2016

Fourier series for singular measures and the Kaczmarz algorithm

John Edward Herr
Iowa State University

Fourier series for singular measures and the Kaczmarz algorithm

by

John Edward Herr

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:

Eric Weber, Major Professor

Justin Peters

Yiu Tung Poon

Paul Sacks

Scott Hansen

Iowa State University

Ames, Iowa

2016

DEDICATION

I would like to dedicate this thesis to my mother Barbara, without whose unfailing love and support I would not have completed this undertaking. I would also like to thank my brothers Stephen, Philip, and David, my friends Nate, Tim, Will, Kevin, and other innumerable family and friends for their love and companionship. I thank in a special way Professor Weber, who is as great a major professor as a graduate student could ask for. Finally, I would like to thank God and His servant, Saint Thomas Aquinas. Everything contained in this dissertation is "so much straw."

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I would like to express my appreciation to Dr. Eric Weber for acting as my major professor the past three years and being my mentor and partner in this research. I would also like to thank Dr. Palle Jorgensen from the Univeristy of Iowa for his collaboration.

ABSTRACT

Using the Kaczmarz algorithm, we obtain a Fourier series formulation for functions in the L^2 space of singular measures on the unit circle. This formula is applied to the problem of finding reproducing kernel Hilbert spaces inside the classical Hardy space, where the norm is instead that of boundary integration with respect to a singular measure. We also give some conditions ensuring that these subspaces bear some essential semblances to the classical Hardy space.

CHAPTER 1. PRELIMINARIES

1.1 Frame Theory

The usual way of studying a separable Hilbert space $\mathbb{H}$ is by fixing an orthonormal basis $\{x_n\}_{n=0}^{\infty}$. Such a basis is easy to use, both because expansions $f = \sum_{n=0}^{\infty} \langle f, x_n \rangle x_n$ in terms of it are unique, and because such expansions, having orthogonal terms, tend to cancel out and simplify when used in computations. However, orthonormal bases have a major drawback when used in real-world applications. If a signal $f \in V$ is transmitted by sending the information $\{\langle f, x_n \rangle\}_{n=0}^{\infty}$, and one of the terms in the sequence is corrupted or missing, an entire dimension of the data is lost, and so the signal received can vary wildly from the one sent.

A method of mediating the effect of error in transmission is to use instead a sequence of vectors that is somewhat redundant. This way errors in sampling or sending one coefficient $\langle f, x_{n_0} \rangle$ are compensated for by other coefficients in the sequence that also bear information about the dimension that was sampled or transmitted incorrectly. This type of sequence is known as a frame, which can be thought of as a generalization of an orthogonal basis. We give a brief introduction to frame theory here.

Definition 1. A sequence $\{x_n\}_{n=0}^{\infty}$ in a Hilbert space $\mathbb{H}$ is said to be *Bessel* if there exists a constant $B > 0$ such that for any $\phi \in \mathbb{H}$,

$$\sum_{n=0}^{\infty} |\langle \phi, x_n \rangle|^2 \leq B \|\phi\|^2. \tag{1.1}$$

This is equivalent to the existence of a constant $D > 0$ such that

$$\left\| \sum_{n=0}^{K} c_n x_n \right\| \leq D \sqrt{\sum_{n=0}^{K} |c_n|^2}$$

for any finite sequence $\{c_0, c_1, \ldots, c_K\}$ of complex numbers. The sequence is called a *frame* if in addition there exists a constant $A > 0$ such that for any $\phi \in \mathbb{H}$,

$$A\|\phi\|^2 \leq \sum_{n=0}^{\infty} |\langle \phi, x_n \rangle|^2 \leq B\|\phi\|^2. \tag{1.2}$$

If $A = B$, then the frame is said to be *tight.* If $A = B = 1$, then $\{x_n\}_{n=0}^{\infty}$ is a *Parseval frame.* The constant A is called the *lower frame bound* and the constant B is called the *upper frame bound* or *Bessel bound.*

Definition 2. Let $\{x_n\}_{n=0}^{\infty}$ be a frame in a Hilbert space $\mathbb{H}$. A frame $\{y_n\}_{n=0}^{\infty}$ in $\mathbb{H}$ is a dual frame of $\{x_n\}_{n=0}^{\infty}$ if

$$\sum_{n=0}^{\infty} \langle \phi, x_n \rangle y_n = \phi \text{ for all } \phi \in \mathbb{H}. \tag{1.3}$$

A given frame will generally have many dual frames, but every frame possesses a unique canonical dual frame, constructed as follows:

Due to condition (1.1), the linear operator $\theta : \mathbb{H} \to \ell^2(\mathbb{N}_0)$ given by $\theta(\phi) = \{\langle \phi, x_n \rangle\}$ is bounded. θ is called the **analysis operator** of the frame $\{x_n\}_{n=0}^{\infty}$. Its adjoint, $\theta^* : \ell^2(\mathbb{N}_0) \to \mathbb{H}$, satisfies $\theta^*\{c_n\} = \sum_{n=0}^{\infty} c_n x_n$. We then can form the **frame operator** $S : \mathbb{H} \to \mathbb{H}$ by $S = \theta^*\theta$, so that $S\phi = \sum_{n=0}^{\infty} \langle \phi, x_n \rangle x_n$.

The frame operator is injective, since

$$\begin{aligned} S\phi = \theta^*\theta\phi = 0 &\implies \langle \theta^*\theta\phi, \phi \rangle = 0 \\ &\implies \langle \theta\phi, \theta\phi \rangle = 0 \\ &\implies 0 = \|\theta\phi\|^2 \geq A\,\|\phi\|^2 \\ &\implies 0 = \|\phi\| \\ &\implies \phi = 0. \end{aligned}$$

It is also surjective, since $S = S^*$ implies $\text{Ran}(S) = \text{Ker}(S^*)^\perp = \text{Ker}(S)^\perp = \mathbb{H}$. We claim that $\left\{S^{-1}x_n\right\}_{n=0}^{\infty}$ is a dual frame of $\{x_n\}_{n=0}^{\infty}$. Observe that for any $\phi \in \mathbb{H}$,

$$\|\phi\|^2 = \left\|SS^{-1}\phi\right\|^2$$

$$
\begin{aligned}
&\le \|S\|^2 \left\|S^{-1}\phi\right\|^2 \\
&\le \frac{\|S\|^2}{A} \sum_{n=0}^{\infty} \left|\left\langle S^{-1}\phi, x_n\right\rangle\right|^2 \\
&= \frac{\|S\|^2}{A} \sum_{n=0}^{\infty} \left|\left\langle \phi, S^{-1}x_n\right\rangle\right|^2 \\
&\le \frac{\|S\|^2 B}{A} \left\|S^{-1}\phi\right\|^2 \\
&\le \frac{\|S\|^2 \left\|S^{-1}\right\|^2 B}{A} \|\phi\|^2 .
\end{aligned}
$$

Multiplying through by $\frac{A}{\|S\|^2}$, we have that

$$
\frac{A}{\|S\|^2} \|\phi\|^2 \le \sum_{n=0}^{\infty} \left|\left\langle \phi, S^{-1}x_n\right\rangle\right|^2 \le \left\|S^{-1}\right\|^2 B \|\phi\|^2 .
$$

This shows that $\left\{S^{-1}x_n\right\}_{n=0}^{\infty}$ satisfies condition (1.2) and hence is a frame. It is dual to $\{x_n\}_{n=0}^{\infty}$, because

$$
x = S(S^{-1}x) = \sum_{n=0}^{\infty} \left\langle S^{-1}x, x_n\right\rangle x_n = \sum_{n=0}^{\infty} \left\langle x, S^{-1}x_n\right\rangle x_n.
$$

Thus, every frame $\{x_n\}_{n=0}^{\infty}$ comes with a canonical dual frame $\left\{S^{-1}x_n\right\}_{n=0}^{\infty}$.

One can show that a Parseval frame is its own canonical dual. In general, a frame can have many dual frames. If $\{y_n\}_{n=0}^{\infty}$ is a frame with synthesis operator θ_y^*, and $\{x_n\}_{n=0}^{\infty}$ is a frame with analysis operator θ_x, then the two frames will be dual if and only if $\theta_y^*\theta_x = I$. Thus a multiplicity of dual frames can arise if θ_x has a multiplicity of left inverses.

Observe that if $\theta_y^*\theta_x = I$, then by taking the adjoint of both sides, we obtain $\theta_x^*\theta_y = I$. Thus if $\{x_n\}$ and $\{y_n\}$ are frames and

$$
\sum_{n=0}^{\infty} \langle \phi, x_n\rangle y_n = \phi \text{ for all } \phi \in \mathbb{H},
$$

then it is also necessarily true that

$$
\sum_{n=0}^{\infty} \langle \phi, y_n\rangle x_n = \phi \text{ for all } \phi \in \mathbb{H}. \tag{1.4}
$$

Thus, frame duality is symmetric.

It is true, of course, that orthogonal bases are (Parseval) frames, but the class of frames is always much larger than this, including many sequences with some useful redundancy. For example, one need only take an orthonormal basis and form a new sequence by melding weighted copies of it to form a frame with redundancy.

While redundancy is the main advantage frames have over orthonormal bases, it is not the only reason they may be sought. In a great many cases, it does not suffice merely to find any orthonormal basis or frame. Rather, the sequence, to be useful, must consist of very specific types of elements.

In this dissertation, one type of Hilbert space we will examine will be the L^2-spaces of measures on the 1-torus $[0, 1) \equiv \mathbb{T}$. For these Hilbert spaces, we specifically desire orthonormal bases or frames consisting of complex exponential functions, that is, functions of the form $e_\lambda(x) := e^{2\pi i\lambda x}$ for some $\lambda \in \mathbb{R}$. In some cases, orthogonal bases of complex exponentials can be found, but in many cases they can be proven not to exist at all. If an orthogonal basis of complex exponentials does not exist, it may still be possible for a frame of complex exponentials to exist. However, this has proven to be a very difficult problem. In [JP98], it was shown that the ternary Cantor measure on $[0, 1)$ does not admit an orthogonal basis of complex exponential functions, but the question of whether it admits an exponential frame is still an open problem at the time of writing.

Both orthonormal bases and frames are sought because they enable the reconstruction of Hilbert space elements through formula (1.3). In particular, if the elements of the frame are complex exponentials, (1.3) gives us a sort of Fourier series expansion of a function. Due to the common difficulty of finding an exponential frame, it would be useful for there to be an alternative to a frame with a similar sort of reconstruction property. Fortunately, there is a type of sequence called an "effective sequence" that has just this sort of attribute. Two of the central goals of this thesis are to show that these effective sequences are a useful alternative to frames, and to show that we can easily find effective sequences consisting of complex exponential functions in the case of L^2-spaces for singular Borel probability measures on $[0, 1)$.

1.2 Effective Sequences

In this section, we give necessary background material concerning effective sequences, which are the core motivators of the discoveries of Chapter 2.

Let $\{\varphi_n\}_{n=0}^{\infty}$ be a linearly dense sequence of unit vectors in a Hilbert space $\mathbb{H}$. Given any element $x \in \mathbb{H}$, we may define a sequence $\{x_n\}_{n=0}^{\infty}$ in the following manner:

$$x_0 = \langle x, \varphi_0 \rangle \varphi_0$$
$$x_n = x_{n-1} + \langle x - x_{n-1}, \varphi_n \rangle \varphi_n.$$

If $\lim_{n\to\infty} \|x - x_n\| = 0$ regardless of the choice of x, then the sequence $\{\varphi_n\}_{n=0}^{\infty}$ is said to be effective.

The above formula is known as the Kaczmarz algorithm. In 1937, Stefan Kaczmarz [Kac37] proved the effectivity of linearly dense periodic sequences in the finite-dimensional case. In 2001, these results were extended to infinite-dimensional Banach spaces under certain conditions by Kwapień and Mycielski [KM01]. These two also gave the following formula for the sequence $\{x_n\}_{n=0}^{\infty}$, which we state here for the Hilbert space setting: Define

$$\begin{aligned} g_0 &= \varphi_0 \\ g_n &= \varphi_n - \sum_{i=0}^{n-1} \langle \varphi_n, \varphi_i \rangle g_i. \end{aligned} \tag{1.5}$$

Then

$$x_n = \sum_{i=0}^{n} \langle x, g_i \rangle \varphi_i. \tag{1.6}$$

As shown by [KM01], and also more clearly for the Hilbert space setting by [HS05], we have

$$\|x\|^2 - \lim_{n\to\infty} \|x - x_n\|^2 = \sum_{n=0}^{\infty} |\langle x, g_n \rangle|^2,$$

from which it follows that $\{\varphi_n\}_{n=0}^{\infty}$ is effective if and only if

$$\sum_{n=0}^{\infty} |\langle x, g_n \rangle|^2 = \|x\|^2. \tag{1.7}$$

That is to say, $\{\varphi_n\}_{n=0}^{\infty}$ is effective if and only if the associated sequence $\{g_n\}_{n=0}^{\infty}$ is a Parseval frame.

If $\{\varphi_n\}_{n=0}^{\infty}$ is effective, then (1.6) implies that for any $x \in \mathbb{H}$, $\sum_{i=0}^{\infty} \langle x, g_i \rangle \varphi_i$ converges to x in norm, and as noted $\{g_n\}_{n=0}^{\infty}$ is a Parseval frame. This does not mean that $\{g_n\}_{n=0}^{\infty}$ and $\{\varphi_n\}_{n=0}^{\infty}$ are dual frames, since $\{\varphi_n\}_{n=0}^{\infty}$ need not even be a frame. However, $\{\varphi_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ are pseudo-dual in the following sense, first given by Li and Ogawa in [LO01]:

Definition 3. Let $\mathbb{H}$ be a separable Hilbert space. Two sequences $\{\varphi_n\}$ and $\{\varphi_n^\star\}$ in $\mathbb{H}$ form a pair of *pseudoframes* for $\mathbb{H}$ if for all $x, y \in \mathbb{H}$, $\langle x, y\rangle = \sum_n \langle x, \varphi_n^\star\rangle\langle\varphi_n, y\rangle$.

All frames are pseudoframes, but not the converse as we shall see. Observe that if $x, y \in \mathbb{H}$ and $\{\varphi_n\}_{n=0}^\infty$ is effective, then

$$\begin{aligned}\langle x, y\rangle &= \left\langle \sum_{m=0}^\infty \langle x, g_m\rangle\varphi_m, y\right\rangle \\ &= \sum_{m=0}^\infty \langle x, g_m\rangle \left\langle \varphi_m, y\right\rangle,\end{aligned}$$

and so $\{\varphi_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$ are pseudo-dual.

Of course, since $\{g_n\}_{n=0}^\infty$ is a Parseval frame, it is a true dual frame for itself.

Because in this dissertation we will be promoting effective sequences as an alternative to frames for reconstruction, a more generalized notion of reconstructive sequence duality is merited. Thus, we introduce the following definitions:

Definition 4. Given a Hilbert space $\mathbb{H}$ and two sequences $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ in $\mathbb{H}$, if we have

$$\sum_{n=0}^\infty \langle f, x_n\rangle y_n = f \tag{1.8}$$

with convergence in norm for all $f \in \mathbb{H}$, then $\{x_n\}_{n=0}^\infty$ is said to be dextrodual to $\{y_n\}_{n=0}^\infty$ (or, "a dextrodual of $\{y_n\}_{n=0}^\infty$"), and $\{y_n\}_{n=0}^\infty$ is said to be levodual to $\{x_n\}_{n=0}^\infty$.

In the parlance of frame theory, if S_y is the synthesis operator of $\{y_n\}$ and A_x is the analysis operator of $\{x_n\}$, then $\{x_n\}$ is dextrodual to $\{y_n\}$ if $S_y A_x = I$. However, a sequence does not need to be a frame to have a dextrodual. (For example, in the next chapter we will exhibit a Parseval frame $\{g_n\}$ that by (2.1) will be seen to be dextrodual to another sequence $\{e_n\}$ that is not even Bessel.).

Unlike frame duality, which we know is symmetric, it is unknown whether dextroduality need be symmetric. If at least one of the sequences $\{x_n\}$ and $\{y_n\}$ is not a frame, it is possible for either S_y or A_x to be unbounded, in which case the argument leading to equation (1.4) no longer works. We surmise it is not symmetric in general, but we lack an example.

1.3 Classical Complex and Harmonic Analysis

In this paper, we will be dealing with measures μ on the unit circle. The unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and its topology shall be identified with $[0,1)$ via the relation $\xi = e^{2\pi i x}$ for $\xi \in \mathbb{T}$ and $x \in [0,1)$. We will regard the measures μ as being supported on $[0,1)$. A function $f(\xi)$ defined on $\mathbb{T}$ (for example, a boundary function) may be regarded as being in $L^2(\mu)$ if $f(e^{2\pi i x}) \in L^2(\mu)$ as a function of x. For aesthetics, the inner product (norm) in $L^2(\mu)$ will be denoted $\langle \cdot, \cdot \rangle_\mu$ ($\|\cdot\|_\mu$) rather than $\langle \cdot, \cdot \rangle_{L^2(\mu)}$ ($\|\cdot\|_{L^2(\mu)}$). The subscript will be suppressed where context suffices.

Because of the importance of the complex exponential functions, as well as to emphasize them as elements of the Hilbert spaces we will be working with, we shall henceforth often use the notation $e_\lambda(x) := e^{2\pi i \lambda x}$.

The following classical results will be used when we discuss the proof of the Kwapień-Mycielski Theorem in 1.7:

Let h^1 denote the space of complex-valued harmonic functions u on $\mathbb{D}$ satisfying

$$\sup_{0<r<1} \int_0^1 |u(re^{2\pi i\theta})| \, d\theta < \infty.$$

By $M(\mathbb{T})$, denote the space of complex-valued Borel measures on $\mathbb{T} \equiv [0,1)$.

Remark 1. It is sometimes more elegant to use the notation

$$\int_{\mathbb{T}} F(\xi) \, d\mu(\xi) := \int_0^1 F(e^{2\pi i x}) \, d\mu(x)$$

for an integral against a measure μ on $[0,1)$. This $\mathbb{T}$ (as opposed to $[0,1)$) notation will be used later on certain occasions, where it will allow a more compact presentation that is easier on the eyes. Standardization to just one notation is surprisingly difficult, so we beseech the reader's tolerance. A measure $\nu \in M(\mathbb{T})$ can be thought of as a measure either on $\mathbb{T}$ or $[0,1)$, since these two sets are identified via $\xi = e^{2\pi i x}$. We will use the $\mathbb{T}$ notation in the following three theorems to get the reader accustomed.

Theorem (Riesz-Herglotz)**.** *The Poisson integral acts as an isometric isomorphism from $M(\mathbb{T})$ onto h^1. (See [Pav14], page 7.)*

Theorem (Herglotz)**.** *Let u be a harmonic function on $\mathbb{D}$, $u \geq 0$. Then there exists a unique Borel measure μ on $\mathbb{T}$, $\mu \geq 0$, such that*

$$u(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2}\, d\mu(\xi), \qquad z \in \mathbb{D}.$$

(See [Nik02], page 42.)

Theorem (Fatou)**.** *Let $\mu \in M(\mathbb{T})$ and let $\xi \in \mathbb{T}$ be a Lebesgue point of μ. Then the Poisson integral of μ,*

$$u(z) := \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2}\, d\mu(\xi),$$

has a nontangential limit at ξ, and this limit coincides with the derivative $\frac{d\mu}{dm}(\xi)$. In particular,

$$\lim_{r \to 1^-} u(r\xi) = \frac{d\mu_a}{dm}(\xi)$$

almost everywhere with respect to Lebesgue measure m on $\mathbb{T}$. Here μ_a denotes the absolutely continuous part of μ. (See [Nik02], page 39.)

For the following treatment of Bochner's Theorem, we credit Rudin's eminent text [Rud90].

Definition 5. Let G be a locally compact abelian group. A *positive-definite function* on G is a function $\phi : G \to \mathbb{C}$ such that for every finite sequence $x_1, \ldots, x_N \in G$,

$$\sum_{m,n=1}^{N} c_n \overline{c_m} \phi(x_n - x_m) \geq 0$$

for all choices $c_1, \ldots, c_N \in \mathbb{C}$.

If G is a locally compact abelian group, then a *character* on G is a function $\gamma : G \to \mathbb{C}$ satisfying $|\gamma(x)| = 1$ for all $x \in G$ and $\gamma(x \bullet y) = \gamma(x)\gamma(y)$, where $\bullet$ is the group operation on G. We denote by $M(G)$ the set of bounded, regular, complex-valued measures on G.

Let Γ_G denote the set of continuous characters on G. Then Γ_G is itself a locally compact abelian group under the operation $*$ defined by $(\gamma_1 * \gamma_2)(x) = \gamma_1(x)\gamma_2(x)$.

Theorem (Bochner)**.** *A continuous function ϕ on G is positive-definite if and only if there is a nonnegative measure $\nu \in M(\Gamma)$ such that*

$$\phi(x) = \int_{\Gamma_G} \gamma(x)\, d\nu(\gamma)$$

for all $x \in G$.

