Numerical computation of transonic flow in nozzles with small throat radii of curvature

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Numerical computation of transonic flow in nozzles with small throat radii of curvature

by

Richard Rodney Wear

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NOMENCLATURE

A  flow area

c  chord

CF  influence coefficient

CD  discharge coefficient

e  relative velocity error

E  velocity error or residual

f  mass flux

g  positive definite error function

M  Mach number

p  fluid pressure

q  flow velocity

R  radius or radius of curvature

R̃  non-dimensional throat radius of curvature, \( R_c/R_t \)

s  streamline segment

S  arc-length ratio function

SR  arc-length ratio

T  arithmetic term

V  non-dimensional flow velocity

x  axial coordinate

y  vertical or radial coordinate

z  streamline node displacement

γ  ratio of specific heats

η  natural coordinate perpendicular to velocity

θ  flow angle
\( \mu \) angle of rotation
\( \xi \) natural coordinate parallel to velocity
\( \rho \) fluid density
\( \phi \) velocity potential
\( \psi \) stream function

Superscripts
(n) \( n \)th iteration, not a power
\( \epsilon \) parameter defining either two-dimensional or axisymmetric flow
\( \cdot \) denotes differential
\( \bar{\cdot} \) average value
\( \vec{\cdot} \) vector
\( \sim \) non-dimensional
\( \sim \) arc segment
\( * \) sonic conditions

Subscripts
A mesh corner point
B mesh corner point
C mesh corner point
D mesh corner point
AB streamline segment
AC potential segment
BD potential segment
CD streamline segment
c curvature at throat
max maximum value possible
M denotes Mth step along axis

MP mid-point

o stagnation conditions

t nozzle throat

upper upper streamtube

lower lower streamtube

1 just above mid-cell node

2 just below mid-cell node

12 average of conditions at points 1 and 2

Other

\frac{d}{d(\ )} derivative with respect to ( )

\frac{d^2}{d(\ )^2} second derivative with respect to ( )

\frac{\partial}{\partial (\ )} partial derivative with respect to ( )

\frac{\partial^2}{\partial (\ )^2} second partial derivative with respect to ( )

\Delta(\ ) denotes small quantity

\nabla del operator

\nabla \cdot (\ ) divergence of ( )

\nabla \times (\ ) curl of ( )

\Sigma summation
INTRODUCTION

Transonic Nozzle Flow

Expansion of a gas from subsonic to supersonic velocities through a converging-diverging nozzle has been of interest to research workers for over a century. A general picture of such a flow field is shown in Figure 1. The flow accelerates in the converging subsonic section, passes through the sonic line in the throat region, and accelerates further in the diverging supersonic exhaust nozzle.

The sonic line, however, has a much greater significance than just being the locus of points at which the flow Mach number is unity. Mathematically, the flow upstream of the sonic line is characterized, for steady flow, by elliptic equations for which the solution is of the boundary value type. Downstream of the sonic line the flow is described by hyperbolic equations for which the solution is of the initial value type. The governing equations are therefore mixed, and change from elliptic to hyperbolic across the sonic line.

The method of characteristics has been used with great success in treating supersonic nozzle flow with given initial data. However, good initial data depend upon a knowledge of the flow field in the throat region. For many applications the assumption of one-dimensional flow in the throat region or the use of series approximations have been used to generate the initial data for supersonic calculations. However, in the case of a throat with small radius of curvature, these approximations become less accurate since the two-dimensional effects become more and more important. The sharp wall curvature at the throat causes acceleration of the gas at the walls. Thus there is a significant difference in the throat region between the
Figure 1. General picture of transonic nozzle flow
velocities at the nozzle walls and the velocity along the axial centerline of the nozzle. This makes the flow non-one-dimensional and causes curvature of the sonic line. The mass flux peaks at $M = 1$ and there is less flow through the actual throat with its curved sonic line than through the one-dimensional model with uniform flow. The ratio of the actual mass flow to the one-dimensional mass flow is defined as the discharge coefficient, $C_D$, which is numerically less than one. It follows, then, that the thrust coefficient for the actual flow is less than predicted by the one-dimensional flow model.

Determination of the flow in the throat region requires a knowledge of the flow in both the subsonic and transonic regions. It is not possible to specify the value of the stream function at the wall in advance of the solution. This is unlike incompressible flow in which the stream function at the wall may be arbitrarily specified. For the transonic flow problem the value of the discharge coefficient (and thus the stream function at the wall) depends on the nature of the velocity distribution in the throat and this value must be determined as part of the complete flow solution.

The solution of the general transonic nozzle problem therefore consists of finding the velocity distribution throughout the entire subsonic and transonic portions of the nozzle. This solution must satisfy the basic gasdynamic equations while maximizing the mass flow through the nozzle.

Review of Previous Work

Investigations of choked flow through ducts and nozzles were first undertaken almost 150 years ago. An excellent survey and bibliography of work through 1962 has been given by Hall and Sutton (12). The character of
the expansion and acceleration of a compressible gas and its subsequent choking were recognized as early as 1839 by St. Venant and Wantzel, although it was not until the late 1800's that Hugoniot and Reynolds included the concept of speed of sound in the analysis of flow in the throat. One-dimensional flow in converging-diverging nozzles was studied in the first few years of this century by Stodola, Lorenz, and Prandtl. The one-dimensional flow model has been further studied and the analysis applied countless times to many actual flow problems. It remains today as an important method of treating certain nozzle problems.

It is difficult to categorize the analysis of flows with two dimensions because of the many and varied approaches which have been undertaken. Most approaches, however, will fit into one of the following four categories: series expansion, relaxation methods, unsteady solutions, and streamtube-continuity methods.

**Series expansion methods**

Series expansion methods are themselves many and varied. They include both direct and indirect approaches to the study of both two-dimensional and axisymmetric flows. In most cases they represent approximate solutions to the local flow in the neighborhood of the throat. Excellent surveys of this work are given by Hall (11), Hall and Sutton (12), and Holt (14).

**Indirect solutions** The first solution for flow with two dimensions was a series expansion by Meyer in 1908 (19). He solved the indirect problem for two-dimensional flow with a linear axial velocity distribution. He expanded the velocity potential in a double power series in terms of \(x\) and \(y\), the flow field coordinates, and found that the sonic line is parabolic
for such a flow.

Another indirect approach for two-dimensional flow is the hodograph method in which the differential equations become linear when transformed into the hodograph plane. The various techniques and their histories are outlined by Hall and Sutton (12). Either the stream function or the velocity potential is expanded in a series in which the independent variables are functions of the flow velocity and the flow angle.

Holt (14) applied the method of integral relations to the two-dimensional indirect problem. The velocity components, u and v, are formed in series expansions in terms of the flow field coordinates, x and y. Holt also discusses application to axisymmetric flows and to the direct flow problem. Belotserkovskii and Chuskin (3) discuss work done in Russia on the application of the method of integral relations to solve both indirect and direct flow problems in axisymmetric and two-dimensional nozzles.

Hopkins and Hill (15, 16) have solved several indirect problems in axisymmetric flow by using the space coordinates x and y as dependent variables and expanding each in a series in terms of a non-dimensional stream function parameter. By using empirically generated equations for the axis velocity distribution, the authors are able to obtain "nozzle-shaped streamlines" which can be used to represent flow in nozzles with small throat radii of curvature.

**Direct solutions** The direct solutions using series methods all involve expansions about the local flow conditions at the throat. The influence of the upstream geometry and the convergence of the series is limited largely by the curvature of the wall at the throat. In general, these methods are limited to simple geometries and large values of \( R \), the non-dimensional ratio.
of throat radius of curvature to throat radius.

In 1930 Taylor (33) solved a direct problem for two-dimensional flow with circular arc walls by expanding the velocity potential in a double power series (terminated at the fourth order terms) and then using the wall geometry to find the unknown coefficients in the series. The geometry corresponded to a value for $\tilde{R}$ of four.

Sauer (30) analyzed the two-dimensional case by retaining only the dominant terms in the differential equations. A series solution of this simplified case terminates itself and there are no higher order terms. However, this simplifying assumption gives answers good for only large values of $\tilde{R}$ (2, 15).

By a method of successive approximations Oswatitsch and Rothstein (23) used perturbation theory to expand a two-dimensional flow as a perturbation of the one-dimensional solution.

One of the most general and useful expansion methods was performed by Hall (11). He solved both two-dimensional and axisymmetric flow in nozzles with parabolic, circular-arc, and hyperbolic profiles. He expanded the velocity components in terms of inverse powers of $\tilde{R}$. He also gives expansions for the velocity magnitude, the flow angle, and the discharge coefficient. Excellent agreement is shown between his results and the results shown in the earlier series solutions. The application of Hall's solution, however, is also limited by the throat curvature, for the series converges only for values of $\tilde{R}$ greater than unity.

An attempt was made by Kliegal and Levine (18) to extend the applicability of Hall's method to nozzles with smaller throat curvature. They investigated flow in circular-arc profile nozzles and showed that a toroidal
coordinate system, unlike Hall's cylindrical coordinate system, possesses the advantage that the wall boundary is a constant coordinate line. The expansion parameter for this coordinate system is $1/(1 + \tilde{R})$. Kliegal and Levine transform Hall's series into a new series in terms of the new expansion parameter, and in addition, they point out and correct two small errors in Hall's work. The advantage of the new series is that convergence is now extended to all values of $\tilde{R}$ greater than zero. Comparisons between theory and experiment for a nozzle with $\tilde{R} = 0.625$ is made and excellent agreement is claimed. However, more extensive comparisons made by Cuffel et al. (6) show that while this method predicts axial velocities reasonably well, the predictions of velocities on the wall are much less successful.

Kliegal and Levine also claim that the case $\tilde{R} = 0$ corresponds to flow through an axisymmetric sharp edged orifice and that their series solution would yield an approximate solution of this orifice flow as $\tilde{R}$ approaches zero. This claim must be discounted, however, for the case $\tilde{R} = 0$ in the toroidal coordinates would actually correspond to the physically unrealistic case of flow with an upper wall consisting of a single point. Furthermore, from the equations and curves presented, it appears that physically reasonable finite answers are not obtained as $\tilde{R}$ approaches zero.

