A FORMALISM FOR ACOUSTICAL TRAVELTIME TOMOGRAPHY IN HETEROGENEOUS ANISOTROPIC MEDIA

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INTRODUCTION

In many cases, acoustical traveltime tomography is applied by assuming isotropy of media. Indeed, isotropy makes the tomographic problem to be treated more simply. It should be admitted that some tomographic results are successfully achieved with the assumption of isotropy. Nevertheless, the simplification of isotropy is of inherent limitation. In fact, most of materials, such as rocks, composites, and biological tissues are of velocity anisotropy in varying degrees. Therefore in recent years, the investigations on anisotropic tomography attracted much attention, which not only can remove the unavoidable discrepancy produced by isotropic assumption, but also provide very useful information on the internal distribution of the physical properties in the materials being investigated.

In heterogeneous anisotropic ray tomography, one of the problems is concerned with ray traveling. But, it is difficult to directly calculate ray velocity for general anisotropy, where ray vector and phase slowness vector are reciprocally coupled and not always parallel to each other. For making the tomographic problem soluble, one has to resort to some approximations. For example, Pratt and Chapman (1992) [1] presented a perturbation approach for 2-D weakly anisotropic tomography in which isotropic ray tracing was used. Jansen et al (1992) [2] gave a modification to simultaneous iterative reconstruction technique (SIRT) algorithm for imaging weakly anisotropic velocity by assuming the equivalence between the group velocity and phase velocity, and so on.

In this paper, an alternative method is proposed for 2-D anisotropic traveltime tomography based on the anisotropy described by elliptical slowness curve of the quasi-P
METHODOLOGY

Formulation

According to Fermat’s principle, the first-arrival traveltimes of the qP wave along the ray path $\Gamma$ in general media are measured by the following formula:

$$ t = \int_{\Gamma} u(x, z) \, dr $$

where $u$ is the ray, or group slowness (reciprocal of group velocity) of qP wave and $dr$ is the incremental length along the ray path. In isotropic traveltomeography, the equation (1) is straightforward and has been extensively applied. But for anisotropic traveltomeography the situation is not so simple. In anisotropic media, a ray velocity vector $v$ and phase slowness vector $p$ are related by

$$ v \cdot p = 1 $$

which shows that the ray and slowness surface are polar reciprocals of each other [3]. The relation between the phase slowness vector and the elastic parameters can be directly described by Christoffel equations (also called slowness surface equations) [3]

$$ F(p) = C_{ijkl} p_i p_j - \rho \delta_{ik} = 0 $$

where $C_{ijkl}$ is the fourth-order elastic tensor, $\rho$ the density and $\delta_{ik}$ the Kronecker function. $p_j$ ($p_i$) are the components of $p$. Unfortunately, there exists no explicit equation, analogous to Eq. (3), that relates the ray velocity and the elastic constants. Therefore, it is convenient to express the traveltome integral (1) in terms of the phase slowness vector $p$ using Eq (2)

$$ t = \int_{\Gamma} p \cdot dr $$

where $dr$ is the incremental vector along the ray direction.

In tomography procedure, the target domain is discretized into $n$ rectangular cells, thus Eq. (4) becomes

$$ t_i = \sum_{j=1}^{n} (p_{ij} \cdot r_{ij}) \,, \, i = 1, \ldots, m $$

where $t_i$ is the total traveltome along the $i$th ray, $p_{ij}$ and $r_{ij}$ are the slowness vector and the position vector (parallel to ray direction) of the $i$th ray path through the $j$th cell, respectively. $m$ is the total number of rays used. Because of the coupling between $v$ and $p$, the traveltome calculation in Eq. (5) must be performed either by determining $p_{ij}$ from given $r_{ij}$ or determining $r_{ij}$ from $p_{ij}$ governed by Eq. (3). We prefer using the former way because the calculated $r_{ij}$ from latter way is generally not at the specified point.

For qP wave in 2-D case, an elliptical slowness curve can be used as a good approximation of an actual one, which is valid for cubic, hexagonal, tetragonal and
orthorhombic symmetry systems [4]. Then the explicit connection between \( r_{ij} \) and \( p_{ij} \) can be obtained with elliptical anisotropy described by introducing parameterization procedure.