In section 1.7, Bochner's Theorem will be applied to the situation $G = \mathbb{Z}$. The character group of $\mathbb{Z}$ is isomorphic to $[0,1) \equiv \mathbb{T}$. One isomorphism from $[0,1)$ to $\Gamma_{\mathbb{Z}}$ is given by $t \mapsto \gamma_t$, where $\gamma_t : \mathbb{Z} \to \mathbb{C}$ is defined by $\gamma_t(n) = e^{-2\pi int}$. Thus, for a positive definite function $\phi : \mathbb{Z} \to \mathbb{C}$, Bochner's Theorem ensures a unique nonnegative measure ν on $[0,1)$ such that

$$\phi(n) = \int_0^1 e^{-2\pi int}\, d\nu(t).$$

Definition 6. If μ is a finite Borel measure on $[0,1)$ and $f(z)$ is an analytic function on $\mathbb{D}$, we say that $f^\star \in L^2(\mu)$ is an $L^2(\mu)$-boundary function of f if

$$\lim_{r\to 1^-} \left\| f^\star(x) - f(re^{2\pi ix}) \right\|_\mu = 0.$$

If a function possesses an $L^2(\mu)$-boundary function, then clearly that boundary function is unique. The $L^2(\mu)$-boundary function of a function $f : \mathbb{D} \to \mathbb{C}$ shall be denoted $f^\star_\mu$, but we omit the subscript when context precludes ambiguity.

Definition 7. The *Fourier-Stieltjes transform* of a finite Borel measure μ on $[0,1)$, denoted $\widehat{\mu}$, is defined by

$$\widehat{\mu}(x) := \int_0^1 e^{-2\pi ixy}\, d\mu(y).$$

Definition 8. A function $b \in H^\infty(\mathbb{D})$ (the space of bounded holomorphic functions on $\mathbb{D}$) is said to be *inner* if the radial limits $b^*(e^{2\pi ix}) := \lim_{r\to 1^-} b(re^{2\pi ix})$ exist for almost all $x \in [0,1)$ with respect to Lebesgue measure and $|b^*(e^{2\pi ix})| = 1$ for almost all x.

Definition 9. Given a positive Borel measure ν on $[0,1)$, define the *Cauchy transform* from $L^1(\nu)$ to the set of functions defined on $\mathbb{C} \setminus \mathbb{T}$ by

$$C_\nu f(z) := \int_0^1 \frac{f(z)}{1 - ze^{-2\pi ix}}\, d\nu(x).$$

Definition 10. The *Poisson integral* of a measure μ on $[0,1)$ is a function $u(z)$ on $\mathbb{D}$ given by

$$u(z) := \int_0^1 \frac{1-|z|^2}{|e^{2\pi ix} - z|^2} d\mu(x).$$

There is a one-to-one correspondence between the nonconstant inner functions b and the finite nonnegative singular Borel measures μ on $[0,1)$ given by the Herglotz representation:

$$\operatorname{Re}\left(\frac{1+b(z)}{1-b(z)}\right) = \int_0^1 \frac{1-|z|^2}{|e^{2\pi ix} - z|^2}\, d\mu(x). \tag{1.9}$$

(See [Her11].) For a singular measure μ and an inner function b related in this way, we will say that μ is the measure "corresponding" to b, or that b is the inner function "corresponding" to μ.

Lemma 1. *If μ is a Borel probability measure on $[0,1)$, then $G(z) := 1 - \frac{1}{C_\mu 1(z)}$ is a holomorphic function on $\mathbb{D}$ and*

$$Re\left(\frac{1+G(z)}{1-G(z)}\right) = \int_0^1 \frac{1-|z|^2}{|e^{2\pi i x}-z|^2}\, d\mu(x),$$

where the function on the left is the unique nonnegative harmonic function on $\mathbb{D}$ associated to μ by the Herglotz representation theorem. In particular, if μ is a singular Borel probability measure, then the inner function b corresponding to μ is given by $b(z) = 1 - \frac{1}{C_\mu 1(z)}$.

Proof. Note that $C_\mu 1(z)$ is well-defined in $\mathbb{D} = \{z : |z| < 1\}$, because if $|z| < 1$, then $|1 - ze^{-2\pi i x}| \geq |1| - |ze^{-2\pi i x}| = 1 - |z| > 0$. Furthermore, for a fixed $|z| < 1$, we have

$$C_\mu 1(z) = \int_0^1 \frac{1}{1 - ze^{-2\pi i x}}\, d\mu(x)$$
$$= \int_0^1 \sum_{n=0}^{\infty} \left(ze^{-2\pi i x}\right)^n \, d\mu(x)$$
$$= \sum_{n=0}^{\infty} z^n \int_0^1 e^{-2\pi i n x}\, d\mu(x) \text{ [Because the series converges absolutely where } \left|ze^{-2\pi i n x}\right| = |z| < 1.]$$
$$= \sum_{n=0}^{\infty} \widehat{\mu}(n) z^n.$$

The existence of the Taylor series representation shows that $C_\mu 1(z)$ is holomorphic on $\mathbb{D}$.

Since the linear fractional transformation $z \mapsto 1/(1-z)$ maps $\mathbb{D}$ onto $\{z : \mathrm{Re}(z) > 1/2\}$, and since μ is a probability measure, we have

$$\mathrm{Re}(C_\mu 1(z)) = \int_0^1 \mathrm{Re}\left(\frac{1}{1 - ze^{-2\pi i x}}\right) d\mu(x) \geq \int_0^1 \frac{1}{2}\, d\mu(x) = \frac{1}{2}$$

for all $z \in \mathbb{D}$. (Note: It is in the above equation that the important fact that $\mu \geq 0$ comes into play.) The inverse map of $z \mapsto 1/(1-z)$ is $z \mapsto 1 - 1/z$. It follows that if we define G by $G(z) := 1 - \frac{1}{C_\mu 1(z)}$, then $|G(z)| < 1$ for all $z \in \mathbb{D}$.

Because $\mathrm{Re}(C_\mu 1(z)) \geq \frac{1}{2} \neq 0$ for all $z \in \mathbb{D}$, it follows that $C_\mu 1(z) \neq 0$ on $\mathbb{D}$, and hence $G(z)$ is holomorphic on $\mathbb{D}$. Then because $|G(z)| < 1$ on $\mathbb{D}$, $(1 + G(z))/(1 - G(z))$ is holomorphic on $\mathbb{D}$.

We compute that

$$\begin{aligned}
\mathrm{Re}\left(\frac{1+G(z)}{1-G(z)}\right) &= \mathrm{Re}\left(\frac{1+(1-1/C_\mu 1(z))}{1-(1-1/C_\mu 1(z))}\right) \\
&= \mathrm{Re}(2C_\mu 1 - 1) \\
&= \mathrm{Re}\left(2\int_{\mathbb{T}} \frac{d\mu(\xi)}{1-z\overline{\xi}} - 1\right) \\
&= \mathrm{Re}\left(\int_{\mathbb{T}} \frac{2}{1-z\overline{\xi}} - 1\, d\mu(\xi)\right) \qquad \text{[Since } \mu \text{ is a probability measure.]} \\
&= \int_{\mathbb{T}} 2\mathrm{Re}\left(\frac{1}{1-z\overline{\xi}}\right) - 1\, d\mu(\xi).
\end{aligned}$$

For $|\xi| = 1$, we have that

$$\begin{aligned}
2\mathrm{Re}\left(\frac{1}{1-z\overline{\xi}}\right) - 1 &= \left(\frac{1}{1-z\overline{\xi}} + \frac{1}{1-\overline{z}\xi}\right) - 1 \\
&= \frac{1-\overline{z}\xi + 1 - z\overline{\xi}}{|1-z\overline{\xi}|^2} - 1 \\
&= \frac{2-\overline{z}\xi - z\overline{\xi}}{|\xi - z|^2} - 1 \\
&= \frac{2-\overline{z}\xi - z\overline{\xi} - |\xi|^2 + \overline{z}\xi + z\overline{\xi} - |z|^2}{|\xi - z|^2} \\
&= \frac{2-|\xi|^2 - |z|^2}{|\xi - z|^2} \\
&= \frac{1-|z|^2}{|\xi - z|^2}.
\end{aligned}$$

From the above computations, we obtain that for all $z \in \mathbb{D}$,

$$\int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2}\, d\mu(\xi) = \mathrm{Re}\left(\frac{1+G(z)}{1-G(z)}\right).$$

The function $\mathrm{Re}\left(\frac{1+G(z)}{1-G(z)}\right)$ is harmonic as the real part of a holomorphic function. Observe that

$$\begin{aligned}
\mathrm{Re}\left(\frac{1+G(z)}{1-G(z)}\right) &= \frac{1}{2}\left(\frac{1+G(z)}{1-G(z)} + \frac{1+\overline{G(z)}}{1-\overline{G(z)}}\right) \\
&= \frac{1}{2}\left(\frac{1-\overline{G(z)} + G(z) - |G(z)|^2 + 1 - G(z) + \overline{G(z)} - |G(z)|^2}{|1-G(z)|^2}\right) \\
&= \frac{1-|G(z)|^2}{|1-G(z)|^2}.
\end{aligned}$$

Since $|G(z)| < 1$ on $\mathbb{D}$, this shows that $\mathrm{Re}\left(\frac{1+G(z)}{1-G(z)}\right) \geq 0$. Thus, by Herglotz's Theorem, $\mathrm{Re}\left(\frac{1+G(z)}{1-G(z)}\right)$ is the unique nonnegative harmonic function on $\mathbb{D}$ satisfying

$$\mathrm{Re}\left(\frac{1+G(z)}{1-G(z)}\right) = \int_{\mathbb{T}} \frac{1-|z|^2}{|e^{2\pi ix}-z|^2}\, d\mu(x).$$

If μ is singular, then $\frac{d\mu_a}{dm} = 0$, and so by Fatou's Theorem, we have that for Lebesgue-almost-every $x \in [0,1)$,

$$0 = \lim_{r\to 1^-} \int_0^1 \frac{1-\left|re^{2\pi ix}\right|^2}{\left|re^{2\pi ix}-\xi\right|^2}\, d\mu(\xi) = \lim_{r\to 1^-} \mathrm{Re}\left(\frac{1+G(re^{2\pi ix})}{1-G(re^{2\pi ix})}\right) = \lim_{r\to 1^-} \frac{1-|G(re^{2\pi ix})|^2}{|1-G(re^{2\pi ix})|^2}.$$

Because the denominator on the right is bounded, the numerator must tend to 0, and hence $|G^*(x)| = 1$. So, for Lebesgue-almost-every $x \in [0,1)$, $|G^*(x)| = 1$, and so G is an inner function. Thus by the Herglotz representation theorem, $b(z) = G(z) = 1 - 1/C_\mu 1(z)$ must be the unique inner function corresponding to μ. This completes the proof. □

Corollary 1. *A nonnegative singular Borel measure μ on $[0,1)$ is a probability measure if and only if its corresponding inner function b satisfies $b(0) = 0$.*

Proof. Suppose μ is a probability measure. Then by Lemma 1, $b(z) = 1 - 1/C_\mu 1(z)$. Hence,

$$b(0) = 1 - 1/C_\mu 1(0) = 1 - \int_0^1 \frac{1}{1-0}\, d\mu(x) = 1 - 1 = 0.$$

Conversely, suppose $b(0) = 0$. Then

$$1 = \mathrm{Re}\left(\frac{1+b(0)}{1-b(0)}\right) = \int_0^1 \frac{1-|0|^2}{\left|0-e^{2\pi ix}\right|^2}\, d\mu(x) = \int_0^1 1\, d\mu(x) = \|\mu\|\,,$$

and so μ is a probability measure. □

1.4 Reproducing Kernel Hilbert Spaces

Let $\mathbb{H}$ be a Hilbert space whose members are complex-valued functions defined on a domain E. For $x \in E$, let $\ell_x : \mathbb{H} \to \mathbb{C}$ denote the linear functional of point-evaluation at x. That is,

$$\ell_x(f) = f(x) \text{ for all } f \in \mathbb{H}.$$

If ℓ_x is continuous (equivalently, bounded in operator norm), then the Riesz Representation Theorem implies the existence of an element $k_x \in \mathbb{H}$ such that for all $f \in \mathbb{H}$,

$$f(x) = \ell_x(f) = \langle f, k_x \rangle\,.$$

If ℓ_x is bounded for all $x \in E$, then $\mathbb{H}$ is said to be a **reproducing kernel Hilbert space**, with kernel $k : E \times E \to \mathbb{C}$ defined by

$$k(x, y) = k_x(y).$$

We will usually use the notation $k_x(y)$ rather than $k(x, y)$ in order to emphasize that when x is fixed, we get a function of y that is a member of $\mathbb{H}$.

Clearly, any subspace of a reproducing kernel Hilbert space is also a reproducing kernel Hilbert space, because the point evaluation functionals remain bounded. However, the kernel of the subspace is different, and is obtained by orthogonally projecting the kernel of the full space onto the subspace.

Some examples of reproducing kernel Hilbert spaces include all finite-dimensional Hilbert spaces, Bergman spaces, de Branges spaces (including the Paley-Wiener space), and the Hardy space, which we will talk about in section 1.5. In fact, a 1-to-1 correspondence between reproducing kernel Hilbert spaces and functions called positive matrices is given by the Moore-Aronszajn Theorem, which we will discuss in section 1.6.

In chapter 4, we shall make use of the following known theorem, which can be found in [Pau08]:

Theorem (Papadakis)**.** *Let $\mathbb{H}$ be a RKHS on a domain E with reproducing kernel $K(x, y)$. Then $\{f_s : s \in S\} \subseteq \mathbb{H}$ is a Parseval frame for $\mathbb{H}$ if and only if $K(x, y) = \sum_{s\in S} f_s(x)\overline{f_s(y)}$, where the series converges pointwise.*

1.5 The Hardy Space

The classical Hardy space H^2 consists of those holomorphic functions f defined on $\mathbb{D}$ satisfying

$$\|f\|_{H^2}^2 := \sup_{0<r<1} \int_0^1 |f(re^{2\pi ix})|^2 \, dx < \infty. \tag{1.10}$$

The Hardy space is a Hilbert space with the above norm. It is well-known that an equivalent description of H^2 is as the space of holomorphic functions on $\mathbb{D}$ with square-summable coefficients:

$$H^2 = \left\{ \sum_{n=0}^{\infty} c_n z^n \;\middle|\; \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\},$$

where the norm is then equivalently given by

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |c_n|^2.$$

In addition, each $f \in H^2$ possesses an $L^2(dx)$-boundary function $f^\star_{dx} \in L^2([0,1))$. In fact, as a consequence of Fatou's Theorem (see [Koo98], II:B.1), each $f \in H^2$ possesses radial limits $f^*(x) := \lim_{r\to 1^-} f(re^{2\pi i x})$ for Lebesgue-almost-every $x \in [0,1)$, and $f^*(x)$ is in the equivalence class of $f^\star_{dx}$ in $L^2(dx)$, i.e.

$$\lim_{r\to 1^-} \int_0^1 |f(re^{2\pi i x}) - f^*(x)|^2 \, dx = 0. \tag{1.11}$$

Thus, even though the notation $f^*(x)$ in this dissertation properly refers to the radial limit of $f(z)$, we will adopt the convention of writing $f^*(x)$ wherever $f^\star_{dx}$ would be appropriate, a nod to the conventions of classical texts. In the case of non-Lebesgue measures, we will always write $f^\star(x)$ for the L^2-boundary of $f(z)$.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are two members of H^2, the inner product of f and g in H^2 can be described in two ways:

$$\langle f, g\rangle_{H^2} = \sum_{n=0}^{\infty} a_n \overline{b_n} = \int_0^1 f^*(x)\overline{g^*(x)} \, dx. \tag{1.12}$$

Because the point-evaluation functionals on the Hardy space are bounded, the Hardy space is a reproducing kernel Hilbert space. By the Cauchy Integral Formula, its kernel is the classical Szegő kernel $k(z,w) =: k_z$, defined by

$$k_z(w) := \frac{1}{1 - \overline{z}w}.$$

Note that

$$k_z^*(x) = \frac{1}{1 - \overline{z}e^{2\pi i x}}.$$

We then have

$$f(z) = \langle f, k_z\rangle_{H^2} = \int_0^1 f^*(e^{2\pi i x})\overline{k_z^*(e^{2\pi i x})} \, dx$$

for all $f \in H^2$. In particular,

$$k_z(w) := \int_0^1 k_z^*(e^{2\pi ix})\overline{k_w^*(e^{2\pi ix})}\,dx. \tag{1.13}$$

Equation (1.13) shows that the Szegő kernel reproduces itself with respect to what is, by some definition, its boundary. The measure dx used to define k_z^* in (1.11) is Lebesgue measure, as is the measure dx in (1.13). Chapter 3 of this dissertation shows that among the functions in the Hardy space, there are a host of other kernels that reproduce with respect to their boundaries. However, these boundary functions will not be taken with respect to Lebesgue measure, but with respect to a given singular measure, and the integration of these boundary functions will also done with respect to this singular measure. Will we ask two main questions: Which kernels does the Hardy space contain that reproduce by boundary functions with respect to a given measure, and with respect to which measures will a candidate for a kernel reproduce by boundary functions?

Remark 2. Because there are other senses of the Hardy space than the one described here, what we have just called H^2 is more completely denoted $H^2(\mathbb{D})$. Another sense of the Hardy space will be described in section 1.8, where the more complete notation will be used in order to make the distinction. However, where context will allow, we will use the more compact notation H^2 to refer to whatever Hardy space we are discussing, which will usually be $H^2(\mathbb{D})$.

1.6 Positive Matrices and the Moore-Aronszajn Theorem

Definition 11. A positive matrix (in the sense of E. H. Moore) on a domain E is a function $K(z,w) : E \times E \to \mathbb{C}$ such that for all finite sequences $\zeta_1, \zeta_2, \ldots, \zeta_n \in E$, the matrix

$$(K(\zeta_j, \zeta_i))_{ij}$$

is positive semidefinite. We will usually write $K_z(w)$ instead of $K(z,w)$, to emphasize that each fixed z yields a function in w. Given a positive matrix $K_z(w)$, we will use the bare notation K to refer to the set $\{K_z : z \in E\}$ of functions from E to $\mathbb{C}$ comprising it, though sometimes we will use K to refer to the positive matrix itself as a function from $E \times E$ to $\mathbb{C}$.

Our interest is in positive matrices on $E = \mathbb{D}$, and more specifically those residing in $H^2(\mathbb{D})$. Recall that the classical Hardy space is a reproducing kernel Hilbert space. We therefore desire to find subspaces of the Hardy space that not only are Hilbert spaces with respect to the $L^2(\mu)$-boundary norm, but are in fact reproducing kernel Hilbert spaces with respect to this norm. The classical Moore-Aronszajn Theorem in [Aro50] connects positive matrices to reproducing kernel Hilbert spaces:

Theorem (Moore-Aronszajn)**.** *To every positive matrix $K_z(w)$ on a domain E there corresponds one and only one class of functions on E with a uniquely determined quadratic form in it, forming a Hilbert space and admitting $K_z(w)$ as a reproducing kernel. This class of functions is generated by all functions of the form $\sum_{k=1}^{n} \xi_k K_{z_k}(w)$, with norm given by*

$$\left\| \sum_{k=1}^{n} \xi_k K_{z_k}(w) \right\|^2 = \sum_{i,j=1}^{n} K_{z_j}(z_i) \overline{\xi_i} \xi_j.$$

Conversely, every reproducing kernel of a Hilbert space of functions on a common domain is a positive matrix. Reproducing kernel Hilbert spaces of finite dimensions are classified by the following theorem, the finite-dimensional Moore-Aronszajn theorem:

Theorem (Finite-Dimensional Moore-Aronszajn)**.** *If F is an n-dimensional class of functions and $w_1(x), \ldots, w_n(x)$ are linearly-independent in F, then any $f, g \in F$ can be represented uniquely as*

$$f(x) = \sum_{k=1}^{n} \zeta_k w_k(x) \qquad g(x) = \sum_{k=1}^{n} \eta_k w_k(x).$$

Then F becomes a RKHS with $\langle f, g \rangle = \sum_{i,k=1}^{n} \alpha_{ij} \zeta_i \overline{\eta_j}$ for any positive definite matrix (α_{ij}). (α_{ij}) is then the Gram matrix of w_k in this Hilbert space, and the reproducing kernel is

$$K(z, w) = \sum_{i,j=1}^{n} \beta_{ij} w_i(w) \overline{w_j(z)},$$

where $(\beta_{ij}) = \overline{(\alpha_{ij})^{-1}}$. Moreover, the kernel of any finite-dimensional class arises in this way.

1.7 The Kwapień-Mycielski Theorem

The results of Chapter 2 depend on an excellent theorem obtained by Stanisław Kwapień and Jan Mycielski in [KM01] in 2001. To understand the theorem, we first need to know what it means for a sequence to be "stationary."

Definition 12. A sequence $\{x_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathbb{H}$ is said to be *stationary* if $\langle x_{k+m}, x_{l+m}\rangle = \langle x_k, x_l\rangle$ for any positive integers k, l, m.

Given a stationary sequence $\{x_n\}_{n=1}^{\infty}$, define a function $\phi : \mathbb{Z} \to \mathbb{C}$ by $\phi(m) = \langle x_k, x_{k+m}\rangle$, where by stationarity of $\{x_n\}_{n=1}^{\infty}$, specifying that $k > \max\{0, -m\}$ is enough to ensure ϕ is well-defined. Let $z_1, \ldots, z_N \in \mathbb{Z}$ and let $c_1, \ldots, c_N \in \mathbb{C}$. Let $M = \max\{|z_j| : 1 \leq j \leq N\} + 1$. Then

$$\begin{aligned}
\sum_{k,l=1}^{N} c_k \overline{c_l} \phi(z_k - z_l) &= \sum_{k,l=1}^{N} c_k \overline{c_l} \langle x_M, x_{M+z_k-z_l}\rangle \\
&= \sum_{k,l=1}^{N} c_k \overline{c_l} \langle x_{M+z_l}, x_{M+z_k}\rangle \\
&= c^* \mathcal{G} c,
\end{aligned}$$

where $c = \begin{bmatrix} c_1 & \cdots & c_N \end{bmatrix}^T$ and $\mathcal{G}$ is the Gram matrix of $x_{M+z_1}, \ldots, x_{M+z_N}$. Since Gram matrices are positive-semidefinite, we have that $c^*\mathcal{G}c \geq 0$, and so ϕ is a positive-definite function.

