \mathsf{Df}_3$ with elements e and q such that the Németi reduct $\mathfrak{J}\mathfrak{B}$ is isomorphic to $\mathfrak{A}$.* □

PROOF Since p_0 and p_1 are functions (by the representability of $\mathfrak{A}$), we use standard functional notation, writing $p_k(x) = y$ in place of $\langle x, y\rangle \in p_k$. We may assume that p_0 and p_1 have the same domain. Let $\mathfrak{B}'$ be the full cylindric set algebra of dimension 3 on V and $\mathfrak{B}$ be its Df_3 reduct. Choose parameters:

$$e = \left\{\mathbf{x} \in {}^3V : x_0 = x_1 = x_2\right\}$$

and

$$q = \left\{\mathbf{x} \in {}^3V : p_0(x_0) = x_1, p_1(x_0) = x_2\right\}.$$

For $i = i_0 \cdots i_n \in 2^*$ and $x \in V$ we define $p_i(x) = p_{i_n} \cdots p_{i_0}(x)$, so that according to Definition 32 we have:

$$\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle} = \{\mathbf{x} : p_i(x_\kappa) = p_j(x_\lambda)\},$$

$$1 = \{\mathbf{x} : x_0 \in \operatorname{Dom} p_0\},$$

and J is the powerset of 1 (since $\mathfrak{B}' \in \mathsf{RCA}_3 \models \mathsf{s}_0^1\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_2 x = \mathsf{c}_1\mathsf{c}_2 x$).

Let $h\colon J \to A$ be defined by:

$$h(X) := \{\langle p_0(x_0), p_1(x_0)\rangle \ : \mathbf{x} \in X\}.$$

Note that h is well defined since for $\mathbf{x} \in X \in J$, we must also have $\mathbf{x} \in \dot{1}$, so $x_0 \in \operatorname{Dom} p_i$. For any $R \in A$ the element

$$h^{-1}(R) := \left\{\mathbf{x} \in \dot{1} : \langle p_0(x_0), p_1(x_0)\rangle \in R\right\}$$

maps to R under h, so h is onto. We note that $h^{-1}(R)$ is guaranteed to exist since (1.2) and (1.4) guarantee that for each $\langle u, v\rangle \in {}^2V$ there is some unique $x \in V$ such that $p_0(x) = u$ and $p_1(x) = v$. Now suppose that $h(X) = h(Y)$ for some $X, Y \in J$. Then observe:

$$\begin{aligned}
\mathbf{x} \in X &\Leftrightarrow \langle p_0(x_0), p_1(x_0)\rangle \in h(X)\\
&\Leftrightarrow \langle p_0(x_0), p_1(x_0)\rangle \in h(Y)\\
&\Leftrightarrow \mathbf{x} \in Y.
\end{aligned}$$

Thus h is also injective and hence bijective, as required.

All that remains is to show that h respects the operations of $\mathfrak{JB}$. In what follows we sometimes write $X_0 := \{x_0 : \mathbf{x} \in X\}$, so that $x_0 \in X_0$ iff $\mathbf{x} \in X$ (since $J \subseteq \mathrm{Nr}_1 A$ by definition). For every $X, Y \in J$ we have:

$$\begin{aligned}
h(X + Y) &= \{\langle p_0(x_0), p_1(x_0)\rangle : \mathbf{x} \in X + Y\}\\
&= \{\langle p_0(x_0), p_1(x_0)\rangle : \mathbf{x} \in X\} \cup \{\langle p_0(x_0), p_1(x_0)\rangle : \mathbf{x} \in Y\}\\
&= h(X) \cup h(Y),
\end{aligned}$$

and:

$$\begin{aligned}
h(\dot{-}X) &= \{\langle p_0(x_0), p_1(x_0)\rangle : \mathbf{x} \in \dot{-}X\}\\
&= \{\langle p_0(x_0), p_1(x_0)\rangle : \mathbf{x} \in \dot{1} \cdot -X\}\\
&= \{\langle p_0(x_0), p_1(x_0)\rangle : x_0 \in \operatorname{Dom} p_0, x_0 \notin X_0\}\\
&= \{\langle p_0(x_0), p_1(x_0)\rangle : x_0 \in \operatorname{Dom} p_0\} \setminus \{\langle p_0(x_0), p_1(x_0) : x_0 \in X_0\}\\
&= {}^2V \setminus h(X)\\
&= {\sim}h(X).
\end{aligned}$$

The second to last equality again follows from Equations (1.2) and (1.4). Then we have that h respects the Boolean substructure of $\mathfrak{JB}$. We now turn our attention to the relational operations. Observe that for every $X \in J$ we have $x \in X_0$ iff $\langle p_0(x), p_1(x)\rangle \in h(X)$, and that $\mathsf{s}^0_{\lambda\langle i\rangle} X = \left\{\mathbf{x} \in {}^3V : p_i(x_\lambda) \in X_0\right\}$. For every $X, Y \in J$ we have:

$$\begin{aligned}
h(\breve{X}) &= h\left(\mathbf{c}_1\left(\mathsf{s}^0_1 X \cdot \epsilon_{01}\right)\right)\\
&= h\left(\mathbf{c}_1\{\mathbf{x} : x_1 \in X_0, p_0(x_0) = p_1(x_1), p_1(x_0) = p_0(x_1)\}\right)
\end{aligned}$$

$$
\begin{aligned}
&= h\left(\mathsf{c}_1\left\{\mathbf{x} : \langle p_0(x_1), p_1(x_1)\rangle \in h(X), p_0(x_0) = p_1(x_1), p_1(x_0) = p_0(x_1)\right\}\right)\\
&= h\left(\left\{\mathbf{x} : \langle p_1(x_0), p_0(x_0)\rangle \in h(X)\right\}\right)\\
&= h\left(\left\{\mathbf{x} : \langle p_0(x_0), p_1(x_0)\rangle \in h(X)^{-1}\right\}\right)\\
&= h(X)^{-1},
\end{aligned}
$$

and:

$$
\begin{aligned}
&h(X \circ Y)\\
&= h\left(\mathsf{c}_1\left(\mathsf{s}^0_{1\langle 0\rangle} X \cdot \mathsf{s}^0_{1\langle 1\rangle} Y \cdot \delta_{01}\right)\right)\\
&= h\left(\mathsf{c}_1\left\{\mathbf{x} : p_0(x_1) \in X_0, p_1(x_1) \in Y_0, p_0(x_0) = p_0p_0(x_1), p_1(x_0) = p_1p_1(x_1), p_0p_1(x_1) = p_1p_0(x_1)\right\}\right)\\
&= h\left(\mathsf{c}_1\left\{\mathbf{x} : \langle p_0p_0(x_1), p_1p_0(x_1)\rangle \in h(X), \langle p_0p_1(x_1), p_1p_1(x_1)\rangle \in h(Y),\right.\right.\\
&\qquad\qquad\left.\left. p_0(x_0) = p_0p_0(x_1), p_1(x_0) = p_1p_1(x_1), p_0p_1(x_1) = p_1p_0(x_1)\ \right\}\right)\\
&= h\left(\mathsf{c}_1\left\{\mathbf{x} : \langle p_0(x_0), p_1p_0(x_1)\rangle \in h(X), \langle p_0p_1(x_1), p_1(x_0)\rangle \in h(Y),\right.\right.\\
&\qquad\qquad\left.\left. p_0(x_0) = p_0p_0(x_1), p_1(x_0) = p_1p_1(x_1), p_0p_1(x_1) = p_1p_0(x_1)\ \right\}\right)\\
&= h\left(\left\{\mathbf{x} : \exists y \in V[\langle p_0(x_0), y\rangle \in h(X), \langle y, p_1(x_0)\rangle \in h(Y)]\right\}\right)\\
&= h\left(\left\{\mathbf{x} : \langle p_0(x_0), p_1(x_0)\rangle \in h(X) \mid h(Y)\right\}\right)\\
&= h(X) \mid h(Y)
\end{aligned}
$$

Thus h respects the relational operations of $\mathfrak{JB}$ as desired. Finally, we verify that h maps the identity of $\mathfrak{JB}$ to the identity of $\mathfrak{Re}V$, i.e. that $h(\dot{1}\text{'}) = Id_V$.

$$
\begin{aligned}
h(\dot{1}\text{'}) &= h\left(\left\{\mathbf{x} : p_0(x_0) = p_1(x_0)\right\}\right)\\
&= h\left(\left\{\mathbf{x} : \langle p_0(x_0), p_1(x_0)\rangle \in Id_V\right\}\right)\\
&= Id_V
\end{aligned}
$$

Thus h is an isomorphism from $\mathfrak{JA}$ to $\mathfrak{A}$, as was to be shown. ■

Next we turn to proving several lemmas which will make the proof of Theorem 34 much shorter and much more readable.

2.2 Preliminary lemmas

For convenience we maintain the numbering system of [HMT71], but emphasize that the structure $\langle A, +, ^-, \mathsf{c}_\kappa, \mathsf{d}_{\kappa\lambda}\rangle_{\kappa,\lambda\in 3}$ is *not necessarily* a CA_3. Although Ax and the definitions of $\mathsf{d}_{\kappa\lambda}$ do a good enough job at simulating diagonal elements to establish that $\mathfrak{J}\mathfrak{A} \in \mathsf{RA}$, we have not fully recovered all of the axioms of CA_3. In particular, the axiom C_7 would require an infinite number of assumptions on our parameters because each element of the algebra would require a separate assumption.

Observe that by Definition 32, when $\kappa, \lambda \in 3$ and $i, j \in 2^*$ we have $\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle} = \mathsf{d}_{\lambda\langle j\rangle\kappa\langle i\rangle}$ (similar to `1.3.1`). We shall use this fact throughout the remainder of this work without annotation. We begin by establishing some well-known CA_3 identities.

Lemma 36 *All of the identities from Table 2.1 hold in* $\mathfrak{A}$. □

Table 2.1 Important CA_α identities which hold in $\mathfrak{A}$.

$\mathsf{c}_\kappa x = 0 \Leftrightarrow x = 0$		`1.2.1`
$\mathsf{c}_\kappa \mathsf{c}_\kappa x = \mathsf{c}_\kappa x$		`1.2.3`
$\mathsf{c}_\kappa (x + y) = \mathsf{c}_\kappa x + \mathsf{c}_\kappa y$		`1.2.6(ii)`
$x \leq y \Rightarrow \mathsf{c}_\kappa x \leq \mathsf{c}_\kappa y$		`1.2.7`
$\mathsf{c}_\kappa x = x \Leftrightarrow \mathsf{c}_\kappa \overline{x} = \overline{x}$		`1.2.12(i)`
$\mathsf{d}_{\kappa\lambda} = \mathsf{d}_{\lambda\kappa}$		`1.3.1`
$\mathsf{c}_\kappa \mathsf{d}_{\kappa\lambda} = 1$		`1.3.2`
$\mathsf{s}^\kappa_\lambda (x + y) = \mathsf{s}^\kappa_\lambda x + \mathsf{s}^\kappa_\lambda y$		`1.5.3(i)`
$\mathsf{s}^\kappa_\lambda \mathsf{c}_\kappa x = \mathsf{c}_\kappa x$		`1.5.8(i)`
$\mathsf{s}^\mu_\lambda \mathsf{s}^\kappa_\mu \mathsf{c}_\mu x = \mathsf{s}^\kappa_\lambda \mathsf{c}_\mu x$	$\kappa \neq \lambda$	`1.5.11(i)`
$\mathsf{s}^\lambda_\kappa \mathsf{s}^\kappa_\lambda \mathsf{c}_\lambda \mathsf{c}_\mu x = \mathsf{s}^\mu_\kappa \mathsf{s}^\kappa_\mu \mathsf{c}_\lambda \mathsf{c}_\mu x$	$\kappa \neq \lambda, \mu$	`1.5.11(ii)`

Proof Axioms C_0 through C_4 must hold since $\mathfrak{A} \in \mathsf{Df}_3$. Thus the proofs of `1.2.1` through `1.2.12(i)` found in [HMT71] remain valid for $\mathfrak{A}$ (indeed, for every algebra in Df_3). Theorem `1.3.1` holds by definition of $\mathsf{d}_{\kappa\lambda}$. Axiom C_5 is guaranteed by Ax_0 and C_6 holds by the following argument.

$$\begin{aligned}
\mathsf{d}_{\lambda\mu} &= \mathsf{c}_\kappa e && \text{D32} \\
&\leq \mathsf{c}_\kappa \left(\mathsf{c}_\mu e \cdot \mathsf{c}_\lambda e\right) && \mathrm{C}_2, \texttt{1.2.7} \\
&\leq \mathsf{c}_\kappa \left(\mathsf{d}_{\kappa\lambda} \cdot \mathsf{d}_{\kappa\mu}\right) && \text{D32} \\
\mathsf{c}_\kappa \left(\mathsf{d}_{\kappa\lambda} \cdot \mathsf{d}_{\kappa\mu}\right) &\leq \mathsf{c}_\kappa \mathsf{d}_{\lambda\mu} && \mathrm{Ax}_1, \texttt{1.2.7} \\
&= \mathsf{c}_\kappa \mathsf{c}_\kappa e = \mathsf{c}_\kappa e = \mathsf{d}_{\lambda\mu} && \texttt{1.2.3}, \text{D32}
\end{aligned}$$

The proof of `1.5.3(i)` found in [HMT71] depends only on the axioms C_0 through C_6, so it remains valid in $\mathfrak{A}$ as well. We offer alternative proofs for those identities which require axiom C_7: namely `1.3.2`, `1.5.8(i)`, and `1.5.11`. As to `1.3.2`, we have that $\mathsf{c}_\kappa \mathsf{d}_{\kappa\lambda} = \mathsf{d}_{\lambda\lambda} = 1$ by definition (32) and Ax_0. The proof of `1.5.8(i)` in [HMT71] does not depend directly on C_7, but rather on `1.3.2`, so it remains valid in $\mathfrak{A}$ as well.

To see `1.5.11(i)`, suppose that $\kappa \neq \lambda$. If $\mu = \kappa$ or $\mu = \lambda$ then the result is obvious by definition (32), so assume κ, λ, μ are distinct.

$$\begin{aligned}
\mathsf{s}^{\mu}_{\lambda}\mathsf{s}^{\kappa}_{\mu}\mathsf{c}_{\mu}x &= \mathsf{c}_{\mu}\left(\mathsf{d}_{\mu\lambda} \cdot \mathsf{c}_{\kappa}\left(\mathsf{d}_{\kappa\mu} \cdot \mathsf{c}_{\mu}x\right)\right) && \text{D32}\\
&= \mathsf{c}_{\mu}\mathsf{c}_{\kappa}\left(\mathsf{d}_{\mu\lambda} \cdot \mathsf{d}_{\kappa\mu} \cdot \mathsf{c}_{\mu}x\right) && \text{C}_3\\
&= \mathsf{c}_{\kappa}\left(\mathsf{c}_{\mu}\left(\mathsf{d}_{\mu\lambda} \cdot \mathsf{d}_{\kappa\mu}\right) \cdot \mathsf{c}_{\mu}x\right) && \text{C}_4, \text{C}_3\\
&= \mathsf{c}_{\kappa}\left(\mathsf{d}_{\kappa\lambda} \cdot \mathsf{c}_{\mu}x\right) && \text{C}_6\\
&= \mathsf{s}^{\kappa}_{\lambda}\mathsf{c}_{\mu}x && \text{D32}
\end{aligned}$$

As to `1.5.11(ii)`, suppose $\kappa \neq \lambda, \mu$. If $\lambda = \mu$ then the result is trivial, so again we assume that κ, λ, μ are distinct.

$$\begin{aligned}
\mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\lambda}\mathsf{c}_{\lambda}\mathsf{c}_{\mu}x &= \mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\mu}_{\lambda}\mathsf{s}^{\kappa}_{\mu}\mathsf{c}_{\lambda}\mathsf{c}_{\mu}x && \text{C}_4, \texttt{1.5.11(i)}\\
&= \mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\mu}_{\lambda}\mathsf{c}_{\lambda}\mathsf{s}^{\kappa}_{\mu}\mathsf{c}_{\mu}x && \text{C}_3, \text{C}_4, \text{D32}\\
&= \mathsf{s}^{\mu}_{\kappa}\mathsf{c}_{\lambda}\mathsf{s}^{\kappa}_{\mu}\mathsf{c}_{\mu}x && \texttt{1.5.11(i)}\\
&= \mathsf{s}^{\mu}_{\kappa}\mathsf{s}^{\kappa}_{\mu}\mathsf{c}_{\lambda}\mathsf{c}_{\mu}x && \text{C}_3, \text{C}_4, \text{D32}
\end{aligned}$$
■

Next we investigate some further properties of the substitution operations. As suggested by the notation, the operations $\mathsf{s}^{\lambda}_{\kappa}$ and $\mathsf{s}^{\kappa}_{\lambda}$ are natural inverses of one another. Indeed using axiom C_7 one can prove that $\mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\lambda}x = \mathsf{s}^{\lambda}_{\kappa}x$ [HMT71, `1.5.10(v)`] and further, that if $\lambda \notin \Delta(x)$ then $\mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\lambda}x = x$ by `1.5.8(i)`. Remarkably, this is the only use of axiom C_7 which is required in the proof of [Ném85, Proposition 2.10] (on p. 49). The definition of J_κ is the key to recovering this identity and avoiding the use of axiom C_7.

If $\{\kappa, \lambda, \mu\} = 3$ and $\Delta(x) \subseteq \{\kappa\}$, then by `1.5.11(ii)` we have that $\mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\lambda}x = \mathsf{s}^{\mu}_{\kappa}\mathsf{s}^{\kappa}_{\mu}x$, thus the definition of J_κ is unambiguous. We can generally make use of the inequality $x \leq \mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\lambda}x$ for $x \in \mathrm{Nr}_{\{\kappa\}}A$. Remarkably, when $x = \mathsf{s}^{\lambda}_{\kappa}u$ for some $u \in \mathrm{Nr}_{\{\lambda\}}A$, we have $x = \mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\lambda}x$ (thus $x \in J_\kappa$). In other words, once we perform a single substitution on a 1-dimensional element "the damage is done" and subsequent substitutions cease to create more difficulties. This is particularly advantageous due to the definitions of $x \circ y$ and $\breve{x}$, which we shall see to be the images of a substitution in §2.3.