The application of this method (and other direct series expansion methods) to actual flow situations is further limited by the geometry of such nozzles. As the throat radius of curvature of a nozzle with a circular-arc throat becomes smaller, the circular-arc portion takes up a smaller portion of the throat region and the wall geometry of the upstream and downstream portions influences the flow to a greater extent. It is this inability of the series expansion to account for the entire wall geometry in the throat
region that causes the deviation of the predictions from measured data.

**Relaxation methods**

Relaxation techniques have long been used to solve Laplace's equation obtained from the boundary value problems of heat transfer and incompressible aerodynamics. The first attempt to apply relaxation techniques to the equations describing transonic flow was made by Emmons in 1944 (8). He attacked the direct problem of two-dimensional flow through a converging-diverging nozzle with hyperbolic profile. Although the wall curvature was moderate ($R \approx 2.4$), he obtained solutions for three cases: compressible but completely subsonic; continuously accelerating transonic flow; and transonic flow followed by a normal shock wave in the downstream section. The nozzle-shaped flow region was transformed into a rectangular region upon which a rectangular grid was placed. The differential equation for the stream function was written in finite-difference form and a relaxation technique was used to reduce the stream function errors, or residuals, at each mesh point to zero. Because of the singularity at the sonic line, difficulties were encountered with convergence at the throat. However, by using a finer mesh and by relaxing the density rather than the stream function in this region, a solution was obtained for all regions of the flow.

A recent application of a relaxation process was published by Prozan and Kooker in 1970 (27). In this case the nozzle shaped flow region is also transformed into a fixed rectangular grid and the differential equations expressing continuity and irrotationality are written in finite difference form. The residuals involved with these two equations are incorporated into a positive definite error function at each grid point. Error minimization techniques are used to reduce these errors to zero by suitably adjusting the
velocity components at each grid point. A finite difference method is used to approximate the partial derivatives needed in the steepest descent method which is used to predict the new values of the velocity components in each iteration. This technique is applied to nozzles with hyperbolic profiles defined by values of \( \tilde{R} \) as low as 0.5. In addition, the method is applied to the geometry of a Jet Propulsion Laboratory (JPL) nozzle whose test results are reported in (2) and (6). This nozzle has a conical entrance and conical exit and a circular-arc throat profile with \( \tilde{R} = 0.625 \). Predicted Mach number contours are compared to experimental contours with excellent agreement. The Mach number distributions along the centerline and on the wall also show excellent agreement. Limitations on the application of this method to nozzles with even smaller throat curvatures are reported to be due to the fine grid and long computer run times needed for such cases. Typical run times for the cases considered in this study are reported to be 5 to 10 minutes on an IBM 7094 computer.

**Time-dependent methods**

A third method used to solve compressible flow problems is the unsteady finite-difference technique. The unsteady flow equations are hyperbolic for subsonic as well as supersonic flow. Any reasonable estimate of flow properties may be used as starting data for the solution of the initial value problem. The equations are integrated forward in time and the asymptotic solution is considered to yield the corresponding steady-state flow.

Saunders (31) analyzed flow in an axisymmetric nozzle with moderate throat curvature (\( \tilde{R} = 3.0 \)). He obtained the asymptotic solution with approximately 45 minutes of computation time on a CDC 3200 computer.

Migdal *et al.* (22), using the Moretti time-dependent procedure, analyzed
the JPL nozzle discussed previously (2, 6). Their computed wall pressure
distribution showed excellent agreement with test data. Computer run time
was less than 5 minutes on an IBM 360/75 computer.

Streamtube-continuity methods

Streamtube-continuity methods attempt to solve for flow fields by
solving for the streamlines. An early application of this method was per­
formed by Uchida in 1948 (34). He wrote the partial differential equation
for the stream function with respect to curvilinear coordinates which are
hopefully near the desired streamlines. Small terms are neglected which
leads to an ordinary differential equation which is integrated graphically.
The results are used in an iterative technique until the stream function
distribution converges. An alternate technique allows the coordinates to
change in each iteration and hopefully converge to the streamlines. The
streamline modifications are carried out graphically. The author compares
the results of his techniques with experimental data for two-dimensional
flow past circular-arc walls. Fair agreement is obtained, although it is
difficult to assess the accuracy and stability of such a method since it
depends so heavily on graphical techniques.

A related technique was proposed by Katsanis in 1968 (17) in which he
solves elliptic type equations by the use of quasi-orthogonals. Gradient
tiles (analogous to streamlines) are assumed inside a curvilinear region.
These gradient lines are intersected by quasi-orthogonals which remain
fixed during the solution. The governing differential equations are
written in natural coordinates, and, as in the previous approach by Uchida,
certain terms are small if the quasi-orthogonals are nearly orthogonal.
Gradient values are approximated and used in a numerical integration to
obtain new estimates of the gradient lines. This technique is iterated until the lines converge to a solution. Corrections are sometimes required to prevent divergence. This technique has the advantage over that of Uchida in that it is easily adapted to computer calculations. Katsanis also mentions application to compressible flow problems, but he has not considered application to the problem of mixed equations with complex curvature.

A streamtube-continuity method was advanced by Ringleb in the early 1960's (28, 29). He uses geometrical construction of potentials with known streamlines to solve both compressible and incompressible two-dimensional and axisymmetric flow fields. The streamlines and potentials are approximated piecewise by segments of circular arcs. The four corners of a cell of the circular-arc mesh are constrained by a theorem of geometry which states that "the corners of a rectangle formed by circular arcs are situated on a circle". This theorem is used in the construction of the circular arc network as shown in Figure 2 from (28). The continuity and vorticity equations are written in finite difference form and, using geometrical areas and lengths computed during the network construction, the velocity distribution on the walls may be computed. Ringleb analyzed several different flow geometries with graphical computation and coarse grids and obtained excellent agreement with analytical velocity distributions. He also describes how to determine intermediate streamlines in a computed network by an iterative application of the geometrical construction.

Chou and Mortimer applied Ringleb's method to axisymmetric flow in a converging-diverging nozzle with a hyperbolic wall profile (5). The nose points generated on the axis and wall are then used to compute an inter-
Figure 2. Example of Ringleb's method as applied to flow between confocal parabolas
mediate streamline. The velocity distributions on the axis, wall, and intermediate streamline compare very well with the solution predicted by the series method of Oswatitsch and Rothstein. The throat curvature, however, is rather large, with $\bar{R} = 5$.

It is doubtful that application of Ringleb's method could or should be made to nozzles with less than moderate values of $\bar{R}$. A basic assumption of the method is that a circular arc can approximate a potential in all portions of the flow field. At the geometric throat the circular arc normal to both the axis and the wall is a vertical straight line, and the minimum flow area will be the geometric throat area. The equations used in the method will calculate choked flow in this throat area, and the flow solution thus corresponds to the case with a discharge coefficient of unity. An intermediate streamline calculated by the method will have a zero flow angle at the throat. The error associated with these results is very small for large values of $\bar{R}$, but as $\bar{R}$ becomes small the true flow field deviates from the model. The true streamlines reach a zero flow angle downstream of the throat. Consequently, a potential in the geometric throat region must be S-shaped to some degree, and only approaches a straight line in the limit as $\bar{R}$ approaches infinity. The method should therefore be limited to configurations in which the walls have nearly equal curvature.

Purpose of This Study

Of all the methods discussed, only two are applicable to nozzles with small throat radius of curvature. These two are the relaxation (error-minimization) technique employed by Prozan and Kooker (27) and the time-
dependent technique as typified by Migdal et al. (22). Both have been compared with test data for a nozzle with $\bar{R} = 0.625$, and both give excellent results. The only drawback to these methods is the excessive amount of computer time necessary to achieve good results, which for a fine grid require run times on the order of 5 to 10 minutes.

The method advanced by Ringleb appears very attractive, for it has the capability of predicting excellent velocity distributions for certain flow fields with a rather coarse grid. Its drawback, however, is that it is limited to cases with streamlines of nearly equal curvature.

The purpose of this study is to investigate the application of a streamtube-continuity method to transonic flow in both two-dimensional and axisymmetric nozzles with small throat radii of convergence. The primary goal is to obtain realistic results with a smaller expenditure of computer time and with a coarse grid.

A relaxation approach technique is used in this study, since the streamline shape is iterated or relaxed to reduce velocity errors. The streamline geometry of the problem is highly non-linear and conventional stability and convergence criteria are not applicable. The technique is applied to two-strip or one internal streamline solutions. Convergence is demonstrated through the investigation of flow in three different nozzle configurations.

The results obtained from these cases indicate that the goal has been reached. Good agreement with analytical and experimental data is obtained with a coarse grid and with substantially less computer time than the more complex finite-difference methods require.
STREAMLINE RELAXATION

Potential Flow

The potentials in flow with large curvature of the walls and streamlines cannot be modeled adequately by a circular arc as proposed by Ringleb. For example, some incompressible potential flows calculated by Birkhoff and Zarantello (4) show considerable changes of curvature and inflection. Figure 3 shows an example from (4) which exhibits such curvature. However, it appears that while one arc is not adequate, it would be possible to use several circular arcs to piecewise approximate each potential. In doing so, the internal streamlines must be calculated as part of the problem solution; they cannot be calculated afterwards as in Ringleb's method.

The problem, then, becomes one of the determination of the curvilinear mesh made up of streamlines and the orthogonal potentials. With such a network it should be possible to utilize the advantage of a coarse grid as demonstrated by Ringleb. The method presented in this study solves for the streamlines and potentials in an iterative fashion; in each iteration the streamlines are adjusted or "relaxed" so as to reduce the errors associated with their previous positions.

Basic Equations

The flow is assumed to be steady, inviscid, adiabatic and non-heat-conducting, and irrotational. This idealized isentropic flow is governed by the following equations:

\[ \nabla \cdot (\rho \mathbf{q}) = 0 \] (1)
Jet from 60° nozzle

half of flow

Figure 3. Example of a streamline-potential curvilinear mesh (from (4))
\[ \nabla \cdot \vec{q} = 0 \]  
\[ \frac{p}{\rho^2} = \frac{p_0}{\rho_0^2} \]  
\[ \frac{q^2}{2} + \frac{[\gamma/(\gamma-1)] p}{\rho} = \frac{[\gamma/(\gamma-1)] p_0}{\rho_0} \]  

which express, respectively, conservation of mass, irrotationality, isentropicity, and conservation of momentum of isentropic flow in the direction of the flow. The last equation is Bernoulli's compressible equation.