It follows by Bochner's Theorem that if we define a_m for $m \in \mathbb{Z}$ by $a_m := \phi(m) := \langle x_k, x_{k+m}\rangle$, where k is a positive integer such that $k > -m$, then there exists a unique positive measure σ on $[0, 1)$, called the *spectral measure* of the stationary sequence, such that

$$a_m = \int_0^1 e^{-2\pi i m x}\, d\sigma(x) \qquad \text{for each } m \in \mathbb{Z}. \tag{1.14}$$

(Note that $a_{-m} = \overline{a_m}$ for all $m \in \mathbb{Z}$.) A stationary sequence consists of unit vectors if and only if its spectral measure is a probability measure, and it is orthonormal if and only if its spectral measure coincides with the normalized Lebesgue measure on $\mathbb{T}$.

Theorem (Kwapień & Mycielski)**.** *A stationary sequence of unit vectors that is linearly dense in a Hilbert space is effective if and only if its spectral measure either coincides with the normalized Lebesgue measure or is singular with respect to Lebesgue measure.*

While the proof in [KM01] gives sufficient detail for the reader to fill in the gaps, the filling of these gaps is a task unto itself. Furthermore, differences in notation and minor typographical errors tend to add undue confusion. We therefore will re-give the proof of the theorem here, filling in details and fixing any discrepancies:

Proof. Let $\{\phi_k\}_{k=1}^{\infty}$ be a stationary sequence of unit vectors in a Hilbert space $\mathbb{H}$. Let σ be its spectral probability measure on $[0,1)$, and let $\{a_m\}_{m\in\mathbb{Z}}$ be defined as above. By Lemma 1 and its proof, we have that $C_\sigma 1(z) = \sum_{n=0}^{\infty} z^n \int_0^1 e^{-2\pi i n x}\, d\sigma(x) = \sum_{n=0}^{\infty} a_n z^n$. We also have that $\frac{1}{C_\sigma 1(z)}$ and $G(z) := 1 - \frac{1}{C_\sigma(z)}$ are holomorphic on $\mathbb{D}$ with $|G(z)| < 1$. Thus there exists a power series representation

$$\frac{1}{C_\sigma 1(z)} = \sum_{n=0}^{\infty} c_n z^n$$

valid for all $z \in \mathbb{D}$. Now, observe that because $|G(z)| < 1$ for all $z \in \mathbb{D}$, we have

$$\left|\frac{1}{C_\sigma 1(z)}\right| = \left|\frac{1}{C_\sigma 1(z)} - 1 + 1\right| \le \left|\frac{1}{C_\sigma 1(z)} - 1\right| + 1 = |G(z)| + 1 < 2$$

for all $z \in \mathbb{D}$. It follows that

$$\sup_{0<r<1} \left(\int_0^1 \left| \frac{1}{C_\sigma 1(re^{2\pi i\theta})} \right|^2 d\theta \right)^{\frac{1}{2}} \le \sup_{0<r<1} \left(\int_0^1 4\, d\theta \right)^{\frac{1}{2}} = 2 < \infty,$$

which implies that $\frac{1}{C_\sigma 1(z)}$ is a member of the Hardy space H^2 on the unit disk, with

$$\sum_{n=0}^{\infty} |c_n|^2 = \left\| \frac{1}{C_\sigma 1(z)} \right\|_{H^2}^2 \le 4 < \infty.$$

Because

$$C_\sigma 1(0) = \int_{\mathbb{T}} \frac{\sigma(dw)}{1-0} = 1,$$

we must have

$$a_0 = c_0 = 1. \tag{1.15}$$

In addition, for $z \in \mathbb{D}$ we have

$$\begin{aligned} 1 &= C_\sigma 1(z) \cdot \frac{1}{C_\sigma 1(z)} \\ &= \left(\sum_{n=0}^{\infty} a_n z^n \right) \cdot \left(\sum_{k=0}^{\infty} c_k z^k \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n c_k z^{n+k} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_k a_{n-k} \right) z^n. \end{aligned}$$

Thus, by uniqueness of power series, we must have

$$\sum_{k=0}^{n} c_k a_{n-k} = 0 \quad \text{for all } n \ge 1. \tag{1.16}$$

Recall that we may associate to the sequence $\{\phi_n\}_{n=1}^{\infty}$ a sequence $\{g_n\}_{n=1}^{\infty}$ defined recursively by

$$g_1 = \phi_1 \text{ and } g_k = \phi_k - \sum_{n=1}^{k-1} \langle \phi_k, \phi_n \rangle g_n.$$

We claim that $g_k = \sum_{i=1}^{k} \overline{c_{k-i}} \phi_i$ for all $k \geq 1$. Since $g_1 = \phi_1$ and $c_0 = 1$, this is obviously true for the case $k = 1$. Suppose the equation holds for the cases $k = 1, 2, \ldots, K$. Then we have

$$\begin{aligned}
g_{K+1} &= \phi_{K+1} - \sum_{n=0}^{K} \langle \phi_{K+1}, \phi_n \rangle g_n \\
&= \phi_{K+1} - \sum_{n=0}^{K} \langle \phi_{K+1}, \phi_n \rangle \left(\sum_{i=1}^{n} \overline{c_{n-i}} \phi_i \right) \\
&= \phi_{K+1} - \sum_{n=0}^{K} a_{n-K-1} \left(\sum_{i=1}^{n} \overline{c_{n-i}} \phi_i \right) \\
&= \phi_{K+1} - \sum_{n=0}^{K} \sum_{i=1}^{n} \overline{c_{n-i}} a_{n-K-1} \phi_i \\
&= \phi_{K+1} - \sum_{i=1}^{K} \left(\sum_{n=i}^{K} \overline{c_{n-i}} a_{n-K-1} \right) \phi_i \\
&= \phi_{K+1} - \sum_{i=1}^{K} \overline{\left(\sum_{n=i}^{K} c_{n-i} \overline{a_{n-K-1}} \right)} \phi_i \\
&= \phi_{K+1} - \sum_{i=1}^{K} \overline{\left(\sum_{n=i}^{K} c_{n-i} a_{K+1-n} \right)} \phi_i \\
&= \phi_{K+1} - \sum_{i=1}^{K} \overline{\left(\sum_{n=0}^{K-i} c_n a_{K+1-i-n} \right)} \phi_i \\
&= \phi_{K+1} - \sum_{i=1}^{K} \overline{\left(\left(\sum_{n=0}^{K+1-i} c_n a_{K+1-i-n} \right) - c_{K+1-i} a_0 \right)} \phi_i \\
&= \phi_{K+1} - \sum_{i=1}^{K} \left(-\overline{c_{K+1-i}} \right) \phi_i && \text{[by (1.15) and (1.16)]} \\
&= \overline{c_0} \phi_{K+1} + \sum_{i=1}^{K} \overline{c_{K+1-i}} \phi_i && \text{[by (1.15)]} \\
&= \sum_{i=1}^{K+1} \overline{c_{K+1-i}} \phi_i.
\end{aligned}$$

Therefore, by strong induction we have $g_k = \sum_{i=1}^{k} \overline{c_{k-i}} \phi_i$ for all $k \geq 1$. Then for each $n \geq 1$

and $x \in \mathbb{H}$, we have

$$L_n(x) := \sum_{k=1}^{n} \langle x, g_k \rangle \phi_k = \sum_{k=1}^{n} \left\langle x, \sum_{i=1}^{k} \overline{c_{k-i}} \phi_i \right\rangle \phi_k = \sum_{k=1}^{n} \left(\sum_{i=1}^{k} c_{k-i} \langle x, \phi_i \rangle \right) \phi_k. \tag{1.17}$$

Recall by (1.6) that if $\{x_n\}_{n=0}^{\infty}$ is the sequence associated to x by the Kaczmarz algorithm via the recursion relation

$$x_0 = 0 \text{ and } x_n = x_{n-1} + \langle x - x_{n-1}, \phi_n \rangle \phi_n,$$

then

$$x_n = \sum_{k=1}^{n} \langle x, g_k \rangle \phi_k =: L_n(x).$$

Therefore, the sequence $\{\phi_n\}_{n=1}^{\infty}$ is effective if and only if $\lim_{n\to\infty} \|x - L_n(x)\|_H = 0$ for all $x \in \mathbb{H}$. Since $\{\phi_n\}_{n=1}^{\infty}$ is linearly dense in H and L_n is linear, $\{\phi_n\}_{n=1}^{\infty}$ is effective if and only if $\lim_{n\to\infty} L_n(\phi_j) = \phi_j$ for all $j = 1, 2, \ldots$.

First note that for any $n \geq 1$, we have

$$\begin{aligned}
L_n(\phi_1) &= \sum_{k=1}^{n} \left(\sum_{i=1}^{k} c_{k-i} \langle \phi_1, \phi_i \rangle \right) \phi_k && \text{[by (1.17)]} \\
&= \sum_{k=1}^{n} \left(\sum_{i=1}^{k} c_{k-i} a_{i-1} \right) \phi_k \\
&= \sum_{k=1}^{n} \left(\sum_{i=0}^{k-1} c_i a_{k-1-i} \right) \phi_k \\
&= \sum_{k=1}^{n} \left(\begin{cases} 0 & \text{if } k \geq 2 \\ c_0 a_0 & \text{if } k = 1 \end{cases} \right) \phi_k && \text{[by (1.16)]} \\
&= c_0 a_0 \phi_1 \\
&= \phi_1. && \text{[by (1.15)]}
\end{aligned}$$

Thus, we certainly have $\lim_{n\to\infty} L_n(\phi_j) = \phi_j$ in the $j = 1$ case.

We claim that for all $n \geq j \geq 2$,

$$L_n(\phi_j) - \phi_j = \sum_{i=1}^{j-1} \overline{a_i} \left(\sum_{l=0}^{n+i-j} c_l \phi_{l+j-i} \right). \tag{1.18}$$

We will show this using induction on n. Fix $j \geq 2$. First, consider the base case $n = j$. We compute:

$$L_n(\phi_j) - \phi_j = L_j(\phi_j) - \phi_j$$

$$
\begin{aligned}
&= \sum_{k=1}^{j} \left(\sum_{i=1}^{k} c_{k-i} \langle \phi_j, \phi_i \rangle \right) \phi_k - \phi_j && \text{[by (1.17)]} \\
&= \sum_{k=1}^{j} \left(\sum_{i=1}^{k} c_{k-i} a_{i-j} \right) \phi_k - \phi_j \\
&= \sum_{k=1}^{j-1} \left(\sum_{i=1}^{k} c_{k-i} a_{i-j} \right) \phi_k + \left(\sum_{i=1}^{j} c_{j-i} a_{i-j} \right) \phi_j - \phi_j \\
&= \sum_{i=1}^{j-1} \sum_{k=i}^{j-1} c_{k-i} a_{i-j} \phi_k + \left(c_0 a_0 + \sum_{i=1}^{j-1} c_{j-i} a_{i-j} \right) \phi_j - \phi_j \\
&= \sum_{i=1}^{j-1} \sum_{k=i}^{j-1} c_{k-i} a_{i-j} \phi_k + \sum_{i=1}^{j-1} c_{j-i} a_{i-j} \phi_j \\
&= \sum_{i=1}^{j-1} \sum_{k=i}^{j} c_{k-i} a_{i-j} \phi_k \\
&= \sum_{i=1}^{j-1} \overline{a_{j-i}} \sum_{k=i}^{j} c_{k-i} \phi_k \\
&= \sum_{w=1}^{j-1} \overline{a_w} \sum_{k=j-w}^{j} c_{k-j+w} \phi_k \\
&= \sum_{w=1}^{j-1} \overline{a_w} \sum_{k=0}^{w} c_k \phi_{k+j-w}.
\end{aligned}
$$

By relabeling w with i and k with l, this establishes (1.18) for the case $n = j$. Now, assume that (1.18) holds for the case $n = N \geq j$. Then

$$
\begin{aligned}
&L_{N+1}(\phi_j) - \phi_j \\
&= \sum_{k=1}^{N+1} \left(\sum_{i=1}^{k} c_{k-i} \langle \phi_j, \phi_i \rangle \right) \phi_k - \phi_j \\
&= \sum_{k=1}^{N} \left(\sum_{i=1}^{k} c_{k-i} a_{i-j} \right) \phi_k + \left(\sum_{i=1}^{N+1} c_{N+1-i} a_{i-j} \right) \phi_{N+1} - \phi_j \\
&= L_N(\phi_j) + \left(\sum_{i=1}^{N+1} c_{N+1-i} a_{i-j} \right) \phi_{N+1} - \phi_j \\
&= \sum_{i=1}^{j-1} \overline{a_i} \left(\sum_{l=0}^{N+i-j} c_l \phi_{l+j-i} \right) + \left(\sum_{i=1}^{N+1} c_{N+1-i} a_{i-j} \right) \phi_{N+1} && \text{[by the induction hypothesis]} \\
&= \sum_{i=1}^{j-1} \overline{a_i} \left(\sum_{l=0}^{N+i-j} c_l \phi_{l+j-i} \right) + \left(\sum_{w=0}^{N} c_w a_{N+1-j-w} \right) \phi_{N+1}
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{i=1}^{j-1} \overline{a_i} \left(\sum_{l=0}^{N+i-j} c_l \phi_{l+j-i} \right) + \left(\sum_{w=N+2-j}^{N} c_w a_{N+1-j-w} \right) \phi_{N+1} && \text{[by (1.16)]} \\
&= \sum_{i=1}^{j-1} \overline{a_i} \left(\sum_{l=0}^{N+i-j} c_l \phi_{l+j-i} \right) + \left(\sum_{w=0}^{j-2} c_{w+N+2-j} a_{-1-w} \right) \phi_{N+1} \\
&= \sum_{i=1}^{j-1} \overline{a_i} \left(\sum_{l=0}^{N+i-j} c_l \phi_{l+j-i} \right) + \left(\sum_{w=1}^{j-1} c_{w+N+1-j} \overline{a_w} \right) \phi_{N+1} \\
&= \sum_{i=1}^{j-1} \overline{a_i} \left(\sum_{l=0}^{N+i-j} c_l \phi_{l+j-i} + c_{i+N+1-j} \phi_{N+1} \right) \\
&= \sum_{i=1}^{j-1} \overline{a_i} \left(\sum_{l=0}^{N+1+i-j} c_l \phi_{l+j-i} \right).
\end{aligned}$$

This establishes (1.18) for $n = N+1$, so by induction this establishes (1.18) for all $n \geq j \geq 2$.

Now, if the spectral measure σ of $\{\phi_n\}_{n=1}^{\infty}$ is Lebesgue measure, then $a_i = 0$ for all $i \geq 1$ by (1.14). This means $\{\phi_n\}_{n=1}^{\infty}$ is orthonormal, and it follows trivially (or by (1.18)) that it is effective.

So suppose the spectral measure σ of $\{\phi_n\}_{n=1}^{\infty}$ is not Lebesgue measure. We will show that, under this assumption, $\{\phi_n\}_{n=1}^{\infty}$ is effective if and only if its spectral measure is singular with respect to Lebesgue measure, which will complete the proof.

We claim that the sequence $\{\phi_n\}_{n=1}^{\infty}$ is effective if and only if $\lim_{n\to\infty} \|x - L_n(x)\|_H = 0$ for all $x \in \mathbb{H}$. First, suppose that $\{\phi_n\}_{n=1}^{\infty}$ is effective. Then $\lim_{n\to\infty} L_n(\phi_j) = \phi_j$ for all $j \geq 1$. Since σ is not Lebesgue measure, by (1.14) we must have $a_i \neq 0$ for some $i \geq 1$, or else we would violate the uniqueness of the spectral measure. Let i_0 be the smallest positive integer i such that $a_i \neq 0$. Then taking $j = i_0 + 1$ in equation (1.18), we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} L_n(\phi_{i_0+1}) - \phi_{i_0+1} \\
&= \lim_{n\to\infty} \sum_{i=1}^{i_0} \overline{a_i} \left(\sum_{l=0}^{n+i-i_0-1} c_l \phi_{l+i_0+1-i} \right) \\
&= \lim_{n\to\infty} \overline{a_{i_0}} \sum_{l=0}^{n-1} c_l \phi_{l+1}.
\end{aligned}$$

This implies that

$$0 = \lim_{n\to\infty} \sum_{l=0}^{n-1} c_l \phi_{l+1} = \sum_{l=0}^{\infty} c_l \phi_{l+1}.$$

Conversely, suppose that $\sum_{l=0}^{\infty} c_l\phi_{l+1} = 0$. Because the sequence $\{\phi_n\}_{n=1}^{\infty}$ is stationary, for any integer $m \geq 0$, the map $\phi_{l+1} \mapsto \phi_{l+1+m}$ extends to an isometry of H into itself. Thus, for any $j > i$, we have

$$\lim_{n\to\infty} \left\| \sum_{l=0}^{n+i-j} c_l\phi_{l+j-i} \right\| = \lim_{n\to\infty} \left\| \sum_{l=0}^{n+i-j} c_l\phi_{l+1} \right\| = \left\| \sum_{l=0}^{\infty} c_l\phi_{l+1} \right\| = 0.$$

Hence, $\lim_{n\to\infty} \sum_{l=0}^{n+i-j} c_l\phi_{l+j-i} = 0$. Then by (1.18), for any $j \geq 2$ we have that

$$\lim_{n\to\infty} L_n(\phi_j) - \phi_j = \sum_{i=1}^{j-1} \overline{a_i} \lim_{n\to\infty} \sum_{l=0}^{n+i-j} c_l\phi_{l+j-i} = \sum_{i=1}^{j-1} \overline{a_i} \cdot 0 = 0.$$

Since we have also computed that $\lim_{n\to\infty} L_n(\phi_1) = \phi_1$, we have established that $\{\phi_n\}_{n=1}^{\infty}$ is effective if and only if $\sum_{l=0}^{\infty} c_l\phi_{l+1} = 0$.

Equivalently, by defining a sequence $\{r_n\}_{n=0}^{\infty}$ of real numbers by

$$r_n^2 = \left\| \sum_{l=0}^{n} c_l\phi_{l+1} \right\|^2,$$

$\{\phi_n\}_{n=1}^{\infty}$ is effective if and only if $r_n \to 0$. We will prove by induction that $\{r_n\}_{n=0}^{\infty}$ satisfies

$$r_n^2 = 1 - |c_1|^2 - |c_2|^2 - \cdots - |c_n|^2.$$

Indeed, for $n = 0$, $r_0^2 = \|c_0\phi_1\|^2 = \|\phi_1\|^2 = 1$. For $n \geq 1$, we have

$$\begin{aligned}
r_n^2 &= \left\| \sum_{l=0}^{n} c_l\phi_{l+1} \right\|^2 \\
&= \left\langle \sum_{l=0}^{n} c_l\phi_{l+1}, \sum_{l=0}^{n} c_l\phi_{l+1} \right\rangle \\
&= \left\langle \sum_{l=0}^{n-1} c_l\phi_{l+1} + c_n\phi_{n+1}, \sum_{l=0}^{n-1} c_l\phi_{l+1} + c_n\phi_{n+1} \right\rangle \\
&= \left\langle \sum_{l=0}^{n-1} c_l\phi_{l+1}, \sum_{l=0}^{n-1} c_l\phi_{l+1} \right\rangle + \left\langle \sum_{l=0}^{n-1} c_l\phi_{l+1}, c_n\phi_{n+1} \right\rangle + \left\langle c_n\phi_{n+1}, \sum_{l=0}^{n-1} c_l\phi_{l+1} \right\rangle + \langle c_n\phi_{n+1}, c_n\phi_{n+1} \rangle \\
&= r_{n-1}^2 + 2\mathrm{Re}\left(\left\langle \sum_{l=0}^{n-1} c_l\phi_{l+1}, c_n\phi_{n+1} \right\rangle \right) + \|c_n\phi_{n+1}\|^2 \\
&= r_{n-1}^2 + 2\mathrm{Re}\left(\overline{c_n} \sum_{l=0}^{n-1} c_l a_{n-l} \right) + |c_n|^2 \\
&= r_{n-1}^2 + 2\mathrm{Re}\left(-\overline{c_n} c_n a_0\right) + |c_n|^2 \text{ [by (1.16)]}
\end{aligned}$$

$$= r_{n-1}^2 - 2|c_n|^2 + |c_n|^2$$

$$= r_{n-1}^2 - |c_n|^2,$$

which concludes the induction step. Recall that

$$G(z) = 1 - \frac{1}{C_\sigma 1(z)} = 1 - \sum_{n=0}^{\infty} c_n z^n = \sum_{n=1}^{\infty} c_n z^n.$$

Therefore,

$$\sum_{n=1}^{\infty} |c_n|^2 = \int_0^1 |G^*(x)|^2 \, dx.$$

It follows that $\{\phi_n\}_{n=1}^{\infty}$ is effective if and only if

$$1 = \lim_{n\to\infty} 1 - r_n^2 = \sum_{n=1}^{\infty} |c_n|^2 = \int_{\mathbb{T}} |G^*(x)|^2 \, dx. \tag{1.19}$$

Recall that $|G(z)| < 1$ for $z \in \mathbb{D}$. Since $H_r(t) := G(re^{2\pi it})$ converges to $H(t) := G^*(t)$ in the Lebesgue $L^2([0,1))$ norm as $r \to 1^-$, it converges in measure. Thus the discrete sequence $\{H_{1-1/k}(t)\}_{k=1}^{\infty}$ converges in measure to $H(t)$ and hence has a subsequence that converges to $H(t)$ pointwise almost everywhere. Since $|H_{1-1/k}(t)| = |G((1-1/k)e^{2\pi it})| < 1$, it follows that $|H(t)| = |G^*(t)| \leq 1$ almost everywhere.