Lemma 37 *Let $\kappa \neq \lambda$. If $\Delta(x) \subseteq \{\kappa\}$ then $x \leq \mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\lambda}x$. Moreover, if $x = \mathsf{s}^{\lambda}_{\kappa}u$ for some $u \in \mathrm{Nr}_{\{\lambda\}}A$, we have $\mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\lambda}x = x \in J_\kappa$.* □

Proof Suppose first that $\Delta(x) \subseteq \{\kappa\}$.

$$\begin{aligned}
x &= \mathsf{s}^{\kappa}_{\lambda}x && \texttt{1.5.8(i)}\\
&= \mathsf{s}^{\kappa}_{\lambda}\left(\mathsf{d}_{\kappa\lambda} \cdot x\right) && \textsf{BA}
\end{aligned}$$

$$\begin{aligned} &\leq \mathsf{s}^{\kappa}_{\lambda}\mathsf{c}_{\lambda}\left(\mathsf{d}_{\kappa\lambda}\cdot x\right) && \mathrm{C}_2, \texttt{1.2.7} \\ &= \mathsf{s}^{\kappa}_{\lambda}\mathsf{s}^{\lambda}_{\kappa}x && \mathrm{D32} \end{aligned}$$

Now suppose further that $x = \mathsf{s}^{\lambda}_{\kappa}u$ for some u with $\Delta(u) \subseteq \{\lambda\}$ and let $\{\kappa, \lambda, \mu\} = 3$.

$$\begin{aligned} \mathsf{s}^{\lambda}_{\kappa}\mathsf{s}^{\kappa}_{\lambda}x &= \mathsf{s}^{\mu}_{\kappa}\mathsf{s}^{\kappa}_{\mu}x && \texttt{1.5.11(ii)} \\ &= \mathsf{s}^{\mu}_{\kappa}\mathsf{s}^{\kappa}_{\mu}\mathsf{s}^{\lambda}_{\kappa}u && \mathrm{Def} \\ &= \mathsf{s}^{\mu}_{\kappa}\mathsf{s}^{\lambda}_{\mu}u && \texttt{1.5.11(i)} \\ &= \mathsf{s}^{\lambda}_{\kappa}u = x && \texttt{1.5.11(i)}, \mathrm{Def} \end{aligned}$$

■

As an immediate consequence we can observe that if $x \in J_\kappa$ then $\mathsf{s}^{\kappa}_{\lambda}x \in J_\lambda$.

Our recovery of identity `1.5.10(v)` via the definition of J_κ comes at a price. In [Ném85], the universe of the Németi reduct is $\{\dot{1} \cdot x : \mathsf{c}_1\mathsf{c}_2x = x\}$ which is quite easily seen to be closed under Boolean operations. That this is also true of J_κ is not a trivial fact. In fact, it was the realization and proof of Equation (2.8) below which first convinced me beyond a doubt that Theorem 34 was an attainable result. Once again, using axiom C_7 it is straightforward to (equivalently) show that $\mathsf{s}^{\kappa}_{\lambda}\overline{x} = (\mathsf{s}^{\kappa}_{\lambda}x)^{-}$ [HMT71, `1.5.3(ii)`]. An alternative proof follows.

Lemma 38 *The structure $\langle J_\kappa, +, ^{-}\rangle$ is a Boolean algebra.* □

Proof We only need establish closure under the Boolean operations. Closure of J_κ under $+$ follows immediately from `1.5.3(i)`. For closure under complementation, we prove the following identities. Let $\{\kappa, \lambda, \mu\} = 3$ and $x \in J_\kappa$.

$$(\mathsf{d}_{\kappa\mu})^{-} + \mathsf{d}_{\lambda\kappa} = (\mathsf{d}_{\kappa\mu})^{-} + \mathsf{d}_{\lambda\mu} \tag{2.6}$$

$$(\mathsf{s}^{\kappa}_{\lambda}x)^{-} = \mathsf{s}^{\mu}_{\lambda}\big(\mathsf{s}^{\kappa}_{\mu}x\big)^{-} \tag{2.7}$$

$$(\mathsf{s}^{\kappa}_{\lambda}x)^{-} = \mathsf{s}^{\kappa}_{\lambda}\overline{x} \tag{2.8}$$

Equation (2.6) follows quickly from C_6 and the Boolean structure of $\mathfrak{Nr}_{\{\kappa\}}\mathfrak{A}$. We prove inequality in one direction, however, as κ, λ, μ are arbitrary this is sufficient to establish equality.

$$\begin{aligned} (\mathsf{d}_{\kappa\mu})^{-} + \mathsf{d}_{\lambda\kappa} &\geq (\mathsf{d}_{\kappa\mu})^{-} + \mathsf{d}_{\kappa\mu}\cdot\mathsf{d}_{\lambda\mu} && \mathrm{C}_6 \\ &= \Big((\mathsf{d}_{\kappa\mu})^{-} + \mathsf{d}_{\kappa\mu}\Big)\cdot\Big((\mathsf{d}_{\kappa\mu})^{-} + \mathsf{d}_{\lambda\mu}\Big) && \mathsf{BA} \\ &= (\mathsf{d}_{\kappa\mu})^{-} + \mathsf{d}_{\lambda\mu} && \mathsf{BA} \end{aligned}$$

Equation (2.7) is a technical consequence of (2.6) which is quite useful in the proof of (2.8). It bears some resemblance to `1.5.11(i)`, but owing to the negations involved the proof is much more complex.

$$
\begin{aligned}
(\mathsf{s}^\kappa_\lambda x)^- &= (\mathsf{c}_\kappa\,(\mathsf{d}_{\kappa\lambda}\cdot x))^- && \text{D32}\\
&= \mathsf{c}_\mu\left(\mathsf{d}_{\lambda\mu}\cdot(\mathsf{c}_\kappa\,(\mathsf{d}_{\kappa\lambda}\cdot x))^-\right) && \texttt{1.5.8(i)},\texttt{1.2.12(i)}\\
&= \mathsf{c}_\mu\Big((\mathsf{d}_{\lambda\mu})^- + \mathsf{c}_\kappa\,(\mathsf{d}_{\kappa\lambda}\cdot x)\Big)^- && \mathsf{BA}\\
&= \mathsf{c}_\mu\Big(\mathsf{c}_\kappa\,\Big((\mathsf{d}_{\lambda\mu})^- + \mathsf{d}_{\kappa\lambda}\cdot x\Big)\Big)^- && \texttt{1.2.12(i)},\texttt{1.2.6(ii)}\\
&= \mathsf{c}_\mu\Big(\mathsf{c}_\kappa\,\Big(\Big((\mathsf{d}_{\lambda\mu})^- + \mathsf{d}_{\kappa\lambda}\Big)\cdot\Big((\mathsf{d}_{\lambda\mu})^- + x\Big)\Big)\Big)^- && \mathsf{BA}\\
&= \mathsf{c}_\mu\Big(\mathsf{c}_\kappa\,\Big(\Big((\mathsf{d}_{\lambda\mu})^- + \mathsf{d}_{\kappa\mu}\Big)\cdot\Big((\mathsf{d}_{\lambda\mu})^- + x\Big)\Big)\Big)^- && (2.6)\\
&= \mathsf{c}_\mu\Big(\mathsf{c}_\kappa\,\Big((\mathsf{d}_{\lambda\mu})^- + \mathsf{d}_{\kappa\mu}\cdot x\Big)\Big)^- && \mathsf{BA}\\
&= \mathsf{c}_\mu\Big((\mathsf{d}_{\lambda\mu})^- + \mathsf{c}_\kappa\,(\mathsf{d}_{\kappa\mu}\cdot x)\Big)^- && \texttt{1.2.12(i)},\texttt{1.2.6(ii)}\\
&= \mathsf{c}_\mu\left(\mathsf{d}_{\lambda\mu}\cdot(\mathsf{c}_\kappa\,(\mathsf{d}_{\kappa\mu}\cdot x))^-\right) && \mathsf{BA}\\
&= \mathsf{s}^\mu_\lambda(\mathsf{s}^\kappa_\mu x)^- && \text{D32}
\end{aligned}
$$

Finally, we are prepared to prove (2.8). Let $u = \mathsf{s}^\kappa_\lambda x$, so that $\mathsf{s}^\lambda_\kappa u = x$ (by definition of J_κ).

$$
\begin{aligned}
(\mathsf{s}^\kappa_\lambda x)^- &= \mathsf{s}^\mu_\lambda(\mathsf{s}^\kappa_\mu x)^- && (2.7)\\
&= \mathsf{s}^\mu_\lambda(\mathsf{s}^\kappa_\mu\mathsf{s}^\lambda_\kappa u)^- && \text{Def}\\
&= \mathsf{s}^\mu_\lambda(\mathsf{s}^\lambda_\mu u)^- && \texttt{1.5.11(i)}\\
&= \mathsf{s}^\mu_\lambda\mathsf{s}^\kappa_\mu(\mathsf{s}^\lambda_\kappa u)^- && (2.7)\\
&= \mathsf{s}^\mu_\lambda\mathsf{s}^\kappa_\mu\overline{x} && \text{Def}\\
&= \mathsf{s}^\kappa_\lambda\overline{x} && \texttt{1.5.11(i)}
\end{aligned}
$$

Applying Equation (2.8) twice (along with the definition of J_κ) is sufficient to show closure under complementation.

$$
\begin{aligned}
\overline{x} &= \left(\mathsf{s}^\lambda_\kappa\mathsf{s}^\kappa_\lambda x\right)^- && \text{D32}\\
&= \mathsf{s}^\lambda_\kappa(\mathsf{s}^\kappa_\lambda x)^- && (2.8)\\
&= \mathsf{s}^\lambda_\kappa\mathsf{s}^\kappa_\lambda\overline{x} && (2.8)
\end{aligned}
$$

Thus $\overline{x} \in J_\kappa$ by definition. ■

Remark 39 Equation (2.6) is (a dual version of) an algebraization of the logically valid "$x = y$ and $x \neq z$ iff $x = y$ and $y \neq z$". Equation (2.8) is a weaker formulation of axiom C_7, holding only for

$x \in J_\kappa$. Rewriting axiom C_7 using substitutions yields $\mathsf{s}^\kappa_\lambda x \cdot \mathsf{s}^\kappa_\lambda \overline{x} = 0$, which follows immediately from Equation (2.8) when $x \in J_\kappa$. Conversely, (2.8) can be easily derived (for all $x \in A$ as opposed to $x \in J_\kappa$) using C_7 by [HMT71, `1.5.3(ii)`]. $\square$

Next, we establish that the dimension sets of $\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}$, $\delta_{\kappa\langle i\rangle\lambda\langle j\rangle}$, $\epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle}$, and $\mathsf{s}^\kappa_{\lambda\langle i\rangle} x$ are as suggested by the notation. When we invoke the following lemma, we simply use Δ as our annotation.

Lemma 40 ($\boldsymbol{\Delta}$) *If $x \in \left\{\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}, \epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle}, \delta_{\kappa\langle i\rangle\lambda\langle j\rangle}\right\}$ then $\Delta(x) \subseteq \{\kappa, \lambda\}$. For $x \in \mathrm{Nr}_{\{\kappa\}} A$, $\Delta\left(\mathsf{s}^\kappa_{\lambda\langle i\rangle} x\right) \subseteq \{\lambda\}$.* $\square$

Proof For the first part, it suffices to show that $\Delta\left(\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}\right) \subseteq \{\kappa, \lambda\}$ since $\Delta(x \cdot y) \subseteq \Delta(x) \cup \Delta(y)$ (by C_3). We proceed by case analysis on the definition of $\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}$, examining each of (2.1) through (2.5) from Definition 32. In case (2.5), we note that $\mathsf{d}_{\kappa\langle i\rangle\kappa\langle j\rangle} = \mathsf{c}_{\kappa+_3 1}\left(\mathsf{d}_{\kappa+_3 1\langle\rangle\kappa\langle j\rangle} \cdot \mathsf{d}_{\kappa+_3 1\langle\rangle\kappa\langle j\rangle}\right)$, so $\kappa +_3 1 \notin \Delta\left(\mathsf{d}_{\kappa\langle i\rangle\kappa\langle j\rangle}\right)$ by `1.2.3`. This reduces the proof to the case where $\kappa \neq \lambda$ (by C_3 and C_4). In this case $\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}$ is defined by one of (2.1)-(2.4). We have already established for $\{\kappa, \lambda, \mu\} = 3$ that $\mathsf{c}_\mu \mathsf{d}_{\kappa\lambda} = \mathsf{d}_{\kappa\lambda}$ (by Definition 32) so the result holds for (2.1). In the case of (2.3) or (2.4), we have $\mu \notin \Delta\left(\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}\right)$ by `1.2.3`. Finally in the case of (2.2). Note that $\Delta(\mathsf{p}_k) \subseteq \{0, 1\}$ by Definition 32. Now we must use a case analysis for each possible choice of κ and λ—this can be found in Table 2.2. Finally, we note that when $\kappa \neq \lambda$, $\kappa \notin \Delta\left(\mathsf{s}^\kappa_\lambda x\right)$ by `1.2.3` and Definition 32. Applying this principle to each of the possibilities for κ and λ concludes the proof that $\Delta(\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}) \subseteq \{\kappa, \lambda\}$.

Now suppose $\Delta(u) \subseteq \{\kappa\}$. Then $\mathsf{s}^\kappa_{\lambda\langle i\rangle} u = \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\lambda\langle i\rangle} \cdot u\right)$ by Definition 32, so $\kappa \notin \mathsf{s}^\kappa_{\lambda\langle i\rangle} u$ by `1.2.3`. Further, if $\mu \neq \kappa, \lambda$

$$
\begin{aligned}
\mathsf{c}_\mu \mathsf{s}^\kappa_{\lambda\langle i\rangle} u &= \mathsf{c}_\mu \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\lambda\langle i\rangle} \cdot u\right) && \mathrm{D}32\\
&= \mathsf{c}_\kappa \mathsf{c}_\mu\left(\mathsf{d}_{\kappa\lambda\langle i\rangle} \cdot u\right) && \mathrm{C}_4\\
&= \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\lambda\langle i\rangle} \cdot u\right) && \mathrm{C}_3, \Delta.
\end{aligned}
$$

Thus $\mu \notin \Delta\left(\mathsf{s}^\kappa_{\lambda\langle i\rangle} u\right)$ either, concluding the proof. ■

Now we show that the auxiliary dimension in the definition of $\mathsf{d}_{\kappa\langle i\rangle\kappa\langle j\rangle}$ need not be $\kappa +_3 1$. This is hardly surprising, indeed we chose it quite arbitrarily. We require this fact in order to fully exploit the symmetry between dimensions. It won't be used directly during the course of our main proof, but rather the consequences which follow.

Proposition 41 *For $\mu \neq \kappa \in 3$ and $i, j \in S_3$, $\mathsf{d}_{\kappa\langle i\rangle\kappa\langle j\rangle} = \mathsf{c}_\mu\left(\mathsf{d}_{\mu\kappa\langle i\rangle} \cdot \mathsf{d}_{\mu\kappa\langle j\rangle}\right)$.* $\square$

κ	λ	$\mathsf{p}_k(\kappa, \lambda)$	
0	1	$\mathsf{s}_0^0\mathsf{s}_1^1\mathsf{p}_k$	$= \mathsf{p}_k$
0	2	$\mathsf{s}_0^0\mathsf{s}_2^1\mathsf{p}_k$	$= \mathsf{s}_2^1\mathsf{p}_k$
1	0	$\mathsf{s}_1^2\mathsf{s}_0^1\mathsf{s}_2^0\mathsf{p}_k$	
1	2	$\mathsf{s}_1^0\mathsf{s}_2^1\mathsf{p}_k$	
2	0	$\mathsf{s}_2^2\mathsf{s}_0^1\mathsf{s}_2^0\mathsf{p}_k$	$= \mathsf{s}_0^1\mathsf{s}_2^0\mathsf{p}_k$
2	1	$\mathsf{s}_2^2\mathsf{s}_1^1\mathsf{s}_2^0\mathsf{p}_k$	$= \mathsf{s}_2^0\mathsf{p}_k$

Table 2.2 Case analysis for the substitution convention

Proof Let $\{\kappa, \lambda, \mu\} = 3$. Then depending on the choice of λ and μ, $\mathsf{d}_{\kappa\langle i\rangle\kappa\langle j\rangle}$ must be the left or right hand side of the following inequality. That is, $\kappa +_3 1 \in \{\lambda, \mu\}$.

$$\begin{aligned}
\mathsf{c}_\lambda\left(\mathsf{d}_{\lambda\kappa\langle i\rangle} \cdot \mathsf{d}_{\lambda\kappa\langle j\rangle}\right) &= \mathsf{c}_\lambda\mathsf{c}_\mu\left(\mathsf{d}_{\mu\lambda} \cdot \mathsf{d}_{\lambda\kappa\langle i\rangle} \cdot \mathsf{d}_{\lambda\kappa\langle j\rangle}\right) && \texttt{1.5.8(i)} \\
&\leq \mathsf{c}_\lambda\mathsf{c}_\mu\left(\mathsf{d}_{\mu\kappa\langle i\rangle} \cdot \mathsf{d}_{\mu\kappa\langle j\rangle}\right) && \mathrm{Ax}_1, \texttt{1.2.7} \\
&= \mathsf{c}_\mu\left(\mathsf{d}_{\mu\kappa\langle i\rangle} \cdot \mathsf{d}_{\mu\kappa\langle j\rangle}\right) && \Delta
\end{aligned}$$

The inequality in the opposite direction follows from a symmetric argument, so equality holds. ■

The following is a generalization of `1.5.8(i)` to our projectional substitutions. Specifically, we give conditions under which the operation $\mathsf{s}^\kappa_{\lambda\langle i\rangle}$ fixes an element of A.

Proposition 42 *If $\kappa \neq \lambda$, $\kappa \notin \Delta(x)$, and $x \leq \mathsf{d}_{\lambda\langle i\rangle\lambda\langle i\rangle}$, then $\mathsf{s}^\kappa_{\lambda\langle i\rangle}x = x$.* □

Proof Suppose x satisfies the above hypotheses.

$$\begin{aligned}
x &= x \cdot \mathsf{d}_{\lambda\langle i\rangle\lambda\langle i\rangle} && \mathsf{BA} \\
&= x \cdot \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\lambda\langle i\rangle}\right) && 41 \\
&= \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\lambda\langle i\rangle} \cdot x\right) && \mathrm{C}_3, \kappa \notin \Delta(x) \\
&= \mathsf{s}^\kappa_{\lambda\langle i\rangle}x && \mathrm{D}32
\end{aligned}$$
■

Remark 43 The hypothesis $x \leq \mathsf{d}_{\lambda\langle i\rangle\lambda\langle i\rangle}$ from Proposition 42 may seem overly restrictive at first glance, but recall that $\mathsf{d}_{\lambda\langle i\rangle\lambda\langle i\rangle}$ is meant to simulate $\{\mathbf{x} : P_i(x_\lambda) = P_i(x_\lambda)\}$. In other words, the hypothesis only guarantees that projection P_i is defined on the λ-coordinates of x. □

The following Proposition 44 will mostly be used in conjunction with Ax_2. The reason it is required is that the left hand side of equations in Ax_2 will not show up frequently in our calculations. As we shall see however, they are upper bounds for the projectional diagonal elements which we **do** use frequently.