In the principal coordinate system shown in Figure 1, the parametrized vectors of phase slowness and ray velocity are given as follows:

\[
\begin{bmatrix}
  p_1 \\
  p_2 
\end{bmatrix} = \begin{bmatrix}
  p_a \cos \phi \\
  p_b \sin \phi 
\end{bmatrix}, \quad \begin{bmatrix}
  v_1 \\
  v_2 
\end{bmatrix} = \begin{bmatrix}
  p_a^{-1} \cos \phi \\
  p_b^{-1} \sin \phi 
\end{bmatrix} \tag{6}
\]

where \( \phi \) is the eccentric angle of a parameterized ellipse, \( p_1 \) and \( p_2 \) are the components of slowness vector along wave propagation direction, \( p_a \) and \( p_b \) are the horizontal and vertical slownesses along the principal axes, \( v_1 \) and \( v_2 \) are the components of ray velocity, respectively. Clearly, the ray velocity and phase slowness vectors are related in terms of the angle \( \phi \), which can be analytically determined based on following formula:

\[
tg 4\phi = \frac{p_b}{p_a} z \frac{z}{x} \tag{7}
\]

where \( x, z \) are the components of position vector \( r \). Therefore, the associated phase slowness vector can be conveniently acquired with given ray direction from Eq. (6) and Eq. (7). Finally, the first-arrival time and the corresponding raypath can be determined based on Fermat’s principle and network theory [5] once the problem of finding \( p_{ij} \) from \( r_{ij} \) is solved.

Now, we formulate the inverse problem. According to previous discussions, there are two parameters per cell, i.e. both slownesses along the principal axes of elliptical slowness curve. Therefore, replacing the \( p_{ij} \) in Eq. (5) with the parametrized slowness vectors in Eq. (6), we can transform Eq. (5) into the following matrix form

\[
t = As \tag{8}
\]

where \( t \) is the traveltime column vector whose component is \( t_i \), \( s \) is the model slowness 2n-vector \( s^T = (s_1, \ldots, s_n, s_{n+1}, \ldots, s_{2n}) \), with \( s_k \) being the horizontal and vertical components of the slowness for \( 1 \leq k \leq n \) and \( n+1 \leq k \leq 2n \), respectively (a superscript T means the transpose). The elements of the \( m \times 2n \) matrix \( A \) are always greater than or equal to zero and are calculated by
\[
A_{ik} = \begin{cases}
x_{ij} \cos \phi_{ij} & 1 \leq k = j \leq n \\
z_{ij} \sin \phi_{ij} & 1 + n \leq k + n \leq 2n
\end{cases}
\]  

(9)

where \(x_{ij}, z_{ij}\) are the coordinate components of the \(r_{ij}\), and \(\phi_{ij}\) is the angle of the \(i\)th ray through the \(j\)th cell determined by Eq. (7)

**Weighted Conjugate Gradient Algorithm**

For a reference model \(s_0\), we now seek the weighted, damped least squares solution of Eq. (8) by minimizing the following functional

$$
F(s) = (t - As)^T W(t - As) + \mu(s - s_0)^T Q(s - s_0)
$$  

(10)

where \(W\) and \(Q\) are the data and model weighting matrices respectively, \(\mu\) is the damping factor which controls the trade-off between the data misfit and the constrained term of model in Eq. (10). In fact, minimizing the functional \(F(s)\) in Eq. (10) can also be cast into finding the least squares solution of the following augmented linear system

$$
\bar{A}\Delta s = \bar{t}
$$  

(11)

where \(\bar{A} = \begin{bmatrix} \frac{1}{W^2} A & 0 \\ \frac{1}{\mu Q^2} \end{bmatrix}\), \(\bar{t} = \begin{bmatrix} \frac{1}{W^2}(t - As_0) \\ 0 \end{bmatrix}\), and \(\Delta s = s - s_0\).