Thus, by (1.19) we have that $\{\phi_n\}_{n=1}^{\infty}$ is effective if and only if $|G^*(t)| = 1$ Lebesgue-almost-everywhere on $[0,1)$. By definition, this last condition means G is an inner function. We will show that G being an inner function is equivalent to σ being a singular measure, which will complete the proof.

Assume $G(z)$ is inner. By Lemma 1,

$$\mathrm{Re}\left(\frac{1+G(z)}{1-G(z)}\right) = \int_0^1 \frac{1-|z|^2}{|\xi - z|^2} \, d\sigma(\xi).$$

We have $G(0) = 1 - \frac{1}{C_\sigma 1(0)} = 1 - 1 = 0$. Because G is inner, we have that for almost all (with respect to Lebesgue measure) $\xi \in \mathbb{T}$, $\lim_{r\to 1^-} |G(r\xi)| = 1$. If $G^*(x) := \lim r \to 1^- G(re^{2\pi ix}) = 1$ on a subset of $\mathbb{T}$ of positive Lebesgue measure, then we would have $G(z) \equiv 1$ on $\mathbb{D}$, which would contradict $G(0) = 0$. Hence, $G^*(x) \neq 1$ almost everywhere on $\mathbb{T}$. Then by Fatou's Theorem, we have that for almost all $\xi \in \mathbb{T}$,

$$\frac{d\sigma_a}{dm}(\xi) = \lim_{r\to 1^-} \mathrm{Re}\left(\frac{1+G(r\xi)}{1-G(r\xi)}\right) = \lim_{r\to 1^-} \frac{1-|G(r\xi)|^2}{|1-G(r\xi)|^2} = 0.$$

This shows that σ is singular.

Conversely, suppose σ is singular. Then by Lemma 1, G is inner.

Thus, G is inner if and only if σ is singular, which completes the proof. □

1.8 Toeplitz Operators

Recall that every member $f \in H^2(\mathbb{D})$ has a Lebesgue-measure boundary function $f^* \in L^2([0,1))$, and that $\|f\|_{H^2} = \|f^*\|_{L^2([0,1))}$. This correspondence allows us to identify $H^2(\mathbb{D})$ with a subspace of $L^2([0,1))$ via $f \leftrightarrow f^*$. Let us call this subspace $H^2([0,1))$. It is well-known that the exponential functions $\left\{e^{2\pi inx}\right\}_{n\in\mathbb{Z}}$ form an orthonormal basis of $L^2([0,1))$. It is then not difficult to see that $H^2([0,1)) = \overline{span}\left(\left\{e^{2\pi inx}\right\}_{n=0}^{\infty}\right)$. That is, $H^2([0,1))$ is the subspace of $L^2([0,1))$ spanned by the exponentials of nonnegative integral powers. As some authors put it, $H^2([0,1))$ is the closure of the polynomials in $L^2([0,1))$.

By P_+ we will mean the orthogonal projection onto $H^2([0,1))$ in $L^2([0,1))$, the "+" being a nod to the fact that $H^2([0,1))$ is that half of $L^2([0,1))$ spanned by the nonnegative exponentials. If $f \in L^2([0,1))$, then of course $f = \sum_{n\in\mathbb{Z}} \langle f, e_n\rangle e_n$. We then have $P_+f = \sum_{n=0}^{\infty} \langle f, e_n\rangle e_n$.

Suppose $\varphi \in L^\infty[([0,1))$. It is clear that we can define a multiplication operator $M_\varphi : L^2([0,1)) \to L^2([0,1))$ by $M_\varphi f = \varphi \cdot f$. However, note that M_φ need not leave $H^2([0,1))$ invariant. Following M_φ with the orthogonal projection P_+ back into $H^2([0,1))$, however, will. That is why the Toeplitz operator is defined as follows:

Definition 13. Let $\varphi \in L^\infty([0,1))$. The Toeplitz operator with symbol φ, denoted T_φ, is a bounded linear operator from $H^2([0,1))$ to $H^2([0,1))$ defined by $T_\varphi f = P_+ M_\varphi \restriction_{H^2([0,1))}$.

That is, if $f \in H^2([0,1))$, we have $T_\varphi f = P_+(\varphi \cdot f)$. It is well-known that T_φ satisfies $\|T_\varphi\| = \|\varphi\|_\infty$, and the adjoint satisfies $(T_\varphi)^* = T_{\overline{\varphi}}$.

The Toeplitz operator can, of course, be defined on $H^2(\mathbb{D})$ as well. Let us formalize this. Let $\iota : H^2(\mathbb{D}) \to H^2([0,1))$ be the bijection defined by $\iota(f) = f^*$. Then the Toeplitz operator $\tilde{T}_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ may be defined by $\tilde{T}_\varphi = \iota^{-1} P_+ M_\varphi \iota$. That is, $\tilde{T}_\varphi f = F$, where $F^* = P_+(\varphi \cdot f^*)$. Through an abuse of notation, we will usually just write T_φ in both cases, leaving to context whether we mean it to be an operator on $H^2([0,1)$ or $H^2(\mathbb{D})$.

1.9 de Branges-Rovnyak Spaces

The space $H^\infty(\mathbb{D})$, the bounded holomorphic functions on $\mathbb{D}$, is a subspace of $H^2(\mathbb{D})$. Then $\iota(H^\infty(\mathbb{D})) =: H^\infty([0,1))$ is a subspace of $H^2([0,1))$. Indeed, $H^\infty([0,1)) = H^2([0,1)) \cap L^\infty([0,1))$. Thus, if we select a function $b \in H^\infty(\mathbb{D})$, it may be identified with $\iota(b) = b^* \in L^\infty([0,1))$. Hence, through an additional abuse of notation, we have a Toeplitz operator $T_b : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ defined by

$$T_b = \iota^{-1} P_+ M_{b^*} \iota.$$

By $T_{\overline{b}}$, we of course mean

$$T_{\overline{b}} = \iota^{-1} P_+ M_{\overline{b^*}} \iota.$$

The operator $I - T_b T_{\overline{b}} = I - T_b T_b^*$ is positive semidefinite, and so admits a unique nonnegative square root $(I - T_b T_{\overline{b}})^{1/2} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$. From this operator, we can define a new Hilbert space that is a subset of $H^2(\mathbb{D})$, but has (in general) a different norm:

Definition 14. The de Branges-Rovnyak space with symbol $b \in H^\infty(\mathbb{D})$ is the set

$$\mathcal{H}(b) := \operatorname{Ran}((I - T_b T_{\overline{b}})^{1/2}),$$

equipped with the norm that makes $(I - T_b T_{\overline{b}})^{1/2}$ a coisometry from $H^2(\mathbb{D})$ to $\mathcal{H}(b)$.

An additional property of Toeplitz operators noted in [Sar94] is that if $\varphi, \psi \in L^\infty([0,1))$, and at least one of them is in $H^\infty([0,1))$, then $T_{\overline{\psi}} T_\varphi = T_{\overline{\psi}\varphi}$. It follows that for the special case that b is an inner function, $T_{\overline{b}} T_b = T_{\overline{b}b} = T_1 = I$, the identity on $H^2(\mathbb{D})$. In this case, we observe that

$$\begin{aligned}(I - T_b T_{\overline{b}})(I - T_b T_{\overline{b}}) &= I - T_b T_{\overline{b}} - T_b T_{\overline{b}} + T_b T_{\overline{b}} T_b T_{\overline{b}} \\ &= I - 2 T_b T_{\overline{b}} + T_b I T_{\overline{b}} \\ &= I - T_b T_{\overline{b}}.\end{aligned}$$

Thus, in the case that b is inner, $I - T_b T_{\overline{b}}$ is its own unique nonnegative square root, and hence also a projection. It is an orthogonal projection since it is self-adjoint. So in the case that b is inner, $\mathcal{H}(b)$ is an ordinary subspace of $H^2(\mathbb{D})$. In fact, [Sar94] notes that in the case that b is inner, $\mathcal{H}(b)$ is in fact the space $H^2(\mathbb{D}) \ominus bH^2(\mathbb{D})$. The spaces $bH^2(\mathbb{D})$, by a theorem of Beurling from [Beu48], are exactly the forward-shift-invariant subspaces of $H^2(\mathbb{D})$. Thus the de Branges-Rovnyak spaces $\mathcal{H}(b)$, for b inner, are exactly the backward-shift-invariant subspaces of $H^2(\mathbb{D})$.

We make a final note that by II-6 of [Sar94], if b is any inner function and u is any nonconstant function in the unit ball of H^∞, then $\mathcal{H}(bu) = \mathcal{H}(b) \oplus b\mathcal{H}(u)$.

CHAPTER 2. FOURIER SERIES FOR SINGULAR MEASURES AND THE KACZMARZ ALGORITHM

For a Borel probability measure μ, a spectrum is a sequence $\{\lambda_n\}_{n\in I}$ such that the functions $\{e^{2\pi i\lambda_n x} : n \in I\} \subset L^2(\mu)$ constitute an orthonormal basis. If μ possesses a spectrum, we say μ is spectral, and then every $f \in L^2(\mu)$ possesses a (nonharmonic) Fourier series of the form $f(x) = \sum_{n\in I}\langle f(x), e^{2\pi i\lambda_n x}\rangle e^{2\pi i\lambda_n x}$.

In [JP98], Jorgensen and Pedersen considered the question of whether measures induced by iterated function systems on $\mathbb{R}^d$ are spectral. Remarkably, they demonstrated that the quaternary Cantor measure μ_4 is spectral. Equally remarkably, they also showed that no three exponentials are orthogonal with respect to the ternary Cantor measure μ_3, and hence μ_3 is not spectral. The lack of a spectrum for μ_3 motivated subsequent research to relax the orthogonality condition, instead searching for an exponential frame or Riesz basis, since an exponential frame would provide a Fourier series (see [DS52]) similar to the spectral case. Though these searches have yielded partial results, it is still an open question whether $L^2(\mu_3)$ possesses an exponential frame. It is known that there exist singular measures without exponential frames. In fact, Lai [Lai12] showed that self-affine measures induced by iterated function systems with no overlap cannot possess exponential frames if the probability weights are not equal.

In this chapter, we demonstrate that the Kaczmarz algorithm educes another potentially fruitful substitute for exponential spectra and exponential frames: the "effective" sequences defined by Kwapień and Mycielski [KM01]. We show that the nonnegative integral exponentials in $L^2(\mu)$ for any singular Borel probability measure μ are such an effective sequence and that this effectivity allows us to define a Fourier series representation of any function in $L^2(\mu)$. This recovers a result of Poltoratskiĭ [Pol93] concerning the normalized Cauchy transform.

2.1 Fourier Series from the Kaczmarz Algorithm

Our main result is as follows:

Theorem 1. *If μ is a singular Borel probability measure on $[0,1)$, then the sequence $\{e_n\}_{n=0}^{\infty}$ is effective in $L^2(\mu)$. As a consequence, any element $f \in L^2(\mu)$ possesses a Fourier series*

$$f(x) = \sum_{n=0}^{\infty} c_n e^{2\pi i n x},$$

where

$$c_n = \int_0^1 f(x)\overline{g_n(x)}\, d\mu(x),$$

and $\{g_n\}_{n=0}^{\infty}$ is the sequence associated to $\{e_n\}_{n=0}^{\infty}$ via Equation (1.5). The sum converges in norm, and Parseval's identity $\|f\|^2 = \sum_{n=0}^{\infty} |c_n|^2$ holds.

Our proof proceeds in a series of lemmas. First, in order to show completeness of $\{e_n\}_{n=0}^{\infty}$, we appeal to the well-known theorem of Frigyes and Marcel Riesz [RR16]:

Theorem (F. and M. Riesz)**.** *Let μ be a complex Borel measure on $[0,1)$. If*

$$\int_0^1 e^{2\pi i n x}\, d\mu(x) = 0$$

for all $n \in \mathbb{N}$, then μ is absolutely continuous with respect to Lebesgue measure.

From this theorem, we prove the desired lemma:

Lemma 2. *If μ is a singular Borel measure on $[0,1)$, then $\{e_n\}_{n=0}^{\infty}$ is linearly dense in $L^2(\mu)$.*

Proof. Assume, for the sake of contradiction, that $\overline{\text{span}}(\{e_n\}_{n=0}^{\infty}) \neq L^2(\mu)$. Then there exists some $f \in L^2(\mu)$ such that $f \in \overline{\text{span}}(\{e_n\}_{n=0}^{\infty})^{\perp}$. Then for any $n \in \mathbb{N}$, we have

$$\int_0^1 e^{2\pi i n x}\overline{f(x)}\, d\mu(x) = 0.$$

By the F. and M. Riesz Theorem, this implies that $\overline{f} d\mu$ is absolutely continuous with respect to Lebesgue measure $d\lambda$. Since $\overline{f} d\mu << d\lambda$ and $\overline{f} d\mu \perp d\lambda$, it follows by uniqueness in Lebesgue's Decomposition Theorem that $\overline{f} d\mu \equiv 0$. Thus, $f = 0$ almost everywhere with respect to μ, which is a contradiction. Therefore, $\overline{\text{span}}(\{e_n\}_{n=0}^{\infty}) = L^2(\mu)$. □

Recall the Kwapień-Mycielski Theorem from [KM01] discussed in Chapter 1:

Theorem (Kwapień and Mycielski)**.** *A stationary sequence of unit vectors that is linearly dense in a Hilbert space is effective if and only if its spectral measure either coincides with the normalized Lebesgue measure or is singular with respect to Lebesgue measure.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. By Lemma 2, the sequence $\{e_n\}_{n=0}^{\infty}$ is linearly dense in $L^2(\mu)$. It consists of unit vectors, because μ is a probability measure. We see that for all $k, l, m \in \mathbb{N}_0$,

$$\langle e_{k+m}, e_{l+m} \rangle = \int_{[0,1)} e^{2\pi i(k-l)x} \, d\mu(x) = \langle e_k, e_l \rangle.$$

Thus, $\{e_n\}_{n=0}^{\infty}$ is stationary in $L^2(\mu)$, and moreover, μ is its spectral measure. It then follows from the theorem of Kwapień and Mycielski that $\{e_n\}_{n=0}^{\infty}$ is effective in $L^2(\mu)$.

Since $\{e_n\}_{n=0}^{\infty}$ is effective, given any $f \in L^2(\mu)$, we have that the Kaczmarz algorithm sequence defined recursively by

$$f_0 = \langle f, e_0 \rangle e_0$$

$$f_n = f_{n-1} + \langle f - f_{n-1}, e_n \rangle e_n$$

satisfies

$$\lim_{n\to\infty} \|f - f_n\| = 0.$$

We recall that

$$f_n = \sum_{i=0}^{n} \langle f, g_i \rangle e_i,$$

where the sequence $\{g_n\}_{n=0}^{\infty}$ is the sequence associated to the sequence $\{e_n\}_{n=0}^{\infty}$ by (1.5). Hence,

$$f = \sum_{i=0}^{\infty} \langle f, g_i \rangle e_i. \tag{2.1}$$

Setting $c_n = \langle f, g_n \rangle = \int_0^1 f(x)\overline{g_n(x)} \, d\mu(x)$ yields

$$f(x) = \sum_{n=0}^{\infty} c_n e^{2\pi i n x}, \tag{2.2}$$

where the convergence is in norm. Furthermore, since $\{e_n\}_{n=0}^{\infty}$ is effective, by (1.7) $\{g_n\}_{n=0}^{\infty}$ is a Parseval frame. Thus,

$$\sum_{n=0}^{\infty}|c_n|^2 = \sum_{n=0}^{\infty}|\langle f, g_n\rangle|^2 = \|f\|^2.$$

This completes the proof. □

Since the ternary Cantor measure μ_3 is a singular probability measure, Theorem 1 demonstrates that any $f \in L^2(\mu_3)$ possesses a Fourier series of the form prescribed by the theorem. This comes despite the fact that μ_3 does not possess an orthogonal basis of exponentials. It is still unknown whether $L^2(\mu_3)$ even possesses an exponential frame.

The sequence $\{e_n\}_{n=0}^{\infty}$ of exponentials is effective in $L^2(\mu)$ for all singular Borel probability measures μ, but it is Bessel in none of them. Indeed, if it were Bessel, μ would be absolutely continuous rather than singular by Theorem 3.10 of [DHW14]. Therefore, it is not possible for $\{e_n\}_{n=0}^{\infty}$ to be a frame in $L^2(\mu)$. However, by a remark in [LO01], since $\{e_n\}_{n=0}^{\infty}$ is pseudo-dual to the (in this case Parseval) frame $\{g_n\}_{n=0}^{\infty}$, the upper frame bound for $\{g_n\}_{n=0}^{\infty}$ implies a lower frame bound for $\{e_n\}_{n=0}^{\infty}$.

Moreover, some of the examples in [Lai12] of measures that do not possess an exponential frame are singular, and hence if we normalize them to be probability measures, Theorem 1 applies.

2.2 Elucidation of Fourier Coefficients

We shall give a somewhat more explicit formula for the coefficients c_n. We will require a lemma to do this, but first we discuss some notation:

Remark on Notation. Recall that a composition of a positive integer n is an ordered arrangement of positive integers that sum to n. Whereas for a partition the order in which the terms appear does not matter, two sequences having the same terms but in a different order constitute different compositions. We will think of compositions of n as tuples of positive integers whose entries sum to n. The set of compositions of n will be denoted P_n. In other words,

$$P_n := \left\{(p_1, p_2, \ldots, p_k) \mid k \in \mathbb{N}, (p_1, p_2, \ldots, p_k) \in \mathbb{N}^k, p_1 + p_2 + \cdots + p_k = n\right\}.$$

Thus, we have $P_1 = \{(1)\}$, $P_2 = \{(2), (1,1)\}$, $P_3 = \{(3), (1,2), (2,1), (1,1,1)\}$, etc. The length of a tuple $p \in P_n$ will be denoted $l(p)$, i.e. $p = (p_1, p_2, \ldots, p_{l(p)}) \in \mathbb{N}^{l(p)}$.

Lemma 3. *Let μ be a Borel probability measure on $[0,1)$ with Fourier-Stieltjes transform $\widehat{\mu}$. Define $\alpha_0 = 1$, and for $n \geq 1$, let*

$$\alpha_n = \sum_{p \in P_n} (-1)^{l(p)} \prod_{j=1}^{l(p)} \widehat{\mu}(p_j).$$

Let $\{g_n\}_{n=0}^{\infty}$ be as defined in (1.5). Then for all $n \in \mathbb{N}_0$,

$$g_n = \sum_{j=0}^{n} \overline{\alpha_{n-j}} e_j.$$

Proof. Clearly, $g_0 = e_0$ and $g_1 = e_1 - \langle e_1, e_0 \rangle e_0 = e_1 - \overline{\widehat{\mu}(1)} e_0$. We have that $P_1 = \{(1)\}$, so

$$\alpha_1 = (-1)^1 \widehat{\mu}(1) = -\widehat{\mu}(1).$$

So, the conclusion holds for $n = 0, 1$. Suppose that the conclusion holds up to some $n \in \mathbb{N}$. We have that

$$\begin{aligned}
g_{n+1} &= e_{n+1} - \sum_{j=0}^{n} \langle e_{n+1}, e_j \rangle g_j \\
&= e_{n+1} - \sum_{j=0}^{n} \overline{\widehat{\mu}(n+1-j)} g_j \\
&= e_{n+1} - \sum_{j=0}^{n} \overline{\widehat{\mu}(n+1-j)} \left(\sum_{k=0}^{j} \overline{\alpha_{j-k}} e_k \right) \\
&= e_{n+1} - \sum_{j=0}^{n} \sum_{k=0}^{j} \overline{\widehat{\mu}(n+1-j)} \overline{\alpha_{j-k}} e_k \\
&= e_{n+1} - \sum_{k=0}^{n} \sum_{j=k}^{n} \overline{\widehat{\mu}(n+1-j)} \overline{\alpha_{j-k}} e_k.
\end{aligned}$$

Thus, it remains only to show that

$$\alpha_{n+1-k} = -\sum_{j=k}^{n} \widehat{\mu}(n+1-j) \alpha_{j-k}.$$

We have:

$$-\sum_{j=k}^{n} \widehat{\mu}(n+1-j) \alpha_{j-k} = -\sum_{j=k}^{n} \widehat{\mu}(n+1-j) \sum_{p \in P_{j-k}} (-1)^{l(p)} \prod_{w=1}^{l(p)} \widehat{\mu}(p_w)$$

$$= \sum_{j=k}^{n} \sum_{p \in P_{j-k}} (-1)^{l(p)+1} \widehat{\mu}(n+1-j) \prod_{w=1}^{l(p)} \widehat{\mu}(p_w)$$
$$= \sum_{j=1}^{n+1-k} \sum_{p \in P_{n-k+1-j}} (-1)^{l(p)+1} \widehat{\mu}(j) \prod_{w=1}^{l(p)} \widehat{\mu}(p_w).$$

The last equality is obtained by reindexing the sum $j \mapsto n+1-j$. Now, if $p = (p_1, \ldots, p_{l(p)}) \in P_n$, then it is obvious that $p_1 \in \{1, 2, \ldots, n\}$ and $(p_2, p_3, \ldots, p_{l(p)}) \in P_{n-p_1}$ (where we define $P_0 = \varnothing$). Conversely, if $p_1 \in \{1, 2, \ldots, n\}$ and $(p_2, p_3, \ldots, p_{l(p)}) \in P_{n-p_1}$, then clearly $(p_1, p_2, \ldots, p_{l(p)}) \in P_n$. Thus, it follows that

$$-\sum_{j=k}^{n} \widehat{\mu}(n+1-j)\alpha_{j-k} = \sum_{p \in P_{n+1-k}} (-1)^{l(p)} \prod_{w=1}^{l(p)} \widehat{\mu}(p_w) = \alpha_{n+1-k}.$$

This completes the proof. □

Remark. Lemma 3 can easily be generalized to any Hilbert space setting in which the $\{g_n\}_{n=0}^{\infty}$ are induced by a stationary sequence $\{\varphi_n\}_{n=0}^{\infty}$ simply by replacing $\widehat{\mu}(m)$ with a_m in all instances, where the a_m are as defined after Definition 12.