New non-dimensional velocity and density are defined as

\[ V = \frac{q}{q_{\text{max}}} \]  
\[ \tilde{\rho} = \frac{\rho}{\rho_0} \]  

where \( q_{\text{max}} \) is the maximum velocity to which the flow can be accelerated. This maximum velocity corresponds to a state of zero temperature and internal energy, and can be calculated from Bernoulli's equation as

\[ q_{\text{max}} = \sqrt{\frac{2\gamma}{\gamma-1} \frac{p_0}{\rho_0}} \]  

The relation between \( V \) and the Mach number, \( M \), is

\[ V^2 = \frac{\gamma-1}{2} \frac{M^2}{1 + \frac{\gamma-1}{2} M^2} \]  

and \( V^* \), the value of \( V \) at sonic velocity, is

\[ V^* = \sqrt{\frac{\gamma-1}{\gamma+1}} \]  

The equations may be expressed in terms of a natural, or streamline...
oriented, coordinate system by introducing the stream function and velocity potential (24, 32). Written in terms of the non-dimensional velocity and density, the defining equations for these new variables are:

\[ \frac{d\psi}{d\eta} = \zeta \nu y^\epsilon \quad (10) \]
\[ \frac{d\phi}{d\xi} = \nu \quad (11) \]

where \( \eta \) and \( \xi \) are coordinates of a streamline-potential oriented coordinate system. The parameter \( \epsilon \) determines the type of flow: \( \epsilon = 0 \) corresponds to two-dimensional flow and \( \psi \) represents mass flow per foot of depth; \( \epsilon = 1 \) corresponds to axisymmetric flow and \( \psi \) represents mass flow per radian.

Equations 3, 4, and 7 may be used to express \( \zeta \) in terms of \( \nu \) so that Equation 10 may be rewritten as

\[ \frac{d\psi}{d\eta} = \nu (1 - \nu^2)^{\frac{1}{r-1}} y^\epsilon \quad (12) \]

Consider the flow in a stream tube bounded by streamlines between which a mass flow of \( \Delta\psi \) passes, as shown in Figure 4. Let the potentials be represented by circular arcs orthogonal to each streamline. The average flux, \( \bar{f} \), passing through any potential is related to the flow area of the potential, \( A \), by

\[ \Delta\psi = \bar{f} A \quad (13) \]

This non-dimensional flux is related to the average velocity in the stream tube, \( \bar{V} \), by

\[ \bar{f} = \bar{V}(1 - \bar{V}^2)^{\frac{1}{r-1}} \quad (14) \]
Figure 4. Streamtube in potential flow
and any local flux is related to a local velocity by the more general expression

\[ f = V(1-V^2)^{\frac{1}{2}} \]  \hspace{1cm} (15)

This functional relationship is shown in Figure 5. At low values of \( V \) the characteristic linear relationship of incompressible flow is observed, followed by the non-linear relationship of compressible flows for slightly higher velocities until, at the sonic velocity, the flux reaches its maximum. As the velocity increases to supersonic speeds, the flux decreases to a value of zero at the maximum velocity. An inherent problem in transonic flow research is indicated by this figure, for the velocity is a double-valued function of the flux. A technique for determining the correct branch of the curve is necessary in any solution involving the density-velocity relationship.

For a given stream tube with mass flow \( \Delta \psi \), the average flux through any potential can be found from Equation 13 if its flow area is calculated. The average velocity can then be calculated from Equation 14. This is nothing more than the conventional calculation of one-dimensional flow velocities through use of the continuity equation.

The effect of the extra flow dimension is introduced through the irrotationality of the flow. Rather than using Equation 2, the irrotationality may also be expressed as the absence of circulation around any closed path, as defined by

\[ \oint \vec{A} \cdot d\vec{s} = 0 \]  \hspace{1cm} (16)
Figure 5. Mass flux-velocity relationship

\[ f = V(1 - V^2)^{\frac{\gamma}{2}} \]
This equation can be applied to each curvilinear cell defined by two streamlines and two potentials as shown in Figure 6. Following Ringleb's nomenclature, the corners of the cell are lettered, and, since flow is always normal to all potentials, Equation 16 simplifies to

\[ \int_{AB} \mathbf{V} \cdot d\mathbf{s} - \int_{CD} \mathbf{V} \cdot d\mathbf{s} = 0 \]  

Defining mean values for the velocities on the streamlines allows Equation 17 to be simplified as

\[ \overline{V}_{AB} S_{AB} = \overline{V}_{CD} S_{CD} \equiv \Delta \phi \]  

where \( \overline{V}_{AB} \) and \( \overline{V}_{CD} \) are the mean velocities along the arc lengths \( S_{AB} \) and \( S_{CD} \), and, by definition of the potential from Equation 11, the product of each mean velocity and arc length must equal the magnitude of the potential difference.

Streamtube Velocity Distribution

Ringleb (28, 29) and Chou and Mortimer (5) choose to solve for the velocity distribution in a stepwise fashion. Starting with a known potential (arc AC) and a point B, they find point D by graphically constructing the circle through A, B, C as indicated in Figure 2. All arc lengths are measured and the flow area BD is calculated. They then use Equations 13 and 14 to find \( V_{BD} \), the average velocity through arc BD. If the curvatures of AB and CD are nearly equal, the following approximation may be made:

\[ \frac{(V_{AC} + V_{BD})}{2} = V_{BD} \]
Figure 6. Ringleb's curvilinear cell
and Equation 18 may be rewritten as

\[
\left(\frac{V_A + V_B}{2}\right) S_{AB} = \left(\frac{V_C + V_D}{2}\right) S_{CD}
\]

Equations 19 and 20 may now be solved for the unknown flow velocities \(V_B\) and \(V_D\). A new point B may be picked and the next cell may be calculated in a like manner until the entire flow field is determined. It is apparent that the velocities on the initial potential must be known correctly or the solution will not proceed smoothly through the throat.

There is a difficulty inherent in this approach which is not mentioned by Ringleb and which is not explained by Chou and Mortimer. Chou and Mortimer state, "If an incorrect position of the trial equipotential line or incorrect initial velocities are assumed, then the velocity along the walls will vary erratically, and the spacing of the equipotential lines \((\Delta S, \Delta S')\) will also fluctuate erratically". They do not state that the probable reason for these fluctuations is an instability in the basic method of solution. This instability will be discussed in greater detail later, but it is present because the method is basically an initial value solution of a boundary value problem. Because of the instability, any errors in the calculation of the potential spacing by the use of the circumscribed circle and in the calculation of the velocities will propagate as the solution progresses, leading to the fluctuations mentioned by Chou and Mortimer. These errors can be reduced by use of walls with small curvature, by decreasing the spacing of the potentials, and by careful calculation, although the instability is still present.

This instability can be avoided by using another approach in calculating the streamtube velocity distribution. First, previous potential
nodes on the walls are not used in calculating a new potential. There is only one circular arc from point B that is orthogonal to both streamlines, and this arc can be calculated independent of the previous potential. This calculation thus accounts for the fact that the streamline segment CD may differ from a circular arc, whereas this distinction is neglected in Ringleb's circumscribing circle approach. Although errors may be present in this new calculation, their effect is not propagated downstream. The second change is in the velocities that are calculated. The midpoint AB of arc AB is picked (see Figure 7), and the circular arc from this point which is orthogonal to CD is calculated, with the intersection point defined as point CD. The flow area defined by this arc is calculated and used to calculate a mean flow velocity through this new potential, $V_{MP}$. The equations used by Ringleb are now applied to this new geometry to define new mean velocities on the streamlines:

$$V_{AB} S_{AB} = V_{CD} S_{CD}$$

$$V_{AB} + V_{CD} = 2V_{MP}$$

These equations may be easily solved for $V_{AB}$ and $V_{CD}$, the velocities at points AB and CD, as

$$V_{AB} = V_{MP} \left( \frac{2SR}{1 + SR} \right)$$

$$V_{CD} = V_{MP} \left( \frac{2}{1 + SR} \right)$$

where SR is the arc length ratio, the ratio of $S_{CD}$ to $S_{AB}$. These new velocities do not depend on the streamline velocities of the previous cell,
Figure 7. Curvilinear cell with midpoint calculations
so errors will not be propagated downstream.

These modifications to Ringleb's approach will render the calculations stable and eliminate the fluctuations. By recognizing that the mass flux is actually defined by the straight line potential at the throat, it becomes apparent that this method is actually a one-dimensional method with two-dimensional geometrical corrections, and it is not, as stated by Chou and Mortimer, numerical integration of the flow equations along natural coordinates.

The accuracy of the geometrical correction to predict the effect of the two dimensions depends on the ratio of arc lengths of the cell. The streamline curvatures and slope changes should be nearly equal to obtain greatest accuracy. This method could be applied to any streamtube to determine its velocity distribution, provided the \( \Delta \psi \) is consistent with the geometrical throat.

**Multiple Streamtubes**

Consider a nozzle with several internal streamlines as shown in Figure 8. The method outlined in the preceding section can be applied to each streamtube to determine its velocity distribution. Also, by approximating each potential segment by piecewise circular arcs, it is possible to account for varying curvatures and possible inflection points for each potential.

Several attempts were made to calculate both direct and indirect nozzle flows by attempting to solve for the internal curvilinear mesh using the geometrical concept of Ringleb and an initial guess at a starting potential. All attempts were unsuccessful, as streamline oscillations quickly
Figure 8. Nozzle with multiple streamtubes and potentials
built up which could not be made to converge to a smooth solution. The failure of these solutions to converge is believed to be due to an instability inherent in such an application of Ringleb's method. Basically, any attempt to solve the flow by straightforward numerical integration from an initial potential is an attempt at applying an initial value method to a boundary value problem. Such methods are inherently unstable, for any errors will be amplified and propagated with the solution. Hayes and Probstein (13) state that, "as far as elliptic differential equations are concerned, the initial value problem is improperly posed and leads to an unstable solution when treated by finite differences". They go on to show a numerical example in which slight inaccuracies in initial conditions can lead to large deviations in the solution. This instability is also pointed out by Gravalos et al. (9) in their work on the solution of the blunt body problem.