Hence, the application of the original conjugate gradient (CG) algorithm [6] to the normal equations \(\bar{A}^T \Delta s = \bar{A}^T \bar{t}\) resulted from Eq (11) is straightforward where \(\bar{A}^T \bar{A}\) is always symmetrical and non-negative. In practice, we do not need to take extra storage and work of \(\bar{A}^T \bar{A}\) and \(\bar{A}^T \bar{t}\) in the following steps. An explicit CG algorithm to solve weighted least squares problem is: set \(\Delta s_0 = 0\), put \(p_0 = r_0 = A^T W(t - As_0)\), then for \(k = 0, 1, 2, \cdots\).

$$
q_k = \bar{A}p_k
$$

$$
\alpha_{k+1} = \frac{(r_k, r_k)}{(q_k^T Wq_k + \mu p_k^T Qp_k)}
$$

$$
\Delta s_{k+1} = \Delta s_{k} + \alpha_{k+1} p_k
$$

$$
r_{k+1} = r_k - \alpha_{k+1} (\bar{A}^T Wq_k + \mu Qp_k)
$$

$$
\beta_{k+1} = \frac{(r_{k+1}, r_{k+1})}{(r_k, r_k)}
$$

$$
p_{k+1} = r_{k+1} + \beta_{k+1} p_k
$$  

(12)

In the algorithm of Eq (12), main operations are involved in matrix-vector products and the vector-vector products. It is still convenient to take full advantage of the sparseness of
matrix A even if weighting matrices and damping factor are incorporated into the algorithm. The above algorithm is easily implemented with small modifications of available CG code.

Theoretically, the CG method can converge to the least-squares solution in n steps at most, n being the number of unknowns. Practically, it is unhelpful to perform n iterations because of finite precision arithmetic. The numerical studies show that the convergence of iterative solution depends upon the distribution of singular value of matrix. It is therefore useful if the singular value range of the inverse problem is known in advance. In our formulation of anisotropic traveltime tomography, the elements of the matrix in (8) are non-negative, then the positive, diagonal weighting matrices W and Q can be constructed with modifications to those given in isotropic traveltime tomography [7]. It is very important to choose such weighting matrices which naturally normalize the singular value of the inverse problem. Clearly, the damping factor $\mu$ now should lie in the interval $[0, 1]$. Additionally, we relate the CG iteration number with the damping factor for stabilizing the iterative solution. In the below synthetic example, we let the damping factor linearly increase with the iteration number from a small, initial value in order to filter out smaller singular values’ components in the iteration process, which have negative effects on the convergence of the solution. Of course, a more detailed discussion on the above problems is beyond the scope of the paper.

APPLICATIONS

A Synthetic Data for Heterogeneous Anisotropic Model

The model for this example consists of a layer and an inclined belt, which are characterized by the elliptical anisotropy. The layer is assigned by the horizontal velocity of $3000 \text{m/s}$ and the vertical velocity of $2800 \text{m/s}$. The inclined belt is assigned by the horizontal velocity of $4000 \text{m/s}$ and the vertical velocity of $3800 \text{m/s}$. Background velocity is $4500 \text{m/s}$. The target model is shown in Figure 2a. The model contains 20x20 square cells. The 2400 observed traveltimes are yielded to the target model assuming four-sided view geometry. Therefore the dimensions of the matrix are $2400 \times 800$.

Obviously, a sequence of linearized steps are needed to approach the true solution because of strong inhomogeneity in the problem. Such steps are called global iterations. Correspondingly, the local iterations are defined as solving each linear system of equations via iterative method. In the reconstruction, an initial, isotropic model of $4500 \text{m/s}$ is chosen, and the regularizing parameter is designed to linearly increase with the local iteration number from the initial value 0.001. We then perform the tomographic reconstructions on this example with the local iterations of 5 and 100 separately. But both of the results are reconstructed through the global iterations of 10.