It should be pointed out that the sequence of scalars $\{\alpha_n\}_{n=0}^{\infty}$ depends only on the measure μ. In a general Hilbert space setting where we may not have stationarity, an expansion of the $\{g_n\}$ in terms of the sequence $\{\varphi_n\}$ to which they are associated by (3) can be described by using inversion of an infinite lower-triangular Gram matrix. For a treatment, see [HS05].

Definition 15. Define a Fourier transform of $f \in L^2(\mu)$ by

$$\mathcal{F}f(y) = \widehat{f}(y) := \int_0^1 f(x) e^{-2\pi i y x}\, d\mu(x). \tag{2.3}$$

Observe that

$$|\mathcal{F}f(y)| = |\langle f, e_y \rangle| \leq \|f\|_\mu \cdot \|e_y\|_\mu = \|f\|_\mu.$$

Thus $\mathcal{F}$ is a linear operator from $L^2(\mu)$ to $L^\infty(\mathbb{R})$ with operator norm $\|\mathcal{F}\| = 1$.

Corollary 2. *Assume the conditions and definitions of Theorem 1. Then the coefficients c_n may be expressed*

$$c_n = \sum_{j=0}^{n} \alpha_{n-j} \widehat{f}(j),$$

and as a result

$$f(x) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \alpha_{n-j} \widehat{f}(j) \right) e^{2\pi i n x},$$

where the convergence is in norm.

Proof. We compute:

$$c_n = \langle f, g_n \rangle = \left\langle f, \sum_{j=0}^{n} \overline{\alpha_{n-j}} e_j \right\rangle = \sum_{j=0}^{n} \alpha_{n-j} \widehat{f}(j).$$

The second formula then follows by substitution into (2.2). □

While we have Parseval's identity $\|f\|^2 = \sum_{n=0}^{\infty} |c_n|^2$ as demonstrated by Theorem 1, in general the lack of the Bessel condition means that $\|f\|^2 \lesssim \sum_{n=0}^{\infty} |\widehat{f}(n)|^2$ does not hold. In fact, Proposition 3.10 in [DHSW11] demonstrates an example of a function where $\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 = +\infty$.

2.3 Non-Uniqueness of Fourier Coefficients

We begin with an example. In [JP98], it was shown that the quaternary Cantor measure μ_4 possesses an orthonormal basis of exponentials. This basis is $\{e_\lambda\}_{\lambda \in \Lambda}$, where the spectrum Λ is given by

$$\Lambda = \left\{ \sum_{n=0}^{k} \alpha_n 4^n : \alpha_n \in \{0, 1\}, k \in \mathbb{N}_0 \right\} = \{0, 1, 4, 5, 16, 17, 20, 21, \dots\}.$$

As a result, any vector $f \in L^2(\mu_4)$ may be written as

$$f = \sum_{\lambda \in \Lambda} \langle f, e_\lambda \rangle e_\lambda,$$

where the convergence is in the $L^2(\mu_4)$ norm. Notice that if we define a sequence of vectors $\{h_n\}_{n=0}^{\infty}$ by

$$h_n = \begin{cases} e_n & \text{if } n \in \Lambda \\ 0 & \text{otherwise,} \end{cases}$$

we have that

$$\sum_{n=0}^{\infty} \langle f, h_n \rangle e_n = \sum_{\lambda \in \Lambda} \langle f, e_\lambda \rangle e_\lambda = f.$$

On the other hand, since μ_4 is a singular probability measure, by Theorem 1 we also have

$$f = \sum_{n=0}^{\infty} c_n e_n = \sum_{n=0}^{\infty} \langle f, g_n \rangle e_n.$$

It can easily be checked that $h_0 = g_0 = e_0$ and $h_1 = g_1 = e_1$, but that $g_2 \neq h_2 = 0$. Thus, the sequences $\{g_n\}$ and $\{h_n\}$ yield different expansions for general $f \in L^2(\mu_4)$.

We can again use the Kaczmarz algorithm to generate a large class of sequences $\{h_n\}$ such that $\sum \langle f, h_n \rangle e_n = f$ in the $L^2(\mu)$ norm as follows.

Theorem 2. *Suppose μ and λ are nonnegative Borel probability measures on $[0, 1)$ such that $\mu << \lambda$, and suppose there exist constants A and B such that*

$$0 < A \leq \frac{d\mu}{d\lambda} \leq B$$

on supp $\left(\frac{d\mu}{d\lambda}\right) := \left\{x \in [0, 1) \mid \frac{d\mu}{d\lambda}(x) \neq 0\right\}$. If $\{h_n\}$ is the canonical sequence associated to λ by (1.5), then for all $f \in L^2(\mu)$,

$$f = \sum_{n=0}^{\infty} \left\langle f, \frac{h_n}{\frac{d\mu}{d\lambda}} \right\rangle_{\mu} e_n \tag{2.4}$$

in the $L^2(\mu)$ norm. Moreover, $\left\{\frac{h_n}{\frac{d\mu}{d\lambda}}\right\}$ is a frame in $L^2(\mu)$ with bounds no worse than $\frac{1}{B}$ and $\frac{1}{A}$. Furthermore, if λ' also satisfies the hypotheses, then $\lambda' \neq \lambda$ implies $\left\{\frac{h'_n}{\frac{d\mu}{d\lambda'}}\right\} \neq \left\{\frac{h_n}{\frac{d\mu}{d\lambda}}\right\}$ in $L^2(\mu)$.

Proof. Let $M :=$ supp $\left(\frac{d\mu}{d\lambda}\right)$, and define $\tilde{f} := f \cdot \chi_M$. First, we observe that

$$\int_{[0,1)} \left| \frac{h_n}{\frac{d\mu}{d\lambda}} \right|^2 d\mu = \int_M \frac{|h_n|^2}{\left(\frac{d\mu}{d\lambda}\right)^2} \frac{d\mu}{d\lambda} d\lambda \leq \int_M \frac{|h_n|^2}{A} d\lambda < \infty.$$

This shows that $\left\{\frac{h_n}{\frac{d\mu}{d\lambda}}\right\} \subset L^2(\mu)$. Now, suppose $f \in L^2(\mu)$. Then

$$A \int_{[0,1)} |\tilde{f}|^2 \, d\lambda \leq \int_{[0,1)} \left|\tilde{f}\right|^2 \frac{d\mu}{d\lambda} \, d\lambda \leq B \int_{[0,1)} |\tilde{f}|^2 \, d\lambda.$$

Therefore,

$$\frac{1}{B} \int_{[0,1)} |f|^2 \, d\mu \leq \int_{[0,1)} \left|\tilde{f}\right|^2 \, d\lambda \leq \frac{1}{A} \int_{[0,1)} |f|^2 \, d\mu < \infty.$$

Thus, $f \in L^2(\mu) \implies \tilde{f} \in L^2(\lambda)$. Now, we compute that for any $f \in L^2(\mu)$,

$$\left\langle f, \frac{h_n}{\frac{d\mu}{d\lambda}} \right\rangle_{\mu} = \int_{[0,1)} f \cdot \overline{\left(\frac{h_n}{\frac{d\mu}{d\lambda}}\right)} \frac{d\mu}{d\lambda} \, d\lambda = \int_{[0,1)} \tilde{f} \cdot \overline{h_n} \, d\lambda = \langle \tilde{f}, h_n \rangle_{\lambda}.$$

Then because $\{h_n\}$ is a Parseval frame in $L^2(\lambda)$, the previous two computations show that

$$\frac{1}{B}\|f\|_\mu^2 \leq \|\tilde{f}\|_\lambda^2 = \sum_{n=0}^{\infty} |\langle \tilde{f}, h_n\rangle_\lambda|^2 = \sum_{n=0}^{\infty} \left| \left\langle f, \frac{h_n}{\frac{d\mu}{d\lambda}} \right\rangle_\mu \right|^2 \leq \frac{1}{A}\|f\|_\mu^2, \tag{2.5}$$

and hence that $\left\{ \frac{h_n}{\frac{d\mu}{d\lambda}} \right\}$ is a frame in $L^2(\mu)$ with bounds no worse than $\frac{1}{B}$ and $\frac{1}{A}$. Now, we have that

$$\begin{aligned}
&\lim_{k\to\infty} \left\| f - \sum_{n=0}^{k} \left\langle f, \frac{h_n}{\frac{d\mu}{d\lambda}} \right\rangle_\mu e_n \right\|_\mu^2 \\
&= \lim_{k\to\infty} \int_{[0,1)} \left| f - \sum_{n=0}^{k} \left\langle f, \frac{h_n}{\frac{d\mu}{d\lambda}} \right\rangle_\mu e_n \right|^2 \frac{d\mu}{d\lambda}\, d\lambda \\
&= \lim_{k\to\infty} \int_{[0,1)} \left| \tilde{f} - \sum_{n=0}^{k} \left\langle \tilde{f}, h_n \right\rangle_\lambda e_n \right|^2 \frac{d\mu}{d\lambda}\, d\lambda \\
&\leq \lim_{k\to\infty} B \int_{[0,1)} \left| \tilde{f} - \sum_{n=0}^{k} \left\langle \tilde{f}, h_n \right\rangle_\lambda e_n \right|^2 d\lambda \\
&= B \lim_{k\to\infty} \left\| \tilde{f} - \sum_{n=0}^{k} \left\langle \tilde{f}, h_n \right\rangle_\lambda e_n \right\|_\lambda^2 \\
&= 0,
\end{aligned}$$

which shows that $\left\{ \frac{h_n}{\frac{d\mu}{d\lambda}} \right\}$ is dextrodual to $\{e_n\}$ in $L^2(\mu)$.

It remains only to show that different measures λ generate different sequences $\left\{ \frac{h_n}{\frac{d\mu}{d\lambda}} \right\}$ in $L^2(\mu)$. Therefore, suppose λ' is another singular Borel probability measure on $[0,1)$ such that $\mu << \lambda'$ and $0 < A' \leq \frac{d\mu}{d\lambda'} \leq B'$ on $\operatorname{supp}\left(\frac{d\mu}{d\lambda'}\right)$. Let $\{h_n'\}$ be the canonical sequence associated to $\{e_n\}$ in $L^2(\lambda')$ by (1.5). Suppose $\lambda' \neq \lambda$. We wish to show that $\left\{ \frac{h_n}{\frac{d\mu}{d\lambda}} \right\} \neq \left\{ \frac{h_n'}{\frac{d\mu}{d\lambda'}} \right\}$ in $L^2(\mu)$.

If $\frac{d\mu}{d\lambda} \neq \frac{d\mu}{d\lambda'}$ in $L^2(\mu)$, then $\frac{h_0}{\frac{d\mu}{d\lambda}} = \frac{e_0}{\frac{d\mu}{d\lambda}} \neq \frac{e_0}{\frac{d\mu}{d\lambda'}} = \frac{h_0'}{\frac{d\mu}{d\lambda'}}$ in $L^2(\mu)$. Therefore, assume that $\frac{d\mu}{d\lambda} = \frac{d\mu}{d\lambda'}$ in $L^2(\mu)$. By virtue of the F. and M. Riesz Theorem, since $\lambda \neq \lambda'$, there must exist an integer n such that $\widehat{\lambda}(n) \neq \widehat{\lambda'}(n)$. Following [HS05], we define a lower-triangular Gram matrix G of the nonnegative integral exponentials by

$$(G)_{ij} = \begin{cases} \langle e_i, e_j\rangle = \widehat{\lambda}(j-i) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases},$$

and then the inverse of this matrix determines the sequence $\{h_n\}$ associated to $\{e_n\}$ in $L^2(\lambda)$ via $h_n = \sum_{i=0}^{n} \overline{\alpha_{n-i}} e_i$ where $\alpha_{n-i} = \overline{(G^{-1})_{ni}}$. See [HS05] for details. (G and G^{-1} are stratified since $\{e_n\}$ is stationary.) Therefore, the sequences of scalars $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\alpha'_n\}_{n=0}^{\infty}$ induced by λ and λ', respectively, in Lemma 3 differ. Let n be the smallest positive integer such that $\alpha_n \neq \alpha'_n$. Then since $\frac{d\mu}{d\lambda} = \frac{d\mu}{d\lambda'}$, we have

$$\frac{h'_n}{\frac{d\mu}{d\lambda'}} - \frac{h_n}{\frac{d\mu}{d\lambda}} = \frac{\sum_{j=0}^{n} \left(\overline{\alpha'_{n-j}} - \overline{\alpha_{n-j}} \right) e_j}{\frac{d\mu}{d\lambda}} = \frac{(\overline{\alpha_n - \alpha'_n}) e_0}{\frac{d\mu}{d\lambda}} \neq 0.$$

Thus, $\left\{ \frac{h_n}{\frac{d\mu}{d\lambda}} \right\}$ and $\left\{ \frac{h'_n}{\frac{d\mu}{d\lambda'}} \right\}$ are distinct sequences in $L^2(\mu)$. □

Remark 3. We note that any convex combination of sequences $\{h_n\}$ that satisfy Equation (2.4) will again satisfy that equation. Also note that if $\frac{d\mu}{d\lambda}$ is constant on its support, then $\left\{ \frac{h_n}{\frac{d\mu}{d\lambda}} \right\}$ is a tight frame in $L^2(\mu)$.

In general, for a fixed $f \in L^2(\mu)$ the set of coefficient sequences $\{d_n\}$ that satisfy $f = \sum_{n=0}^{\infty} d_n e_n$ can be parametrized by sequences $\{\gamma_n\}$ of scalars satisfying $\sum_{n=0}^{\infty} \gamma_n e_n = 0$ via $d_n = \langle f, g_n \rangle_\mu + \gamma_n$. Clearly, Theorem 2 is not a complete description of all Fourier series expansions for f.

2.4 Connection to the Normalized Cauchy Transform

By Abel summability, the series $\sum_{n=0}^{\infty} \langle f, g_n \rangle e_n$ given by Theorem 1 is the boundary function of the analytic function $\sum_{n=0}^{\infty} \langle f, g_n \rangle z^n$ on $\mathbb{D}$. This function is in the Hardy space since the coefficients are square summable. An intriguing connection between the Kaczmarz algorithm and de Branges-Rovnyak spaces is given by the observations that follow.

Definition 16. Given a positive Borel measure ν on $[0, 1)$, define a map V_ν, called the normalized Cauchy transform, from $L^1(\nu)$ to the functions defined on $\mathbb{C} \setminus \mathbb{T}$ by

$$V_\nu f(z) := \frac{C_\nu f(z)}{C_\nu 1(z)} = \frac{\int_0^1 \frac{f(e^{2\pi i x})}{1 - z e^{-2\pi i x}} \, d\nu(x)}{\int_0^1 \frac{1}{1 - z e^{-2\pi i x}} \, d\nu(x)}.$$

(Here C_ν is the Cauchy transform from Definition 9.)

Suppose μ is, more specifically than ν, a singular Borel probability measure. Sarason proved in [Sar90] that V_μ maps $L^2(\mu)$ isometrically to the de Branges-Rovnyak space $\mathcal{H}(b)$, where $b(z)$ is the inner function associated to μ via the Herglotz representation theorem. In [Pol93] Poltoratskiĭ shows that V_μ is the inverse of a unitary operator that is a rank one perturbation of the unilateral shift as given by Clark [Cla72], and hence V_μ is unitary.

Proposition 1. *Assume the hypotheses of Theorem 1. Then for $z \in \mathbb{D}$,*

$$V_\mu f(z) = \sum_{n=0}^{\infty} \langle f, g_n \rangle z^n.$$

Proof. We have already seen in the proof of Lemma 1 that $C_\mu 1(z)$ and $\frac{1}{C_\mu 1(z)}$ are analytic on $\mathbb{D}$. It was also seen there that

$$C_\mu 1(z) = \sum_{n=0}^{\infty} \widehat{\mu}(n) z^n.$$

Writing $1/C_\mu 1(z) = \sum_{n=0}^{\infty} q_n z^n$, we have $1 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} q_k \widehat{\mu}(n-k) \right) z^n$, and so $\sum_{k=0}^{n} q_k \widehat{\mu}(n-k) = 0$ for all $n \geq 1$, and $1 = q_0 \widehat{\mu}(0)$. Now, since μ is a probability measure, we have that $\widehat{\mu}(0) = 1$. Hence, $q_0 = 1$. By (1.5), we see that

$$g_0 = e_0$$

and

$$g_n = e_n - \sum_{k=0}^{n-1} \overline{\widehat{\mu}(n-k)} g_k.$$

Therefore, we have that

$$g_0 = e_0 = \overline{q_0} e_0.$$

Now, suppose that for some $n \geq 1$, we have that $g_k = \sum_{j=0}^{k} \overline{q_{k-j}} e_j$ for all $k = 0, 1, \ldots, n-1$. We compute:

$$\begin{aligned} g_n &= e_n - \sum_{k=0}^{n-1} \overline{\widehat{\mu}(n-k)} g_k \\ &= e_n - \sum_{k=0}^{n-1} \overline{\widehat{\mu}(n-k)} \sum_{j=0}^{k} \overline{q_{k-j}} e_j \\ &= e_n - \sum_{k=0}^{n-1} \sum_{j=0}^{k} \overline{\widehat{\mu}(n-k) q_{k-j}} e_j \end{aligned}$$

$$
\begin{aligned}
&= e_n - \sum_{j=0}^{n-1} \left(\overline{\sum_{k=j}^{n-1} \widehat{\mu}(n-k) q_{k-j}} \right) e_j \\
&= e_n - \sum_{j=0}^{n-1} \left(\overline{\sum_{k=0}^{n-j-1} \widehat{\mu}(n-j-k) q_k} \right) e_j \\
&= e_n - \sum_{j=0}^{n-1} \left(\overline{\sum_{k=0}^{n-j} \widehat{\mu}(n-j-k) q_k} - \overline{\widehat{\mu}(0) q_{n-j}} \right) e_j \\
&= e_n + \sum_{j=0}^{n-1} \overline{q_{n-j}} e_j \\
&= \sum_{j=0}^{n} \overline{q_{n-j}} e_j .
\end{aligned}
$$

Thus by strong induction, $g_n = \sum_{j=0}^{n} \overline{q_{n-j}} e_j$ for all n. The q_n are unique by Gaussian elimination, so in fact $q_n = \alpha_n$ for all n, the α_n as in Lemma 3. Hence,

$$\frac{1}{C_\mu 1(z)} = \sum_{n=0}^{\infty} \alpha_n z^n .$$

It is also clear that

$$\int_0^1 \frac{f(e^{2\pi i x})}{1 - z e^{-2\pi i x}} \, d\mu(x) = \sum_{n=0}^{\infty} \langle f, e_n \rangle z^n .$$

Therefore, we have

$$
\begin{aligned}
\frac{\int_0^1 \frac{f(e^{2\pi i x})}{1-ze^{-2\pi i x}} \, d\mu(x)}{\int_0^1 \frac{1}{1-ze^{-2\pi i x}} \, d\mu(x)} &= \left(\sum_{n=0}^{\infty} \langle f, e_n \rangle z^n \right) \left(\sum_{m=0}^{\infty} \alpha_m z^m \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \langle f, \overline{\alpha_{n-j}} e_j \rangle \right) z^n \\
&= \sum_{n=0}^{\infty} \langle f, g_n \rangle z^n .
\end{aligned}
$$

□

Two of the main results in [Pol93] are Theorems 2.5 and 2.7, which together show that the Fourier series of $V_\mu f(z)$ converges to f in the $L^2(\mu)$ norm provided that μ is singular. Combining this together with Proposition 1 recovers our Theorem 1. Adding Clark's result that implies that V_μ is unitary, and we recover the Plancherel identity.