This instability, as mentioned earlier, is believed to account for the fluctuations and oscillations discussed by Chou and Mortimer (5). These difficulties could be eliminated for their single-streamtube nozzle analysis by using the revised method of the preceding section which is not an initial value approach and in which velocity errors and mesh errors are not propagated.

In a multiple streamtube approach the streamlines and potentials are not known but must be computed as part of the solution. The method used in this study seeks to do so by starting with an initial streamline distribution which is relaxed iteratively until a final curvilinear mesh is obtained which yields convergent velocity distributions.

The method may be illustrated by considering the two-strip case shown in Figure 9. A starting potential is defined by specifying the nodes on
Figure 9. Calculation of potentials for a two-strip case
this potential where it is intersected by the axis, the internal streamline, and the wall. The potential spacing is specified by defining the axis points for the potentials. The streamline is defined initially by specifying a number of points on an assumed streamline.

The accuracy of the resulting flow solution will depend in part on three possible sources of error in the above data. First, the starting potential is probably not correct. The error associated with this discrepancy can be minimized by placing the potential far from the throat where the potential shape is better known. Second, the number of strips, or streamtubes, needed depends on the amount of wall curvature. Increasing the number of strips will reduce the error in the geometrical approximation. Third, the coarseness of the potential grid introduces some error in computing arc lengths and reduces the resolution of the computed velocity distribution.

The streamline is defined by spline functions between the specified points. Detailed explanation of these functions and a short computer program to generate such curves is given by Greville (10). The spline functions used in this method are third-degree polynomials between adjacent points. The coefficients of each polynomial are found such that the polynomials to the left and right of each defining point pass through each point and their first and second derivatives are equal at each point. The curvature is assumed to vary linearly between adjacent points. The spline functions therefore "join smoothly" at each point and the resulting streamline is not only smooth but is represented by an easily calculated function for use in interpolation.

With the streamline thus defined, the potentials between the axis and
the streamline may be found by calculating, from each axis mesh point, a circular arc normal to both the axis and the streamline. Potentials are also erected from points midway between the mesh points, and the flow area through these potentials is calculated. The arc lengths associated with each curvilinear cell are also computed. These computations are carried out for all points on the axis until all curvilinear cells in the first strip have been calculated. The minimum flow area in the strip and its location are also computed.

The calculations of the next streamtube proceeds in the same way except that now the newly computed nodes on the streamline are used in computing new circular arcs which are orthogonal to both the streamline and the wall (see Figure 9). In a case with a greater number of strips, the calculations would proceed through each strip successively until the wall was reached.

The minimum flow areas in each strip may be summed to give the total throat area available for choked flow. This may be compared to the total flow area available through initial potential to determine the total mass flow through the nozzle. The assumption is made that the velocities are equal at all nodes on the initial potential. With this velocity, the mass flow may be found in each streamtube and thus the stream function values on each streamline are known.

The velocity distribution in each streamtube may now be computed because the flow areas and arc length ratios of each curvilinear cell have been calculated. Calculations proceed by starting at the initial potential and working completely through the first strip, calculating a mid-point velocity, \( v_{MP} \), and upper and lower velocities, \( v_{CD} \) and \( v_{AB} \), for each cell.
The second strip is then computed in a like manner, with three velocities being calculated for each cell. If additional strips are present, each is computed in turn.

Consider again Figure 9. After the above calculations have been completed, observe that there are two velocities associated with each mid-cell node on the internal streamline. One is $V_{CD}$ from the calculation of the lower strip and the other is $V_{AB}$ from the upper strip calculation. The differences between these velocities serve as a measure of the error involved in the streamline position and they are used to correct the streamline shape. A solution is reached when these velocity errors, or residuals, are all less than some prescribed tolerance.

Streamline Residuals and Relaxation

Each mid-cell node on each internal streamline has associated with it a residual, or velocity error, due to differences in the calculated velocities in the adjacent streamtubes. These residuals can be used to estimate displacements of the mid-cell nodes which would lead to a new streamline shape which will hopefully have smaller errors associated with it. Conventional relaxation solutions use fixed grid points at which residuals are reduced, or relaxed, to zero through systematic recalculation of the dependent variables. The present method can aptly be called a streamline relaxation method because residuals are also reduced, but now by recalculation, or relaxation, of the streamline shapes.

The displacements of the mid-cell nodes are obtained by calculating influence coefficients or correction factors for each point. Consider a small displacement normal to the streamline as shown in Figure 10.
Figure 10. Geometrical effect of streamline displacement

--- Original mesh

--- Displaced mesh
Assume that the other mid-cell nodes remain fixed and that new spline functions are used to define the new streamline. The new streamline shape causes the potentials to also be altered, and from Figure 10 two geometrical changes can be seen, both of which affect the velocity distribution. First, the flow areas in each cell are changed which will alter the average velocity through the cell, and, second, the ratio of arc lengths for each cell is changed which changes the velocity distribution across each cell. Consider the lower cell and assume the flow is subsonic. The increase in flow area due to the displacement will reduce the mass flux and lower the average velocity. The displacement also causes an increase in the arc length ratio, which, by Equations 22, will lower the ratio of streamline velocity to average velocity. Both of these effects act together to lower the velocity at the streamline in the lower cell. The effect is just opposite in the upper cell, as both geometrical changes act to increase the streamline velocity and lower the wall velocity in the upper cell.

For supersonic flow the two geometrical effects act in opposition. For the lower cell, the arc length ratio change would still tend to lower the streamline velocity, but now the area increase would tend to increase the streamline velocity. The effects are in opposition in the upper cell also. The net change in velocity, whether positive or negative, will depend very strongly on the mesh geometry and the flow velocities.

A quantitative estimate of the influence coefficients can be obtained by differentiating the mass flow equations in each streamtube with respect to node movement. Define the two velocities across the streamline as $V_1$ and $V_2$ as in Figure 11 and define a displacement, $z$, normal to the stream-
Figure 11. Streamline velocities and displacement
The derivative of the general mass flow equation expressed by Equation 13 is

\[ O = \frac{\int dA}{dz} + A \frac{d\bar{F}}{dz} \]  

(23)

since the same mass flow, \( d\psi \), flows through the displaced streamtube. Now let \( \bar{V} \) represent the average velocity in the cell and, from Equations 22, this average velocity can be expressed in terms of the streamline velocity, \( V \), by

\[ \bar{V} = S V \]  

(24)

where \( S \) is some function of the arc length ratio, depending on whether \( V \) is the upper or lower cell velocity. Using the chain rule of differentiation, Equation 23 can be expressed as

\[ O = S A \frac{d\bar{F}}{d\bar{V}} \frac{d\bar{V}}{dz} + A V \frac{d\bar{F}}{d\bar{V}} \frac{dS}{d\bar{V}} \frac{d\bar{V}}{dz} + \bar{F} \frac{dA}{dz} \]  

(25)

This equation can be solved for \( CF \), the generalized influence coefficient.

\[ CF = \frac{d\bar{V}}{d\bar{V}} = - \frac{S \left[ \frac{A d\bar{F}/d\bar{V}}{\bar{F} dA/d\bar{V}} \right]}{1 + V (dS/d\bar{V}) \left[ \frac{A d\bar{F}/d\bar{V}}{\bar{F} dA/d\bar{V}} \right]} \]  

(26)

For the upper cell and lower cells, the \( S \) functions are

\[ S_{\text{upper}} = \frac{1}{2} \left( \frac{1 + SR}{SR} \right)_{\text{upper}} \]  

(27)

\[ S_{\text{lower}} = \frac{1}{2} \left( \frac{1 + SR}{SR} \right)_{\text{lower}} \]  

(28)

The influence coefficients for the upper and lower cells become, respectively
The velocity residual, \( E \), may be defined as the difference between \( V_1 \) and \( V_2 \), and the overall influence coefficient is

\[
CF = \frac{d A}{d E}
\]

which may be expressed in terms of \( CF_1 \) and \( CF_2 \) as

\[
CF = - \frac{(CF_1)(CF_2)}{CF_1 - CF_2}
\]

The evaluation of \( CF_1 \) and \( CF_2 \) requires the calculation of several derivatives. The derivative \( d \bar{r} / d \bar{v} \) is found by differentiating Equation 14, and \( dA/dz \) is simply

\[
\frac{dA}{dz} = \pm y^e
\]

where the positive sign is used for the lower cell and the negative sign for the upper cell. The derivative \( dSR/dz \) is much more complicated, for it involves, for the two-strip case discussed here, the geometry of the streamline, the axis, and the wall. For two cells in the middle of a multi-strip case, the geometries of three streamlines are involved. Two derivatives are actually involved for each cell, for the arc length ratio is affected by displacements of both the top and bottom mid-cell nodes.
An exact analytical derivation of these derivatives was not possible because of the complexities of the geometry, but an approximate model of the streamline displacement was formulated and it was used to generate the $\frac{dSR}{dz}$ derivatives. This model and the equations for the derivatives are given in Appendix A.

A first order estimate of the displacement necessary to reduce the residual to zero can be obtained from

$$ z = -E \cdot CF $$

(34)

This approximation can be improved by iterating Equation 34 as in a Newton-Raphson technique to obtain an improved $z$ estimate

$$ z^{(n+1)} = z^{(n)} - E^{(n)} \cdot CF^{(n)} $$

(35)

This requires evaluating new estimates of velocities, areas, arc length ratios, and influence coefficients at the start of each iteration. However, by using Equation 35, the non-linear effects are more fully included.