Proposition 44 *For $\kappa, \lambda \in 3$ and $il, j \in S_3$, $\mathsf{d}_{\kappa\langle il\rangle\lambda\langle j\rangle} \leq \mathsf{d}_{\kappa\langle i\rangle\kappa\langle i\rangle}$.* □

Proof Choose any $\mu \neq \kappa, \lambda$.

$$\begin{aligned}
\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle} &= \mathsf{c}_\mu\left(\mathsf{d}_{\mu\kappa\langle i\rangle} \cdot \mathsf{d}_{\mu\lambda\langle j\rangle}\right) && \text{D32, 41} \\
&\leq \mathsf{c}_\mu\left(\mathsf{d}_{\mu\kappa\langle i\rangle}\right) && \mathsf{BA}, \mathtt{1.2.7} \\
&= \mathsf{d}_{\kappa\langle i\rangle\kappa\langle i\rangle} && 41
\end{aligned}$$

Now, let $k \in 2$ and $ik \in S_3$.

$$\begin{aligned}
\mathsf{d}_{\kappa\langle ik\rangle\kappa\langle ik\rangle} &= \mathsf{c}_\mu\left(\mathsf{d}_{\mu\kappa\langle ik\rangle}\right) && 41 \\
&= \mathsf{c}_\mu\mathsf{c}_\lambda\left(\mathsf{d}_{\kappa\langle i\rangle\lambda} \cdot \mathsf{d}_{\lambda\langle k\rangle\mu}\right) && \text{D32} \\
&\leq \mathsf{c}_\mu\mathsf{c}_\lambda\left(\mathsf{d}_{\kappa\langle i\rangle\lambda}\right) && \mathsf{BA}, \mathtt{1.2.7} \\
&= \mathsf{c}_\lambda\left(\mathsf{d}_{\kappa\langle i\rangle\lambda}\right) && \Delta \\
&= \mathsf{d}_{\kappa\langle i\rangle\kappa\langle i\rangle} && 41
\end{aligned}$$

Applying these recursively yields the desired inequality. ■

Remark 45 As regarding the previous proof: informally the first argument says "if $P_i(x_\kappa) = P_j(x_\lambda)$ then $P_i(x_\kappa) = P_i(x_\kappa)$." Intuitively "$P_i(x_\kappa) = P_i(x_\kappa)$" simply expresses that P_i is defined on x_κ. This is not necessarily true of all elements as the functions P_i are not necessarily (or desirably) entire. The second argument could be read as "if $P_{i_n}P_{i_{n-1}}\ldots P_{i_0}(x_\kappa)$ exists then $P_{i_{n-1}}\ldots P_{i_0}(x_\kappa)$ exists as well." □

Next, Proposition 46 establishes an identity similar to those in Ax_1 for the terms $\epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle}$ and $\delta_{\kappa\langle i\rangle\lambda\langle j\rangle}$. This is not surprising, as these terms are simply Boolean combinations of the projectional diagonal elements. Proposition 47 is a similar result relating to projectional substitutions.

Proposition 46 *When $\kappa, \lambda, \mu \in 3$ and $i \in S_2, j \in S_1$; we have the following:*

$$\begin{aligned}
\mathsf{d}_{\mu\kappa\langle i\rangle} \cdot \delta_{\mu\lambda\langle j\rangle} &\leq \delta_{\kappa\langle i\rangle\lambda\langle j\rangle} \\
\mathsf{d}_{\mu\lambda\langle j\rangle} \cdot \delta_{\kappa\langle i\rangle\mu} &\leq \delta_{\kappa\langle i\rangle\lambda\langle j\rangle} \\
\mathsf{d}_{\mu\kappa\langle i\rangle} \cdot \epsilon_{\mu\lambda\langle j\rangle} &\leq \epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle}
\end{aligned}$$ □

Proof These follow quickly by applying Ax_1 to the definitions of the terms.

$$\begin{aligned}
\mathsf{d}_{\mu\kappa\langle i\rangle} \cdot \delta_{\mu\lambda\langle j\rangle} &= \mathsf{d}_{\mu\kappa\langle i\rangle} \cdot \mathsf{d}_{\mu\langle 0\rangle\lambda\langle j00\rangle} \cdot \mathsf{d}_{\mu\langle 1\rangle\lambda\langle j11\rangle} \cdot \mathsf{d}_{\lambda\langle j01\rangle\lambda\langle j10\rangle} && \text{D32} \\
&\leq \mathsf{d}_{\kappa\langle i0\rangle\lambda\langle j00\rangle} \cdot \mathsf{d}_{\kappa\langle i1\rangle\lambda\langle j11\rangle} \cdot \mathsf{d}_{\lambda\langle j01\rangle\lambda\langle j10\rangle} && \mathrm{Ax}_1 \\
&= \delta_{\kappa\langle i\rangle\lambda\langle j\rangle} && \text{D32}
\end{aligned}$$

$$\begin{aligned}
\mathsf{d}_{\mu\lambda\langle j\rangle}\delta_{\kappa\langle i\rangle\mu} &= \mathsf{d}_{\mu\lambda\langle j\rangle}\cdot\mathsf{d}_{\kappa\langle i0\rangle\mu\langle 00\rangle}\cdot\mathsf{d}_{\kappa\langle i1\rangle\mu\langle 11\rangle}\cdot\mathsf{d}_{\mu\langle 01\rangle\mu\langle 10\rangle} && \text{D32}\\
&\le \mathsf{d}_{\kappa\langle i0\rangle\lambda\langle j00\rangle}\cdot\mathsf{d}_{\kappa\langle i1\rangle\lambda\langle j11\rangle}\cdot\mathsf{d}_{\lambda\langle j10\rangle\lambda\langle j01\rangle} && \text{Ax}_1\\
&= \delta_{\kappa\langle i\rangle\lambda\langle j\rangle} && \text{D32}\\
\mathsf{d}_{\mu\kappa\langle i\rangle}\cdot\epsilon_{\mu\lambda\langle j\rangle} &= \mathsf{d}_{\mu\kappa\langle i\rangle}\cdot\mathsf{d}_{\mu\langle 0\rangle\lambda\langle j1\rangle}\cdot\mathsf{d}_{\mu\langle 1\rangle\lambda\langle j0\rangle} && \text{D32}\\
&\le \mathsf{d}_{\kappa\langle i0\rangle\lambda\langle j1\rangle}\cdot\mathsf{d}_{\kappa\langle i1\rangle\lambda\langle j0\rangle} && \text{Ax}_1\\
&= \epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle} && \text{D32} \quad \blacksquare
\end{aligned}$$

Proposition 47 *Let $\{\kappa,\lambda,\mu\}=3$; $x\in J_\kappa$; and $i,j\in S_3$. Then $\mathsf{s}^\kappa_{\lambda\langle i\rangle}x\cdot\mathsf{d}_{\lambda\langle i\rangle\mu\langle j\rangle}\le\mathsf{s}^\kappa_{\mu\langle i\rangle}x$. Further, if $ji\in S_3$ then $\mathsf{s}^\kappa_{\lambda\langle i\rangle}x\cdot\mathsf{d}_{\lambda\mu\langle j\rangle}\le\mathsf{s}^\kappa_{\mu\langle ji\rangle}x$.* □

Proof Assume $i,j\in S_3$ for the first inequality, and further that $ji\in S_3$ for the second inequality.

$$\begin{aligned}
\mathsf{s}^\kappa_{\lambda\langle i\rangle}x\cdot\mathsf{d}_{\lambda\langle i\rangle\mu\langle j\rangle} &= \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\lambda\langle i\rangle}\cdot x\right)\cdot\mathsf{d}_{\lambda\langle i\rangle\mu\langle j\rangle} && \text{D32}\\
&= \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\lambda\langle i\rangle}\cdot x\cdot\mathsf{d}_{\lambda\langle i\rangle\mu\langle j\rangle}\right) && \Delta,\mathrm{C}_3\\
&\le \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\mu\langle j\rangle}\cdot x\right) && \text{Ax}_1,\mathtt{1.2.7}\\
&= \mathsf{s}^\kappa_{\mu\langle j\rangle}x && \text{D32}\\
\mathsf{s}^\kappa_{\lambda\langle i\rangle}x\cdot\mathsf{d}_{\lambda\mu\langle j\rangle} &= \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\lambda\langle i\rangle}\cdot x\right)\cdot\mathsf{d}_{\lambda\mu\langle j\rangle} && \text{Def}\\
&= \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\lambda\langle i\rangle}\cdot x\cdot\mathsf{d}_{\lambda\mu\langle j\rangle}\right) && \Delta,\mathrm{C}_3\\
&\le \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\mu\langle ji\rangle}\cdot x\right) && \text{Ax}_1,\mathtt{1.2.7}\\
&= \mathsf{s}^\kappa_{\mu\langle ji\rangle}x && \text{D32} \quad \blacksquare
\end{aligned}$$

There is quite a bit of symmetry between cylindrified dimensions. In fact, we can cylindrify dimensions interchangeably provided we are careful not to change the dimension sets of the respective terms.

Lemma 48 *Let $\{\kappa,\lambda,\mu\}=3$ and let $X(\lambda)$ be a combination (via the $\cdot$ operation) of some elements: $\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}$, $\delta_{\kappa\langle i\rangle\lambda\langle j\rangle}$, $\epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle}$, and $\mathsf{s}^\kappa_{\lambda\langle i\rangle}x$ (where $x\in J_\kappa$)—then $\mathsf{c}_\lambda(X(\lambda))=\mathsf{c}_\mu(X(\mu))$.* □

Proof A simple inductive argument using Ax_1 along with 46 and 47 shows that $\Delta(X(\lambda))\subseteq\{\lambda,\kappa\}$ and $\mathsf{d}_{\lambda\mu}\cdot X(\lambda)\le X(\mu)$.

$$\begin{aligned}
\mathsf{c}_\lambda X(\lambda) &= \mathsf{c}_\lambda\mathsf{c}_\mu(\mathsf{d}_{\lambda\mu}\cdot X(\lambda)) && \mathtt{1.5.8(i)}\\
&\le \mathsf{c}_\lambda\mathsf{c}_\mu(X(\mu)) && \text{Induction}\\
&\le \mathsf{c}_\mu(X(\mu)) && \Delta
\end{aligned}$$

The two possible choices of $\{\lambda,\mu\}$ yield opposite inequalities, so equality must hold. ■

Finally, we establish that x can be "recovered" from $\mathsf{s}^{\kappa}_{\lambda\langle i\rangle}x$ when $x \in J_\kappa$. This is a generalized and algebraic version of `(6)` from [Ném85, Proposition 2.10]. The proof there invokes C_7 to obtain $\mathsf{s}^2_0\mathsf{s}^0_2 x = x$, which we have by the definition of J_0.

Lemma 49 *Let $\kappa \neq \lambda$, $x \in J_\kappa$, and $i \in S_3$. Then $\mathsf{s}^{\kappa}_{\lambda\langle i\rangle}x \cdot \mathsf{d}_{\kappa\lambda\langle i\rangle} \leq x$.* □

Proof Let $\{\kappa, \lambda, \mu\} = 3$.

$$\begin{aligned}
\mathsf{s}^{\kappa}_{\lambda\langle i\rangle}x \cdot \mathsf{d}_{\kappa\lambda\langle i\rangle} &= \mathsf{c}_\mu\left(\mathsf{d}_{\kappa\mu}\cdot \mathsf{s}^{\kappa}_{\lambda\langle i\rangle}x \cdot \mathsf{d}_{\kappa\lambda\langle i\rangle}\right) && \texttt{1.5.8(i)}\\
&\leq \mathsf{c}_\mu\left(\mathsf{d}_{\kappa\mu}\cdot \mathsf{d}_{\mu\lambda\langle i\rangle}\cdot \mathsf{s}^{\kappa}_{\lambda\langle i\rangle}x\right) && \mathrm{Ax}_1, \mathsf{BA}, \texttt{1.2.7}\\
&\leq \mathsf{c}_\mu\left(\mathsf{d}_{\kappa\mu}\cdot \mathsf{d}_{\mu\lambda\langle i\rangle}\cdot \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\lambda\langle i\rangle}\cdot x\right)\right) && \mathrm{D}32\\
&\leq \mathsf{c}_\mu\left(\mathsf{d}_{\kappa\mu}\cdot \mathsf{c}_\kappa\left(\mathsf{d}_{\mu\lambda\langle i\rangle}\cdot \mathsf{d}_{\kappa\lambda\langle i\rangle}\cdot x\right)\right) && \mathrm{C}_3\\
&\leq \mathsf{c}_\mu\left(\mathsf{d}_{\kappa\mu}\cdot \mathsf{c}_\kappa\left(\mathsf{d}_{\kappa\mu}\cdot x\right)\right) && \mathrm{Ax}_1\\
&\leq \mathsf{s}^{\mu}_{\kappa}\mathsf{s}^{\kappa}_{\mu}x \leq x && \mathrm{D}32
\end{aligned}$$
■

We are now fully prepared for the proof of our main theorem.

2.3 $\mathsf{Q}^+\mathsf{RA}$ reducts in Df_3

We first prove that J is closed under all the operations of $\mathfrak{JA}$, and thusly that $\mathfrak{JA}$ is an algebra. By Ax_1 and Lemma 37 we have that $\dot{1} \in J$, and since $\langle J_0, +, ^-\rangle \in \mathsf{BA}$, we have that $\langle J, +, \dot{-}\rangle \in \mathsf{BA}$ as well (since it is a relativization to $\dot{1}$). As to closure under the relational operations we note that by Definition 32, $x \in J$ if and only if $\mathsf{s}^2_0\mathsf{s}^0_2 x = x$ and $x \leq \dot{1}$. Let $x, y \in J$ be arbitrary.

$$\begin{aligned}
x \circ y &\leq \mathsf{s}^2_0\mathsf{s}^0_2(x\circ y) && 37\\
&\leq \mathsf{s}^2_0\mathsf{c}_0\left(\mathsf{d}_{02}\cdot \mathsf{c}_1\left(\mathsf{s}^0_{1\langle 0\rangle}x \cdot \mathsf{s}^0_{1\langle 1\rangle}y \cdot \delta_{01}\right)\right) && \mathrm{D}32\\
&\leq \mathsf{s}^2_0\mathsf{c}_0\mathsf{c}_1\left(\mathsf{d}_{02}\cdot \mathsf{s}^0_{1\langle 0\rangle}x \cdot \mathsf{s}^0_{1\langle 1\rangle}y \cdot \delta_{01}\right) && \Delta, \mathrm{C}_3\\
&\leq \mathsf{c}_2\left(\mathsf{d}_{02}\cdot \mathsf{c}_0\mathsf{c}_1\left(\mathsf{s}^0_{1\langle 0\rangle}x \cdot \mathsf{s}^0_{1\langle 1\rangle}y \cdot \delta_{21}\right)\right) && 46, \texttt{1.2.7}, \mathrm{D}32\\
&\leq \mathsf{c}_2\mathsf{c}_1\left(\mathsf{d}_{02}\cdot \mathsf{s}^0_{1\langle 0\rangle}x \cdot \mathsf{s}^0_{1\langle 1\rangle}y \cdot \delta_{21}\right) && \mathrm{C}_4, \Delta, \mathrm{C}_3\\
&\leq \mathsf{c}_1\left(\mathsf{s}^0_{1\langle 0\rangle}x \cdot \mathsf{s}^0_{1\langle 1\rangle}y \cdot \delta_{01}\right) && 46, \mathrm{C}_2, \mathrm{C}_4, \Delta\\
&\leq x \circ y && \mathrm{D}32\\
&\leq \mathsf{d}_{0\langle 0\rangle 0\langle 0\rangle}\cdot \mathsf{d}_{0\langle 1\rangle 0\langle 1\rangle} = \dot{1} && \mathsf{BA}, 44, \Delta
\end{aligned}$$

$$\begin{aligned}
\breve{x} &\leq \mathsf{s}_0^2\mathsf{s}_2^0\breve{x} && 37\\
&\leq \mathsf{s}_0^2\mathsf{c}_0\left(\mathsf{d}_{02}\cdot\mathsf{c}_1\left(\mathsf{s}_1^0 x\cdot\epsilon_{01}\right)\right) && \mathrm{D}32\\
&\leq \mathsf{s}_0^2\mathsf{c}_0\mathsf{c}_1\left(\mathsf{d}_{02}\cdot\mathsf{s}_1^0 x\cdot\epsilon_{01}\right) && \Delta, \mathrm{C}_3\\
&\leq \mathsf{c}_2\left(\mathsf{d}_{02}\cdot\mathsf{c}_0\mathsf{c}_1\left(\mathsf{s}_1^0 x\cdot\epsilon_{21}\right)\right) && 46, \texttt{1.2.7}, \mathrm{D}32\\
&\leq \mathsf{c}_2\mathsf{c}_1\left(\mathsf{d}_{02}\cdot\mathsf{s}_1^0 x\cdot\epsilon_{21}\right) && \mathrm{C}_4, \Delta, \mathrm{C}_3\\
&\leq \mathsf{c}_1\left(\mathsf{s}_1^0 x\cdot\epsilon_{01}\right) && 46, \mathrm{C}_4, \Delta\\
&\leq \breve{x} && \mathrm{D}32\\
&\leq \mathsf{d}_{0\langle 0\rangle 0\langle 0\rangle}\cdot\mathsf{d}_{0\langle 1\rangle 0\langle 1\rangle} = \mathrm{1} && \mathsf{BA}, 44, \Delta
\end{aligned}$$

We shall prove conditions I-IV from [JT51, Definition 4.1] to show that $\mathfrak{J}\mathfrak{A}\in\mathsf{RA}$. These conditions are well-known to be equivalent to the RA axioms. For proof, we refer the reader to [CT51, Theorem 2.2] or [Mad06, *314*].

(I) $\langle J, +, \dot{-}\rangle \in \mathsf{BA}$.

(II) $(x\circ y)\circ z = x\circ(y\circ z)$ for any $x, y, z\in J$.

(III) $\mathrm{1'}\circ x = x = x\circ\mathrm{1'}$ for every $x\in J$.

(IV) $(x\circ y)\cdot z = 0 \Leftrightarrow (\breve{x}\circ z)\cdot y = 0 \Leftrightarrow (z\circ\breve{y})\cdot x = 0$ for any $x, y, z\in J$.

Condition (I) has already been established (by Lemma 38), so we begin with condition (II).