Poltoratskiĭ's results are more general than our Theorem 1 in the following way: if μ has an absolutely continuous component and a singular component, then for any $f \in L^2(\mu)$, the Fourier series of $V_\mu f$ converges to f in norm with respect to the singular component. The Fourier series cannot in general converge to f with respect to the absolutely continuous component of μ since the nonnegative exponentials are incomplete. It is unclear whether for such a μ every f can be expressed in terms of a bi-infinite Fourier series. For singular μ, our Theorem 1 guarantees norm convergence of the Fourier series of $V_\mu f$ to f as do Poltoratskiĭ's results. However, Poltoratskiĭ also comments in [Pol93] that the Fourier series converges pointwise μ-a.e. to f.

2.5 A Shannon Sampling Formula

In [Str00], Strichartz introduces a sampling formula for functions that are bandlimited in a generalized sense. He considers functions whose spectra are contained in a certain compact set K that is the support of a spectral measure μ. If F is a strongly K-bandlimited function, then he shows that it has an expression

$$F(x) = \sum_{\lambda \in \Lambda} F(\lambda) \widehat{\mu}(x - \lambda),$$

where Λ is a spectrum for $L^2(\mu)$.

We will now prove a similar sampling formula for analogously bandlimited functions. Our formula does not rely on an exponential basis and hence holds even for non-spectral singular measures. (Indeed, it even holds for singular measures devoid of exponential frames.) The price paid for not using an exponential sequence dual to itself is that the samples $F(\lambda)$ are replaced by the less tidy $\sum_{j=0}^{n} \alpha_{n-j} F(j)$.

Theorem 3. *Let μ be a singular Borel probability measure on $[0, 1)$. Let $\{\alpha_i\}_{i=0}^{\infty}$ be the sequence of scalars induced by μ by Lemma 3. Suppose $F : \mathbb{R} \to \mathbb{C}$ is of the form*

$$F(y) = \int_0^1 f(x) e^{-2\pi i y x} \, d\mu(x)$$

for some $f \in L^2(\mu)$. Then

$$F(y) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \alpha_{n-j} F(j) \right) \widehat{\mu}(y - n),$$

where the series converges uniformly in y.

Proof. By Theorem 1, f may be expressed $f = \sum_{n=0}^{\infty} c_n e_n$, the convergence occurring in the $L^2(\mu)$ norm. We compute:

$$\begin{aligned}
F(y) &= \int_0^1 f(x) e^{-2\pi i y x}\, d\mu(x) \\
&= \langle f, e_y \rangle \\
&= \left\langle \sum_{n=0}^{\infty} c_n e_n, e_y \right\rangle \\
&= \sum_{n=0}^{\infty} c_n \langle e_n, e_y \rangle \\
&= \sum_{n=0}^{\infty} c_n \widehat{\mu}(y-n).
\end{aligned}$$

Recall from Corollary 2 that

$$c_n = \sum_{j=0}^{n} \alpha_{n-j} \widehat{f}(j) = \sum_{j=0}^{n} \alpha_{n-j} F(j),$$

where the α_n are defined by Lemma 3. Combining these computations, we obtain that for any $y \in \mathbb{R}$,

$$F(y) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \alpha_{n-j} F(j) \right) \widehat{\mu}(y-n). \tag{2.6}$$

Let $S_k := \sum_{n=0}^{k} c_n e_n$. Since $S_k \to f$ in the $L^2(\mu)$ norm and the Fourier transform $\mathcal{F} : L^2(\mu) \to L^\infty(\mathbb{R})$ is bounded, $\{\mathcal{F} S_k\} \to \mathcal{F} f$ in $L^\infty(\mathbb{R})$. Then because $\mathcal{F} S_k(y) = \sum_{n=0}^{k} c_n \widehat{\mu}(y-n)$, we have that $\sum_{n=0}^{\infty} c_n \widehat{\mu}(y-n)$ and hence (2.6) converges uniformly in y to $\mathcal{F} f(y)$. □

It should be noted that, in contradistinction to the sampling formula of Strichartz, the convergence of the series in Equation (2.6) does not follow from the Cauchy-Schwarz inequality, because it is possible that $\sum_{n=0}^{\infty} |\widehat{\mu}(y-n)|^2 = \infty$.

CHAPTER 3. SINGULAR MEASURE REPRODUCING KERNELS

In the first chapter, we showed how the Kaczmarz algorithm can be used to compute Fourier series that converge in $L^2(\mu)$ for any singular Borel probability measure μ on $[0,1)$. In this chapter, we apply this knowledge to the problem of finding positive matrices on $\mathbb{D}$ that possess singular measure boundaries and reproduce themselves with respect to these boundaries, similar to the way the Hardy space's Szegő kernel reproduces itself with respect to Lebesgue measure boundaries. Formally, we are concerned with identifying members of two special sets of interest:

Definition 17. Let μ be a nonnegative Borel measure on $[0,1)$. We define $\mathcal{K}(\mu)$ to be the set of positive matrices K on $\mathbb{D}$ such that for each fixed $z \in \mathbb{D}$, K_z possesses an $L^2(\mu)$-boundary $K_z^\star$, and $K_z(w)$ reproduces itself with respect to integration of these $L^2(\mu)$-boundaries, i.e.

$$K_z(w) = \int_0^1 K_z^\star(x)\overline{K_w^\star(x)}\, d\mu(x)$$

for all $z, w \in \mathbb{D}$.

Definition 18. Let K be a positive matrix on $\mathbb{D}$. We define $\mathcal{M}(K)$ to be the set of nonnegative Borel measures μ on $[0,1)$ such that for each fixed $z \in \mathbb{D}$, K_z possesses an $L^2(\mu)$-boundary $K_z^\star$, and $K_z(w)$ reproduces itself with respect to integration of these $L^2(\mu)$-boundaries.

The quaternary Cantor measure μ_4 is the restriction of the $\frac{1}{2}$-dimensional Hausdorff measure to the quaternary Cantor set. Likewise, the ternary Cantor measure μ_3 is the restriction of the $\frac{\ln(2)}{\ln(3)}$-dimensional Hausdorff measure to the ternary Cantor set. In [JP98], Jorgensen and Pedersen showed that the quaternary Cantor measure is spectral. That is, there exists a set $\Lambda \subset \mathbb{Z}$ such that the set of complex exponentials $\left\{e^{2\pi i\lambda x}\right\}_{\lambda\in\Lambda}$ is an orthonormal basis of $L^2(\mu_4)$. From this, Dutkay and Jorgensen [DJ11] constructed a kernel G_Γ inside H^2 that reproduces itself both in H^2 and with respect to $L^2(\mu_4)$ boundary integration. Thus $G_\Gamma \in \mathcal{K}(\mu_4)$.

In [JP98], it was also shown that μ_3 is not spectral. Thus, it is not possible to construct a kernel for μ_3 in the same way. It is still unknown whether μ_3 even contains a set of complex exponentials that is a frame. Despite this seeming impediment, in this chapter we will show not only that $\mathcal{K}(\mu_3)$ is nonempty, but that it contains infinitely many members within H^2. In fact, we will show this for all singular Borel probability measures on $[0, 1)$.

3.1 Mu-Kernels That Are Also H^2 Kernels

Let b be an inner function such that $b(0) = 0$. Recall from section 1.9 that we have a de Branges-Rovnyak space $\mathcal{H}(b) = H^2 \ominus bH^2$, residing as an ordinary subspace inside the Hardy space H^2. This subspace is an invariant subspace of the backward shift operator S^*, where $S^*f(z) = \frac{f(z)-f(0)}{z}$. At the same time, we remember from section 1.3 that b corresponds to a unique singular Borel probability measure μ on $[0, 1)$. The triangle of relationships will be complete when we identify how μ is related to $\mathcal{H}(b)$. As it turns out, this relationship will yield for us an important member of $\mathcal{K}(\mu)$.

We begin by obtaining a result concerning the normalized Cauchy transform V_μ, which was given by Definition 16. Recall that when μ is a singular Borel probability measure, V_μ is an isometry of $L^2(\mu)$ onto $\mathcal{H}(b)$. In [Pol93], it was shown that if $f \in L^2(\mu)$ for a finite nonnegative regular singular measure μ, then f is an $L^2(\mu)$-boundary of $V_\mu f(z)$. We give an alternate proof of this for the case that μ is a singular Borel probability measure:

Proposition 2. *If μ is a singular Borel probability measure and $f \in L^2(\mu)$, then f is an $L^2(\mu)$-boundary function of $V_\mu f(z)$. Consequently, for any $F \in \mathcal{H}(b)$, $V_\mu^{-1}F = F^\star$.*

Proof. Let $f \in L^2(\mu)$. Since the sum in (2.1) is summable in $L^2(\mu)$, it is Abel summable, and hence by Proposition 1 we have that

$$V_\mu f(re^{2\pi i x}) = \sum_{n=0}^{\infty} \langle f, g_n \rangle r^n e_n(x)$$

converges in $L^2(\mu)$ for all $0 < r < 1$. By Abel summability,

$$\lim_{r\to 1^-} V_\mu f(re^{2\pi i x}) = \lim_{r \to 1^-} \sum_{n=0}^{\infty} \langle f, g_n \rangle r^n e_n = \sum_{n=0}^{\infty} \langle f, g_n \rangle e_n = f$$

in the $L^2(\mu)$ norm. Hence, f is an $L^2(\mu)$-boundary function of $V_\mu f(z)$.

Now if $F \in \mathcal{H}(b)$, then by bijectivity of V_μ, there exists a unique $f \in L^2(\mu)$ such that $V_\mu f(z) = F(z)$. Then f is an $L^2(\mu)$-boundary of $V_\mu f(z) = F(z)$, and since an $L^2(\mu)$-boundary is unique, we have $F^\star = f$. Hence, $V_\mu^{-1} F = F^\star$. □

Corollary 3. *If μ is a singular Borel probability measure with corresponding inner function b, then for any $f(z), j(z) \in \mathcal{H}(b)$, we have*

$$\langle f, j\rangle_{\mathcal{H}(b)} = \langle f^\star, j^\star\rangle_\mu, \tag{3.1}$$

where $f^\star$ and $j^\star$ are the $L^2(\mu)$-boundary functions of f and j, respectively.

Proof. Since V_μ is an isometry from $L^2(\mu)$ to $\mathcal{H}(b)$, Proposition 2 implies

$$\langle f, j\rangle_{\mathcal{H}(b)} = \langle V_\mu^{-1} f, V_\mu^{-1} j\rangle_\mu = \langle f^\star, j^\star\rangle_\mu.$$

□

Thus, functions in $\mathcal{H}(b)$ not only have Lebesgue boundaries, but also $L^2(\mu)$-boundaries, and the norm of $\mathcal{H}(b)$ is equal to boundary integration with respect to either boundary/measure pair. As an ordinary subspace of H^2, $\mathcal{H}(b)$ is of course a reproducing kernel Hilbert space. Let $k_z(w) \in H^2$ denote the Szegő kernel of H^2. That is,

$$k_z(w) := \frac{1}{1 - \overline{z} w}. \tag{3.2}$$

It is known (see [Sar94]) that the kernel of $\mathcal{H}(b)$ is given by

$$k_z^b(w) = (1 - \overline{b(z)} b(w)) k_z(w). \tag{3.3}$$

We give the following alternative form for V_μ:

Theorem 4. *Let μ be a singular Borel probability measure with corresponding inner function b and associated sequence $\{g_n\}_{n=0}^\infty \subset L^2(\mu)$ defined by (1.5). Then*

$$k_z^b(w) = \frac{1 - \overline{b(z)} b(w)}{1 - \overline{z} w} = \sum_{m=0}^\infty \sum_{n=0}^\infty \langle g_n, g_m\rangle_\mu \overline{z}^n w^m. \tag{3.4}$$

Proof. By Lemma 1 the inner function b satisfies

$$b(z) = 1 - \frac{1}{C_\mu 1(z)}. \tag{3.5}$$

As explained in the proof of Proposition 1, the scalar sequence $\{\alpha_n\}$ from (3) satisfies

$$\frac{1}{C_\mu 1(z)} = \sum_{n=0}^{\infty} \alpha_n z^n, \tag{3.6}$$

so that

$$b(z) = -\sum_{n=1}^{\infty} \alpha_n z^n. \tag{3.7}$$

Since the sequence $\{g_n\}_{n=0}^{\infty}$ is Bessel, $\sum_{n=0}^{\infty} \overline{z}^n g_n$ converges in $L^2(\mu)$ for all $z \in \mathbb{D}$. Observe that for a fixed $z \in \mathbb{D}$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{z}^n g_n &= \sum_{n=0}^{\infty} \overline{z}^n \left(\sum_{j=0}^{n} \overline{\alpha_{n-j}} e_j \right) \\
&= \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \overline{z}^n \overline{\alpha_{n-j}} e_j \\
&= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \overline{z}^{n+j} \overline{\alpha_n} e_j \\
&= \left(\sum_{n=0}^{\infty} \overline{\alpha_n z^n} \right) \left(\sum_{j=0}^{\infty} \overline{z}^j e_j \right) \\
&= \left(1 + \sum_{n=1}^{\infty} \overline{\alpha_n z^n} \right) \left(\sum_{j=0}^{\infty} \overline{z}^j e_j \right) \\
&= \left(1 - \overline{b(z)} \right) \frac{1}{1 - \overline{z} e_1} \\
&= (1 - \overline{b(z)}) k_z^\star.
\end{aligned}$$

The rearrangement of summation above is justified, because

$$\begin{aligned}
\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \| \overline{z}^{n+j} \overline{\alpha_n} e_j \| &= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} |z|^n |z|^j |\alpha_n| \\
&\leq \sum_{j=0}^{\infty} |z|^j \sqrt{\sum_{n=0}^{\infty} |z^2|^n} \sqrt{\sum_{n=0}^{\infty} |\alpha_n|^2} \\
&= \frac{1}{1 - |z|} \cdot \sqrt{\frac{1}{1 - |z|^2}} \cdot \|1 - b\|_{H^2} < \infty,
\end{aligned}$$

which shows that the sum converges absolutely.

Recall from Proposition 1 that for $f \in L^2(\mu)$, $V_\mu f(w) = \sum_{n=0}^\infty \langle f, g_n\rangle w^n$. Therefore, we have

$$V_\mu\left[\sum_{n=0}^\infty \overline{z}^n g_n\right](w) = \sum_{m=0}^\infty \left\langle \sum_{n=0}^\infty \overline{z}^n g_n, g_m\right\rangle w^m$$
$$= \sum_{m=0}^\infty \sum_{n=0}^\infty \langle g_n, g_m\rangle \overline{z}^n w^m.$$

On the other hand, in [Sar94] it is computed that

$$\begin{aligned}
C_\mu k_z^\star(w) &= \langle k_z^\star, k_w^\star\rangle_\mu \\
&= \int_0^1 \frac{1}{(1-e^{2\pi i x}\overline{z})(1-e^{-2\pi i x}w)}\, d\mu(x) \\
&= \frac{1}{1-\overline{z}w}\int_0^1 \frac{1}{2}\left(\frac{e^{-2\pi i x}+\overline{z}}{e^{-2\pi i x}-\overline{z}} + \frac{e^{2\pi i x}+w}{e^{2\pi i x}-w}\right) d\mu(x) \\
&= \frac{1}{2(1-\overline{z}w)}\left(\frac{1+\overline{b(z)}}{1-\overline{b(z)}} + \frac{1+b(w)}{1-b(w)}\right) \qquad \text{[by the Herglotz integral representation]} \\
&= \frac{1-\overline{b(z)}b(w)}{(1-\overline{b(z)})(1-b(w))(1-\overline{z}w)} \\
&= (1-\overline{b(z)})^{-1}(1-b(w))^{-1}k_z^b(w).
\end{aligned}$$

Therefore, since by [Sar94] $V_\mu f(w) = (1-b(w))C_\mu f(w)$,

$$\begin{aligned}
V_\mu\left[(1-\overline{b(z)})k_z^\star\right](w) &:= (1-b(w))C_\mu\left[(1-\overline{b(z)})k_z^\star\right](w) \\
&= (1-b(w))(1-\overline{b(z)})C_\mu k_z(w) \\
&= k_z^b(w) \\
&= \frac{1-\overline{b(z)}b(w)}{1-\overline{z}w}.
\end{aligned}$$

It follows that

$$\sum_{m=0}^\infty \sum_{n=0}^\infty \langle g_n, g_m\rangle \overline{z}^n w^m = \frac{1-\overline{b(z)}b(w)}{1-\overline{z}w}.$$

□

The preceding observations directly reveal the following:

Theorem 5. *If μ is a singular Borel probability measure on $[0,1)$ with corresponding inner function b, then $k^b \in \mathcal{K}(\mu)$, and $\mu \in \mathcal{M}(k_z^b)$.*

Proof. k_z^b is a reproducing kernel of $\mathcal{H}(b)$ with respect to the H^2 norm. By Corollary 3, it reproduces itself with respect to $L^2(\mu)$-boundary. □

Remark 4. It should be noted that Proposition 2 and Corollary 3 are previously known. See, for example, Clark's influential paper [Cla72] and Sarason's book [Sar94]. Theorem 5 is thus simply a formality. However, it can be proven another way, by combining Theorem 4 with Theorem 8, which is to come. In truth, Theorem 5 holds even when the assumption $\|\mu\| = 1$ is replaced by $\|\mu\| < \infty$, a fact shown in section 4.6 of the appendix. Our Kaczmarz-based methods do not work in that case since the normalized Cauchy transform is not an isometry when $\|\mu\| \neq 1$. However, our methods will be able to demonstrate many members of $\mathcal{K}(\mu)$ not previously known.

Corollary 4. *Assume the hypotheses of Theorem 5. If $V \subseteq \mathcal{H}(b)$ is a closed subspace and P_V is the orthogonal projection onto V, then $P_V k_z^b \in \mathcal{K}(\mu)$.*

Proof. The norm of V is the same as the norm of $\mathcal{H}(b)$, which is the norm of integration with respect to $L^2(\mu)$-boundary, and it is known that the reproducing kernel of a closed subspace is the orthogonal projection of the kernel of the whole space. □

Since the ternary Cantor measure μ_3 is singular with $\|\mu_3\| = 1$, Theorem 5 shows that $\mathcal{K}(\mu_3)$ is nonempty, despite μ_3 being nonspectral. Corollary 4 shows that $\mathcal{K}(\mu_3)$ contains other members as well. We will see that there are many more kernels in $\mathcal{K}(\mu_3)$, including some that lie outside $\mathcal{H}(b)$.

For a singular Borel probability measure μ, the function $k_z^b(z)$ is one example of a function that reproduces itself with respect to its $L^2(\mu)$-boundary. This is a particularly special case, since this function also reproduces itself with respect to its Lebesgue boundary, because the Lebesgue boundary norm and the $L^2(\mu)$-boundary norm are equal on $\mathcal{H}(b)$.

3.2 Mu-Kernels from Wold Decompositions

Let b be an inner function, and let μ be its corresponding singular measure. Since the Toeplitz operator $T_b : H^2 \to H^2$ is an isometry, and $\mathcal{H}(b)$ is a wandering subspace for T_b, the

Wold Decomposition Theorem implies

$$H^2 = \bigoplus_{n=0}^{\infty} T_b^n \mathcal{H}(b).$$

See [LS97] and [Ste99]. Although the Wold Decomposition Theorem is well-known, we offer the following alternative proof for the present situation:

Theorem 6. *Let μ be a finite, positive, singular Borel measure on $[0,1)$, and let b be the inner function corresponding to μ via the Poisson integral. Then for any $f \in H^2$, there exists a unique sequence of functions $\{\phi_n\}_{n=0}^{\infty} \subset \mathcal{H}(b)$ such that*

$$f = \sum_{n=0}^{\infty} \phi_n \cdot b^n.$$

Proof. We know that $k_z^b(w) = \frac{1-\overline{b(z)}b(w)}{1-\overline{z}w}$ is the kernel of $\mathcal{H}(b)$. Thus, $K_z(w) = \overline{b^n(z)}b^n(w)k_z^b(w) \in b^n\mathcal{H}(b)$ for each n. (Indeed, it is easy to see it is the kernel of $b^n\mathcal{H}(b)$.) Now, let

$$J = \overline{\text{span}}\{b^n \cdot \phi : n \in \mathbb{N}_0, \phi \in \mathcal{H}(b)\}.$$

For each $k \in \mathbb{N}$, we have that

$$\begin{aligned}
&\sum_{n=0}^{k-1} \overline{b^n(z)}b^n(w)k_z^b(w) \\
&= \frac{1-\overline{b(z)}b(w)}{1-\overline{z}w} + \frac{\overline{b(z)}b(w) - \overline{b^2(z)}b^2(w)}{1-\overline{z}w} + \cdots + \frac{\overline{b^{k-1}(z)}b^{k-1}(w) - \overline{b^k(z)}b^k(w)}{1-\overline{z}w} \\
&= \frac{1-\overline{b^k(z)}b^k(w)}{1-\overline{z}w}
\end{aligned}$$

is in J. Now, observe that

$$\begin{aligned}
\left\| \frac{1-\overline{b^k(z)}b^k(w)}{1-\overline{z}w} - \frac{1}{1-\overline{z}w} \right\|_{H^2}^2 &= \int_{[0,1)} \frac{|b^k(z)b^{*k}(x)|^2}{|1-\overline{z}e^{2\pi i x}|^2}\, dx \\
&= \int_0^1 \frac{|b^k(z)|^2}{|1-\overline{z}e^{2\pi i x}|^2}\, dx \\
&\leq |b(z)|^{2k} C,
\end{aligned}$$

where $C = \frac{1}{1-|z|} > 0$. Since b is inner, $|b(z)| \leq 1$ on $\mathbb{D}$. Since b is nonconstant, the maximum modulus principle implies $|b(z)| < 1$. Hence, for each fixed $z \in \mathbb{D}$,

$$\lim_{k\to\infty} \frac{1-\overline{b^k(z)}b^k(w)}{1-\overline{z}w} = \frac{1}{1-\overline{z}w}$$

in the H^2-norm. Thus, $\frac{1}{1-\bar{z}w} \in J$ for each fixed $z \in \mathbb{D}$. Since $k_z(w) = \frac{1}{1-\bar{z}w}$ is the kernel of H^2, this implies $J = H^2$.

