INTRODUCTION

Leaky Lamb waves have been used extensively for ultrasonic non-destructive evaluation of elastic properties: the reader is referred to papers by Dayal and Kinra [1,2], Chimenti and Martin [3], Mal et al. [4], and Chimenti and Nayfeh [5]. The principal disadvantage of this method is the high attenuation of the waves in the immersed solid plate due to continuous radiation to the surrounding fluid. As a result, their amplitudes become immeasurably small after a short distance of travel as observed by Dayal and Kinra. This provided the motivation for the present work: a study of the propagation of harmonic waves in a solid plate loaded by a fluid layer of a finite thickness. In a previous work by the authors [6], the dispersion equation for an isotropic solid/fluid bilayer was obtained. It was seen that a thin layer of fluid coupled the symmetric (S) and antisymmetric (A) modes in the solid layer and that along a branch a quasi-symmetric mode changed character to a quasi-antisymmetric mode near a region where previously the S and the A branches crossed but were uncoupled. In the present work the dispersion equation for an orthotropic solid/fluid bilayer is derived. Mode shapes are studied for a graphite-epoxy/water bilayer for the case of equal thickness of the fluid and solid layers. In this case, coupling between fluid and solid modes is observed.

THEORY

For an elastic liquid medium, consider plane-harmonic waves traveling in the $x_1$ direction with an apparent wavenumber $k_1$ and an apparent phase velocity $c_1$. Assuming plane strain conditions, $u_2 = 0$ and $\partial(\cdot)/\partial x_2 = 0$, the displacement and stress expressions satisfying the equations of motion are

$$u_{1F}^{c} = \left( C_1 \cos k_1 \alpha^{F} x_3 + C_2 \sin k_1 \alpha^{F} x_3 \right) e^{ik_1(x_1-c_1 t)} , \quad \alpha^{F} = \left( c_1^2/c_F^2 - 1 \right)^{1/2}$$

$$u_{3F}^{c} = i \alpha^{F} \left( C_1 \sin k_1 \alpha^{F} x_3 - C_2 \cos k_1 \alpha^{F} x_3 \right) e^{ik_1(x_1-c_1 t)}$$

$$\sigma_{33}^{F} = i k_1 \rho_F c_F \left( C_1 \cos k_1 \alpha^{F} x_3 + C_2 \sin k_1 \alpha^{F} x_3 \right) e^{ik_1(x_1-c_1 t)}$$

where $c_F = \left( k_F / \rho_F \right)^{1/2}$ is the phase velocity of longitudinal waves of the fluid. For an
orthotropic solid medium under plane strain conditions, the plane harmonic waves are also written in terms of a standing wave in the $x_3$ direction, propagating in the $x_1$ direction.

$$u_1 = (A_1 \cos k_1 \alpha_1 x_3 + A_2 \sin k_1 \alpha_1 x_3 + B_1 \cos k_1 \alpha_3 x_3 + B_2 \sin k_1 \alpha_3 x_3) e^{i(k_1 (x_3 - ct))}$$  \hspace{1cm} (4)$$

$$u_3 = i \left[(A_1 \sin k_1 \alpha_1 x_3 - A_2 \cos k_1 \alpha_1 x_3) \psi_1 + (B_1 \sin k_1 \alpha_3 x_3 - B_2 \cos k_1 \alpha_3 x_3) \psi_3 \right] e^{i(k_1 (x_3 - ct))}$$  \hspace{1cm} (5)$$

$$\sigma_{33} = i k \left[(A_1 \cos k_1 \alpha_1 x_3 + A_2 \sin k_1 \alpha_1 x_3) \phi_1 + (B_1 \cos k_1 \alpha_3 x_3 + B_2 \sin k_1 \alpha_3 x_3) \phi_3 \right] e^{i(k_1 (x_3 - ct))}$$  \hspace{1cm} (6)$$

$$\sigma_{13} = k \left[-A_1 \sin k_1 \alpha_1 x_3 + A_2 \cos k_1 \alpha_1 x_3 \right] e_1 + \left[-B_1 \sin k_1 \alpha_3 x_3 + B_2 \cos k_1 \alpha_3 x_3 \right] e_3 \right] e^{i(k_1 (x_3 - ct))}$$  \hspace{1cm} (7)$$

where $\alpha$ is the ratio of the wavenumbers in the $x_3$ and $x_1$ directions. Here $\alpha_1$ and $\alpha_3$ correspond to QL and QT waves, respectively.