Proof (Condition II) To begin, we require some technical lemmas. These employ symmetry in the definitions in order to shorten the proof. Both are taken from [Ném85] and [Ném86], but are formulated there in logical terms. To begin, we work with terms of the form $\mathsf{s}^0_{1\langle i\rangle}(x\circ y)$ where $i, j = \{0, 1\}$. The following is proved as `(4)` in [Ném85, p. 45] and as `(5)` in [Ném86, p. 56].

$$\mathsf{c}_0\left(\mathsf{d}_{01\langle i\rangle}\cdot\mathsf{c}_1\left(\mathsf{s}^0_{1\langle i\rangle}x\cdot\mathsf{s}^0_{1\langle j\rangle}y\cdot\delta_{01}\right)\right)\leq\mathsf{c}_2\left(\mathsf{d}_{2\langle j\rangle 1\langle j\rangle}\cdot\mathsf{s}^0_{2\langle i0\rangle}x\cdot\mathsf{s}^0_{2\langle i1\rangle}y\cdot\delta_{1\langle i\rangle 2\langle i\rangle}\right) \tag{2.9}$$

$$\begin{aligned}
&\mathsf{c}_0\left(\mathsf{d}_{01\langle i\rangle}\cdot\mathsf{c}_1\left(\mathsf{s}^0_{1\langle i\rangle}x\cdot\mathsf{s}^0_{1\langle j\rangle}y\cdot\delta_{01}\right)\right)\\
&\leq \mathsf{c}_0\left(\mathsf{d}_{01\langle i\rangle}\cdot\mathsf{c}_2\left(\mathsf{s}^0_{2\langle i\rangle}x\cdot\mathsf{s}^0_{2\langle j\rangle}y\cdot\delta_{02}\right)\right) && 48\\
&\leq \mathsf{c}_0\mathsf{c}_2\left(\mathsf{d}_{01\langle i\rangle}\cdot\mathsf{s}^0_{2\langle i\rangle}x\cdot\mathsf{s}^0_{2\langle j\rangle}y\cdot\delta_{02}\right) && \mathrm{C}_3, \Delta\\
&\leq \mathsf{c}_2\left(\mathsf{s}^0_{2\langle i\rangle}x\cdot\mathsf{s}^0_{2\langle j\rangle}y\cdot\delta_{1\langle i\rangle 2}\right) && 46, \texttt{1.2.7}, \Delta
\end{aligned}$$

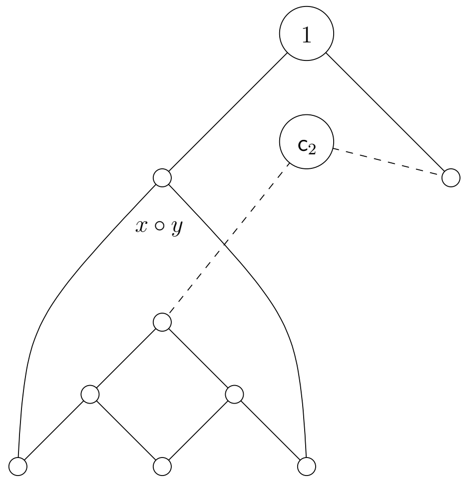

Figure 2.3 An illustration of (2.9) with $i = 0$. Circles containing a cylindrification operation represent cylindrified dimensions. Thus we can see the dimension set of the terms involved is $\{1\}$. Lines emanating from the left side of a point indicate the projection p_0 and similarly p_1 on the right. Dotted lines indicate the right hand side of the inequality.

$$
\begin{aligned}
&\leq \mathsf{c}_2\mathsf{c}_0\left(\mathsf{d}_{02}\cdot \mathsf{s}^0_{2\langle i\rangle}x\cdot \mathsf{s}^0_{2\langle j\rangle}y\cdot \delta_{1\langle i\rangle 2}\right) && \texttt{1.5.8(i)}\\
&\leq \mathsf{c}_2\mathsf{c}_0\left(\mathsf{d}_{02}\cdot \mathsf{s}^0_{2\langle i\rangle}x\cdot \mathsf{s}^0_{2\langle j\rangle}y\cdot \delta_{1\langle i\rangle 0}\right) && 46, \texttt{1.2.7}\\
&\leq \mathsf{c}_0\left(\mathsf{c}_2\left(\mathsf{d}_{02}\cdot \mathsf{s}^0_{2\langle i\rangle}x\cdot \mathsf{s}^0_{2\langle j\rangle}y\right)\cdot \delta_{1\langle i\rangle 0}\right) && \mathrm{C}_4, \mathrm{C}_3, \Delta\\
&\leq \mathsf{c}_0\left(\mathsf{c}_1\left(\mathsf{d}_{01}\cdot \mathsf{s}^0_{1\langle i\rangle}x\cdot \mathsf{s}^0_{1\langle j\rangle}y\right)\cdot \delta_{1\langle i\rangle 0}\right) && 48\\
&\leq \mathsf{c}_0\mathsf{c}_2\left(\mathsf{d}_{2\langle i\rangle 0}\cdot \mathsf{d}_{2\langle j\rangle 1\langle j\rangle}\cdot \mathsf{c}_1\left(\mathsf{d}_{01}\cdot \mathsf{s}^0_{1\langle i\rangle}x\cdot \mathsf{s}^0_{1\langle j\rangle}y\right)\cdot \delta_{1\langle i\rangle 0}\right) && \mathrm{Ax}_2, \mathrm{C}_3\\
&\leq \mathsf{c}_2\left(\mathsf{d}_{1\langle j\rangle 2\langle j\rangle}\cdot \mathsf{c}_1\left(\mathsf{d}_{02\langle i\rangle}\cdot \mathsf{d}_{01}\cdot \mathsf{s}^0_{1\langle i\rangle}x\cdot \mathsf{s}^0_{1\langle j\rangle}y\right)\cdot \delta_{1\langle i\rangle 2\langle i\rangle}\right) && 46, \mathrm{C}_3\\
&\leq \mathsf{c}_2\left(\mathsf{d}_{1\langle j\rangle 2\langle j\rangle}\cdot \mathsf{c}_0\mathsf{c}_1\left(\mathsf{s}^0_{2\langle ii\rangle}x\cdot \mathsf{s}^0_{2\langle ij\rangle}y\right)\cdot \delta_{1\langle i\rangle 2\langle i\rangle}\right) && \mathrm{Ax}_1, 47\\
&\leq \mathsf{c}_2\left(\mathsf{d}_{1\langle j\rangle 2\langle j\rangle}\cdot \mathsf{s}^0_{2\langle ii\rangle}x\cdot \mathsf{s}^0_{2\langle ij\rangle}y\cdot \delta_{1\langle i\rangle 2\langle i\rangle}\right) && \Delta
\end{aligned}
$$

The next lemma enables us to recover the term δ_{01} after invoking Equation (2.9). The following is assumed as $(\mathtt{Ax}_8)$ in [Ném85, p. 48] and proved as (S13) in [Ném86, p. 53].

$$
\begin{aligned}
&\mathsf{d}_{0\langle i\rangle 2\langle iii\rangle}\cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle}\cdot \mathsf{d}_{2\langle iij\rangle 2\langle iji\rangle}\cdot \mathsf{d}_{2\langle ijj\rangle 2\langle ji\rangle}\leq\\
&\qquad \mathsf{c}_1\left(\mathsf{d}_{1\langle i\rangle 2\langle ii\rangle}\cdot \mathsf{c}_0\left(\mathsf{d}_{01\langle j\rangle}\cdot \mathsf{c}_1\left(\mathsf{d}_{1\langle i\rangle 2\langle ij\rangle}\cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle}\cdot \delta_{01}\right)\right)\cdot \delta_{01}\right)
\end{aligned}
\tag{2.10}
$$

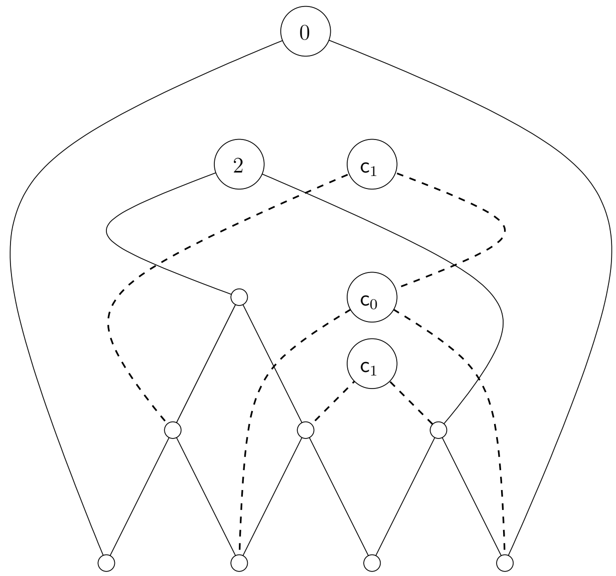

Figure 2.4 An illustration of (2.10) with $i = 0$. When we employ this lemma, x, y, z will be represented by the second row from the bottom. The corresponding figure for $i = 1$ would be a mirrored reflection of this one.

$$
\begin{aligned}
& \mathsf{d}_{0\langle i\rangle 2\langle iii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{2\langle iij\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{2\langle ijj\rangle 2\langle ji\rangle} \\
& \leq \mathsf{d}_{0\langle i\rangle 2\langle iii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{2\langle iij\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{2\langle ijj\rangle 2\langle ji\rangle} \\
& \quad \mathsf{c}_0 \left(\mathsf{d}_{0\langle i\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{c}_1 \left(\mathsf{d}_{1\langle i\rangle 2\langle ii\rangle} \cdot \mathsf{d}_{1\langle j\rangle 0}\right) \cdot \mathsf{c}_1 \left(\mathsf{d}_{1\langle i\rangle 2\langle ij\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle}\right)\right) && \mathrm{Ax}_2 \\
& \leq \mathsf{d}_{0\langle i\rangle 2\langle iii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{2\langle iij\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{2\langle ijj\rangle 2\langle ji\rangle} \cdot \\
& \quad \mathsf{c}_0 \left(\mathsf{d}_{0\langle i\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{c}_1 \left(\mathsf{d}_{1\langle i\rangle 2\langle ii\rangle} \cdot \mathsf{d}_{1\langle j\rangle 0} \cdot \mathsf{c}_1 \left(\mathsf{d}_{1\langle i\rangle 2\langle ij\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle}\right)\right)\right) && \mathrm{C}_3 \\
& \leq \mathsf{d}_{0\langle i\rangle 2\langle iii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{2\langle iij\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{2\langle ijj\rangle 2\langle ji\rangle} \cdot \\
& \quad \mathsf{c}_1\mathsf{c}_0 \left(\mathsf{d}_{0\langle i\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{1\langle i\rangle 2\langle ii\rangle} \cdot \mathsf{d}_{1\langle j\rangle 0} \cdot \mathsf{c}_1 \left(\mathsf{d}_{1\langle i\rangle 2\langle ij\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle}\right)\right) && \mathrm{C}_3, \mathrm{C}_4 \\
& \leq \mathsf{c}_1 \left(\mathsf{d}_{0\langle i\rangle 2\langle iii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{2\langle iij\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{2\langle ijj\rangle 2\langle ji\rangle} \cdot \right. \\
& \quad \left. \mathsf{c}_0 \left(\mathsf{d}_{0\langle i\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{1\langle i\rangle 2\langle ii\rangle} \cdot \mathsf{d}_{1\langle j\rangle 0} \cdot \mathsf{c}_1 \left(\mathsf{d}_{1\langle i\rangle 2\langle ij\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle}\right)\right)\right) && \Delta \\
& \leq \mathsf{c}_1 \left(\mathsf{d}_{0\langle i\rangle 2\langle iii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{2\langle iij\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{1\langle i\rangle 2\langle ii\rangle} \cdot \right. \\
& \quad \left. \mathsf{c}_0 \left(\mathsf{d}_{0\langle i\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{1\langle j\rangle 0} \cdot \mathsf{c}_1 \left(\mathsf{d}_{2\langle ijj\rangle 2\langle ji\rangle} \cdot \mathsf{d}_{1\langle i\rangle 2\langle ij\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle}\right)\right)\right) && \mathrm{C}_3
\end{aligned}
$$

$$\begin{aligned}
&\leq \mathsf{c}_1 \left(\mathsf{d}_{0\langle i\rangle 2\langle iii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{2\langle iij\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{1\langle i\rangle 2\langle ii\rangle} \cdot \right.\\
&\quad \mathsf{c}_0 \left(\mathsf{d}_{1\langle j\rangle 0} \cdot \mathsf{d}_{1\langle ji\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{1\langle jj\rangle 2\langle jj\rangle} \cdot \right.\\
&\quad \left.\left.\mathsf{c}_1 \left(\mathsf{d}_{0\langle i\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{2\langle ijj\rangle 2\langle ji\rangle} \cdot \mathsf{d}_{1\langle i\rangle 2\langle ij\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle} \right)\right)\right) && \mathrm{Ax}_1\\
&\leq \mathsf{c}_1 \left(\mathsf{d}_{0\langle i\rangle 2\langle iii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{2\langle iij\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{1\langle i\rangle 2\langle ii\rangle} \cdot \mathsf{d}_{1\langle ji\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{1\langle jj\rangle 2\langle jj\rangle} \cdot \right.\\
&\quad \left.\mathsf{c}_0 \left(\mathsf{d}_{1\langle j\rangle 0} \cdot \mathsf{c}_1 \left(\mathsf{d}_{0\langle i\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{2\langle ijj\rangle 2\langle ji\rangle} \cdot \mathsf{d}_{1\langle i\rangle 2\langle ij\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle} \right)\right)\right) && \mathrm{C}_3\\
&\leq \mathsf{c}_1 \left(\mathsf{d}_{1\langle i\rangle 2\langle ii\rangle} \cdot \mathsf{c}_0 \left(\mathsf{d}_{01\langle j\rangle} \cdot \mathsf{c}_1 \left(\mathsf{d}_{1\langle i\rangle 2\langle ij\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle} \cdot \delta_{01}\right)\right) \cdot \delta_{01}\right) && \mathrm{Ax}_1
\end{aligned}$$

We are now prepared to prove that $\circ$ is associative in $\mathfrak{JA}$. Let $x, y, z \in J$ be arbitrary.

$$\begin{aligned}
&\mathsf{c}_1 \left(\mathsf{s}^0_{1\langle i\rangle}\mathsf{c}_1 \left(\mathsf{s}^0_{1\langle i\rangle}x \cdot \mathsf{s}^0_{1\langle j\rangle}y \cdot \delta_{01}\right) \cdot \mathsf{s}^0_{1\langle j\rangle}z \cdot \delta_{01}\right)\\
&\leq \mathsf{c}_1 \left(\mathsf{c}_2 \left(\mathsf{d}_{1\langle j\rangle 2\langle j\rangle} \cdot \mathsf{s}^0_{2\langle ii\rangle}x \cdot \mathsf{s}^0_{2\langle ij\rangle}y \cdot \delta_{1\langle i\rangle 2\langle i\rangle}\right) \cdot \mathsf{s}^0_{1\langle j\rangle}z \cdot \delta_{01}\right) && (2.9)\\
&\leq \mathsf{c}_1\mathsf{c}_2 \left(\mathsf{s}^0_{2\langle ii\rangle}x \cdot \mathsf{s}^0_{2\langle ij\rangle}y \cdot \mathsf{s}^0_{1\langle j\rangle}z \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle} \cdot \delta_{1\langle i\rangle 2\langle i\rangle} \cdot \delta_{01}\right) && \mathrm{C}_3\\
&\leq \mathsf{c}_1\mathsf{c}_2 \left(\mathsf{s}^0_{2\langle ii\rangle}x \cdot \mathsf{s}^0_{2\langle ij\rangle}y \cdot \mathsf{s}^0_{2\langle j\rangle}z \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle} \cdot \delta_{1\langle i\rangle 2\langle i\rangle} \cdot \delta_{01}\right) && 47\\
&\leq \mathsf{c}_1\mathsf{c}_2 \left(\mathsf{s}^0_{2\langle ii\rangle}x \cdot \mathsf{s}^0_{2\langle ij\rangle}y \cdot \mathsf{s}^0_{2\langle j\rangle}z \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle} \cdot \mathsf{d}_{1\langle ii\rangle 2\langle iii\rangle} \cdot \mathsf{d}_{1\langle ij\rangle 2\langle ijj\rangle} \cdot \mathsf{d}_{2\langle iij\rangle 2\langle iji\rangle} \cdot \right.\\
&\quad \left.\mathsf{d}_{0\langle i\rangle 1\langle ii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 1\langle jj\rangle} \cdot \mathsf{d}_{1\langle ij\rangle 1\langle ji\rangle} \right) && \mathrm{Ax}_1, \mathrm{D}32\\
&\leq \mathsf{c}_2 \left(\mathsf{s}^0_{2\langle ii\rangle}x \cdot \mathsf{s}^0_{2\langle ij\rangle}y \cdot \mathsf{s}^0_{2\langle j\rangle}z \cdot \mathsf{d}_{0\langle i\rangle 2\langle iii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle jj\rangle} \cdot \mathsf{d}_{2\langle iij\rangle 2\langle iji\rangle} \cdot \mathsf{d}_{2\langle ijj\rangle 2\langle ji\rangle}\right) && \mathrm{Ax}_1, \Delta\\
&\leq \mathsf{c}_2 \left(\mathsf{s}^0_{2\langle ii\rangle}x \cdot \mathsf{s}^0_{2\langle ij\rangle}y \cdot \mathsf{s}^0_{2\langle j\rangle}z \cdot \right.\\
&\quad \left.\mathsf{c}_1 \left(\mathsf{d}_{1\langle i\rangle 2\langle ii\rangle} \cdot \mathsf{c}_0 \left(\mathsf{d}_{01\langle j\rangle} \cdot \mathsf{c}_1 \left(\mathsf{d}_{1\langle i\rangle 2\langle ij\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle} \cdot \delta_{01}\right)\right) \cdot \delta_{01}\right) \right) && (2.10)\\
&\leq \mathsf{c}_2\mathsf{c}_1 \left(\mathsf{s}^0_{2\langle ii\rangle}x \cdot \mathsf{d}_{1\langle i\rangle 2\langle ii\rangle} \cdot \right.\\
&\quad \left.\mathsf{c}_0 \left(\mathsf{d}_{01\langle j\rangle} \cdot \mathsf{c}_1 \left(\mathsf{s}^0_{2\langle ij\rangle}y \cdot \mathsf{d}_{1\langle i\rangle 2\langle ij\rangle} \cdot \mathsf{s}^0_{2\langle j\rangle}z \cdot \mathsf{d}_{1\langle j\rangle 2\langle j\rangle} \cdot \delta_{01}\right) \right) \cdot \delta_{01}\right) && \mathrm{C}_3\\
&\leq \mathsf{c}_2\mathsf{c}_1 \left(\mathsf{s}^0_{1\langle i\rangle}x \cdot \mathsf{c}_0 \left(\mathsf{d}_{01\langle j\rangle} \cdot \mathsf{c}_1 \left(\mathsf{s}^0_{1\langle i\rangle}y \cdot \mathsf{s}^0_{1\langle j\rangle}z \cdot \delta_{01}\right)\right) \cdot \delta_{01}\right) && 47\\
&\leq \mathsf{c}_1 \left(\mathsf{s}^0_{1\langle i\rangle}x \cdot \mathsf{s}^0_{1\langle j\rangle}\mathsf{c}_1 \left(\mathsf{s}^0_{1\langle i\rangle}y \cdot \mathsf{s}^0_{1\langle j\rangle}z \cdot \delta_{01}\right) \cdot \delta_{01}\right) && \mathrm{D}32, \Delta
\end{aligned}$$

Now, if $i = 0$ and $j = 1$ this inequality would read $(x \circ y) \circ z \leq x \circ (y \circ z)$. On the other hand, if $i = 1$ and $j = 0$ it would read $z \circ (y \circ x) \leq (z \circ y) \circ x$. As x, y, z are arbitrary, this is sufficient to establish that $(x \circ y) \circ z = x \circ (y \circ z)$. ♦