We will show that $b^n\mathcal{H}(b) \perp b^k\mathcal{H}(b)$ for all $n \neq k$. Let $f, g \in \mathcal{H}(b)$, and without loss of generality, suppose $0 < k \leq n$. Then by Proposition 6 from the appendix,

$$\begin{aligned}\langle b^n f, b^k g\rangle_{H^2} &= \int_{[0,1)} b^{*n}(x) f^*(x) \left(\overline{b^*(x)}\right)^k \overline{g^*(x)}\, dx \\ &= \int_{[0,1)} |b^*(x)|^{2k} b^{*n-k}(x) f^*(x)\overline{g^*(x)}\, dx \\ &= \int_{[0,1)} b^{*n-k}(x) f^*(x)\overline{g^*(x)}\, dx \\ &= \langle b^{n-k} f, g\rangle_{H^2} \\ &= 0,\end{aligned}$$

because $b^{n-k}f \in bH^2$, $g \in \mathcal{H}(b)$, and $\mathcal{H}(b) = H^2 \ominus bH^2$. Thus, $b^n\mathcal{H}(b) \perp b^k\mathcal{H}(b)$. Hence, the terms $\psi_0, \psi_1 \cdot b, \psi_2 \cdot b^2, \ldots$ for any choices $\psi_n \in \mathcal{H}(b)$ are all orthogonal.

Let $\varepsilon > 0$. Since $H^2 = J$ is the closure of the finite linear span of elements of the form $\psi \cdot b^n$, $\psi \in \mathcal{H}(b)$ and $n \in \mathbb{N}_0$, there exist $\psi_0, \psi_1, \ldots, \psi_k \in \mathcal{H}(b)$ and $n_1, n_2, \ldots, n_k \in \mathbb{N}_0$ such that

$$\left\| f - \sum_{j=0}^{k} \psi_j \cdot b^{n_j} \right\|_{H^2} < \varepsilon.$$

If multiple n_j are the same, those terms collapse via summation into a single term. So since $0 \in \mathcal{H}(b)$, the above sum can be equivalently written $\sum_{n=0}^{K} \psi_n \cdot b^n$ for an appropriate choice of ψ_n, where $K = \max_{0\leq j\leq k} n_j$. So,

$$\left\| f - \sum_{n=0}^{K} \psi_n \cdot b^n \right\|_{H^2} < \varepsilon.$$

Since the subspaces $\mathcal{H}(b), b\mathcal{H}(b), b^2\mathcal{H}(b), \ldots$ are mutually orthogonal, it is clear from the properties of orthogonal projections that for any fixed $k \in \mathbb{N}_0$,

$$\sum_{n=0}^{k} \phi_n \cdot b^n,$$

where $\phi_n \cdot b^n$ is the orthogonal projection of f onto $b^n\mathcal{H}(b)$, is the unique closest element to f of the form $\sum_{n=0}^{k} \psi_n \cdot b^n$, $\psi_n \in \mathcal{H}(b)$. So,

$$\left\| f - \sum_{n=0}^{K} \phi_n \cdot b^n \right\|_{H^2} < \varepsilon.$$

Assume for some $K' > K$ that

$$\left\| f - \sum_{n=0}^{K'} \phi_n \cdot b^n \right\|_{H^2} \geq \varepsilon.$$

This contradicts the fact that $\sum_{n=0}^{K'} \phi_n \cdot b^n$ is the closest element to f of the form $\sum_{n=0}^{k} \psi_n \cdot b^n$, $\psi_n \in \mathcal{H}(b)$, because

$$\left\| f - \sum_{n=0}^{K} \phi_n \cdot b^n \right\|_{H^2} = \left\| f - \sum_{n=0}^{K} \phi_n \cdot b^n - \sum_{n=K+1}^{K'} 0 \cdot b^n_{H^2} \right\|_{H^2} < \varepsilon.$$

Hence, we must have that

$$\left\| f - \sum_{n=0}^{k} \phi_n \cdot b^n \right\|_{H^2} < \varepsilon$$

for all $k \geq K$. Since ε was arbitrary, this proves that

$$\lim_{k\to\infty} \sum_{n=0}^{k} \phi_n \cdot b^n = f$$

in the H^2-norm. Thus, the theorem is proved. □

If a finite number of functions all have $L^2(\mu)$-boundaries, then their sum has $L^2(\mu)$-boundary equal to the sum of the individual $L^2(\mu)$-boundaries. Since each term in the Wold decomposition has an $L^2(\mu)$-boundary, if the Wold decomposition of a function $f \in H^2$ is a finite sum, then f possesses an $L^2(\mu)$-boundary. Thus, the Wold Decomposition shows, among other things, that the set of functions in H^2 possessing $L^2(\mu)$-boundary is dense in H^2.

Proposition 3. *Let $V_0, V_1, \ldots, V_N$ be mutually orthogonal closed subspaces of $\mathcal{H}(b)$ for some inner function b such that $b(0) = 0$ and corresponding singular Borel probability measure μ. Let $k_z^{(n)}(w)$ denote the kernel of V_n. Then the space $W = \bigoplus_{n=0}^{N} b^n V_n$ is a reproducing kernel Hilbert space with respect to the norm of $L^2(\mu)$-boundary integration, and its kernel is $K_z := \sum_{n=0}^{N} \overline{b^n(z)} b^n k_z^{(n)}$. Consequently, $K_z \in \mathcal{K}(\mu)$, and $\mu \in \mathcal{M}(K)$.*

Proof. For any $f \in W$, we may write $f = f_0 + bf_1 + b^2 f_2 + \ldots + b^N f_N$, where $f_n \in V_n$. Then observe that by mutual orthogonality of the spaces $\mathcal{H}(b), b\mathcal{H}(b), b^2\mathcal{H}(b), \ldots, b^N\mathcal{H}(b)$ in H^2, we have

$$\|f\|_{H^2}^2 = \sum_{n=0}^{N} \|b^n f_n\|_{H^2}^2 = \sum_{n=0}^{N} \|f_n\|_{H^2}^2 = \sum_{n=0}^{N} \|f_n\|_{\mathcal{H}(b)}^2 = \sum_{n=0}^{N} \|f_n^\star\|_{\mu}^2.$$

By mutual orthogonality of the spaces $V_0, V_1, \ldots, V_N$ in $\mathcal{H}(b)$, the f_n are orthogonal in $\mathcal{H}(b)$, and hence by Corollary 3 the $f_n^\star$ are orthogonal in $L^2(\mu)$. Furthermore, it is easy to show that $f^\star = (bf)^\star$. Hence,

$$\sum_{n=0}^{N} \|f_n^\star\|_\mu^2 = \sum_{n=0}^{N} \|(b^n f)^\star\|_\mu^2 = \left\| \sum_{n=0}^{N} (b^n f_n)^\star \right\|_\mu^2 = \left\| \left(\sum_{n=0}^{N} b^n f_n \right)^\star \right\|_\mu^2 = \|f^\star\|_\mu^2.$$

This shows that the H^2 norm and the $L^2(\mu)$-boundary norm are equal on W. Hence, the inner products are equal as well by the polarization identity. The proof is completed by noting that by orthogonality,

$$\begin{aligned}
\left\langle f, \sum_{n=0}^{N} \overline{b^n(z)} b^n k_z^{(n)} \right\rangle_{H^2} &= \left\langle \sum_{m=0}^{N} b^m f_m, \sum_{n=0}^{N} \overline{b^n(z)} b^n k_z^{(n)} \right\rangle_{H^2} \\
&= \sum_{n=0}^{N} \langle b^n f_n, \overline{b^n(z)} b^n k_z^{(n)} \rangle_{H^2} \\
&= \sum_{n=0}^{N} b^n(z) \langle f_n, k_z^{(n)} \rangle_{H^2} \\
&= \sum_{n=0}^{N} b^n(z) f_n(z) \\
&= f(z).
\end{aligned}$$

□

3.3 Mu-Kernels That Are not H^2 Kernels

We will now give examples of functions in H^2 that are kernels with respect to their $L^2(\mu)$-boundaries, but not their Lebesgue boundaries.

For a singular Borel probability measure μ on $[0, 1)$, in Chapter 2 we demonstrated a large class of dextroduals of $\{e_n\}_{n=0}^\infty$ via Theorem 2. In this section we will show that these dextroduals can be used to find members of $\mathcal{K}(\mu)$ that are not members of $\mathcal{K}(dx)$.

Theorem 7. *Assume the hypotheses of Theorem 2. The set*

$$J = \left\{ \sum_{n=0}^{\infty} \left\langle f, \frac{h_n}{\frac{d\mu}{d\lambda}} \right\rangle_\mu z^n \;\middle|\; f \in L^2(\mu) \right\}$$

is a closed linear subspace of H^2*. If* $\lambda\left([0,1)\setminus supp\left(\frac{d\mu}{d\lambda}\right)\right)>0$*, then* J *is not backward-shift-invariant. If* $\lambda\left([0,1)\setminus supp\left(\frac{d\mu}{d\lambda}\right)\right)=0$*, then* $J=\mathcal{H}(\lambda)$*.*

Proof. By Theorem 2, $\left\{\frac{h_n}{\frac{d\mu}{d\lambda}}\right\}$ is a frame in $L^2(\mu)$, so that J is a subset of H^2. It is clearly a linear subspace of H^2 by sesquilinearity of the inner product in $L^2(\mu)$. By Equation (2.5),

$$\|f\|_\mu^2 \simeq \left\|\sum_{n=0}^{\infty}\left\langle f,\frac{h_n}{\frac{d\mu}{d\lambda}}\right\rangle_\mu z^n\right\|_{H^2}^2 .$$

Thus, J is a closed subset of H^2 by virtue of $L^2(\mu)$ being closed.

Now, let S^* denote the backward shift operator acting on H^2. Let $\{\alpha_n\}$ be the sequence from (3) satisfying $h_n=\sum_{i=0}^{n}\overline{\alpha_{n-i}}e_i$. Observe that for all $n\in\mathbb{N}_0$,

$$e_1h_n = e_1\sum_{i=0}^{n}\overline{\alpha_{n-i}}e_i=\sum_{i=0}^{n}\overline{\alpha_{n-i}}e_{i+1}=\sum_{i=1}^{n+1}\overline{\alpha_{n+1-i}}e_i=h_{n+1}-\overline{\alpha_{n+1}}e_0.$$

For any $f\in L^2(\lambda)$, it is trivial to see that

$$\frac{f-\langle f,e_0\rangle_\lambda e_0}{e_1}\in L^2(\lambda).$$

We compute that

$$\begin{aligned}\left\langle\frac{f-\langle f,e_0\rangle_\lambda e_0}{e_1},h_n\right\rangle_\lambda &= \langle f-\langle f,e_0\rangle_\lambda e_0, e_1h_n\rangle_\lambda\\ &= \langle f,e_1h_n\rangle_\lambda-\langle f,e_0\rangle_\lambda\langle e_0,e_1h_n\rangle_\lambda\\ &= \langle f,h_{n+1}\rangle_\lambda-\alpha_{n+1}\langle f,e_0\rangle_\lambda-\langle f,e_0\rangle_\lambda\langle e_0,h_{n+1}\rangle_\lambda+\langle f,e_0\rangle_\lambda\alpha_{n+1}\langle e_0,e_0\rangle_\lambda\\ &= \langle f,h_{n+1}\rangle_\lambda,\end{aligned}$$

because $\langle e_0,e_0\rangle_\lambda=1$ and $\langle e_0,h_{n+1}\rangle_\lambda=0$ for all $n\geq 0$. Thus,

$$S^*\left(\sum_{n=0}^{\infty}\langle f,h_n\rangle_\lambda z^n\right)=\sum_{n=0}^{\infty}\left\langle\frac{f-\langle f,e_0\rangle_\lambda e_0}{e_1},h_n\right\rangle_\lambda z^n.$$

As before, let $M=\operatorname{supp}\left(\frac{d\mu}{d\lambda}\right)$. For any $f\in L^2(\mu)$, $\tilde{f}=f\cdot\chi_M$ is the unique member of $L^2(\lambda)$ such that $\left\langle f,\frac{h_n}{\frac{d\mu}{d\lambda}}\right\rangle_\mu=\langle\tilde{f},h_n\rangle_\lambda$ for all $n\geq 0$. We therefore have

$$S^*\left(\sum_{n=0}^{\infty}\left\langle f,\frac{h_n}{\frac{d\mu}{d\lambda}}\right\rangle_\mu z^n\right)=\sum_{n=0}^{\infty}\left\langle\frac{\tilde{f}-\left\langle\tilde{f},e_0\right\rangle_\lambda e_0}{e_1},h_n\right\rangle_\lambda z^n. \tag{3.8}$$

Observe that on $[0,1)\setminus M$,

$$\frac{\tilde{f}-\left\langle \tilde{f},e_0\right\rangle_\lambda e_0}{e_1}=-\left\langle \tilde{f},e_0\right\rangle_\lambda e_{-1}.$$

Let us examine the particular case in which $f=e_0\in L^2(\mu)$. We have

$$\left\langle \tilde{f},e_0\right\rangle_\lambda=\lambda(M),$$

so that on $[0,1)\setminus M$,

$$\frac{\tilde{f}-\left\langle \tilde{f},e_0\right\rangle_\lambda e_0}{e_1}=-\lambda(M)e_{-1}.$$

Since $\lambda(M)>0$, $-\lambda(M)e_{-1}=0$ λ-a.e. on $[0,1)\setminus M$ if and only if $\lambda([0,1)\setminus M)=0$. It follows that if $\lambda([0,1)\setminus M)>0$, there does not exist $w\in L^2(\mu)$ such that

$$\frac{\tilde{f}-\left\langle \tilde{f},e_0\right\rangle_\lambda e_0}{e_1}=w\cdot\chi_M=\tilde{w}$$

in $L^2(\lambda)$. Hence, if $\lambda([0,1)\setminus M)>0$, then J is not backward-shift-invariant.

If $\lambda([0,1)\setminus M)=0$, then it is easy to see that for all $f\in L^2(\mu)$, $\tilde{f}=f$ in $L^2(\lambda)$, and

$$A\|f\|_\lambda^2\le\|f\|_\mu^2\le B\|f\|_\lambda^2,$$

so that $L^2(\lambda)=L^2(\mu)$ as sets of functions. Thus

$$J=\left\{\sum_{n=0}^{\infty}\left\langle f,\frac{h_n}{\frac{d\mu}{d\lambda}}\right\rangle_\mu z^n\;\middle|\;f\in L^2(\mu)\right\}=\left\{\sum_{n=0}^{\infty}\langle f,h_n\rangle_\lambda z^n\;\middle|\;f\in L^2(\lambda)\right\}=\mathcal{H}(\lambda).$$

$\square$

Theorem 8. *Let μ be a Borel measure on $[0,1)$. Let $\{h_n\}\subset L^2(\mu)$ be a Bessel sequence that is dextrodual to $\{e_n\}$. Then for each fixed $z\in\mathbb{D}$,*

$$K_z(w):=\sum_m\sum_n\langle h_n,h_m\rangle_\mu\overline{z}^n w^m$$

is a well-defined function on $\mathbb{D}$. $K_z(w)\in H^2$ and possesses an $L^2(\mu)$-boundary function $K_z^\star$. Moreover,

$$K_z(w)=\langle K_z^\star,K_w^\star\rangle_\mu.$$

Proof. Fix $z \in \mathbb{D}$. Let $N \in \mathbb{N}_0$, and suppose $n > m \geq N$. Then since $\{h_n\}$ is Bessel, we have

$$\left\| \sum_{k=0}^{n} \overline{z}^k h_k - \sum_{k=0}^{m} \overline{z}^k h_k \right\|_\mu = \left\| \sum_{k=m+1}^{n} \overline{z}^k h_k \right\|_\mu \leq C \sqrt{\sum_{k=m+1}^{n} |z|^{2k}} \leq C \sqrt{\sum_{k=N}^{\infty} |z|^{2k}}.$$

As $N \to \infty$, the right side goes to 0, which shows that the sequence $\left\{ \sum_{k=0}^{n} \overline{z}^k h_k \right\}_n$ is Cauchy and hence convergent in $L^2(\mu)$. By continuity of the inner product in $L^2(\mu)$, we then have

$$\begin{aligned} K_z(w) &:= \sum_m \sum_n \langle h_n, h_m \rangle \overline{z}^n w^m \\ &= \sum_m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle w^m. \end{aligned}$$

Observe that since $\{h_n\}$ is Bessel,

$$\sum_{m=0}^{\infty} \left| \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle \right|^2 \leq C' \left\| \sum_n \overline{z}^n h_n \right\|_\mu^2 < \infty,$$

which shows that $K_z(w) \in H^2$.

We claim that $K_z^\star = \sum_n \overline{z}^n h_n$. Because $\{h_n\}$ is dextrodual to $\{e_n\}$, we have

$$\sum_n \overline{z}^n h_n = \sum_m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle e_m.$$

A summable series in a normed linear space is Abel summable. Hence, for all $0 < r \leq 1$, we have that

$$\sum_m r^m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle e_m$$

converges in $L^2(\mu)$, and

$$\begin{aligned} &\lim_{r \to 1^-} \left\| \sum_m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle e_m - \sum_m r^m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle e_m \right\|_\mu \\ &= \lim_{r \to 1^-} \left\| \sum_n \overline{z}^n h_n - \sum_m r^m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle e_m \right\|_\mu \\ &= 0. \end{aligned}$$

Since

$$K_z(re^{2\pi i x}) = \sum_m \left\langle \sum_n \overline{z}^n h_n, h_m \right\rangle r^m e^{2\pi i m x},$$

the above shows that for each $0 < r < 1$, $K_z(re^{2\pi ix}) \in L^2(\mu)$ with respect to the variable x, and that $\sum_n \overline{z}^n h_n$ is the $L^2(\mu)$-boundary function of $K_z(w)$, establishing the claim. We compute that

$$\begin{aligned}\langle K_z^\star, K_w^\star \rangle &= \left\langle \sum_n \overline{z}^n h_n, \sum_m \overline{w}^m h_m \right\rangle \\ &= \sum_m \sum_n \langle h_n, h_m \rangle \, \overline{z}^n w^m \\ &= K_z(w).\end{aligned}$$

□

Corollary 5. *The space J from Theorem 7 equipped with the inner product*

$$\langle f, g \rangle_J := \int_0^1 f_\mu^\star(x) \overline{g_\mu^\star(x)} \, d\mu(x) = \langle f^\star, g^\star \rangle_\mu \tag{3.9}$$

is a reproducing kernel Hilbert space with kernel

$$K_z(w) := \sum_m \sum_n \left\langle \frac{h_n}{\frac{d\mu}{d\lambda}}, \frac{h_m}{\frac{d\mu}{d\lambda}} \right\rangle_\mu \overline{z}^n w^m,$$

and the norm of this space is equivalent to the usual Hardy space norm.

Proof. Since $\left\{ \frac{h_n}{\frac{d\mu}{d\lambda}} \right\}$ is dextrodual to $\{e_n\}$ in $L^2(\mu)$, by Abel summability each member f of J possesses an $L^2(\mu)$-boundary function $f_\mu^\star$. Indeed, it is easy to see that if $\varphi \in L^2(\mu)$, then $\sum_{n=0}^\infty \left\langle \varphi, \frac{h_n}{\frac{d\mu}{d\lambda}} \right\rangle_\mu z^n$ is the unique member of J whose $L^2(\mu)$-boundary is φ.

Observe that since $\left\{ \frac{h_n}{\frac{d\mu}{d\lambda}} \right\}$ is a frame in $L^2(\mu)$, we have

$$\begin{aligned}\langle f, f \rangle_J = 0 &\iff \langle f^\star, f^\star \rangle_\mu = 0 \\ &\iff f^\star = 0 \\ &\iff \left\langle f^\star, \frac{h_n}{\frac{d\mu}{d\lambda}} \right\rangle_\mu = 0 \; \forall n \\ &\iff \sum_{n=0}^\infty \left\langle f^\star, \frac{h_n}{\frac{d\mu}{d\lambda}} \right\rangle_\mu z^n = 0 \\ &\iff f = 0.\end{aligned}$$

Also, $\langle f, f\rangle_J = \langle f^\star, f^\star\rangle_\mu \geq 0$, so $\langle\cdot,\cdot\rangle_J$ is positive-definite. Linearity of $\langle\cdot,\cdot\rangle_J$ in the first argument follows from linearity of the map $f \mapsto f^\star$ and linearity of $\langle\cdot,\cdot\rangle_\mu$ in the first argument. Conjugate symmetry of $\langle\cdot,\cdot\rangle_J$ follows from conjugate symmetry of $\langle\cdot,\cdot\rangle_\mu$. Hence, $\langle\cdot,\cdot\rangle_J$ is an inner product on J.