It was discovered that while this technique was adequate for the greater part of the flow field, in the throat region the equations are so highly non-linear that the first order correction, even with iteration, was not satisfactory. This problem can be easily seen by considering Equation 26. If the average velocity, $\overline{V}$, should happen to be the sonic velocity, the derivative $\frac{d\overline{E}}{d\overline{V}}$ becomes identically zero and the value of $CF$ from Equation 32 is zero. Therefore no displacement is predicted and the node in question is not corrected. A much more common but equally disturbing occurrence is that for nodes near the sonic line the predicted displacement can sometimes be too large, causing the streamtube to become over-choked such that convergence is delayed and possibly not obtained.
This problem may be eliminated by using a second order correction based on retaining an extra term in the Taylor's series expansion for the displacement as a function of the error. The iterative approach is retained.

\[ z^{(m+1)} = z^{(m)} - E^{(m)} \cdot CF^{(m)} + \frac{1}{2} \left( E^{(m)} \right)^2 \cdot CF^{(m)} \]  

(36)

This equation still requires the evaluation of new parameters during each iteration, and in addition requires a new term \( CF' \). This term is the second derivative of the displacement with respect to the residual and can be expressed as

\[ CF' = \frac{d^2 z}{dE^2} = \frac{(d^2 z / dV_z^2) (d^2 z / dV_t^2) - (d^2 z / dV_t^2) (d^2 z / dV_z^2)}{(d^2 z / dV_t^2 - d^2 z / dV_z^2)^3} \]  

(37)

or more simply as

\[ CF' = \frac{(CF_2)^3 \frac{d^2 z}{dV_t^2} - (CF_1)^3 \frac{d^2 z}{dV_z^2}}{(CF_2 - CF_1)^3} \]  

(38)

Two new terms, the second derivatives of \( z \), are present in Equation 38. These terms may be derived by differentiating Equations 29 and 30. The resulting equations again contain new terms, all but one of which can be easily derived. A new term \( d^2 SR/dz^2 \) is present which is unknown. This term, however, was neglected because of the dominant effect of the mass flux peak in this region, as typified by another new term, \( d^2 E/dV^2 \).

This second order iterative technique was used with success to reduce residuals in the subsonic and transonic regions, but a modification was necessary in the supersonic region. As indicated by Figure 11, an increase
in velocity caused by an area increase in the supersonic region is offset by a velocity decrease due to the arc length changes. It is thus possible that the denominator of Equations 30 or 31 can become zero, in which case an infinite displacement is predicted. Several possible techniques are available to avoid this problem.

An attractive approach is a relaxation technique which includes the effect of residual changes at adjacent points due to a local displacement. Such an approach, similar to Southwell's relaxation search discussed in (25), involves the use of a positive definite error function, \( g \), defined as

\[
g = \sum_{M=1}^{N_{\text{stps}}} E_M^2
\]

A Newton-Raphson search technique can then be applied to solve for the displacements to minimize this function:

\[
Z_M^{(n+1)} = Z_M^{(n)} - \frac{\partial g}{\partial Z_M} \bigg|_{Z = Z^{(n)}}
\]

This technique was applied to the streamline relaxation, but satisfactory convergence was not obtained. Convergence was slow in the subsonic region and divergence was obtained in the supersonic region. This divergence was traced, in part, to an inability to predict the geometrical derivatives, such as \( \partial (SR)_M + 1 / \partial Z_M \), with sufficient accuracy. This difficulty was especially acute near the end of the streamline, where information on the endpoint curvature must be specified for the spline functions. This approach, however, would be a good candidate for future applications if a more analytic means of obtaining streamline geometry derivatives can be obtained.

The technique which was finally adopted uses Equations 29 and 30 in
the subsonic region. The second term in the denominator is positive in this region and must be included to prevent overestimation of the displacements. In the supersonic region this term is set equal to zero. This results in an essentially one-dimensional area correction and thus usually underestimates the displacements and slows the convergence somewhat. However, the streamlines in the supersonic region can obtain such a shape that the geometry effects are predominant and correcting with the local error can lead to displacements in the wrong direction. This is a direct result of the inability to accurately predict the geometrical derivatives. To circumvent this difficulty an error averaging scheme is used in which an average error incorporating both local and adjacent errors is used as the residual at each point. Geometry effects are thus incorporated through their effect on adjacent errors. The averaging scheme is discussed later in greater detail.

Additional specifications are needed for the curvatures at the first and last points used to define each streamline. The first points on each streamline are the nodes on the starting potential. Since this potential is far from the throat, a reasonable assumption is that the streamline curvature be defined as zero at these points. It is possible that this error and the probable errors in the starting potential itself might prevent the complete convergence to zero of the residuals in this area. Therefore the mid-cell node on the streamline of the first cell is not used in defining the streamline, but the error is calculated at this point even though it is not reduced to zero. This error ultimately converges to a non-zero value during the solution and may be used as an indication of the accuracy of the shape of the initial potential and the assumption of
zero streamline curvature at these initial nodes.

The curvature at the last point on each streamline must also be specified, since there are no points further downstream to aid in defining the streamline shape. The program by Greville (10) specifies a curvature of zero at this point, but it seems more correct to specify that the curvature at the final point be half that at the preceding point, as this would allow a smooth curve throughout. It was discovered, however, that this constraint on streamline shape would not allow complete convergence for the shapes which were being generated during typical solutions. This difficulty was eliminated by having the program calculate this curvature during the course of the solution. At the end of each iteration, the errors at the last two points on the streamline are compared and the additional increment of curvature between these points which would make these errors equal is calculated. The new difference in curvature is used in the calculation of the spline functions on the next iteration. It was found that this difference in curvature was an additional parameter which could be successively recalculated and which would converge to a final value as the streamline shape converged to its solution. This method of solution is compatible with the hyperbolic nature of the equations in this supersonic region, since additional downstream information is not necessary.
COMPUTER PROGRAM

The streamline relaxation method was programmed for use on the Iowa State University Computation Center IBM 360/65 computer. The program, written in Fortran IV, consists of a main program and 20 subprograms. The main program reads the input data and calls the appropriate subprograms so that, through successive DO loops, each cell, each strip, and each pass through the nozzle is computed.

A brief explanation of the main program and each subprogram is given in Appendix B. A simplified flow chart of the program is also included.

The input information requirements needed to run a nozzle analysis are quite simple. They consist of data on five aspects of the problem: general problem specification, the wall, the starting potential, the axis, and the streamlines. Each of these types of data is explained below in more detail.

**General problem specification**  A geometry parameter IEPS is used to specify two-dimensional (IEPS = 0) or axisymmetric (IEPS = 1) flow. The number of steps along the axis is specified by NSTEPS and the number of strips or stream tubes is specified by NSTRIPS. The gas characteristics are specified by the ratio of specific heats, GAMMA.

**Nozzle wall information**  The wall is characterized in part by the number of equations (NEQNS) needed to define its geometry and by the axial location of the geometric throat (XTHRT). The geometry of each segment of the wall is specified by giving wall equation coefficients, and flexibility is provided by allowing two types of wall equations. If KTYPE(J) = 1, the segment is described by the conic equation

\[ y = A\left(\sqrt{B + Cx + Dx^2} + E\right) \]  

(41)
and if \( \text{KTYPE}(J) = 2 \), the segment is described by the polynomial equation

\[
y = Ax^4 + Bx^3 + Cx^2 + Dx + E
\]  

(42)

The coefficients of the appropriate equation are used as elements in the wall coefficient array \( \text{WALLCO}(JJ) \). Each segment is also characterized by \( \text{XMAX}(J) \), the maximum value of \( x \) for which the wall equation applies. Therefore, for any value of \( x \), the appropriate wall segment may be found, and using the specified equation, the wall coordinate may be calculated.

The above method of defining wall geometry, adapted from (26), is very flexible and permits a large variety of nozzles to be specified easily and accurately. Additional types of equations could be easily incorporated if desired.

**Starting potential information** The \( x \) and \( y \) coordinates \( (X(1,N), Y(1,N)) \) and flow angle \( (\text{TH}(1,N)) \) of the nodes on the starting potential line, including the axis and wall nodes, are specified. In addition, an initial estimate of the flow velocity through the initial potential \( (\text{VSTART}) \) is specified.

**Axis information** The axial location of the corners of the mesh \( (\text{XAXIS}(M)) \), including the axis point of the starting potential, are specified. The program calculates the midpoint of each pair of points to be used, when needed, as mid cell nodes.

**Streamline information** Several points \( (\text{NPTS}) \), each defined by its \( x-y \) coordinates \( \text{XP2D}(I,\text{ISTL}) \) and \( \text{YP2D}(I,\text{ISTL}) \), are used to define each streamline. A parameter \( \text{KNEW}(I) \) is used to specify the type of streamline. If \( \text{KNEW} = 0 \), the points represent an initial guess of the streamline shape. \( \text{KNEW}(I) = 1 \) signifies that the streamline comes as the continuation of a previous problem. In this case additional pieces of information, \( \text{CRVINC}(I) \),
are specified. CRVINC(I) is the increment of curvature between the last two points on each streamline.
RESULTS AND CONCLUSIONS

Emmons Two-Dimensional Nozzle

The first case analyzed concerns the flow in the two-dimensional hyperbolic nozzle used by Emmons (8) in his relaxation solution study. The constants of the wall equation for this nozzle are not given in (8) but the wall coordinates may be estimated from a drawing of the contour and, in turn, these may be used to estimate the equation of the bounding hyperbola. This nozzle is shown in Figure 12. It may be categorized as being of moderate curvature, since the non-dimensional throat radius of curvature $\bar{R}$ has a value of 2.424.

The starting potential, the potential spacing, and the initial streamline estimate for this two-stripe solution are shown in Figure 13. An ellipse which intersects both the axis and the bounding hyperbola orthogonally and which is situated far from the throat is assumed as the starting potential curve. The desired potential spacing is defined by specifying mesh corner points on the axis. The program automatically interpolates between these corner points to define the axis mid-cell nodes. Any reasonable streamline shape may be used as an initial guess, although convergence is delayed by a poor choice. Also, results from previous solutions or series approximations may be used. If available, the streamline from an incompressible solution may be used (5, 34). The streamline shown in Figure 13 was obtained by sketching a smooth curve which starts at the node on the starting potential, passes midway through the geometric throat, and extends into the diverging section. Several coordinates on this curve were input and used by the program to calculate the spline functions which define
Nozzle wall equation:

\[ y = \sqrt{0.325 + 0.450x^2} \]

**Figure 12. Emmons' two-dimensional hyperbolic nozzle**
Figure 13. Starting potential, potential spacing, and initial streamline estimate
the streamline.