Proof (Condition iii) We begin with two technical lemmas in order to exploit some symmetry in the argument. The following are assumed as $(\mathtt{Ax}_9)$ and $(\mathtt{Ax}_{10})$ respectively in [Ném85, p. 50]. Let $\{i, j\} = \{0, 1\}$.

$$\mathsf{s}^0_{1\langle i\rangle}\mathrm{1'} \cdot \delta_{01} \leq \mathsf{d}_{01\langle j\rangle} \tag{2.11}$$

$$
\begin{aligned}
\mathsf{s}^0_{1\langle i\rangle}\dot{1}' \cdot \delta_{01} &\le \mathsf{c}_0\left(\mathsf{d}_{01\langle i\rangle} \cdot \mathsf{d}_{0\langle i\rangle 0\langle j\rangle}\right) \cdot \mathsf{d}_{0\langle i\rangle 1\langle ii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 1\langle jj\rangle} \cdot \mathsf{d}_{1\langle ij\rangle 1\langle ji\rangle} && \text{Def}\\
&\le \mathsf{d}_{1\langle ii\rangle 1\langle ij\rangle} \cdot \mathsf{d}_{0\langle i\rangle 1\langle ii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 1\langle jj\rangle} \cdot \mathsf{d}_{1\langle ij\rangle 1\langle ji\rangle} && \mathrm{Ax}_1, \Delta\\
&\le \mathsf{d}_{0\langle i\rangle 1\langle ij\rangle} \cdot \mathsf{d}_{0\langle j\rangle 1\langle jj\rangle} \cdot \mathsf{d}_{1\langle ij\rangle 1\langle ji\rangle} && \mathrm{Ax}_1\\
&\le \mathsf{d}_{0\langle i\rangle 1\langle ji\rangle} \cdot \mathsf{d}_{0\langle j\rangle 1\langle jj\rangle} && \mathrm{Ax}_1\\
&\le \mathsf{d}_{01\langle j\rangle} && \mathrm{Ax}_3
\end{aligned}
$$

Next we provide a useful upper bound for $\dot{1}$. Since $x \le \dot{1}$ for all $x \in J$, we will have this as an upper bound for all elements of our universe as well. Let $\{i, j\} = \{0, 1\}$.

$$
\dot{1} \le \mathsf{c}_1\left(\mathsf{d}_{01\langle j\rangle} \cdot \dot{1}'_{1\langle i\rangle} \cdot \delta_{01}\right) \tag{2.12}
$$

$$
\begin{aligned}
\dot{1} &= \mathsf{d}_{0\langle i\rangle 0\langle i\rangle} \cdot \mathsf{d}_{0\langle j\rangle 0\langle j\rangle}\\
&\le \mathsf{c}_2\left(\mathsf{d}_{2\langle i\rangle 0\langle i\rangle} \cdot \mathsf{d}_{2\langle j\rangle 0\langle i\rangle}\right) && \mathrm{Ax}_2\\
&\le \mathsf{c}_2\mathsf{c}_1\left(\mathsf{d}_{1\langle i\rangle 2} \cdot \mathsf{d}_{1\langle j\rangle 0} \cdot \mathsf{d}_{2\langle i\rangle 0\langle i\rangle} \cdot \mathsf{d}_{2\langle j\rangle 0\langle i\rangle}\right) && \mathrm{Ax}_2, \mathrm{C}_3\\
&\le \mathsf{c}_2\mathsf{c}_1\left(\mathsf{d}_{01\langle j\rangle} \cdot \mathsf{d}_{1\langle i\rangle 2} \cdot \mathsf{d}_{1\langle ji\rangle 0\langle i\rangle} \cdot \mathsf{d}_{1\langle jj\rangle 0\langle j\rangle} \cdot \mathsf{d}_{2\langle i\rangle 0\langle i\rangle} \cdot \mathsf{d}_{2\langle j\rangle 0\langle i\rangle}\right) && 44, \mathrm{Ax}_1\\
&\le \mathsf{c}_1\left(\mathsf{d}_{01\langle j\rangle} \cdot \mathsf{d}_{1\langle ji\rangle 0\langle i\rangle} \cdot \mathsf{d}_{1\langle jj\rangle 0\langle j\rangle} \cdot \mathsf{d}_{1\langle ii\rangle 0\langle i\rangle} \cdot \mathsf{d}_{1\langle ij\rangle 0\langle i\rangle}\right) && \mathrm{Ax}_1, \Delta\\
&\le \mathsf{c}_1\left(\mathsf{d}_{01\langle j\rangle} \cdot \mathsf{d}_{1\langle ii\rangle 1\langle ij\rangle} \cdot \delta_{01}\right) && \mathrm{Ax}_1, \mathrm{D}32\\
&\le \mathsf{c}_1\left(\mathsf{d}_{01\langle j\rangle} \cdot \mathsf{c}_0\left(\mathsf{d}_{01\langle i\rangle} \cdot \mathsf{d}_{1\langle ii\rangle 1\langle ij\rangle}\right) \cdot \delta_{01}\right) && 44, \Delta, 42\\
&\le \mathsf{c}_1\left(\mathsf{d}_{01\langle j\rangle} \cdot \mathsf{c}_0\left(\mathsf{d}_{01\langle i\rangle} \cdot \mathsf{d}_{0\langle i\rangle 0\langle j\rangle}\right) \cdot \delta_{01}\right) && \mathrm{Ax}_1\\
&\le \mathsf{c}_1\left(\mathsf{d}_{01\langle j\rangle} \cdot \mathsf{s}^0_{1\langle i\rangle}\dot{1}' \cdot \delta_{01}\right) && \mathrm{D}32
\end{aligned}
$$

The proof that $\dot{1}'$ is an identity for $\circ$ is now very short. Once again, let $x \in J$ and $\{i, j\} = \{0, 1\}$.

$$
\begin{aligned}
x &\le x \cdot \mathsf{c}_1\left(\mathsf{d}_{01\langle j\rangle} \cdot \mathsf{s}^0_{1\langle i\rangle}\dot{1}' \cdot \delta_{01}\right) && (2.12)\\
&\le \mathsf{c}_1\left(\mathsf{d}_{01\langle j\rangle} \cdot x \cdot \mathsf{s}^0_{1\langle i\rangle}\dot{1}' \cdot \delta_{01}\right) && \mathrm{C}_3\\
&\le \mathsf{c}_1\left(\mathsf{c}_0\left(\mathsf{d}_{01\langle j\rangle} \cdot x\right) \cdot \mathsf{s}^0_{1\langle i\rangle}\dot{1}' \cdot \delta_{01}\right) && \mathrm{C}_2\\
&\le \mathsf{c}_1\left(\mathsf{s}^0_{1\langle j\rangle}x \cdot \mathsf{s}^0_{1\langle i\rangle}\dot{1}' \cdot \delta_{01}\right) && \mathrm{D}32\\
&\le \mathsf{c}_1\left(\mathsf{s}^0_{1\langle j\rangle}x \cdot \mathsf{d}_{01\langle j\rangle}\right) && (2.11)\\
&\le \mathsf{c}_1 x = x && 49, \Delta
\end{aligned}
$$

The above simultaneously establishes that $x = \dot{1}' \circ x$ (when $i = 0$ and $j = 1$) and that $x = x \circ \dot{1}'$ (when $i = 1$ and $j = 0$). ♦

PROOF (CONDITION IV) We begin with a lemma from [Ném85], stated and proved therein as `(10)` on page 51. Roughly, we generalize the definition of $\breve{x}$ to elements of the form $\mathsf{s}^0_{\lambda\langle i\rangle}x$. Let $\kappa, \lambda = \{1,2\}$ and $i, j \in S_2$.

$$\mathsf{s}^0_{\kappa\langle i\rangle}x \cdot \epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle} \leq \mathsf{s}^0_{\lambda\langle j\rangle}\breve{x} \tag{2.13}$$

$$\begin{aligned}
\mathsf{s}^0_{\kappa\langle i\rangle}x \cdot \epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle} &= \mathsf{c}_0\left(\mathsf{d}_{0\lambda\langle j\rangle} \cdot x\right) \cdot \epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle} && \text{D32}\\
&\leq \mathsf{c}_0\left(\mathsf{d}_{0\lambda\langle j\rangle} \cdot x \cdot \epsilon_{\kappa\langle i\rangle\lambda\langle j\rangle}\right) && \text{C}_3\\
&\leq \mathsf{s}^0_{\lambda\langle j\rangle}\left(x \cdot \epsilon_{\kappa\langle i\rangle 0}\right) && \text{D32, 46}\\
&\leq \mathsf{s}^0_{\lambda\langle j\rangle}\mathsf{c}_\lambda\left(\mathsf{d}_{\lambda\kappa\langle i\rangle} \cdot x \cdot \epsilon_{\kappa\langle i\rangle 0}\right) && 42\\
&\leq \mathsf{s}^0_{\lambda\langle j\rangle}\mathsf{c}_\lambda\left(x \cdot \epsilon_{\lambda 0}\right) && 46\\
&= \mathsf{s}^0_{\lambda\langle j\rangle}\mathsf{c}_1\left(x \cdot \epsilon_{10}\right) && 48\\
&= \mathsf{s}^0_{\lambda\langle j\rangle}\breve{x} && \text{D32}
\end{aligned}$$

We also prove a lemma (not found in [Ném85] or [Ném86]) which allows us to recover the term δ_{01} at one point in our proof.

$$\mathsf{d}_{02\langle 1j\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle 0\rangle} \cdot \epsilon_{1\langle i\rangle 2\langle 1i\rangle} \cdot \delta_{2\langle 0\rangle 2\langle 1\rangle} \leq \delta_{01} \tag{2.14}$$

$$\begin{aligned}
&\mathsf{d}_{02\langle 1j\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle 0\rangle} \cdot \epsilon_{1\langle i\rangle 2\langle 1i\rangle} \cdot \delta_{2\langle 0\rangle 2\langle 1\rangle}\\
&= \mathsf{d}_{02\langle 1j\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle 0\rangle} \cdot \left(\mathsf{d}_{1\langle ii\rangle 2\langle 1ij\rangle} \cdot \mathsf{d}_{1\langle ij\rangle 2\langle 1ii\rangle}\right) \cdot \left(\mathsf{d}_{2\langle 0i\rangle 2\langle 1ii\rangle} \cdot \mathsf{d}_{2\langle 0j\rangle 2\langle 1jj\rangle} \cdot \mathsf{d}_{2\langle 1ij\rangle 2\langle 1ji\rangle}\right) && \text{D32}\\
&\leq \left(\mathsf{d}_{0\langle i\rangle 2\langle 1ji\rangle} \cdot \mathsf{d}_{0\langle j\rangle 2\langle 1jj\rangle}\right) \cdot \left(\mathsf{d}_{1\langle ji\rangle 2\langle 0i\rangle} \cdot \mathsf{d}_{1\langle jj\rangle 2\langle 0j\rangle}\right) \cdot\\
&\qquad \left(\mathsf{d}_{1\langle ii\rangle 2\langle 1ij\rangle} \cdot \mathsf{d}_{1\langle ij\rangle 2\langle 1ii\rangle}\right) \cdot \left(\mathsf{d}_{2\langle 0i\rangle 2\langle 1ii\rangle} \cdot \mathsf{d}_{2\langle 0j\rangle 2\langle 1jj\rangle} \cdot \mathsf{d}_{2\langle 1ij\rangle 2\langle 1ji\rangle}\right) && 44, \text{Ax}_1\\
&= \left(\mathsf{d}_{0\langle i\rangle 2\langle 1ji\rangle} \cdot \mathsf{d}_{2\langle 1ij\rangle 2\langle 1ji\rangle} \cdot \mathsf{d}_{1\langle ii\rangle 2\langle 1ij\rangle}\right) \cdot\\
&\qquad \left(\mathsf{d}_{0\langle j\rangle 2\langle 1jj\rangle} \cdot \mathsf{d}_{2\langle 0j\rangle 2\langle 1jj\rangle} \cdot \mathsf{d}_{1\langle jj\rangle 2\langle 0j\rangle}\right) \cdot \left(\mathsf{d}_{1\langle ij\rangle 2\langle 1ii\rangle} \cdot \mathsf{d}_{2\langle 0i\rangle 2\langle 1ii\rangle} \cdot \mathsf{d}_{1\langle ji\rangle 2\langle 0i\rangle}\right) && \mathsf{BA}\\
&\leq \mathsf{d}_{0\langle i\rangle 1\langle ii\rangle} \cdot \mathsf{d}_{0\langle j\rangle 1\langle jj\rangle} \cdot \mathsf{d}_{1\langle ij\rangle 1\langle ji\rangle} && \text{Ax}_1\\
&= \delta_{01} && \text{D32}
\end{aligned}$$

By `1.2.1`, it is sufficient to show that $\mathsf{c}_0\left((x \circ y) \cdot z\right) = \mathsf{c}_0\left((z \circ \breve{y}) \cdot x\right) = \mathsf{c}_0\left((\breve{x} \circ z) \cdot y\right)$ (since $\mathsf{c}_0 t = 0$ iff $t = 0$). To that end, we establish the following inequalities.

$$\mathsf{c}_0\left((x \circ y) \cdot z\right) \leq \mathsf{c}_0\left((\breve{x} \circ z) \cdot y\right) \tag{2.15}$$

$$\mathsf{c}_0\left((x \circ y) \cdot z\right) \leq \mathsf{c}_0\left((z \circ \breve{y}) \cdot x\right) \tag{2.16}$$

$$\breve{\breve{x}} \leq x \tag{2.17}$$

The proofs of (2.15) and (2.16) are quite similar in character. Let $\{i, j\} = \{0, 1\}$.

$$
\begin{aligned}
&\mathsf{c}_0\left((x \circ y) \cdot z\right) \\
&= \mathsf{c}_0\left(\mathsf{c}_1\left(\mathsf{s}^0_{1\langle 0\rangle}x \cdot \mathsf{s}^0_{1\langle 1\rangle}y \cdot \delta_{01}\right) \cdot z\right) && \text{D32} \\
&\le \mathsf{c}_0\mathsf{c}_1\left(\mathsf{s}^0_{1\langle 0\rangle}x \cdot \mathsf{s}^0_{1\langle 1\rangle}y \cdot z \cdot \delta_{01}\right) && \mathrm{C}_3 \\
&\le \mathsf{c}_0\mathsf{c}_1\mathsf{c}_2\left(\mathsf{s}^0_{1\langle 0\rangle}x \cdot \mathsf{s}^0_{1\langle 1\rangle}y \cdot z \cdot \mathsf{d}_{2\langle 0\rangle 0} \cdot \mathsf{d}_{2\langle 1\rangle 1} \cdot \delta_{01}\right) && \mathrm{Ax}_2, \mathrm{C}_3 \\
&\le \mathsf{c}_2\left(\mathsf{s}^0_{2\langle 10\rangle}x \cdot \mathsf{s}^0_{2\langle 11\rangle}y \cdot \mathsf{s}^0_{2\langle 0\rangle}z \cdot \delta_{2\langle 0\rangle 2\langle 1\rangle}\right) && 47, \mathrm{C}_2, 46, \Delta \\
&\le \mathsf{c}_2\mathsf{c}_0\left(\mathsf{s}^0_{2\langle 10\rangle}x \cdot \mathsf{s}^0_{2\langle 11\rangle}y \cdot \mathsf{s}^0_{2\langle 0\rangle}z \cdot \epsilon_{02\langle 1i\rangle} \cdot \delta_{2\langle 0\rangle 2\langle 1\rangle}\right) && \mathrm{Ax}_2, \mathrm{C}_3 \\
&\le \mathsf{c}_2\mathsf{c}_0\mathsf{c}_1\left(\mathsf{s}^0_{2\langle 10\rangle}x \cdot \mathsf{s}^0_{2\langle 11\rangle}y \cdot \mathsf{s}^0_{2\langle 0\rangle}z \cdot \mathsf{d}_{1\langle j\rangle 2\langle 0\rangle} \cdot \mathsf{d}_{1\langle i\rangle 0} \cdot \epsilon_{02\langle 1i\rangle} \cdot \delta_{2\langle 0\rangle 2\langle 1\rangle}\right) && \mathrm{Ax}_2, \mathrm{C}_3 \\
&\le \mathsf{c}_2\mathsf{c}_1\left(\mathsf{s}^0_{2\langle 10\rangle}x \cdot \mathsf{s}^0_{2\langle 11\rangle}y \cdot \mathsf{s}^0_{2\langle 0\rangle}z \cdot \mathsf{d}_{1\langle j\rangle 2\langle 0\rangle} \cdot \epsilon_{1\langle i\rangle 2\langle 1i\rangle} \cdot \delta_{2\langle 0\rangle 2\langle 1\rangle}\right) && 46, \Delta \\
&\le \mathsf{c}_2\mathsf{c}_1\mathsf{c}_0\left(\mathsf{s}^0_{2\langle 10\rangle}x \cdot \mathsf{s}^0_{2\langle 11\rangle}y \cdot \mathsf{s}^0_{2\langle 0\rangle}z \cdot \mathsf{d}_{02\langle 1j\rangle} \cdot \mathsf{d}_{1\langle j\rangle 2\langle 0\rangle} \cdot \epsilon_{1\langle i\rangle 2\langle 1i\rangle} \cdot \delta_{2\langle 0\rangle 2\langle 1\rangle}\right) && 42 \\
&\le \mathsf{c}_2\mathsf{c}_1\mathsf{c}_0\left(\mathsf{s}^0_{2\langle 10\rangle}x \cdot \mathsf{s}^0_{2\langle 11\rangle}y \cdot \mathsf{s}^0_{1\langle j\rangle}z \cdot \mathsf{d}_{02\langle 1j\rangle} \cdot \epsilon_{1\langle i\rangle 2\langle 1i\rangle} \cdot \delta_{01}\right) && 47, (2.14)
\end{aligned}
$$