$$\alpha_1 = \left( \frac{-B - \sqrt{B^2 - 4AC}}{2A} \right)^{1/2}, \ \alpha_3 = \left( \frac{-B + \sqrt{B^2 - 4AC}}{2A} \right)^{1/2}$$  \hspace{1cm} (8)$$

where

$$A = C_{33} C_{55},$$

$$B = C_{33} \left[C_{11} - \rho c_1^2 \right] + C_{55} \left[C_{55} - \rho c_3^2 \right] - \left(C_{13} + C_{35} \right)^2,$$

$$C = \left(C_{11} - \rho c_1^2 \right) \left(C_{55} - \rho c_3^2 \right).$$  \hspace{1cm} (9)$$

The ratio of the vertical and horizontal displacement amplitudes for each $\alpha$ are

$$W_1 = -\frac{C_{11} - \rho c_1^2 + C_{55} \alpha_1^2}{\alpha_1 (C_{13} + C_{35})}, \ W_3 = \frac{-C_{11} - \rho c_1^2 + C_{55} \alpha_3^2}{\alpha_3 (C_{13} + C_{35})}.$$  \hspace{1cm} (10)$$

Also,

$$f_1 = C_{13} + C_{33} \alpha_1 W_1, \ f_3 = C_{13} + C_{33} \alpha_3 W_3.$$  \hspace{1cm} (11)$$

Note that the vertical and horizontal displacements and the shear and normal stresses are out of phase by $\pi/2$ radians. It is noted that there may be slip at the interface in the $x_1$ direction, i.e., $u_1$ may be discontinuous. Imposing the boundary conditions for a fluid-solid bilayer, a six-by-six coefficient matrix is obtained for the six unknown constants

$$\begin{bmatrix}
0 & 0 & 0 & \cos r(1 + 2\tau) & \sin r(1 + 2\tau) \\
W_1 \sin a_1 & -W_1 \cos a_1 & W_3 \sin a_3 & -W_3 \cos a_3 & -\alpha_1^f \sin r & \alpha_1^f \cos r \\
f_1 \cos a_1 & 0 & f_3 \cos a_3 & 0 & -\frac{1}{2} \rho f c_1^2 \cos r & -\frac{1}{2} \rho f c_1^2 \sin r \\
0 & e_1 \cos a_1 & 0 & e_3 \cos a_3 & 0 & 0 \\
0 & f_1 \sin a_1 & 0 & f_3 \sin a_3 & -\frac{1}{2} \rho f c_1^2 \cos r & -\frac{1}{2} \rho f c_1^2 \sin r \\
e_1 \sin a_1 & 0 & e_3 \sin a_3 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
A_1 \\
A_2 \\
B_1 \\
B_2 \\
C_1 \\
C_2 \\
\end{bmatrix} = 0$$  \hspace{1cm} (12)$$
where

\[ a_1 = k_1 \alpha_1 h, \quad a_3 = k_2 \alpha_3 h, \quad r = k_3 \alpha R h, \quad \tau = \frac{a}{h}. \]  \hspace{1cm} (13)

As usual, for a nontrivial solution to exist, the determinant of the coefficient matrix is set to zero yielding the dispersion equation. Using the normalized quantities \((\xi, \Omega)\) defined by

\[ \xi = \frac{2h}{\pi} k, \quad \Omega = \frac{2h}{\pi c_T} \omega \]  \hspace{1cm} (14)

where \(c_r^2 = C_{ss}/\rho\), the dispersion relation is given by

\[ \sigma S_{2h} \sin 2\pi r - r S_h A_h \cos 2\pi r = 0 \]  \hspace{1cm} (15)

where

\[ A_h = a_3 d(a_3, a_1) \cos \alpha_1 \sin \alpha_3 - a_3 d(a_1, a_3) \sin \alpha_1 \cos \alpha_3 \]  \hspace{1cm} (16)

\[ S_h = a_3 d(a_3, a_1) \cos \alpha_1 \sin \alpha_3 - a_3 d(a_1, a_3) \sin \alpha_1 \cos \alpha_3 \]  \hspace{1cm} (17)