Inspection of Figures 2b and 2c shows that the reconstructed image with smaller number of local iterations of 5 is inferior to that with larger number of local iterations of 100. In Figure 2b, although the horizontal velocity components of the layer are recovered, the inclined belt is weakly imaged and the some artifacts appear along both left and right boundaries of the model (left). Besides, the vertical velocity components of the layer vertically elongated, the inclined belt laterally widen, and connections between the layer and the inclined belt is smeared (right). In Figure 2c, the inclined belt tends to be more accurately reconstructed in addition to that the layer are still clearly inverted. In a word, the artifacts appeared in Figure 2b are almost disappeared in Figure 2c.
This example shows that the effects of CG iteration number in reconstruction process. While continuously increasing the number of global iterations, the solutions with larger number (100) of local iteration converge better and faster than the ones with smaller number (5) local iteration using our reconstruction techniques. It should be pointed out that the conventional iterative method without considering the weighting matrices is sensitive to the iteration number, and the corresponding solutions often show instability and even divergence while the increasing the local iteration number. Therefore, it should be cautious to apply the conventional least squares method for inverting traveltime data at each linearized step.

Figure 2. The horizontal velocity (left) and vertical velocity (right) tomograms with two different, local iteration number through the global iterations of 10. (a) Target model (b) The result using local iterations of 5 (c) The result using local iterations of 100.
The Real Data for Underground Rock Wall

The experiment is carried out at the underground plant site of Ertan Hydroelectric Power Project, southwest China. The object areas is a rock wall intervening between the two parallel tunnels (tunnel I and tunnel II). The objective of conducting tomographic imaging is to provide the essential data for further reinforcing the rock wall and excavation planning. The 2077 first-arrival times are collected using a hammer source along the two sides of the rock wall at underground level of 987m. Interval between source positions or receiver positions is 1m.

For anisotropic tomography analysis, the imaging region is parametrized into 28x67 pixels of 1m square. We proceed to invert the data from isotropic, reference image of 2000ms$^{-1}$. From the site knowledge, the principal axis angle of $-10^\circ$, which is the angle between the strike of the rock wall and the horizontal coordinate component of the user, is fixed with the assumption of elliptical anisotropy. Anisotropic tomograms are obtained after 10 global iterations.

It should be noted that the parallel and normal velocities are in the directions parallel and normal, respectively, to the strike of the rock wall for analysis convenience below. The parallel and normal velocity images are shown in Figures 3a and 3b. The most obvious features in the tomograms is significant differences existing between parallel and normal velocities. The interior of the rock wall mainly shows the high velocity except that some low velocity areas appear along the edges and both left and right ends of the rock wall. Comparison of the parallel and normal tomograms shows that the lower normal velocity zones distribute more widely and their patterns are relatively complicated. The low velocity zones between 40-65m, particularly in the normal velocity tomogram, generally correspond

Figure 3. Anisotropic tomograms for the rock wall. (a) parallel velocity (b) normal velocity

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to the positions where some geological events such as rockbursts and collapses happened in varying degrees from site observations. Additionally, there are low velocity zones in a triangle form in the right corner of the tomograms, and a larger area of low velocity in Figure 3b perpendicularly extends into the rock wall approximately between 10-25 m. Possibly, these low velocity zones is related to the consequence of the stress relaxation.

From the site investigation, the structures of the rock wall are mainly controlled by cracks/joints which are approximately parallel to the strike of the rock wall. Therefore, it is further inferred that the velocity anisotropy found in the tomograms may be mostly induced by the structures of cracks/joints according to rock mechanics.

In summary, the tomograms reveal that the rock wall between the two parallel tunnels is anisotropic as well as heterogeneous. Their characteristics discussed above can be reconciled with the site observations and rock mechanics knowledge.

The preliminary result provides a fundamental insight into the evolution and causes of excavating-induced rockbursts. The important information available for geotechnical engineers can be used to minimize the potential risk of rockburst by conditioning the rock prior to next excavations. This example shows that the tomographic technique is a useful tool for inspecting the stability of rock structures.

CONCLUSIONS

We have presented a procedure of traveltime tomography in anisotropic media. The synthetic results show that the developed tomographic procedure successfully reconstructs the velocity structures of heterogeneous anisotropic media. Our method has also been applied to imaging the internal structures of a rock wall at the site of the underground plant of a hydroelectric power (Sichuan, China). The anisotropic velocity tomogram provides important and useful information for the evaluation and reinforcement of the rock wall. It is certainly useful to develop anisotropic tomography based on the anelliptical approximation to the actual slowness curve in the future.

REFERENCES