Now if $i = 0$ and $j = 1$ we have:

$$
\begin{aligned}
\mathsf{c}_0\left((x \circ y) \cdot z\right) &\le \mathsf{c}_2\mathsf{c}_1\mathsf{c}_0\left(\mathsf{s}^0_{2\langle 10\rangle}x \cdot \mathsf{s}^0_{2\langle 11\rangle}y \cdot \mathsf{s}^0_{1\langle 1\rangle}z \cdot \mathsf{d}_{02\langle 11\rangle} \cdot \epsilon_{1\langle 0\rangle 2\langle 10\rangle} \cdot \delta_{01}\right) \\
&\le \mathsf{c}_2\mathsf{c}_1\mathsf{c}_0\left(\mathsf{s}^0_{2\langle 10\rangle}x \cdot y \cdot \mathsf{s}^0_{1\langle 1\rangle}z \cdot \epsilon_{1\langle 0\rangle 2\langle 10\rangle} \cdot \delta_{01}\right) && 49 \\
&\le \mathsf{c}_1\mathsf{c}_0\left(\mathsf{s}^0_{1\langle 0\rangle}\breve{x} \cdot y \cdot \mathsf{s}^0_{1\langle 1\rangle}z \cdot \delta_{01}\right) && (2.13), \Delta \\
&\le \mathsf{c}_0\left(\mathsf{c}_1\left(\mathsf{s}^0_{1\langle 0\rangle}\breve{x} \cdot \mathsf{s}^0_{1\langle 1\rangle}z \cdot \delta_{01}\right) \cdot y\right) && \mathrm{C}_4, \mathrm{C}_3 \\
&\le \mathsf{c}_0\left((\breve{x} \circ z) \cdot y\right) && \text{Def}
\end{aligned}
$$

which concludes the proof of Equation (2.15). If instead $i = 1$ and $j = 0$ then we see:

$$
\begin{aligned}
\mathsf{c}_0\left((x \circ y) \cdot z\right) &\le \mathsf{c}_2\mathsf{c}_1\mathsf{c}_0\left(\mathsf{s}^0_{2\langle 10\rangle}x \cdot \mathsf{s}^0_{2\langle 11\rangle}y \cdot \mathsf{s}^0_{1\langle 0\rangle}z \cdot \mathsf{d}_{02\langle 10\rangle} \cdot \epsilon_{1\langle 1\rangle 2\langle 11\rangle} \cdot \delta_{01}\right) \\
&\le \mathsf{c}_2\mathsf{c}_1\mathsf{c}_0\left(x \cdot \mathsf{s}^0_{2\langle 11\rangle}y \cdot \mathsf{s}^0_{1\langle 0\rangle}z \cdot \epsilon_{1\langle 1\rangle 2\langle 11\rangle} \cdot \delta_{01}\right) && 49 \\
&\le \mathsf{c}_1\mathsf{c}_0\left(x \cdot \mathsf{s}^0_{1\langle 1\rangle}\breve{y} \cdot \mathsf{s}^0_{1\langle 0\rangle}z \cdot \delta_{01}\right) && (2.13), \Delta \\
&\le \mathsf{c}_0\left(\mathsf{c}_1\left(\mathsf{s}^0_{1\langle 0\rangle}z \cdot \mathsf{s}^0_{1\langle 1\rangle}\breve{y} \cdot \delta_{01}\right) \cdot x\right) && \mathrm{C}_4, \mathrm{C}_3 \\
&\le \mathsf{c}_0\left((z \circ \breve{y}) \cdot x\right) && \text{D32}
\end{aligned}
$$

concluding the proof of Equation (2.16). Next we turn our attention to Equation (2.17). From Ax_1 and Ax_3 we see that:

$$
\epsilon_{01} \cdot \epsilon_{20} = \mathsf{d}_{0\langle 0\rangle 1\langle 1\rangle} \cdot \mathsf{d}_{0\langle 1\rangle 1\langle 0\rangle} \cdot \mathsf{d}_{2\langle 0\rangle 0\langle 1\rangle} \cdot \mathsf{d}_{2\langle 1\rangle 0\langle 0\rangle} \le \mathsf{d}_{1\langle 0\rangle 2\langle 0\rangle}\mathsf{d}_{1\langle 1\rangle 2\langle 1\rangle} \le \mathsf{d}_{12}. \tag{2.18}
$$

Now let $x \in J$.

$$
\begin{aligned}
\breve{\breve{x}} &\le \mathsf{c}_1\left(\mathsf{s}^0_1\mathsf{c}_1\left(\mathsf{s}^0_1 x \cdot \epsilon_{01}\right) \cdot \epsilon_{01}\right) && \text{D32}\\
&\le \mathsf{c}_2\left(\mathsf{d}_{02} \cdot \mathsf{c}_1\left(\mathsf{s}^0_1\mathsf{c}_1\left(\mathsf{s}^0_1 x \cdot \epsilon_{01}\right) \cdot \epsilon_{01}\right)\right) && \texttt{1.5.8(i)}\\
&\le \mathsf{c}_2\mathsf{c}_1\left(\mathsf{d}_{02} \cdot \mathsf{s}^0_1\mathsf{c}_1\left(\mathsf{s}^0_1 x \cdot \epsilon_{01}\right) \cdot \epsilon_{01}\right) && \text{C}_3\\
&\le \mathsf{c}_2\mathsf{c}_1\left(\mathsf{d}_{02} \cdot \mathsf{c}_0\left(\mathsf{d}_{01} \cdot \mathsf{c}_1\left(\mathsf{s}^0_1 x \cdot \epsilon_{01}\right)\right) \cdot \epsilon_{21}\right) && 46, \text{D32}\\
&\le \mathsf{c}_2\mathsf{c}_1\left(\mathsf{d}_{02} \cdot \mathsf{c}_0\left(\mathsf{d}_{01} \cdot \mathsf{c}_1\left(\mathsf{s}^0_1 x \cdot \epsilon_{01}\right) \cdot \epsilon_{21}\right)\right) && \text{C}_3\\
&\le \mathsf{c}_2\mathsf{c}_1\left(\mathsf{d}_{02} \cdot \mathsf{c}_0\left(\mathsf{d}_{01} \cdot \mathsf{c}_1\left(\mathsf{s}^0_1 x \cdot \epsilon_{01}\right) \cdot \epsilon_{20}\right)\right) && 46\\
&\le \mathsf{c}_2\mathsf{c}_1\left(\mathsf{d}_{02} \cdot \mathsf{c}_0\left(\mathsf{d}_{01} \cdot \mathsf{c}_1\left(\mathsf{s}^0_1 x \cdot \epsilon_{01} \cdot \epsilon_{20}\right)\right)\right) && \text{C}_3\\
&\le \mathsf{c}_2\mathsf{c}_1\left(\mathsf{d}_{02} \cdot \mathsf{c}_0\left(\mathsf{d}_{01} \cdot \mathsf{c}_1\left(\mathsf{d}_{12} \cdot \mathsf{s}^0_1 x\right)\right)\right) && (2.18)\\
&\le \mathsf{c}_1\mathsf{s}^2_0\mathsf{s}^0_1\mathsf{s}^1_2\mathsf{s}^0_1 x && \text{C}_4, \text{Def}\\
&\le \mathsf{c}_1\mathsf{s}^2_0\mathsf{s}^0_1\mathsf{s}^0_2 x && \texttt{1.5.11(i)}\\
&\le \mathsf{c}_1\mathsf{s}^2_0\mathsf{s}^0_2 x && \texttt{1.5.8(i)}\\
&= \mathsf{c}_1 x = x && \text{D32}, \Delta
\end{aligned}
$$

With the aid of these inequalities, we see that:

$$
\begin{aligned}
\mathsf{c}_0\left((x \circ y) \cdot z\right) &\le \mathsf{c}_0\left((\breve{x} \circ z) \cdot y\right) && (2.15)\\
&\le \mathsf{c}_0\left(\left(\breve{\breve{x}} \circ y\right) \cdot z\right) && (2.15)\\
&\le \mathsf{c}_0\left((x \circ y) \cdot z\right) && (2.17)\\
&\le \mathsf{c}_0\left((z \circ \breve{y}) \cdot x\right) && (2.16)\\
&\le \mathsf{c}_0\left(\left(x \circ \breve{\breve{y}}\right) \cdot z\right) && (2.16)\\
&\le \mathsf{c}_0\left((x \circ y) \cdot z\right) && (2.17)
\end{aligned}
$$

concluding the proof of condition IV. ♦

Conditions I through IV have been proved, thus $\mathfrak{J}\mathfrak{A} \in \mathsf{RA}$ as desired. We begin our proof of Equations (1.1)–(1.3) by examining the elements $\breve{P}_0$ and $\breve{P}_1$. Note that by definition we may write $P_k = \mathsf{d}_{0\langle 0k\rangle 0\langle 1\rangle}$.

$$
\begin{aligned}
\breve{P}_k &= \mathsf{c}_1\left(\mathsf{c}_0\left(\mathsf{d}_{01} \cdot \mathsf{d}_{0\langle 0k\rangle 0\langle 1\rangle}\right) \cdot \epsilon_{01}\right) && \text{D32}\\
&\le \mathsf{c}_1\left(\mathsf{d}_{1\langle 0k\rangle 1\langle 1\rangle} \cdot \mathsf{d}_{0\langle 0\rangle 1\langle 1\rangle} \cdot \mathsf{d}_{0\langle 1\rangle 1\langle 0\rangle}\right) && \text{Ax}_1, \text{D32}\\
&\le \mathsf{d}_{0\langle 1k\rangle 0\langle 0\rangle} && \text{Ax}_1, \Delta\\
&\le \mathsf{c}_1\left(\mathsf{d}_{1\langle 0\rangle 0\langle 1\rangle} \cdot \mathsf{d}_{1\langle 1\rangle 0\langle 0\rangle} \cdot \mathsf{d}_{0\langle 1k\rangle 0\langle 0\rangle}\right) && \text{Ax}_2
\end{aligned}
$$

$$\leq \mathsf{c}_1\left(\epsilon_{01}\cdot \mathsf{d}_{1\langle 0k\rangle 1\langle 1\rangle}\right) \qquad \mathrm{Ax}_1, \mathrm{D}32$$

$$\leq \mathsf{c}_1\left(\mathsf{c}_0\left(\mathsf{d}_{01}\cdot \mathsf{d}_{1\langle 0k\rangle 1\langle 1\rangle}\right)\cdot \epsilon_{01}\right) \qquad \texttt{1.5.8(i)}$$

$$\leq \mathsf{c}_1\left(\mathsf{s}^0_1\mathsf{d}_{0\langle 0k\rangle 0\langle 1\rangle}\cdot \epsilon_{01}\right) = \breve{P}_k \qquad \mathrm{D}32, \mathrm{Ax}_1, \mathrm{D}32$$

Thus $\breve{P}_k = \mathsf{d}_{0\langle 1k\rangle 0\langle 0\rangle}$.

Proof (Equation (1.1))

$$\begin{aligned}
\breve{P}_k \circ P_k &= \mathsf{c}_1\left(\mathsf{s}^0_{1\langle 0\rangle}\mathsf{d}_{0\langle 1k\rangle 0\langle 0\rangle}\cdot \mathsf{s}^0_{1\langle 1\rangle}\mathsf{d}_{0\langle 0k\rangle 0\langle 1\rangle}\cdot \delta_{01}\right) && \mathrm{D}32\\
&\leq \mathsf{c}_1\left(\mathsf{d}_{1\langle 01k\rangle 1\langle 00\rangle}\cdot \mathsf{d}_{1\langle 10k\rangle 1\langle 11\rangle}\cdot \mathsf{d}_{0\langle 0\rangle 1\langle 00\rangle}\cdot \mathsf{d}_{0\langle 1\rangle 1\langle 11\rangle}\cdot \mathsf{d}_{1\langle 01\rangle 1\langle 10\rangle}\right) && \mathrm{Ax}_1, \Delta\\
&\leq \mathsf{c}_1\left(\mathsf{d}_{1\langle 01k\rangle 1\langle 00\rangle}\cdot \mathsf{d}_{1\langle 10k\rangle 1\langle 11\rangle}\cdot \mathsf{d}_{0\langle 0\rangle 1\langle 00\rangle}\cdot \mathsf{d}_{0\langle 1\rangle 1\langle 11\rangle}\cdot \mathsf{d}_{1\langle 01k\rangle 1\langle 10k\rangle}\right) && 44, \mathrm{Ax}_1\\
&\leq \mathsf{c}_1\left(\mathsf{d}_{1\langle 11\rangle 1\langle 00\rangle}\cdot \mathsf{d}_{0\langle 0\rangle 1\langle 00\rangle}\cdot \mathsf{d}_{0\langle 1\rangle 1\langle 11\rangle}\right) && \mathrm{Ax}_1\\
&\leq \mathsf{d}_{0\langle 0\rangle 0\langle 1\rangle} && \mathrm{Ax}_1, \Delta
\end{aligned}$$

Thus $(P_k)^{\smile} \circ P_k \leq 1\text{'}$ as required, ($1\text{'} = \mathsf{d}_{0\langle 0\rangle 0\langle 1\rangle}$ by Definition 32). ♦

Proof (Equation (1.2))

$$\begin{aligned}
1 &\leq \mathsf{c}_1\mathsf{c}_2\left(\mathsf{d}_{1\langle 0\rangle 0\langle 0\rangle}\cdot \mathsf{d}_{1\langle 1\rangle 0}\cdot \mathsf{d}_{2\langle 0\rangle 0}\cdot \mathsf{d}_{2\langle 1\rangle 0\langle 1\rangle}\right) && \mathrm{Ax}_2, \mathrm{C}_3\\
&\leq \mathsf{c}_1\mathsf{c}_2\left(\mathsf{d}_{02\langle 0\rangle}\cdot \mathsf{d}_{1\langle 1\rangle 2\langle 0\rangle}\cdot \mathsf{d}_{1\langle 0\rangle 2\langle 00\rangle}\cdot \mathsf{d}_{2\langle 1\rangle 1\langle 11\rangle}\right) && \mathrm{Ax}_1\\
&\leq \mathsf{c}_1\mathsf{c}_2\left(\mathsf{d}_{02\langle 0\rangle}\cdot \mathsf{c}_0\left(\mathsf{d}_{0\langle 0\rangle 1}\cdot \mathsf{d}_{0\langle 1\rangle 2}\cdot \mathsf{d}_{1\langle 1\rangle 2\langle 0\rangle}\cdot \mathsf{d}_{1\langle 0\rangle 2\langle 00\rangle}\cdot \mathsf{d}_{2\langle 1\rangle 1\langle 11\rangle}\right)\right) && \mathrm{Ax}_2, \mathrm{C}_3\\
&\leq \mathsf{c}_2\left(\mathsf{d}_{02\langle 0\rangle}\cdot \mathsf{c}_0\left(\mathsf{d}_{0\langle 1\rangle 2}\cdot \mathsf{d}_{0\langle 01\rangle 2\langle 0\rangle}\cdot \mathsf{d}_{0\langle 00\rangle 2\langle 00\rangle}\cdot \mathsf{d}_{2\langle 1\rangle 0\langle 011\rangle}\right)\right) && \mathrm{Ax}_1, \Delta\\
&\leq \mathsf{c}_2\left(\mathsf{d}_{02\langle 0\rangle}\cdot \mathsf{c}_1\left(\mathsf{d}_{1\langle 1\rangle 2}\cdot \mathsf{d}_{1\langle 01\rangle 2\langle 0\rangle}\cdot \mathsf{d}_{1\langle 00\rangle 2\langle 00\rangle}\cdot \mathsf{d}_{2\langle 1\rangle 1\langle 011\rangle}\right)\right) && 48\\
&\leq \mathsf{c}_1\mathsf{c}_2\left(\mathsf{d}_{02\langle 0\rangle}\cdot \mathsf{d}_{1\langle 1\rangle 2}\cdot \mathsf{d}_{1\langle 01\rangle 2\langle 0\rangle}\cdot \mathsf{d}_{1\langle 00\rangle 2\langle 00\rangle}\cdot \mathsf{d}_{2\langle 1\rangle 1\langle 011\rangle}\right) && \mathrm{C}_3\\
&\leq \mathsf{c}_1\left(\mathsf{d}_{01\langle 10\rangle}\cdot \mathsf{d}_{1\langle 01\rangle 1\langle 10\rangle}\cdot \mathsf{d}_{1\langle 00\rangle 1\langle 100\rangle}\cdot \mathsf{d}_{1\langle 11\rangle 1\langle 011\rangle}\right) && \mathrm{Ax}_1, \Delta\\
&\leq \mathsf{c}_1\left(\mathsf{d}_{01\langle 10\rangle}\cdot \delta_{01}\right) && \mathrm{Ax}_1, \mathrm{Def}\\
&\leq \mathsf{c}_1\left(\mathsf{c}_0\left(\mathsf{d}_{01\langle 0\rangle}\right)\cdot \mathsf{c}_0\left(\mathsf{d}_{01\langle 1\rangle}\right)\cdot \mathsf{d}_{01\langle 10\rangle}\cdot \delta_{01}\right) && 44, 41\\
&\leq \mathsf{c}_1\left(\mathsf{c}_0\left(\mathsf{d}_{01\langle 0\rangle}\cdot \mathsf{d}_{1\langle 00\rangle 1\langle 010\rangle}\right)\cdot \mathsf{c}_0\left(\mathsf{d}_{01\langle 1\rangle}\cdot \mathsf{d}_{1\langle 11\rangle 1\langle 101\rangle}\right)\cdot \delta_{01}\right) && \mathrm{Ax}_1, \mathrm{C}_3\\
&\leq \mathsf{c}_1\left(\mathsf{c}_0\left(\mathsf{d}_{01\langle 0\rangle}\cdot \mathsf{d}_{0\langle 0\rangle 0\langle 10\rangle}\right)\cdot \mathsf{c}_0\left(\mathsf{d}_{01\langle 1\rangle}\cdot \mathsf{d}_{0\langle 1\rangle 0\langle 01\rangle}\right)\cdot \delta_{01}\right) && \mathrm{Ax}_1\\
&= \breve{P}_0 \circ P_1 && \mathrm{Def}
\end{aligned}$$

The opposite inequality follows from the fact that $\mathsf{RA} \models \breve{P}_0 \circ P_1 \leq 1$, therefore we conclude $\breve{P}_0 \circ P_1 = 1$ as desired. ♦

PROOF (EQUATION (1.3))

$$\begin{aligned}
P_k \circ \breve{P}_k &\leq \mathsf{c}_1 \left(\mathsf{s}^0_{1\langle 0\rangle}\mathsf{d}_{0\langle 0k\rangle 0\langle 1\rangle} \cdot \mathsf{s}^0_{1\langle 1\rangle}\mathsf{d}_{0\langle 1k\rangle 0\langle 0\rangle} \cdot \delta_{01}\right) && \text{Def} \\
&\leq \mathsf{c}_1 \left(\mathsf{d}_{1\langle 00k\rangle 1\langle 01\rangle} \cdot \mathsf{d}_{1\langle 11k\rangle 1\langle 10\rangle} \cdot \delta_{01}\right) && \mathrm{Ax}_1, \Delta \\
&\leq \mathsf{d}_{0\langle 0k\rangle 0\langle 1k\rangle} && \mathrm{Ax}_1, \Delta
\end{aligned}$$

Thus $\left(P_0 \circ \breve{P}_0\right) \cdot \left(P_1 \circ \breve{P}_1\right) \leq \mathsf{d}_{0\langle 00\rangle 0\langle 10\rangle} \cdot \mathsf{d}_{0\langle 01\rangle 0\langle 11\rangle}$, and by Ax_3 this is bounded by $\mathsf{d}_{0\langle 0\rangle 0\langle 1\rangle} = \dot{1}$'. ♦

As $\mathfrak{J}\mathfrak{A} \in \mathsf{RA}$ and Equations (1.1)–(1.3) have been verified, we conclude that $\mathfrak{J}\mathfrak{A} \in \mathsf{Q^+RA}$, as desired.