\[ S_{2h} = a_3 d(a_3, a_1) \cos 2 \alpha_1 \sin 2 \alpha_3 - a_3 d(a_3, a_1) \sin 2 \alpha_1 \cos 2 \alpha_3 \]  \hspace{1cm} (18)

\[ d(a_1, a_3) = \delta(a_1) \epsilon(a_1) \]  \hspace{1cm} (19)

\[ \delta(a_1) = \Omega^2 \kappa_3^2 - \frac{4}{\pi^2} a_1^2 \kappa_3^2 + \xi^2 [\kappa_2^2 (\kappa_3^2 + 1) - \kappa_3^2 \kappa_1^2] \]  \hspace{1cm} (20)

\[ \epsilon(a_1) = \Omega^2 + \frac{4}{\pi^2} a_1^2 \kappa_2^2 - \xi^2 \kappa_1^2 \]  \hspace{1cm} (21)

\[ a_1 = \frac{\pi}{2} \left[ a - (a^2 - b)^{1/2} \right]^{1/2}, \quad a_3 = \frac{\pi}{2} \left[ a + (a^2 - b)^{1/2} \right]^{1/2} \]  \hspace{1cm} (22)

\[ a = \frac{1}{2} \left[ \Omega^2 \left( 1 + \frac{1}{\kappa_3^2} \right) - \xi^2 \left( \kappa_1^2 - \frac{\kappa_2^4}{\kappa_3^4} - 2 \frac{\kappa_2^2}{\kappa_3^2} \right) \right], \quad b = \frac{1}{\kappa_3^2} \left( \Omega^2 - \xi^2 \right) \left( \Omega^2 - \xi^2 \kappa_1^2 \right) \]  \hspace{1cm} (23)

\[ \sigma = \gamma \frac{\pi^2}{8} (\kappa_2^2 + 1) \Omega^2 \left( \Omega^2 - \xi^2 \kappa_1^2 \right) (a^2 - b)^{1/2} \]  \hspace{1cm} (24)

\[ r = \frac{\pi}{2} \left( \frac{\Omega^2}{\kappa_3^2} - \xi^2 \right)^{1/2}, \quad \kappa_r = \frac{\epsilon_r}{c_r} \]  \hspace{1cm} (25)

\[ \gamma = \frac{\rho F}{\rho}, \quad \kappa_1^2 = \frac{C_{11}}{C_{ss}}, \quad \kappa_2^2 = \frac{C_{11}}{C_{ss}}, \quad \kappa_3^2 = \frac{C_{33}}{C_{ss}} \]  \hspace{1cm} (26)

When we let the thickness \(a\) of the fluid layer go to zero \((\tau = 0)\), the dispersion equation (15) correctly reduces to that of a free orthotropic plate given by \(S_h\) (longitudinal or symmetrical modes) and \(A_h\) (flexural or antisymmetrical modes) \([7]\). In addition, by letting \(\kappa_2^2 = \kappa_3^2 = \kappa^2\) and \(\kappa_2^2 = \kappa^2 - 2\) the dispersion equation for an isotropic solid layer is recovered.

**RESULTS AND DISCUSSION**

The numerical results presented in this section are for a Graphite-Epoxy (GE)/water bilayer with \(\tau = 1\). Therefore the numerical results are presented for a case where \(c_r < c_{rT} < c_{QL}\). The material properties are listed in Table 1. The discussion is focused on the \(\alpha_1\)-displacement mode shapes for the first four bilayer modes. Mode 1 (Fig. 2) starts with the \(a_0\) mode for the solid but becomes an interface wave with a phase velocity, \(c_s\).
Table I. Material properties for a water/graphite-epoxy bilayer

<table>
<thead>
<tr>
<th>Material</th>
<th>Material Constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>$\rho_f = 0.997$ g/cc</td>
</tr>
<tr>
<td></td>
<td>$c_p = 1.49$ mm/µs</td>
</tr>
<tr>
<td>Graphite Epoxy</td>
<td>$\rho = 1.40$ g/cc</td>
</tr>
<tr>
<td></td>
<td>$C_{11} = 152$ GPa, $C_{13} = 3.03$ GPa</td>
</tr>
<tr>
<td></td>
<td>$C_{33} = 11.1$ GPa, $C_{33} = 4.86$ GPa</td>
</tr>
<tr>
<td>Bilayer Parameters</td>
<td>$\gamma = 0.714$, $\tau = a/h$, $\kappa_f = 0.799$</td>
</tr>
<tr>
<td></td>
<td>$\kappa_1 = 5.59$, $\kappa_2 = 0.790$, $\kappa_3 = 1.51$</td>
</tr>
</tbody>
</table>

Figure 2. Mode shapes for Mode 1 for a GE/water bilayer with $\tau = 1$.