The solution proceeds by relaxing the mid-cell nodes to reduce the streamline residuals. The streamline and potentials that result after six passes or iterations are shown in Figure 14. The location of the minimum area in each streamtube is also shown.

The convergence of the solution is shown in Table 1. The residuals, or velocity errors, at each mid-cell node are shown for each iteration. Relative errors, in percent, are shown, where the relative error at the $M$th mid-cell node, $e_M$, is related to the residual, $E_M$, by

$$e_M = \left[ \frac{(V_i - V_2)}{(V_i + V_2)/2} \right]_M = \frac{2E_M}{(V_2)_M} \tag{43}$$

It is possible that errors in the starting potential might prevent convergence. For this reason the first two points used to define the streamline are the node on the starting potential and the second mid-cell node. The first mid-cell node is not used to define the streamline and its residual is not relaxed. Instead, this node and its residual are calculated during each iteration and the residual converges to some non-zero value. The magnitude of this residual may be used as an indication of the accuracy of the starting potential. From Table 1, this residual converges to $-0.25\%$, indicating a good choice for the starting potential.

The rest of the residuals become very small as convergence is obtained, although the residual at Node 8 converges slowly because it is near the throat of the lower streamtube.

The flow angle along the streamline is shown in Figure 15 for both the initial estimate and the converged solution. The point of zero flow angle occurs downstream of the geometric throat due to the rotational acceler-
Figure 14. Streamline and potentials for converged solution
Table 1. Relative errors during convergence

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</table>
Figure 15. Flow angle along initial and final streamlines
tion effects (12). Also, the "bump" in initial streamline is smoothed as the solution is obtained.

The predicted velocity distributions along the axis, the internal streamline, and the wall are shown in Figure 16. As expected, sonic velocity is reached upstream of the geometric throat for the flow along the wall and downstream of the geometric throat for the flow along the axis.

These results may be compared with the results of Emmons. The Mach number distributions along both the wall and axis are compared in Figure 17. Differences could arise because of inaccuracies in each solution or from inaccuracies in estimating the nozzle contour used by Emmons. However, very good agreement is obtained between the solutions.

The flow velocities across the geometric throat are not equal to the sonic velocity in the converged solution. The mean velocity is subsonic in the lower streamtube at the geometric throat and supersonic in the upper streamtube. The mass flux is therefore less than the one-dimensional value, \( f_{\max} \), and the discharge coefficient, \( C_D \), is less than one. The discharge coefficient calculated for this solution is 0.9978. This may be compared to the discharge coefficient calculated with a series solution of this moderate curvature nozzle. Application of Hall's three-term series solution (11) yields a value of 0.9964 for \( C_D \). Agreement is good and these high values of discharge coefficient confirm that the flow in nozzles with moderate to large values of throat radius of curvature is nearly one-dimensional.

The results shown in Table 1 required 19 seconds of computer execution time for the six iterations. The program was run under the WATFV compiler on the IBM 360/65 computer at the ISU Computation Center.
Figure 16. Velocity distribution for converged solution
Figure 17. Comparison of Mach number distributions in hyperbolic nozzle
Axisymmetric Nozzle with Moderate Throat Radius of Curvature

The second case to be analyzed is flow in an axisymmetric nozzle described in (2). The nozzle geometry is shown in Figure 18. The radius of curvature of the wall at the throat is moderate ($R = 2.0$), but the wall geometry is complex. Five equations are needed to describe the complete wall contour. The internal streamline will be more complex than in the previous case, since a point of inflection can be expected in the subsonic conical portion of the nozzle.

The starting potential, the potential spacing, and the initial streamline estimate for a two-strip solution is shown in Figure 19. Since the low-speed chamber is cylindrical, a vertical line at $x = -3.0$ inches is used to represent the starting potential. The initial streamline estimate is again obtained by sketching a smooth curve approximately through the middle of the nozzle.

The streamline and potentials obtained after six iterations are shown in Figure 20. As was true in the previous case the minimum areas in each streamtube are again displaced from the geometric throat.

The convergence of the solution is shown in Table 2. The initial relative errors are greater than in the previous case, indicating a less accurate initial streamline. The residual at the first node converges to 1.28%, indicating a reasonable approximation for the starting potential. The remainder of the residuals decrease as the solution progresses, until all are less than 0.24% after the sixth iteration. The residuals in the subsonic portion of the nozzle would eventually approach zero, but the residuals in the diverging exhaust nozzle will converge to small non-zero values because of the error-averaging technique used in obtaining the
Figure 18. Nozzle geometry for Case 2 (from (2))
Figure 19. Starting potential, potential spacing, and initial streamline estimate
Figure 20. Streamline and potentials for converged solution
Table 2. Relative errors during convergence

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supersonic velocities. These values are quite small as is shown in Table 2.

The flow angle along the streamline is shown in Figure 21 for both the initial estimate and the converged solution. The expected subsonic streamline inflection point appears as the minimum point in the flow angle curve. Another point of inflection, however, is indicated in the supersonic section by the appearance of a maximum point in the flow angle curve. This inflection can be noted in Figure 20 as a change in sign of the streamline curvature. This inflection did not appear in the previous case; the flow angle continued to increase in the supersonic portion of the Emmons nozzle.

This supersonic point of inflection was not expected since the flow in the exhaust nozzle is an expanding flow and the flow area is continuously increasing. The phenomenon is real, however, and can be traced to angular rotation imparted to the flow as it is accelerated in the circular arc throat region. This angular motion persists downstream of the circular arc region and causes an overturning of the flow. A compressive turning is thus required to turn this flow back to a more nearly conical flow.

This over-expansion and the subsequent recompression are discussed by Migdal and Kosson (20) as a means of explaining shock formation in conical nozzles. Oblique shock waves have been detected experimentally near the axis of conical nozzles and have also been predicted analytically with method of characteristics programs (1, 7, 20, 21, 27). The analyses show that the shock wave is caused by the crossing of right-running characteristics emanating just downstream of the contour junction of the circular-arc throat with the conical nozzle. This pattern of characteristics from the wall would cause the flow compression and streamline deflection, and it
Figure 21. Flow angle along initial and final streamlines

Flow angle, $\theta$, degrees

Axial distance, $x$, inches

First pass
Sixth pass

$\gamma = 1.4$
explains the point of inflection in the streamline shown in Figures 20 and 21.

Predicted velocity distributions along the axis, the internal streamline, and the wall are shown in Figure 22. The predicted values of wall pressure are compared in Figure 23 with the one-dimensional pressure distribution and with experimental data from (2). The predicted pressures show excellent agreement with the measured pressures. The one-dimensional flow model, although adequate for the subsonic and supersonic portions of the nozzle, predicts low velocities and high pressures in the transonic region, with approximately a 30% error just downstream of the throat.

The discharge coefficient predicted for this case converged to a value of 0.9978 after six iterations.

The results shown in Table 2 required 17.65 seconds of computer execution time.

Axisymmetric Nozzle with Small Throat Radius of Curvature

The third nozzle analyzed is also an axisymmetric nozzle described in (2). The radius of curvature at the throat is small (\( R = 0.625 \)) and the wall contour requires five equations for complete description. The nozzle geometry is shown in Figure 24. Since the exhaust nozzle is a conical section following the circular-arc throat, an internal streamline could be expected to have a point of inflection downstream of the throat as in the last case.

The starting potential, the potential spacing, and the initial streamline estimate for a two-strip solution are shown in Figure 25. The starting potential is a vertical line in the cylindrical portion of the chamber, and the initial streamline estimate is again obtained from a curve drawn through
Throat \( x = 3.573 \)

\( \gamma = 1.4 \)

**Figure 22.** Velocity distribution for converged solution
Figure 23. Static-to-stagnation pressure ratios along nozzle wall
Chamber-to-throat area ratio = 9.76

R / R<sub>t</sub> = 0.625

Nozzle wall

Gas flow

Nozzle centerline

Tangency

x = 0.876 in.

Tangency

x = 2.200 in.

x = 2.683 in.

Tangency

x = 0.310 in.

Throat

x = 2.554 in.

Axial distance, x, inches

Figure 24. Nozzle geometry for Case 3 (from (2))
Figure 25. Starting potential, potential spacing, and initial streamline estimate
the nozzle.

The streamline and potentials obtained after ten iterations are shown in Figure 26. The minimum areas in each streamtube are again displaced from the geometric throat. It is evident that one circular arc could not accurately approximate a complete potential in this nozzle, since the potentials have a reflex shape in the region just downstream of the throat and also in the downstream portion of the cylindrical chamber. The method used in this study, however, does model this complex potential shape.

The convergence of the solution is shown in Table 3. The errors in the subsonic portion of the nozzle exhibit oscillation during their convergence because of the large curvature changes occurring along the streamline. The errors in the supersonic portion of the nozzle again converge to small non-zero values because of the error averaging technique. The error at the first mid-cell node converges to 0.07%, while the errors in the remainder of the nozzle are all less than 0.43% after ten iterations.

The flow angle along the streamline is shown in Figure 27 for both the initial estimate and the converged solution. The overturning of the streamline and its subsequent compressive turning are clearly shown in both Figures 26 and 27.

The acceleration of the flow around this small circular-arc throat leads to over-expansion and, thus, the velocity of the flow at the contour junction is too high. The right-running characteristics from this tangency point are compressive and they slow the expansion of the flow along the conical wall. Figure 28 shows the velocity distributions along the axis, the internal streamline, and the wall. The compression region downstream of the throat is clearly indicated by the wall velocities. Cuffel et al.
Figure 26. Streamline and potentials for converged solution
Table 3. Relative errors during convergence

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Figure 27. Flow angle along initial and final streamlines
Figure 28. Velocity distribution for converged solution
(6) use experimentally measured static pressures to calculate the Mach number distribution in the throat region. Their Mach number data for the wall and axis are compared to the predicted values in Figure 29. The wall velocities show excellent agreement throughout the nozzle. The agreement on the axis is very good in the subsonic portion of the nozzle, but the predictions are consistently high in the exhaust nozzle. This discrepancy is discussed in more detail later.