CHAPTER 3. FIRST-ORDER LOGIC

3.1 First-order logic in $\mathsf{Q}^{+}\mathsf{RA}$

Definition 50 For $\langle \mathfrak{A}, P_0, P_1\rangle \in \mathsf{P}^{+}\mathsf{RA}$ and $\kappa \in \omega$, define:

$$\mathsf{P}_0 := P_0, \qquad \mathsf{P}_{\kappa+1} := P_1 ; \mathsf{P}_\kappa, \qquad \mathsf{u} := 1\text{'} \cdot (P_0 ; 1 \cdot P_1 ; 1)^{-}.$$

Let $X, Y \in \Phi[\mathcal{L}]$, $\mu < \alpha$, and $\kappa, \lambda, \kappa_0, \ldots, \kappa_{\rho(\mu)-1} < \omega$.

$$\mathrm{Dr}(X) := \begin{cases} 1 & \mathrm{In}(X) = \emptyset \\ \mathsf{P}_{\kappa_0} ; \mathsf{u} ; 1 \cdot \ldots \cdot \mathsf{P}_{\kappa_\lambda} ; \mathsf{u} ; 1 & \mathrm{In}(X) = \{\kappa_0, \ldots, \kappa_\lambda\} \end{cases}$$

$$\begin{aligned} \mathrm{Tr}(\mathsf{v}_\kappa = \mathsf{v}_\lambda) &:= (\mathsf{P}_\kappa ; \mathsf{u} \cdot \mathsf{P}_\lambda ; \mathsf{u}) ; 1 \\ \mathrm{Tr}(\mathsf{R}_\mu \mathsf{v}_{\kappa_0} \cdots \mathsf{v}_{\kappa_{\rho(\mu)-1}}) &:= \Big(\mathsf{P}_{\kappa_0} ; \mathsf{u} ; \breve{\mathsf{P}}_0 \cdot \ldots \cdot \mathsf{P}_{\kappa_{\rho(\mu)-1}} ; \mathsf{u} ; \breve{\mathsf{P}}_{\rho(\mu)-1}\Big) ; \mathsf{w}_{\mu+1} ; 1 \\ \mathrm{Tr}(\neg X) &:= (\mathrm{Tr}(X))^{-} \cdot \mathrm{Dr}(X) \\ \mathrm{Tr}(X \vee Y) &:= (\mathrm{Tr}(X) + \mathrm{Tr}(Y)) \cdot \mathrm{Dr}(X \vee Y) \\ \mathrm{Tr}(\exists_{\mathsf{v}_\kappa} X) &:= \Big(\mathsf{P}_{\lambda_0} ; \breve{\mathsf{P}}_{\lambda_0} \cdot \ldots \cdot \mathsf{P}_{\lambda_\mu} ; \breve{\mathsf{P}}_{\lambda_\mu}\Big) ; \mathrm{Tr}(X) && \mathrm{In}(\exists_{\mathsf{v}_\kappa} X) = \{\lambda_0, \ldots, \lambda_\mu\} \\ \mathrm{Tr}(\exists_{\mathsf{v}_\kappa} X) &:= 1 ; \mathrm{Tr}(X) && \mathrm{In}(\exists_{\mathsf{v}_\kappa} X) = \emptyset \end{aligned}$$

□

Theorem 51 ([Mad89a, Theorem 21]) *Let U be a set with $\mathfrak{B}(U) \in \mathsf{TPA}$. Suppose $\mathbf{R} = \langle R_\mu : \mu \in \beta\rangle$ is a countable sequence of relations on U with R_μ having rank ρ_μ, $\mathfrak{U} = \langle U, \mathbf{R}\rangle \in \mathsf{RE}[\mathcal{L}]$, $X \in \Phi[\mathcal{L}]$, and $\mathbf{a} \in {}^{\omega}U$. Define $\mathbf{S} = \langle S_\kappa : \kappa \in \omega\rangle$ thusly:*

1. $S_0 = \Big\{\langle v, v'\rangle \in \ddot{U} \times \ddot{U} : \mathsf{P}_\kappa(v) = a_\kappa \text{ for every } \kappa \in \mathrm{In}(X)\Big\}$,
2. $S_{\mu+1} = \Big\{\langle v, v'\rangle \in \ddot{U} \times \ddot{U} : \big\langle \mathsf{P}_0(v), \ldots, \mathsf{P}_{\rho(\mu)-1}(v)\big\rangle \in R_\mu\Big\}$ *for* $\mu < \alpha$, *and*
3. $S_{\mu+1} = \emptyset$ *whenever* $\alpha \le \mu < \omega$.

Then $\mathfrak{U} \models X[\mathbf{a}]$ if and only if $\mathfrak{B}(U) \models \mathsf{w}_0 \le \mathrm{Tr}(X)[\mathbf{S}]$. In particular, if $X \in \Sigma[\mathcal{L}]$, then $S_0 = \ddot{U} \times \ddot{U}$ and $\mathfrak{U} \models X$ if and only if $\mathfrak{B}(U) \models 1 = \mathrm{Tr}(X)[\mathbf{S}]$. □

Proof Fix any $\mathbf{a} \in {}^{\omega}U$. Define $\mathbf{b} : \ddot{U} \to {}^{\omega}U$ by $(\mathbf{b}(v))_{\kappa} = \mathsf{P}_{\kappa}(v)$ if $\mathsf{P}_{\kappa}(v) \in U$ and $(\mathbf{b}(v))_{\kappa} = a_0$ otherwise. For each $X \in \Phi[\mathcal{L}]$ define $S(X) = \{v \in \ddot{U} : \mathsf{P}_{\kappa}(v) = a_{\kappa} \text{ for } \kappa \in \mathrm{In}(X)\}$. Then $\mathfrak{U} \models X[\mathbf{a}]$ if and only if $\mathfrak{U} \models X[\mathbf{b}(v)]$ for each $v \in S(X)$, since the assignment of variables only matters for those variables which are free in X. Finally, define $D, T : \Phi[\mathcal{L}] \to \mathrm{Sb}\,\ddot{U}$ by

$$D(X) \coloneqq \{v : \mathsf{P}_{\kappa}(v) \in U \text{ for } \kappa \in \mathrm{In}(X)\}$$
$$T(X) \coloneqq \{v \in D(X) : \mathfrak{U} \models X[\mathbf{b}(v)]\}\,.$$

Then $S(X) \subseteq D(X)$, and $\mathfrak{U} \models X[\mathbf{a}]$ if and only if $S(X) \subseteq T(X)$.

By definition, we have that $S_0 = S(X) \times \ddot{U}$ and $\mathrm{Dr}(X) = D(X) \times \ddot{U}$. We proceed by proving that $\mathrm{Tr}(X)[\mathbf{S}] = T(X) \times \ddot{U}$. We use induction on the construction of the formula X, supposing first that X is atomic.

Proof (Case $X = \mathsf{v}_{\kappa} = \mathsf{v}_{\lambda}$) In this case we have $\mathfrak{U} \models X[\mathbf{b}(v)]$ if and only if $\mathsf{P}_{\kappa}(v) = \mathsf{P}_{\lambda}(v)$. Thus

$$T(X) = \{v : \mathsf{P}_{\kappa}(v) = \mathsf{P}_{\lambda}(v) \in U\}\,.$$

We also have in this case that $\mathrm{Tr}(X)$ does not depend on $\mathbf{S}$. Indeed:

$$\mathrm{Tr}(X) = \{\langle v, v' \rangle \,:\, \mathsf{P}_{\kappa}(v) = \mathsf{P}_{\lambda}(v) \in U\}\,.$$

Then clearly $\mathrm{Tr}(X)[\mathbf{S}] = T(X) \times \ddot{U}$ as desired. ♦

Proof (Case $X = \mathsf{R}_{\mu}\mathsf{v}_{\kappa_0} \cdots \mathsf{v}_{\kappa_{\rho(\mu)-1}}$) In this case $\mathfrak{U} \models X[\mathbf{b}(v)]$ iff

$$\left\langle \mathsf{P}_{\kappa_0}(v), \ldots, \mathsf{P}_{\kappa_{\rho(\mu)-1}}(v) \right\rangle \in R_{\mu}.$$

Then

$$T(X) = \left\{v : \left\langle \mathsf{P}_{\kappa_0}(v), \ldots, \mathsf{P}_{\kappa_{\rho(\mu)-1}}(v) \right\rangle \in R_{\mu}\right\}.$$

On the other hand,

$$\begin{aligned}
\mathrm{Tr}(X)[\mathbf{S}] &= \left(\mathsf{P}_{\kappa_0} ; \mathsf{u} ; \breve{\mathsf{P}}_0 \cdot \ldots \cdot \mathsf{P}_{\kappa_{\rho(\mu)-1}} ; \mathsf{u} ; \breve{\mathsf{P}}_{\rho(\mu)-1}\right) ; S_{\mu+1} ; 1 \\
&= \left(\mathsf{P}_{\kappa_0} ; \mathsf{u} ; \breve{\mathsf{P}}_0 \cdot \ldots \cdot \mathsf{P}_{\kappa_{\rho(\mu)-1}} ; \mathsf{u} ; \breve{\mathsf{P}}_{\rho(\mu)-1}\right) ; S_{\mu+1} \\
&= \{\langle v, u \rangle \,:\, \mathsf{P}_{\kappa_i}(v) = \mathsf{P}_i(u) \in U \text{ for } i \in \rho_{\mu}\} \mid \left\{\langle u, v' \rangle \,:\, \left\langle \mathsf{P}_0(u), \ldots, \mathsf{P}_{\rho(\mu)-1}(u) \right\rangle \in R_{\mu}\right\} \\
&= \left\{\langle v, v' \rangle \,:\, \left\langle \mathsf{P}_{\kappa_0}(v), \ldots, \mathsf{P}_{\kappa_{\rho(\mu)-1}}(v) \right\rangle \in R_{\mu}\right\}.
\end{aligned}$$

Again, this is the same as $T(X) \times \ddot{U}$ as desired. ♦

With $\mathrm{Tr}(X)[\mathbf{S}] = T(X) \times \ddot{U}$ established for atomic formulas, we proceed assuming that for some formulas X and Y we have $\mathrm{Tr}(X)[\mathbf{S}] = T(X) \times \ddot{U}$ and $\mathrm{Tr}(Y)[\mathbf{S}] = T(Y) \times \ddot{U}$. Then we must prove that:

$$\mathrm{Tr}(\neg X)[\mathbf{S}] = T(\neg X) \times \ddot{U}$$
$$\mathrm{Tr}(X \vee Y)[\mathbf{S}] = T(X \vee Y) \times \ddot{U}$$
$$\mathrm{Tr}(\exists_{\mathsf{v}_\kappa} X)[\mathbf{S}] = T(\exists_{\mathsf{v}_\kappa} X) \times \ddot{U}.$$

Proof (Case $\neg X$) Notice that for $v \in D(X)$, $\mathfrak{U} \models \neg X[\mathbf{b}(v)]$ if and only if $\mathfrak{U} \not\models X[\mathbf{b}(v)]$ if and only if $v \notin T(X)$. We conclude:

$$\begin{aligned} T(\neg X) &= \{v \in D(X) : v \notin T(X)\} \\ &= (T(X))^- \cap D(X). \end{aligned}$$

Turning our attention to $\mathrm{Tr}(X)$ we see:

$$\begin{aligned} \mathrm{Tr}(\neg X)[\mathbf{S}] &= (\mathrm{Tr}(X))^- \cap \mathrm{Dr}(X)[\mathbf{S}] \\ &= (\mathrm{Tr}(X)[\mathbf{S}])^- \cap (\mathrm{Dr}(X)[\mathbf{S}]) \\ &= \left(T(X) \times \ddot{U}\right)^- \cap \left(D(X) \times \ddot{U}\right) \\ &= \left((T(X))^- \cap D(X)\right) \times \ddot{U}, \end{aligned}$$

as desired. We note that the operation $^-$ is used in two different ways. Since $\mathrm{Tr}(X)$ and $T(X) \times \ddot{U}$ are understood to be subsets of $\ddot{U} \times \ddot{U}$ we mean the same interpretation for their complements. Similarly, since $T(X)$ is a subset of $\ddot{U}$, we mean $(T(X))^-$ to denote $\ddot{U} \setminus T(X)$. ♦

Proof (Case $X \vee Y$) Now $\mathfrak{U} \models X \vee Y[\mathbf{b}(v)]$ if and only if either $\mathfrak{U} \models X[\mathbf{b}(v)]$ or $\mathfrak{U} \models Y[\mathbf{b}(v)]$ if and only if $v \in T(X) \cup T(Y)$. Then:

$$\begin{aligned} T(X \vee Y) &= \{v \in D(X \vee Y) : v \in T(X) \cup T(Y)\} \\ &= (T(X) \cup T(Y)) \cap D(X \vee Y) \end{aligned}$$

By definition $\mathrm{Tr}(X \vee Y)[\mathbf{S}] = (\mathrm{Tr}(X) + \mathrm{Tr}(Y)) \cdot \mathrm{Dr}(X \vee Y)[\mathbf{S}]$. This simplifies to:

$$\begin{aligned} \mathrm{Tr}(X \vee Y)[\mathbf{S}] &= (\mathrm{Tr}(X)[\mathbf{S}] \cup \mathrm{Tr}(Y)[\mathbf{S}]) \cap \mathrm{Dr}(X \vee Y)[\mathbf{S}] \\ &= \left(\left(T(X) \times \ddot{U}\right) \cup \left(T(Y) \times \ddot{U}\right)\right) \cap D(X \vee Y) \times \ddot{U} \\ &= ((T(X) \cup T(Y)) \cap D(X \vee Y)) \times \ddot{U}. \end{aligned}$$

Thus $\mathrm{Tr}(X \vee Y) = T(X \vee Y) \times \ddot{U}$ as well. ♦

Proof (Case $\exists_{\mathsf{v}_\kappa} X$) For the final case we have $\mathfrak{U} \models \exists_{\mathsf{v}_\kappa} X[\mathbf{b}(v)]$ if and only if there is some $\mathbf{c} \in {}^\omega U$ such that $c_\lambda = \mathsf{P}_\lambda(v)$ for each $\lambda \in \mathrm{In}(\exists_{\mathsf{v}_\kappa} X)$ and $\mathfrak{U} \models X[\mathbf{c}]$. The latter condition holds if and only if there is some $u \in \ddot{U}$ such that $\mathsf{P}_\lambda(u) = \mathsf{P}_\lambda(v)$ for each $\lambda \in \mathrm{In}(\exists_{\mathsf{v}_\kappa} X)$ such that $\mathfrak{U} \models X[\mathbf{b}(u)]$. This in turn holds if and only if there is some $u \in \ddot{U}$ such that $\mathsf{P}_\lambda(u) = \mathsf{P}_\lambda(v)$ for each $\lambda \in \mathrm{In}(\exists_{\mathsf{v}_\kappa} X)$ and $u \in T(X)$.

In case $\mathrm{In}(\exists_{\mathsf{v}_\kappa} X) = \emptyset$, the condition $\mathsf{P}_\lambda(u) = \mathsf{P}_\lambda(v)$ for each $\lambda \in \mathrm{In}(\exists_{\mathsf{v}_\kappa} X)$ is vacuously satisfied so the set of all $\langle v, u\rangle$ which satisfy the condition is $1 = \ddot{U} \times \ddot{U}$. Otherwise the set of all such $\langle v, u\rangle$ is denoted by $\prod_{\lambda \in \mathrm{In}(\exists_{\mathsf{v}_\kappa} X)} \mathsf{P}_\lambda ; \breve{\mathsf{P}}_\lambda$. Thus $\mathrm{Tr}(\exists_{\mathsf{v}_\kappa} X)$ is the set of all $\langle v, v'\rangle$ such that for some $u \in \ddot{U}$ we have: $\mathsf{P}_\lambda(u) = \mathsf{P}_\lambda(v)$ for each $\lambda \in \mathrm{In}(\exists_{\mathsf{v}_\kappa} X)$ and $\langle u, v'\rangle \in \mathrm{Tr}(X)$. But by the inductive hypothesis $\langle u, v'\rangle \in \mathrm{Tr}(X)$ if and only if $u \in T(X)$. We therefore conclude that $\mathrm{Tr}(\exists_{\mathsf{v}_\lambda} X) = T(\exists_{\mathsf{v}_\lambda} X) \times \ddot{U}$ as required. ♦

We have therefore established that for any $X \in \Phi[\mathcal{L}]$, $\mathrm{Tr}(X)[\mathbf{S}] = T(X) \times \ddot{U}$. As we have already observed, $S_0 = S(X) \times \ddot{U}$ and $\mathfrak{U} \models X[\mathbf{a}]$ if and only if $S(X) \subseteq T(X)$. The theorem therefore follows from the fact that $S_0 \subseteq \mathrm{Tr}(X)[\mathbf{S}]$ if and only if $S(X) \subseteq T(X)$. The theorem's second assertion clearly follows from the first. ■

The next theorem is also taken from [Mad89a, Theorem 22]. We have modified it slightly (formulated here for $\mathsf{P}^+\mathsf{RA}$ instead of PRA) and corrected an error in the proof.

Theorem 52 *For $X \in \Sigma[\mathcal{L}]$ the following are equivalent:*

1. *X is logically valid,*
2. *$\mathsf{TPA} \models 1 = \mathrm{Tr}(X)$, and*
3. *$\mathsf{P}^+\mathsf{RA} \models 1;\mathsf{u};1 = \mathrm{Tr}(X)$.* □

Proof Recall that in $\mathsf{P}^+\mathsf{RA}$, the conjugated quasi-projections satisfy (1.3) (thus their domains are identical).