Figure 3. Mode shapes for Mode 2 for a GE/water bilayer with $\tau = 1$.
that is less than the phase velocity of bulk waves in the fluid. The mode shapes for mode 1 in the solid layer start with bending and develop into an interface wave. The displacements in the solid layer have been magnified because the fluid displacements are very large as the interface wave develops. The magnification ratios used are: (a) 1:1, (b) 1:1, (c) 1:1, (d) 100:1, (e) 10:1, (f) 10:1, (g) 10:1, and (h) 10:1. The mode shapes for the \( a_0 \) mode (dry solid plate) are shown for comparison purposes. As is well known, the mode shapes change from a simple bending motion to a pair of anti-symmetric Rayleigh waves.

Mode 2 (Fig. 3) starts as the \( s_0 \) mode for the dry solid plate, drops before it reaches the free fluid mode, intersects the \( a_0 \) (solid plate) and \( s_1 \) (fluid plate) modes, and approaches \( s_1 \) (fluid plate) asymptotically. The mode shapes for the solid develop from quasi-symmetric near \( s_0 \) to quasi-antisymmetric as the branch drops towards \( a_0 \). As it

Figure 4. Mode shapes for Mode 3 for a GE/water bilayer with \( \tau = 1 \).

Figure 5. Mode shapes for Mode 4 for a GE/water bilayer with \( \tau = 1 \).
approaches \( s_1 \) (fluid plate), the motion in the solid becomes quiescent. The magnification ratios used are: (a) 1:1, (b) 1:1, (c) 1:1, (d) 1:1, (e) 5:1, (f) 10:1, (g) 20:1, and (h) 40:1. At the intersection of the \( s_1 \) fluid mode and the \( a_0 \) solid mode, the mode shapes in the fluid and solid layers are the same as for a free fluid and a free solid plate. This was verified by plotting the mode shapes at the intersection (not shown for brevity). The mode shapes for the \( s_0 \) mode (solid plate) are shown also for comparison purposes. For the \( s_0 \) branch, the motion develops from an extensional motion to a pair of symmetric Rayleigh waves.

Mode 3 (Fig. 4.) passes through the intersection of \( s_0 \) (solid plate) and \( s_1 \) (fluid plate) and also through the intersection of \( a_0 \) (solid plate) and \( a_1 \) (fluid plate). The mode shape in the solid starts as a quasi-antisymmetric mode and becomes symmetric at its first crossing point, then becomes anti-symmetric at its second crossing point. Finally it becomes quiescent as it approaches the first anti-symmetric mode in water, \( a_{1} \). The mode shape in the fluid first develops into its first symmetric mode between points \( b \) and \( c \), then it develops into the first anti-symmetric mode between \( f \) and \( g \). It remains quasi-antisymmetric after \( g \). The magnification ratios used are (a) 1:1, (b) 1:1, (c) 1:1, (d) 1:1, (e) 1:1, (f) 2:1, (g) 5:1, and (h) 10:1.

Mode 4 (Fig. 5.) starts near \( a_1 \) (solid plate) and drops steeply towards \( s_0 \) (solid plate) but remains above \( s_0 \) until it reaches the intersection of \( s_0 \) and \( a_1 \) (fluid plate). Then it crosses the intersection of \( a_0 \) (solid plate) and \( s_2 \) (fluid plate) and remains above \( s_2 \) (fluid) as it approaches the phase velocity of bulk waves in the fluid. The mode shape in the solid is quasi-symmetric (extensional motion) near \( s_0 \) (solid plate) and is symmetric at the intersection near \( d \). From \( d \) to \( f \) the motion changes from quasi-symmetric to quasi-antisymmetric at a phase velocity near the Rayleigh speed. Thus the motion changes from a pair of symmetric quasi-Rayleigh waves to a pair of anti-symmetric quasi-Rayleigh waves. Then, the motion becomes quiescent as it approaches the second symmetric mode in water, \( s_2 \). The mode shape in the fluid first starts as a quasi-symmetric mode and becomes anti-symmetric as it reaches the first crossing point near \( d \), then it becomes symmetric at the next crossing point near \( f \). The magnification ratios used are (a) 1:1, (b) 1:1, (c) 1:1, (d) 1:1, (e) 2:1, (f) 5:1, (g) 10:1, and (h) 10:1.