Predicted wall pressures are compared in Figure 30 to experimental data from (6) and to the one-dimensional pressure distribution. Excellent agreement between the predictions and the experimental data is obtained. The compression zone along the conical wall causes a pressure plateau downstream of the throat which is of importance since it represents an adverse pressure gradient (6). This can lead to a thickening of the boundary layer which can contribute to flow separation in the exhaust nozzle. This pressure plateau is predicted by the streamline relaxation method, although resolution is poor because of the coarse grid used. The one-dimensional model greatly over-estimates the transonic wall pressures, with an error of almost 100% at the geometric throat, and, in addition, the pressure plateau is completely neglected.

A discharge coefficient of 0.9887 is predicted for this nozzle. This compares well with the experimental value of 0.985 reported in (6).

The results shown in Table 3 required 37 seconds of computer execution time.
Figure 29. Mach number distributions in nozzle
Figure 30. Static-to-stagnation pressure ratios along nozzle wall
Conclusions

Convergence

Convergence of the streamline relaxation method for two-strip solutions has been demonstrated for three different nozzle configurations. Results from these cases show that convergence is affected by several factors. As would be expected, convergence is delayed by a poor choice of initial streamline. Also, as the throat radius of curvature becomes smaller, the number of iterations needed to obtain convergence is increased.

The local velocity error has been used in the subsonic portion of the nozzle to calculate the displacement of a mid-cell node. Convergence of the residuals to very low values is possible, although oscillations in the residuals during convergence occur. Since the use of local errors can lead to divergence in the supersonic portion of the nozzle, an error averaging technique is used. This leads to convergence of the residuals to small non-zero values in the exhaust nozzle. Several averaging techniques are possible. Obviously slightly different results and residual values will be obtained for each technique. The technique used to obtain the results shown in this study defines the residual used for streamline corrections, $e_{SC}$, as follows:

$$
(e_{SC})_M = \begin{cases} 
e_M & (V_{iz} \leq V^*) \\ (e_{M-1} + e_M + e_{M+1})/3 & (V_{iz} > V^*, M < (N_{STEPS}-1)) \\ (e_{M-1} + 2e_M)/2 & (V_{iz} > V^*, M = (N_{STEPS}-1)) \\ (2e_{M-1} + e_M)/2 & (V_{iz} > V^*, M = N_{STEPS}) \end{cases}
$$

(44)

The results obtained for Emmons' nozzle also show that convergence is delayed when the velocity at a mid-cell node is supersonic and the node is close to the choked area of a stream tube (see Figure 14 and Table 1).
This is a consequence of the averaging technique and the influence factor calculation. Displacement of a node in this region not only affects the local flow area, but it also displaces and changes the choked area of the streamtube. This latter effect is not included in the calculation of the influence coefficient and it contributes to the delay in convergence. It was also discovered that convergence is not possible for certain locations of a mid-cell node. For example, if a node were located midway between nodes 9 and 10 in the third case (see Figure 26), it was found that convergence was not obtained because accurate influence coefficients could not be calculated. This was also traced to the change in choked flow area caused by node displacements in this region.

All cases considered in this study are two-strip cases with one internal streamline. The extension to multi-streamtube cases has not been attempted, although the method and the computer program are not restricted to the simpler two-strip problem. However, in light of the convergence pattern displayed in this study, it is expected that additional oscillations and possible divergence might accompany additional strips. Further knowledge of the streamline geometry effects on residuals and the development of a more efficient convergence scheme for both the subsonic and supersonic portions of the nozzle would be helpful in attacking the more complex cases.

Accuracy

The curvilinear meshes used in this study constitute a coarse grid approach, since relatively large potential spacings are used with only two streamtubes. The results, however, are good. Velocity and pressure distribution predicted by the computer program show good to excellent agree-
ment with other analytical solutions and with experimental data.

Residuals at the first mid-cell nodes are small for all three cases, indicating that the starting potentials are approximated with sufficient accuracy. Unlike Ringleb's method, this method can generate potentials with varying curvature and with a reflexed shape. This allows the minimum flow area in each streamtube to be displaced from the geometric throat.

Discharge coefficients predicted by the streamline relaxation method show good agreement with analytic and experimental values, although the predicted values are consistently high. This may be traced to the technique adapted from Ringleb (28) to calculate the flux and velocity distribution in a curvilinear cell. The flux is used to calculate a midpoint velocity which is assumed to be the average of the velocities at the upper and lower mid-cell nodes. However, this technique overestimates the mass flow. At the minimum area the lower streamline velocity is subsonic and the upper streamline velocity is supersonic. Using Ringleb's technique, the average velocity is therefore the sonic velocity and \( \frac{f}{f_{\text{max}}} \) is predicted for the flux in the minimum area. But from Figure 5, it is apparent that for such a flow distribution the mean flux is actually less than \( \frac{f}{f_{\text{max}}} \). A technique to calculate the velocity distribution using the mean flux equation can be derived which will lead to lower discharge coefficients. This technique, however, is more complex and is not included in the present study.

**Execution time**

The computer time required for a solution depends on the coarseness of the grid and the number of iterations necessary to obtain convergence. The results shown here required times varying from 19 and 18 seconds for Cases
1 and 2 to approximately 36 seconds for case 3. This represents a considerable saving of computer time compared to the conventional fixed grid methods discussed earlier.

These results were obtained using the WATV compiler. Additional reductions in execution time can be achieved by using IBM's Fortran G or H compilers which produce codes allowing faster execution speeds.

Shock formation

The results of this study demonstrate the flow conditions which lead to shock formation in conical nozzles. Overexpansion of the flow by the circular-arc throat and subsequent compressive turning by the conical wall are predicted for both the second and third cases. Such flow patterns can lead to the eventual crossing of right-running characteristics which in turn causes the formation of the oblique shock wave reported in (1).

Although the predicted pressure and velocity distributions on the wall show excellent agreement with experimental data for the third case, the axial velocity predictions are too high in the exhaust nozzle. Several factors may contribute to this, including the convergence technique used for supersonic velocities, the large potential spacing compared to the streamline curvature changes, and the number of streamtubes. This last factor may be the most important, since the data in (6) show that the compressive turning of the streamlines causes considerable distortion of the velocity profiles. These profile shapes are not modeled well by the assumed velocity distribution across the two streamtubes, and more streamtubes would be necessary to accurately model this flow pattern.
Viscous effects

This study has analyzed idealized inviscid flow. Although viscosity is important in gasdynamics, there are some flow problems where the viscous effects are extremely small. Back et al. (2) experimentally varied the boundary layer thickness at the nozzle inlet. They tested several nozzles and could discern no effect on the measured wall pressure ratios. Hall and Sutton (12) show that, because of the favorable pressure gradient, boundary layer growth is small in converging-diverging nozzles and may usually be neglected.
RECOMMENDATIONS FOR FURTHER STUDY

The streamline relaxation method described here has been shown to give good results for typical nozzle configurations. However, several areas have been uncovered which would benefit from further analysis.

The convergence of the method in both the subsonic and supersonic portions of the nozzle needs improvement. In the subsonic portion convergence is slow and irregular. In the supersonic portion a technique is needed which will converge the residuals to zero. These improvements would reduce both the iterations and execution time.

A more accurate technique to compute streamline geometry derivatives is needed. The technique of Appendix A is approximate and it does not fully utilize the data available in the spline functions. With more knowledge of streamline geometry it might be possible to adapt a method similar to Southwell's relaxation technique (25). More knowledge of the effect of a mid-cell node displacement on the streamline geometry is also needed so that the effect upon the minimum flow area can be calculated. This would improve convergence near the throat and allow nodes to be placed in regions of the throat that are now off-limits.

The application of the streamline relaxation technique to multitube cases (more than two strips) should be investigated. This capability is needed to more accurately analyze cases with sharp curvature and complex streamline shapes.

Improvements in the method to more accurately calculate velocity distributions and the discharge coefficient have been indicated previously.

The streamline relaxation method can easily be adapted to analyze flow
problems which are now quite cumbersome. For example, by replacing the axis with a lower boundary defined in the same manner as the nozzle wall, it would be possible to analyze flow in plug nozzles and other complex channels.

This method may also be applicable to certain free-boundary problems such as flow from a sharp-edged orifice. For this application the free-boundary would replace the downstream nozzle wall as the outer streamline. The relaxation technique would iterate this free-boundary to satisfy the constant pressure constraint rather than the nozzle flow angle constraint now used.
LITERATURE CITED


ACKNOWLEDGMENTS

The author wishes to express his sincere gratitude to Dr. E. W. Anderson for not only his guidance and direction during this study, but also for the counsel and assistance he has provided from the very beginning of my college career.

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Lastly, the author would like to express his appreciation to his wife, Marianne, for her many contributions to the completion of this project.
APPENDIX A

Streamline Geometry Derivatives

An approximate method for computing the changes in arc length ratios due to streamline displacement may be derived by treating a streamline segment as a circular arc. Define the rotation between points C and D as

$$\mu \equiv \theta_D - \theta_C$$  \hspace{1cm} (A1)

Using this rotation and the chord $c_{CD}$, the radius for the circular arc between C and D with rotation $\mu$ may be found as

$$R = \frac{1}{2} \frac{c_{CD}}{\sin(\mu/2)}$$  \hspace{1cm} (A2)

Extend this circular arc on each side of C and D to define points G and H such that (see Figure A1)

$$\theta_H - \theta_G = 2\mu$$  \hspace{1cm} (A3)

Now assume that as point M, the mid-cell node, moves perpendicular to the streamline, the streamline changes shape but remains a circular arc passing through points G and H. Let $c_{GH}$ denote the chord connecting points G and H. It can be shown that

$$\frac{d\mu}{dz} = -\frac{2}{c_{CD}} \frac{\sin^2\mu}{(1-\cos\mu)}$$

and

$$\frac{dR}{dz} = \frac{\cos\mu}{(1-\cos\mu)}$$  \hspace{1cm} (A4)

Consider now the orthogonal circular arc $\hat{AC}$ approximating the potential.