Proof (1⇒2) Suppose X is logically valid, and let U be a nonempty set such that $\ddot{U} \times \ddot{U} = \ddot{U} \setminus U$ (that is to say, let $\mathfrak{B}(U) \in \mathsf{TPA}$). Let $\mathbf{S}' \in {}^\omega\mathrm{Re}\ddot{U}$. For every $\mu < \omega$ define the relation R_μ as follows.

$$R_\mu := \left\{\left\langle \mathsf{P}_0(v), \ldots, \mathsf{P}_{\rho(\mu)-1}(v)\right\rangle \,:\, v \in \mathrm{Dom}(S'_{\mu+1}), \mathsf{P}_0(v), \ldots, \mathsf{P}_{\rho(\mu)-1}(v) \in U\right\}$$

Let $\mathbf{S}$ be as in Theorem 51 (thus $S_0 = 1$ since $X \in \Sigma[\mathcal{L}]$). The variable $\mathsf{w}_{\mu+1}$ only occurs in $\mathrm{Tr}(X)$ as part of a subterm of the form:

$$(\mathsf{P}_{\kappa_0};\mathsf{u};\breve{\mathsf{P}}_0 \cdot \ldots \cdot \mathsf{P}_{\kappa_{\rho(\mu)-1}};\mathsf{u};\breve{\mathsf{P}}_{\rho(\mu)-1});\mathsf{w}_{\mu+1};1.$$

This subterm denotes the same relation in $\mathfrak{B}(U)$ whether $\mathsf{w}_{\mu+1}$ is assigned to $S'_{\mu+1}$ or $S_{\mu+1}$. Recall from the proof of Theorem 51 that in the latter case this subterm denotes:

$$\left\{\langle v, v'\rangle \;:\; \left\langle \mathsf{P}_{\kappa_0}(v), \ldots, \mathsf{P}_{\kappa_{\rho(\mu)-1}}(v)\right\rangle \in R_{\mu+1}\right\}.$$

In case $\mathsf{w}_{\mu+1}$ is mapped to $S'_{\mu+1}$ the same subterm is mapped to:

$$\begin{aligned}
&\{\langle v, u\rangle \;:\; \mathsf{P}_{\kappa_i}(v) = \mathsf{P}_i(u) \in U \text{ for } i < \rho(\mu)\} \mid \left\{\langle u, v'\rangle \;:\; u \in \mathrm{Dom}(S'_{\mu+1})\right\}\\
&= \left\{\langle v, v'\rangle \;:\text{ for some } u \in \ddot{U},\ \mathsf{P}_{\kappa_i}(v) = \mathsf{P}_i(u) \in U \text{ for } i < \rho(\mu) \text{ and } u \in \mathrm{Dom}(S'_{\mu+1})\right\}\\
&= \left\{\langle v, v'\rangle \;:\text{ for some } u \in \ddot{U},\ \mathsf{P}_{\kappa_i}(v) = \mathsf{P}_i(u) \text{ for } i < \rho(\mu) \text{ and } \left\langle \mathsf{P}_0(u), \ldots, \mathsf{P}_{\rho(\mu)-1}(u)\right\rangle \in R_{\mu+1}\right\}\\
&= \left\{\langle v, v'\rangle \;:\; \left\langle \mathsf{P}_{\kappa_0}(v), \ldots, \mathsf{P}_{\kappa_{\rho(\mu)-1}}(v)\right\rangle \in R_{\mu+1}\right\}.
\end{aligned}$$

Therefore $\mathrm{Tr}(X)[\mathbf{S}] = \mathrm{Tr}(X)[\mathbf{S}']$, so we conclude $\mathfrak{B}(U) \models 1 = \mathrm{Tr}(X)[\mathbf{S}']$ if and only if $\mathfrak{B}(U) \models 1 = \mathrm{Tr}(X)[\mathbf{S}]$. By Theorem 51, the latter condition is equivalent to $\mathfrak{U} \models X$ (where $\mathfrak{U} = \langle U, \mathbf{R}\rangle$), which holds since X is assumed to be logically valid. ♦

Proof (2⇒1) Suppose that $\mathsf{TPA} \models 1 = \mathrm{Tr}(X)$. Let $\mathfrak{U} = \langle U, \mathbf{R}\rangle \in \mathsf{RE}[\mathcal{L}]$. We may assume that U satisfies $\ddot{U} \times \ddot{U} = \ddot{U} \setminus U$, because every $\mathfrak{U} \in \mathsf{RE}[\mathcal{L}]$ is isomorphic to some $\mathfrak{U}' \in \mathsf{RE}[\mathcal{L}]$ where U' is such. Let $\mathbf{S} = \langle S_\kappa : \kappa < \omega\rangle$ be as in Theorem 51, so in particular $S_0 = \ddot{U} \times \ddot{U}$. By hypothesis $\mathfrak{B}(U) \models 1 = \mathrm{Tr}(X)[\mathbf{S}]$, which implies $\mathfrak{U} \models X$ by Theorem 51. ♦

Proof (2⇒3) We prove the contrapositive. Suppose $1;\mathsf{u};1 = \mathrm{Tr}(X)$ fails in some $\mathfrak{B} \in \mathsf{P^+RA}$ (with quasi-projections P_0, P_1). We will construct $\mathfrak{B}(U) \in \mathsf{TPA}$ such that $\mathfrak{B}(U) \not\models 1 = \mathrm{Tr}(X)$. We may assume that $\mathfrak{B}$ is simple by Lemma 25. By Tarski's QRA theorem, $\mathfrak{B}$ is representable. Then $\mathfrak{B} \cong \mathfrak{C} \leq \mathfrak{Re}V$ for some set V by [JT52, Theorem 4.28], so we assume without loss of generality that $\mathfrak{B} \leq \mathfrak{Re}V$. We may further assume that $\ddot{V} \times \ddot{V} = \ddot{V} \setminus V$ since the structure of $\mathfrak{Re}V$ does not depend on the structure of V, but rather only on its cardinality. Let $U = V \setminus \mathrm{Dom}\, P_0$ so that $\ddot{U} \times \ddot{U} = \ddot{U} \setminus U$ (this property is easily seen to be inherited by subsets). If $U = \emptyset$ then $\mathrm{Tr}(X)[\mathbf{S}] = \emptyset = 1;\mathsf{u};1$, a contradiction, so we additionally assume that $U \neq \emptyset$ and conclude $\mathfrak{B} \models 1;\mathsf{u};1 = 1$.

Define a binary operation f on V by $f(x, y) = z$ where $P_0(z) = x$ and $P_1(z) = y$. Such a z exists uniquely, since $\mathfrak{A} \in \mathsf{P^+RA}$. Let W be the closure of U under f. Define $P'_0 = P_0 \cap (W \times W)$ and $P'_1 = P_1 \cap (W \times W)$. Then $\langle W, P'_0, P'_1\rangle$ is isomorphic to $\langle \ddot{U}, \mathsf{p}_0^U, \mathsf{p}_1^U\rangle$ by a recursively defined isomorphism h such that $h(u) = u$ for each $u \in U$ and $h(f(x, y)) = \langle h(x), h(y)\rangle$ for $x, y \in W$. Then we assume without loss of generality that $W = \ddot{U}$ and $P'_i = \mathsf{p}_i^U$ for $i \in 2$, so that $\mathfrak{B}(U) = \mathfrak{Rl}_{\ddot{U}\times\ddot{U}}\mathfrak{Re}V$.

Let $\mathbf{S} \in {}^\omega B$ such that $\mathfrak{B} \not\models 1;\mathsf{u};1 = \mathrm{Tr}(X)[\mathbf{S}]$. For $\kappa < \omega$ let $S'_\kappa = S_\kappa \cap (\ddot{U} \times \ddot{U})$ so that $\mathbf{S}'$ may be viewed as an assignment to $\mathfrak{B}$ or to $\mathfrak{B}(U)$. Let R be the interpretation of $\mathrm{Tr}(X)$ in $\mathfrak{B}(U)$ under assignment $\mathbf{S}'$ and S be the interpretation of $\mathrm{Tr}(X)$ in $\mathfrak{B}$ under assignment $\mathbf{S}$. By monotonicity of the operations of $\mathfrak{B}$, we have $R \subseteq S$.

A simple inductive argument shows that $S;1 = S$ (in fact, that $\mathsf{RA} \models \mathrm{Tr}(X);1 = \mathrm{Tr}(X)$). Further, since $\mathrm{In}(X) = \emptyset$ it must be that $1;S = S$. If $X = Y \vee Z$ for $Y, Z \in \Sigma[\mathcal{L}]$ then this follows from the distributivity of relative product over union [Mad06, *261*]. If $X = \neg Y$ for $Y \in \Sigma[\mathcal{L}]$ then it follows from the fact that $\mathsf{RA} \models 1;(1;\mathsf{w}_0)^- = (1;\mathsf{w}_0)^-$ [Mad06, *305*]. Finally, if $X = \exists_{\mathsf{v}_\kappa} Y$ for some Y with $\mathrm{In}(Y) \subseteq \{\kappa\}$ then it follows from $\mathsf{RA} \models 1;1;\mathsf{w}_0 = 1;\mathsf{w}_0$. We conclude then, that $1;S;1 = S$. But recall that $S \neq 1$ by construction, so since $\mathfrak{B}$ is simple it must be that $R = S = \emptyset$ and $\mathfrak{B}(U) \not\models 1 = \mathrm{Tr}(X)[\mathbf{S}']$ as desired. ♦

Proof (3⇒2) Suppose that $\mathsf{P^+RA} \models 1;\mathsf{u};1 = \mathrm{Tr}(X)$. Every algebra in TPA is a $\mathsf{P^+RA}$ in which $1;\mathsf{u};1 = 1$ holds, so every such algebra satisfies $1 = \mathrm{Tr}(X)$. ♦

Thus we have $1 \Leftrightarrow 2 \Leftrightarrow 3$, as desired. ■

Remark 53 The error in the proof given in [Mad89a] occurs during the proof that 2⇒3. In particular, the assignment $\mathbf{S} \in {}^{\omega}B$ under which $\mathsf{P^+RA} \models 1;\mathsf{u};1 \doteq \mathrm{Tr}(X)[\mathbf{S}]$ fails, is there identified to be $\mathbf{S} \in {}^{\omega}\ddot{U}$. The existence of such an assignment is equivalent to the desired conclusion and cannot be assumed. Simply changing $\ddot{U}$ to B is insufficient to fix the proof, since then the statement $\mathfrak{B}(U) \models 1 \doteq \mathrm{Tr}(X)[\mathbf{S}]$ occurring later in the proof ceases to make sense. □

3.2 First-order logic in Df_3

Definition 54 Define $\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}$, $\dot{1}$, $\dot{-}$, $\circ$, $\breve{}$, and $\dot{1}$' be as in Definition 32—replacing every instance of e with w_0 and every instance of q with w_1. Note that these are Df_3-terms with variables contained in $\{\mathsf{w}_0, \mathsf{w}_1\}$ rather than elements and operations on a particular algebra, $\mathfrak{A} \in \mathsf{Df}_3$.

$$\begin{aligned}
\Omega_0 &:= \sum \left\{ (\mathsf{d}_{\kappa\kappa})^- : \kappa \in 3 \right\} \\
\Omega_1 &:= \sum \left\{ \mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle} \cdot \mathsf{d}_{\lambda\langle jl\rangle\mu\langle k\rangle} \cdot \left(\mathsf{d}_{\kappa\langle il\rangle\mu\langle k\rangle}\right)^- : \kappa, \lambda, \mu \in 3; i, j, k, l \in 2^*; i, jl, kl \in S_3 \right\} \\
\Omega_2 &:= \sum \left\{ \mathsf{d}_{\kappa\langle i\rangle\kappa\langle i\rangle} \cdot \mathsf{d}_{\lambda\langle j\rangle\lambda\langle j\rangle} \cdot \left(\mathsf{c}_\mu \left(\mathsf{d}_{\mu\langle 0\rangle\kappa\langle i\rangle} \cdot \mathsf{d}_{\mu\langle 1\rangle\lambda\langle j\rangle}\right)\right)^- : \mu \neq \kappa, \lambda \in 3; i, j \in S_3 \right\} \\
\Omega_3 &:= \sum \left\{ \mathsf{d}_{\kappa\langle i0\rangle\lambda\langle j0\rangle} \cdot \mathsf{d}_{\kappa\langle i1\rangle\lambda\langle j1\rangle} \cdot \left(\mathsf{d}_{\kappa\langle i\rangle\lambda\langle j\rangle}\right)^- : \kappa, \lambda \in 3; i, j \in S_2 \right\}
\end{aligned}$$

Let $X \in \Phi[\mathcal{L}]$ and let $\{\kappa_i : i < \lambda\} = \mathrm{Rel}\, X$ (the indices of relation symbols appearing in X). Then define $\Psi(X) \in \mathrm{Tm}[\mathsf{Df}_3]$ as follows.

$$\Psi(X) := \mathsf{c}_0\mathsf{c}_1\mathsf{c}_2 \left(\sum_{i=0}^{3} \Omega_i + \sum_{i=0}^{\lambda-1} \left(\left(\mathsf{s}_0^1\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_2\mathsf{w}_{\kappa_i+2}\right) \oplus \mathsf{w}_{\kappa_i+2} + \left(\dot{1} \cdot \mathsf{w}_{\kappa_i+2}\right) \oplus \mathsf{w}_{\kappa_i+2} \right) \right)$$

Let $\mathsf{u} = \dot{1}\text{'} \cdot \left(\mathsf{p}_0 \circ \dot{1} + \mathsf{p}_1 \circ \dot{1}\right)^-$, $\mathsf{P}_0 = \mathsf{p}_0$, and $\mathsf{P}_{\kappa+1} = \mathsf{p}_1 \circ \mathsf{P}_\kappa$. Finally, let Tr' and Dr' be as Tr and Dr are in Definition 50—replacing the relational operations and constants with our Df_3 term operations and constants. □

Proposition 55 *For every simple $\mathfrak{A} \in \mathsf{Df}_3$ and $\mathbf{s} \in {}^\omega A$, the following are equivalent:*

(i) $\mathfrak{A} \models \Psi(X) = 0[\mathbf{s}]$.

(ii) both of the following conditions hold:

(a) $s_{\kappa_i+2} \in J$ *for each* $\kappa_i \in \mathrm{Rel}\, X$.

(b) $e = s_0$ *and* $q = s_1$ *satisfy the equations in* Ax.

(iii) $\mathfrak{A} \not\models \Psi(X) = 1[\mathbf{s}]$. □

Proof The equivalence of (i) and (iii) is a direct consequence of Theorem 30 since $\mathfrak{A}$ is assumed to be simple. By Theorem 30 we have (i) if and only if $\Omega_i = 0$ for $i \leq 3$; and $\left(\mathsf{s}_0^1\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_2 s_{\kappa_i+2}\right) \oplus s_{\kappa_i+2} = 0$; and $\dot{1} \cdot s_{\kappa_i+1} \oplus s_{\kappa_i+1} = 0$ for each $i \in \mathrm{Rel}\, X$. By Theorem 31:

$$\begin{aligned} \dot{1} \cdot s_{\kappa_i+2} \oplus s_{\kappa_i+2} = 0 &\Leftrightarrow \dot{1} \cdot s_{\kappa_i+2} = s_{\kappa_i+2} \\ &\Leftrightarrow s_{\kappa_i+2} \leq \dot{1} \\ \left(\mathsf{s}_0^1\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_2 s_{\kappa_i+2}\right) \oplus s_{\kappa_i+2} = 0 &\Leftrightarrow \mathsf{s}_0^1\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_2 s_{\kappa_i+2} = s_{\kappa_i+2} \\ &\Leftrightarrow s_{\kappa_i+2} \in J_0. \end{aligned}$$

The above are equivalent to item (a) of condition (ii). We similarly deduce that $\Omega_i = 0$ if and only if $e = s_0$ and $q = s_1$ satisfy all of the equations in Ax_i which is item (b) of condition (ii). Thus (ii) is equivalent to (i) as desired. ■

Theorem 56 *$X \in \Sigma[\mathcal{L}]$ is logically valid if and only if $\mathsf{Df}_3 \models \mathrm{L}(X)$ where $\mathrm{L}(X)$ is the equation*

$$1 \circ \mathsf{u} \circ 1 + \Psi(X) \doteq \mathrm{Tr}'(X) + \Psi(X).$$

Proof By Theorem 52 it suffices to show that $\mathsf{Df}_3 \models \mathrm{L}(X)$ if and only if $\mathsf{P^+RA} \models 1;\mathsf{u};1 = \mathrm{Tr}(X)$.

Suppose first that $\mathsf{Df}_3 \not\models \mathrm{L}(X)$ for some X. Then since Df_3 is generated by its simple algebras (Theorem 29) there is some simple $\mathfrak{A} \in \mathsf{Df}_3$ and $\mathbf{s} \in {}^\omega A$ such that $\mathfrak{A} \not\models \mathrm{L}(X)[\mathbf{s}]$. It must be that $\mathfrak{A} \models \Psi(X) \doteq 0[\mathbf{s}]$ (otherwise $\mathfrak{A} \models \Psi(X) \doteq 1[\mathbf{s}]$, hence $\mathfrak{A} \models \mathrm{L}(X)[\mathbf{s}]$ trivially). So we conclude that s_0 and s_1 are diagonal and quasi-projection parameters (respectively) satisfying the equations in Ax and further that $s_{\kappa+2} \in J$ for every R_κ appearing in X by Proposition 55. By Theorem 34 we have that

$\mathfrak{J}\mathfrak{A} = \left\langle A, +, \dot{-}, \circ, \breve{}, \dot{1}' \right\rangle \in \mathsf{Q^+RA}$ so that $\langle \mathfrak{J}\mathfrak{A}, P_0, P_1 \rangle \in \mathsf{P^+RA}$. Since $\mathfrak{A} \not\models \mathrm{L}(X)[\mathbf{s}]$ and $\mathfrak{A} \models \Psi(X) \doteq 0$, it follows that $\mathfrak{A} \not\models 1 \circ \mathsf{u} \circ 1 \doteq \mathrm{Tr}'(X)[\mathbf{s}]$. If we define $\mathbf{s}' \in {}^{\omega}J$ by $s'_\kappa = s_{\kappa+2}$ for $\kappa \in \omega$, then this is equivalent to $\langle \mathfrak{J}\mathfrak{A}, P_0, P_1 \rangle \not\models 1 \circ \mathsf{u} \circ 1 \doteq \mathrm{Tr}(X)[\mathbf{s}']$. We conclude that $\mathsf{P^+RA} \not\models 1 ; \mathsf{u} ; 1 \doteq \mathrm{Tr}(X)$.

On the other hand, suppose $\mathsf{Df}_3 \models \mathrm{L}(X)$. Choose any $\mathfrak{A} \in \mathsf{P^+RA}$. We may assume $\mathfrak{A}$ is simple because $\mathsf{P^+RA}$ is generated by its simple algebras by Lemma 25. Then by [JT52, Theorem 4.28] we have that the $\mathsf{Q^+RA}$ reduct of $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{Re}V$ which is in turn isomorphic to the Németi reduct of some $\mathfrak{B} \in \mathsf{Df}_3$ with diagonal and quasi-projection parameters e and q by Theorem 35. Let $f : \mathfrak{A} \to \mathfrak{J}\mathfrak{B}$ be such an isomorphism. Choose any $\mathbf{S} \in {}^{\omega}A$. Define $\mathbf{T} \in {}^{\omega}J$ by $T_\kappa = f(S_\kappa)$, and $\mathbf{T}'$ by $T'_{\kappa+2} = T_\kappa$, $T'_0 = e$, and $T'_1 = q$. Then $\mathfrak{A} \models 1 ; \mathsf{u} ; 1 = \mathrm{Tr}(X)[\mathbf{S}]$ if and only if $\mathfrak{J}\mathfrak{B} \models 1 \circ \mathsf{u} \circ 1 = \mathrm{Tr}(X)[\mathbf{T}]$ if and only if $\mathfrak{B} \models \mathrm{L}(X)[\mathbf{T}']$. The last condition is true by assumption, so we conclude that $\mathfrak{A} \models 1 ; \mathsf{u} ; 1 = \mathrm{Tr}(X)[\mathbf{S}]$ as was to be shown. ∎

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