![Figure 6. Spring-mass system viewed as the coupling between two systems each having two masses.](image-url)
The coupling between the free fluid and dry solid plate modes has an interesting spring-mass analogy. A four-degree-of-freedom system (Fig. 6) will be viewed as a pair of two-degree-of-freedom systems coupled by a spring \( k_c \). The bottom system has masses \( m_1 \) connected with springs \( k_1 \) and the top system has masses \( m_2 \) and springs \( k_2 \). Let the motions of the two systems be coupled by a spring whose value is lower than the bottom or top spring constants. The four modes are plotted as a function of the mass \( m_2 \) in Fig. 6 as thick lines. The uncoupled natural frequencies are plotted as thin lines. For the four-mass system, modes 3 and 4 approach infinity as \( m_2 \to 0 \) and modes 1 and 2 approach zero as \( m_2 \to \infty \). The natural frequency for mode 1 of the four-mass system starts at a higher value than \( \omega_1 \) (first natural frequency for the bottom system) at \( m_2 = 0 \) because the coupling spring and the two top springs contribute to the bottom system. For mode 1, at \( a \) the bottom masses have higher displacements as seen in Fig. 6. At \( m_2 = m_1 k_2 / k_1 \), point \( c \), mode 1 intersects the crossing point of \( \omega_1 \) and \( \omega_2 \) (first natural frequency of the top system). At this point, the middle masses have the same displacement and the bottom and top systems are vibrating as if they were uncoupled. As \( m_2 \) increases further, \( \omega \) approaches zero and remains above \( \omega_2 \). As \( m_2 \) increases, the top masses have the higher displacements in \( e, f, g, \) and \( h \). All masses for the plotted modes are in-phase (IP) for mode 1. For mode 2, the bottom masses are out-of-phase (OP) and the top masses are IP at \( a \). This is consistent with the proximity of \( \omega \) to \( \omega_1 \) (second natural frequency for the bottom system) and \( \omega_2 \). At point \( b \), \( \omega \) intersects the crossing point given by

\[
m_2 = \frac{(3 - \sqrt{5})/(3 + \sqrt{5})}{m_1 k_2 / k_1}.
\]

The bottom masses vibrate at their second mode and the top masses vibrate at their first mode. From \( c \) to \( d \), the bottom masses change from OP to IP. From \( e \) to \( f \) the top masses change from IP to OP. This happens as \( \omega \) comes closer to \( \omega_1 \) and \( \omega_2 \) (second natural frequency for the top system). At \( g \), the crossing point is given by

\[
m_2 = \frac{(3 + \sqrt{5})/(3 - \sqrt{5})}{m_1 k_2 / k_1}.
\]

Here, the bottom masses vibrate in their first mode and the top masses in their second mode. At \( h \), the natural frequency remains higher than \( \omega_3 \). Similar observations can be made for mode 3. Finally, mode 4 doesn't intersect any crossing point. Both the bottom and top masses are OP from \( a \) to \( h \). Also, the middle masses are OP. From \( a \) to \( c \) the motion seems localized at the top. At \( d \) the displacements for all masses are of the same order of magnitude. From \( e \) to \( h \) the motion becomes localized at the bottom. This is opposite to what happens in mode 1. The natural frequency remains above \( \omega_1 \) and \( \omega_3 \) for all values of \( m_2 \).

**CONCLUSIONS**

The dispersion equation for guided (Lamb-type) waves in a fluid-orthotropic solid bilayer has been derived. The dispersion branches and the associated mode shapes for the first four modes of a GE/water bilayer have been studied in detail for the case of equal thickness of the fluid and solid layers. The branches of the bilayer pass through the points of intersection of the isolated solid layer and the isolated fluid layer. At these points the motion of the solid and the fluid in the bilayer become uncoupled.

A simple analogy using four masses and five springs is presented. It is shown that this elementary example captures most of the essential physics of the guided waves in the fluid-solid bilayer.

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REFERENCES