Three equations connect points G, C, and A.

$$y_C = y_A - (x_C - x_A) \cot \left(\frac{\theta_A + \theta_C}{2}\right)$$  \hspace{1cm} (A5)
Figure A1. Streamline deflection for upper node displacement
As M moves to M_p, point C must move to C_p in order that the orthogonality of the circular arc be maintained. Since arc CD can also be written as

\[ S_{CD} = S_{CM} + S_{MD} \]  \( \text{(A8)} \)

it is possible to define

\[ \frac{dS_{CD}}{dz} = \frac{dS_{CM}}{dz} + \frac{dS_{MD}}{dz} \]  \( \text{(A9)} \)

But the first term can be broken down as

\[ \frac{dS_{CM}}{dz} = \frac{d}{dz} [R(\theta_m - \theta_c)] \]  \( \text{(A10)} \)

or

\[ \frac{dS_{CM}}{dz} = \frac{\mu}{2} \frac{dR}{dz} - \frac{R}{2} \frac{d\theta_c}{dz} \]  \( \text{(A11)} \)

since \( \theta_m \) will remain constant under the displacement. But \( d\theta_c/dz \) can be evaluated by differentiating Equations A5, A6, and A7 and using

\[ \frac{d\theta_c}{dz} = -\frac{d\mu}{dz} \]  \( \text{(A12)} \)

Combination of these terms into Equation A11 results in

\[ \frac{dS_{CM}}{dz} = \frac{1}{(1 - \cos \mu)} \left\{ \frac{\mu}{2} \cos \mu + \cos \mu \frac{T_1}{T_1} + \sin \mu \frac{T_2}{T_2} \right\} \]  \( \text{(A13)} \)

where

\[ T_1 = -\cos \theta_c - \frac{AC}{2R \cos (\theta_a + \theta_c)} + \tan (\frac{\theta_a + \theta_c}{2}) \sin \theta_c \]  \( \text{(A14)} \)

\[ T_2 = \tan (\frac{\theta_a + \theta_c}{2}) \left[ \cos \theta_c - \cos (\theta_c - \frac{\mu}{2}) \right] - \left[ \sin \theta_c - \sin (\theta_c - \frac{\mu}{2}) \right] \]  \( \text{(A15)} \)

\[ T_3 = \cos (\theta_c - \frac{\mu}{2}) + \tan (\frac{\theta_a + \theta_c}{2}) \sin (\theta_c - \frac{\mu}{2}) \]  \( \text{(A16)} \)
As the curvature of the arc between C and D approaches zero, the Equation A13 becomes indefinite. Evaluation as $\mu$ approaches zero may be made using L'Hospital's rule to yield the limiting form of Equation A13:

$$\frac{dS_{CD}}{dz} = \frac{1}{4} \left\{ T_5 - \frac{3}{4} \tan\left(\theta_c + \theta_e\right) \cos \theta_c + \frac{3}{4} \sin \theta_c \right\}$$

(A17)

where

$$T_4 = -\cos \theta_c - \tan\left(\theta_c + \theta_e\right) \sin \theta_c$$

(A18)

$$T_5 = \frac{AC}{c_H \cos\left(\frac{\theta_c + \theta_e}{2}\right)}$$

(A19)

Equation A17 can be used for small values of $\mu$ and Equation A13 for large values.

Similar equations can be written for arc MD to predict $dS_{MD}/dz$. A special limiting equation must also be derived for small values of $\mu$.

These results and results from Equation A13 can be used in Equation A9 to predict $dS_{CD}/dz$. Since $S_{AB}$ remains constant, the change in arc length ratio is

$$\frac{d(S_{SR})}{dz} = \frac{1}{S_{AB}} \frac{dS_{CD}}{dz}$$

(A20)

for the streamline deflection due to an upper node displacement.

Another derivative must also be calculated corresponding to the arc length ratio change due to a lower node displacement. A geometric model for this deflection is shown in Figure A2. Using this model, the same equations used before may be applied and it is possible to predict $dS_{AB}/dz$ from

$$\frac{dS_{AB}}{dz} = \frac{dS_{AM}}{dz} + \frac{dS_{MB}}{dz}$$

(A21)

Again, limiting forms of the derived equations are necessary for small curvatures.
Figure A2. Streamline deflection due to lower node displacement
In this model $s_{CD}$ remains constant, so the arc length ratio derivative becomes

$$\frac{dSR}{dz} = -\frac{s_{CD}}{s_{AB}^2} \frac{ds_{AB}}{dz}$$

(A22)
Subroutines and functions

The main program is used to read input data and set up the successive loops to calculate, in turn, a cell, a strip, and a complete pass through the nozzle. The calculations are mainly carried out in subroutines and functions, which in turn use other subroutines and functions to complete the solution. A brief description of each subroutine and function is given below.

PRMTRS This subroutine calculates gasdynamic parameters such as $V^*$ and $f_{\text{max}}$ and defines convergence tolerance parameters.

WALL The wall $y$ coordinate is calculated for a given $x$ value. In addition, the slope, the angle, and the second-derivative at this wall point are also calculated.

SPLINE The spline functions used to define each streamline are calculated using the desired sets of streamline points. The method and program given in (10) is used with a modification to allow the desired curvature increment between the last two points. The spline function parameters are placed in two dimensional arrays.

STMLNE For a given streamline coordinate $x$, the $y$ coordinate, slope, flow angle, and curvature of the desired streamline are calculated. If the streamline is the axis, all calculated values are zero. If the streamline is the wall, subroutine WALL is called. If the streamline is internal, the arrays calculated by SPLINE are used with the interpolation procedures of (10).
LETTER The position arrays are used to define the position coordinates and flow angles of the corner points A, B, and C of the curvilinear cell.

CRSCRB The circumscribed circle passing through points A, B, and C is calculated.

FINDD The fourth point of the curvilinear cell is calculated by solving for the circular arc from point B which is orthogonal to both streamlines. An initial estimate is made by calculating the point on the circumscribed circle which satisfies the theorem of (28) that "The corners of a rectangle formed by circular arcs are situated on a circle". A Newton-Raphson technique is used to find this point. This value is used as an initial guess for FINTUN which calculates the final coordinates and flow angle of point D and places these values in appropriate arrays. In addition, the chords and arc lengths for BD and CD are calculated.

FINTUN The streamline spline functions and the potential circular arc equations are used in a Newton-Raphson iterative technique to solve for point D. The initial guess is obtained from FINDD and the final result returned to FINDD.

ARC This function calculates the length of a circular arc of given chord and angular rotation.

AREA This function calculates the streamtube flow area through the segment of the potential. The potential segment is a circular arc. The flow area is area per foot if two-dimensional and area per radian if axisymmetric.

AREAPR This subroutine calculates the derivative dA/dS, which is the rate of change of flow area of the streamtube in the direction of the
MINARE This subroutine checks the area derivatives at the entrance and exit of each cell. If the area derivative becomes positive, the minimum flow area in the streamtube is in that cell. This area and its location is calculated using a Newton-Raphson technique which utilizes the spline function equations for the bounding streamlines and the area derivatives from AREAPR.

FMEAN This function calculates the mean flux determined by two velocities.

V4FBAR This function calculates the velocity corresponding to a given flux. A parameter (ISIGN) is used to specify whether the subsonic (ISIGN = -1) or supersonic (ISIGN = +1) branch is desired. For flux values less than 95% $f_{max}$, a first-order Newton-Raphson technique is used:

$$\sqrt{(n+1)} = \sqrt{(n)} - \frac{f}{(df/dV)^{(n)}}$$

(B1)

Since the slope approaches zero near the sonic velocity, a different technique is employed if the flux is within 5% of $f_{max}$. The flux is expanded in a second order Taylor series or a function of the velocity. The quadratic formula is used to solve for the velocity to give

$$\sqrt{(n+1)} = \sqrt{(n)} - \frac{(df/dV)^{(n)}}{(d^2f/dV^2)^{(n)}} + (ISIGN) \cdot \sqrt{\left[\frac{(df/dV)^{(m)}}{(d^2f/dV^2)^{(m)}}\right]^2 - \frac{2(f^{(m)} f)_{(m)}}{(d^2f/dV^2)^{(m)}}}$$

(B2)

Sonic conditions are used as the initial guesses in this formula.

FDUBL The second derivative $d^2f/dV^2$ used in V4FBAR is calculated.

VLCTYS The average flow velocity in the cell is found by using the
mean flux and an initial velocity guess to call on V4FBAR.

**EMACH** The Mach number corresponding to a given non-dimensional velocity is computed.

**MID** This subroutine computes each cell by calling the above subroutines and functions. The potential from the mid-cell node on the lower streamline segment is erected and its flow area calculated. The area derivatives are calculated to see if the minimum flow area is in the particular cell. The mean flux is calculated and, with initial guesses, the average velocity is calculated. Using the arc length ratio, the velocities and Mach numbers at the top and bottom of the cell may finally be calculated.

**CRFCTR** The geometrical derivatives used to calculate the arc length ratio changes are calculated using the methods described in Appendix A.

**MOVE** This subroutine is called after all strips and all geometric parameters have been calculated. Each mid-cell node on each internal streamline is treated in turn. The velocities on each side of the node are recomputed and used to calculate the residual. It is necessary to recompute these velocities because the new minimum flow areas calculated by MID may alter slightly the mass flux in each strip, and the velocities in the throat region are especially sensitive to mass flux changes. The derivatives calculated in CRFCTR and the velocities and fluxes in each cell are used to calculate the influence coefficients $C_{F1}$ and $C_{F2}$ of each adjoining cell and the combined coefficient $C_F$. The derivatives and terms used for the second order correction are computed. The desired residual is then found. For the subsonic region this value will be zero, but for supersonic
sonic flow it may be non-zero in order to aid convergence. A second-order iterative technique is then employed to predict the displacement of the mid-cell node needed to reduce the residual to the desired value. The final step is to compute the new predicted mid-cell node positions for use in the next pass through the nozzle.

**Simplified flow chart**

A simplified flow chart describing the sequence of calculations used in the streamline relaxation program is given in Figure B1.
Figure Bl. Streamline-relaxation program flow chart
Figure B1. (Continued)
Calculate total area for choked flow, total mass flow, and the discharge coefficient

Calculate mass flux in streamtubes

Calculate the new velocities and the residual at the mid-cell node

Calculate influence coefficient

Calculate mid-cell node-displacement

Calculate new coordinates of mid-cell node

Is this the last cell in streamtube?

No

Yes

Figure B1. (Continued)
Figure B1. (Concluded)