Equip J with the norm induced by $\langle\cdot,\cdot\rangle_J$. Clearly, $\|f\|_J = \|f^\star\|_\mu$. Thus, if $\{f_k\}$ is Cauchy in J, $\{f_k^\star\}$ is Cauchy in $L^2(\mu)$. Thus, since $L^2(\mu)$ is complete, there exists some $y \in L^2(\mu)$ such that $f_k^\star \to y$. Let $f = \sum_{n=0}^\infty \left\langle y, \frac{h_n}{\frac{d\mu}{d\lambda}}\right\rangle_\mu z^n$ be the unique member of J whose $L^2(\mu)$-boundary is y. We have

$$\lim_{k\to\infty}\|f_k - f\|_J = \lim_{k\to\infty}\|(f_k - f)^\star\|_\mu = \lim_{k\to\infty}\|f_k^\star - y\|_\mu = 0.$$

Hence, J is complete, and therefore a Hilbert space, under $\langle\cdot,\cdot\rangle_{\mu^\star}$.

Let $f = \sum_{n=0}^\infty \left\langle f^\star, \frac{h_n}{\frac{d\mu}{d\lambda}}\right\rangle_\mu z^n \in J$. Observe that by Theorem 2,

$$\begin{aligned}\frac{1}{B}\|f\|_J^2 &= \frac{1}{B}\|f^\star\|_\mu^2 \\ &\leq \sum_{n=0}^\infty \left|\left\langle f^\star, \frac{h_n}{\frac{d\mu}{d\lambda}}\right\rangle\right|^2 = \|f\|_{H^2}^2 \\ &\leq \frac{1}{A}\|f^\star\|_\mu^2 = \frac{1}{A}\|f\|_J^2,\end{aligned}$$

showing that $\|\cdot\|_J$ and $\|\cdot\|_{H^2}$ are equivalent norms on J.

Because $\{\frac{h_n}{\frac{d\mu}{d\lambda}}\}$ is a frame on $L^2(\mu)$, it is Bessel on $L^2(\mu)$, and it follows from Theorem 8 that $K_z(w)$ is well-defined on $\mathbb{D}$, possesses an $L^2(\mu)$-boundary, and reproduces itself with respect to that boundary. For each $z \in \mathbb{D}$, as shown in the proof of Theorem 8, $\sum_{n=0}^\infty \overline{z}^n \frac{h_n}{\frac{d\mu}{d\lambda}} \in L^2(\mu)$, and

$$K_z(w) = \sum_{m=0}^\infty \left\langle \sum_{n=0}^\infty \overline{z}^n \frac{h_n}{\frac{d\mu}{d\lambda}}, \frac{h_m}{\frac{d\mu}{d\lambda}}\right\rangle_\mu w^m.$$

Thus, $K_z(w) \in J$ for each $z \in \mathbb{D}$.

It remains only to show that $\{K_z(w) : z \in \mathbb{D}\}$ is complete in J. Let b denote the inner function corresponding to the measure λ from Theorem 7 (where $\mu = \frac{d\mu}{d\lambda}\, d\lambda$). Since $\{h_n\}$ is the canonical sequence associated to λ via the Kaczmarz algorithm in (1.5), it follows as in the proof of Theorem 4 that

$$\sum_{n=0}^\infty \overline{z}^n \frac{h_n}{\frac{d\mu}{d\lambda}} = \frac{1}{\frac{d\mu}{d\lambda}}(1 - \overline{b(z)})\frac{1}{1 - \overline{z}e_1}, \tag{3.10}$$

where the convergence occurs absolutely in norm. Therefore, if $\left\{\frac{1}{\frac{d\mu}{d\lambda}} \cdot \frac{1}{1-\overline{z}e_1} : z \in \mathbb{D}\right\}$ is dense in $L^2(\mu)$, then by linearity $\{K_z(w) : z \in \mathbb{D}\}$ is dense in J. So suppose $f \in L^2(\mu)$. Then $V_\mu f(z) \in \mathcal{H}(d)$, where d is the inner function corresponding to μ. The kernel functions of $\mathcal{H}(d)$ are of the form

$$k_z^d(w) = \frac{1 - \overline{d(z)}d(w)}{1 - \overline{z}w}. \tag{3.11}$$

These kernels are, of course, linearly-dense in $\mathcal{H}(d)$, and since the norm on $\mathcal{H}(d)$ corresponds to the $L^2(\mu)$-boundary norm, we must have that the boundary functions of the kernels, $\{\left(k_z^d\right)^\star : z \in \mathbb{D}\}$, are linearly-dense in $L^2(\mu)$. As remarked in [Pol93], the radial limits of d satisfy $d^*(e^{2\pi i x}) = 1$ for μ-almost all x. Thus the $L^2(\mu)$-boundary of $k_z^d(w)$ is

$$(k_z^d)^\star(x) = \frac{1 - \overline{d(z)}}{1 - \overline{z}e^{2\pi i x}}. \tag{3.12}$$

Suppressing the constant $1 - \overline{d(z)}$, we see that $\left\{\frac{1}{1-\overline{z}e_1} : z \in \mathbb{D}\right\}$ is dense in $L^2(\mu)$. Because $\frac{d\mu}{d\lambda} \cdot f \in L^2(\mu)$ for any $f \in L^2(\mu)$, this implies $\left\{\frac{1}{\frac{d\mu}{d\lambda}} \cdot \frac{1}{1-\overline{z}e_1} : z \in \mathbb{D}\right\}$ is dense in $L^2(\mu)$, which completes the proof. □

The following observation is immediate:

Corollary 6. *Under the hypotheses and notations of Theorem 2 and Corollary 5, we have that* $\mu \in \mathcal{M}(K)$, *and* $K \in \mathcal{K}(\mu)$.

As a final note, observe that by Theorem 7, if $\lambda\left([0,1) \setminus \operatorname{supp}\left(\frac{d\mu}{d\lambda}\right)\right) > 0$, then because J is not backward-shift-invariant,

$$K(z,w) := \sum_m \sum_n \left\langle \frac{h_n}{\frac{d\mu}{d\lambda}}, \frac{h_m}{\frac{d\mu}{d\lambda}} \right\rangle_\mu \overline{z}^n w^m$$

is a positive matrix such that $\mu \in \mathcal{M}(K)$, but $J \neq \mathcal{H}(b)$. Indeed $J \neq \mathcal{H}(u)$ for any inner function u, because $\mathcal{H}(u)$ is backward-shift-invariant.

3.4 The Set $\mathcal{M}(K)$

Starting with a singular Borel probability measure μ, we have seen several examples of positive matrices $K_z(w)$ that reproduce with respect to $L^2(\mu)$-boundary integration. Reproducing in this way potentially has desirable application, but it may happen in practice that we are more tied to a particular positive matrix than we are a measure. Thus, it is natural for us to ask a question in the opposite direction: Given a positive matrix $K \subset H^2(\mathbb{D})$, for which Borel measures μ does $K_z(w)$ reproduce with respect to $L^2(\mu)$-boundary integration? In other words, which measures are in $\mathcal{M}(K)$? For a given K, it is *a priori* possible that $\mathcal{M}(K) = \varnothing$, though we know of no examples yet. As we have seen, though, this is thankfully not always the case, and the following results give us some more insight.

Theorem 9. *Let V be a closed subspace of H^2, and let K be the reproducing kernel of V. If*

$$\overline{\cup_{n=0}^{\infty} S^{*n} V} \neq H^2,$$

then there exists a singular measure $\mu \in \mathcal{M}(K)$. Indeed, to each inner function b such that $b(0) = 0$ there corresponds a distinct such measure.

Proof. $\overline{\cup_{n=0}^{\infty} S^{*n} V}$ is the smallest closed S^*-invariant subspace containing V. Every proper closed S^*-invariant subspace of H^2 is a de Branges-Rovnyak space $\mathcal{H}(u)$ for some inner function u. Let b be an inner function satisfying $b(0) = 0$. Then bu is an inner function satisfying $(bu)(0) = 0$. Let μ be the singular Borel probability measure corresponding to bu. By Corollary 3, the H^2 norm on $\mathcal{H}(bu)$ is equal to the norm of $L^2(\mu)$-boundary integration. We also recall that since u is inner and b, as an inner function, is in the unit ball of H^∞, we have that $\mathcal{H}(bu) = \mathcal{H}(u) \oplus u\mathcal{H}(b)$. This implies $V \subseteq \mathcal{H}(u) \subseteq \mathcal{H}(bu)$. Thus since K reproduces with respect to the H^2 norm inside $\mathcal{H}(bu)$, it reproduces with respect to the $L^2(\mu)$-boundary norm. Hence, $\mu \in \mathcal{M}(K)$.

If $b' \neq b$ is another inner function satisfying $b'(0) = 0$, then $bu \neq b'u$, which by uniqueness in the Herglotz representation implies μ is not the same as the inner function corresponding to $b'u$, whence we get distinctness. □

Lemma 4. *Let ν and μ be finite Borel measures on $[0, 1)$, and suppose $\nu = \nu_a + \nu_s$ is the Lebesgue decomposition of ν with respect to μ. If $\mu, \nu \in \mathcal{M}(K)$ and $\frac{d\nu_a}{d\mu}$ is bounded, then the affine hull of ν and μ intersected with the set of nonnegative Borel measures is contained in $\mathcal{M}(K)$.*

Proof. Suppose $\nu, \mu \in \mathcal{M}(K)$ with $\frac{d\nu_a}{d\mu} \leq \beta$, and let $\lambda \in \mathbb{R}$ such that $\lambda\mu + (1-\lambda)\nu$ is a nonnegative Borel measure. For each z, let $K^{\star}_{z,\mu} : [0,1) \to \mathbb{C}$ be a representative of the equivalence class of the $L^2(\mu)$-boundary of K_z, and likewise let $K^{\star}_{z,\nu}$ be a representative of the equivalence class of the $L^2(\nu)$ boundary of K_z. Since $\nu_s \perp \mu$, there exist disjoint Borel subsets A and B of $[0,1)$ such that $A \cup B = [0,1)$, $\nu_s(E) = 0$ for all $E \subseteq B$, and $\mu(E) = 0$ for all $E \subseteq A$.

Define $H_z = K^{\star}_{z,\mu} \cdot \chi_B + K^{\star}_{z,\nu} \cdot \chi_A$. It is obvious that $H_z \equiv_{L^2(\mu)} K^{\star}_{z,\mu}$. We claim that $H_z \equiv_{L^2(\nu)} K^{\star}_{z,\nu}$ as well. We compute:

$$\begin{aligned}
&\lim_{r\to1^-} \int_0^1 |K_z(re_x) - H_z(x)|^2 \, d\nu(x) \\
&= \lim_{r\to1^-} \left(\int_{[0,1)} |K_z(re_x) - H_z(x)|^2 \frac{d\nu_a}{d\mu} \, d\mu(x) + \int_{[0,1)} |K_z(re_x) - H_z(x)|^2 \, d\nu_s(x) \right) \\
&= \lim_{r\to1^-} \left(\int_B |K_z(re_x) - H_z(x)|^2 \frac{d\mu_a}{d\mu} \, d\mu(x) + \int_A |K_z(re_x) - H_z(x)|^2 \, d\nu_s(x) \right) \\
&= \lim_{r\to1^-} \left(\int_B |K_z(re_x) - K^{\star}_{z,\mu}(x)|^2 \frac{d\nu_a}{d\mu} \, d\mu(x) + \int_A |K_z(re_x) - K^{\star}_{z,\nu}(x)|^2 \, d\nu_s(x) \right).
\end{aligned}$$

Since $\mu(A) = 0$, we have $\nu_a(A) = 0$, and hence

$$\lim_{r\to1^-} \int_A |K_z(re_x) - K^{\star}_{z,\nu}(x)|^2 \, d\nu_s(x) = \lim_{r\to1^-} \int_A |K_z(re_x) - K^{\star}_{z,\nu}(x)|^2 \, d\nu(x) = 0.$$

Thus,

$$\begin{aligned}
\lim_{r\to1^-} \int_0^1 |K_z(re_x) - H_z(x)|^2 \, d\nu(x) &= \lim_{r\to1^-} \int_B |K_z(re_x) - K^{\star}_{z,\mu}(x)|^2 \frac{d\nu_a}{d\mu} \, d\mu(x) \\
&\leq \beta \lim_{r\to1^-} \int_B |K_z(re_x) - K^{\star}_{z,\mu}(x)|^2 \, d\mu(x) \\
&= 0.
\end{aligned}$$

Therefore, $H_z \equiv_{L^2(\nu)} K^{\star}_{z,\nu}$. Now observe that

$$\begin{aligned}
&\lim_{r\to1^-} \int_{[0,1)} |K_z(re_x) - H_z(x)|^2 \, d[\lambda\mu + (1-\lambda)\nu](x) \\
&= \lim_{r\to1^-} \left(\lambda \int_{[0,1)} |K_z(re_x) - H_z(x)|^2 \, d\mu(x) + (1-\lambda) \int_{[0,1)} |K_z(re_x) - H_z(x)|^2 \, d\nu(x) \right) \\
&= \lim_{r\to1^-} \left(\lambda \int_{[0,1)} |K_z(re_x) - K^{\star}_{z,\mu}|^2(x) \, d\mu(x) + (1-\lambda) \int_{[0,1)} |K_z(re_x) - K^{\star}_{z,\nu}(x)|^2 \, d\nu(x) \right)
\end{aligned}$$

$= 0.$

Thus, H_z is the $L^2(\rho)$ boundary of K_z, where $\rho = \lambda\mu + (1-\lambda)\nu$. We see that

$$\begin{aligned}\int_{[0,1)} H_z \overline{H_w}\, d\rho &= \lambda \int_{[0,1)} H_z \overline{H_w}\, d\mu + (1-\lambda) \int_0^1 H_z \overline{H_w}\, d\nu \\ &= \lambda \int_{[0,1)} K^\star_{z,\mu} \overline{K^\star_{w,\mu}}\, d\mu + (1-\lambda) \int_{[0,1)} K^\star_{z,\nu} \overline{K^\star_{w,nu}}\, d\nu \\ &= \lambda K_z(w) + (1-\lambda) K_z(w) \\ &= K_z(w).\end{aligned}$$

Thus, K_z reproduces itself with respect to $L^2(\rho)$ boundary. Hence, $\rho \in \mathcal{M}(K)$. □

Given that $\mathcal{H}(b)$ is so (relatively) well understood, it is perhaps a more interesting question to ask what happens when a positive matrix lies outside of $\mathcal{H}(b)$. For a nonconstant inner function b, let μ_n denote the unique singular Borel measure on $[0,1)$ corresponding to b^n via the Poisson integral. Given a positive matrix $K_z(w)$ and an inner function b, for which n, if any, is $\mu_n \in \mathcal{M}(K)$? We propose to begin a study of this question here. We begin by revealing the relationship between μ's family of Clark measures and the measures μ_n.

Lemma 5. *Let $b : \mathbb{D} \to \mathbb{D}$, and let $n \in \mathbb{N}$. Then for all $z \in \mathbb{D}$,*

$$\frac{1}{n} \sum_{j=0}^{n-1} \frac{1 + e^{-2\pi i j/n} b(z)}{1 - e^{-2\pi i j/n} b(z)} = \frac{1 + b^n(z)}{1 - b^n(z)}.$$

Proof. For $z \in \mathbb{D}$ such that $b(z) = 0$, the equality is obvious. So suppose $z \in \mathbb{D}$ is such that $b(z) \neq 0$. We have

$$\begin{aligned}\sum_{j=0}^{n-1} \frac{1 + e^{-2\pi i j/n} b(z)}{1 - e^{-2\pi i j/n} b(z)} &= \sum_{j=0}^{n-1} \frac{e^{2\pi i j/n} + b(z)}{e^{2\pi i j/n} - b(z)} \\ &= \sum_{j=0}^{n-1} \frac{e^{2\pi i j/n}}{e^{2\pi i j/n} - b(z)} + \sum_{j=0}^{n-1} \frac{b(z)}{e^{2\pi i j/n} - b(z)} \\ &= \sum_{j=0}^{n-1} \frac{1}{1 - e^{-2\pi i j/n} b(z)} - \sum_{j=0}^{n-1} \frac{1}{1 - \frac{e^{2\pi i j/n}}{b(z)}}.\end{aligned}$$

Observe that

$$\sum_{j=0}^{n-1} \frac{1}{1 - e^{-2\pi i j/n} b(z)} = \sum_{j=0}^{n-1} \sum_{l=0}^{\infty} (e^{-2\pi i j/n} b(z))^l$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty}\sum_{j=0}^{n-1}(e^{-2\pi ijl/n})b^l(z) \\
&= \sum_{l=0}^{\infty} b^l(z)\sum_{j=0}^{n-1}(e^{-2\pi il/n})^j \\
&= \sum_{l=0}^{\infty} b^l(z)\begin{cases} \frac{1-e^{-2\pi il}}{1-e^{-2\pi il/n}} & \text{if } l \neq 0 \bmod n \\ n & \text{if } l = 0 \bmod n \end{cases} \\
&= \sum_{l=0}^{\infty} b^l(z)\begin{cases} 0 & \text{if } l \neq 0 \bmod n \\ n & \text{if } l = 0 \bmod n \end{cases} \\
&= n\sum_{l=0}^{\infty} b^{nl}(z) \\
&= \frac{n}{1-b^n(z)}.
\end{aligned}
$$

A similar computation shows that

$$\sum_{j=0}^{n-1}\frac{1}{1-\frac{e^{2\pi ij/n}}{b(z)}} = \frac{n}{1-\frac{1}{b^n(z)}}.$$

Hence,

$$
\begin{aligned}
\frac{1}{n}\sum_{j=0}^{n-1}\frac{1+e^{-2\pi ij/n}b(z)}{1-e^{-2\pi ij/n}b(z)} &= \frac{1}{1-b^n(z)} - \frac{1}{1-\frac{1}{b^n(z)}} \\
&= \frac{1}{1-b^n(z)} - \frac{b^n(z)}{b^n(z)-1} \\
&= \frac{1+b^n(z)}{1-b^n(z)}.
\end{aligned}
$$

□

Lemma 6. *Given an inner function b, if μ_n is the singular measure associated to b^n, then we have*

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1}\sigma_{e^{2\pi ij/n}},$$

where σ_α is the singular measure corresponding to the inner function $\overline{\alpha}b$.

Proof. By Lemma 5, we have

$$\operatorname{Re}\left(\frac{1+b^n(z)}{1-b^n(z)}\right) = \operatorname{Re}\left(\frac{1}{n}\sum_{j=0}^{n-1}\frac{1+e^{-2\pi ij/n}b(z)}{1-e^{-2\pi ij/n}b(z)}\right)$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} \mathrm{Re}\left(\frac{1+e^{-2\pi ij/n}b(z)}{1-e^{-2\pi ij/n}b(z)}\right)$$
$$= \frac{1}{n}\sum_{j=0}^{n} \int_0^1 \frac{1+|z|^2}{|z-\xi|^2}\, d\sigma_{e^{2\pi ij/n}}(\xi)$$
$$= \int_0^1 \frac{1+|z|^2}{|z-\xi|^2}\, d\left[\frac{1}{n}\sum_{j=0}^{n-1}\sigma_{e^{2\pi ij/n}}\right](\xi).$$

This shows that $\frac{1}{n}\sum_{j=0}^{n-1}\sigma_{e^{2\pi ij/n}}$ is the inner function corresponding to b^n via the Herglotz representation theorem, which completes the proof. ☐

Theorem 10. *Let $K_z(w)$ be a positive matrix and let b be an inner function. Let m, n, and q be positive integers such that $n = qm$. Let*

$$\rho = \frac{q}{(q-1)n}\sum_{\substack{j=0\\ q\nmid j}}^{n-1} \sigma_{e^{2\pi ij/m}}.$$

If two of the measures μ_m, μ_n, and ρ are in $\mathcal{M}(K)$, then so is the third.

Proof. By Lemma 5, we have

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1}\sigma_{e^{2\pi ij/n}}$$
$$= \frac{1}{n}\left(\sum_{\substack{j=0\\ q\mid j}}^{n-1}\sigma_{e^{2\pi ij/(qm)}} + \sum_{\substack{j=0\\ q\nmid j}}^{n-1}\sigma_{e^{2\pi ij/n}}\right)$$
$$= \frac{1}{n}\left(\sum_{j=0}^{m-1}\sigma_{e^{2\pi ij/m}} + \sum_{\substack{j=0\\ q\nmid j}}^{n-1}\sigma_{e^{2\pi ij/n}}\right)$$
$$= \frac{1}{q}\mu_m + \frac{q-1}{q}\rho.$$

So, each of the measures μ_n, μ_m, and ρ is in the affine hull of the other two.

The Clark measures $\{\sigma_\alpha : \alpha \in \mathbb{T}\}$ are mutually singular [Pol93]. It follows that μ_m and ρ, since they are sums of Clark measures that do not share a common Clark measure, are mutually singular. Hence, if $\rho = \rho_a + \rho_s$ is the Lebesgue decomposition of ρ with respect to μ_m, we must have $\rho_a = 0$, and hence $\frac{d\rho_a}{d\mu_m} = 0$.

So the Radon-Nikodym derivative of the part of ρ absolutely continuous to μ_m is bounded. Furthermore, it is clear that μ_m and ρ are absolutely continuous with respect to μ_n with respective Radon-Nikodym derivatives $\frac{d\mu_m}{d\mu_n} \equiv \frac{1}{q}$ and $\frac{d\rho}{d\mu_n} \equiv \frac{q-1}{q}$. Therefore, by Lemma 4, if two of the three measures are in $\mathcal{M}(K)$, so is the third. $\square$

CHAPTER 4. SUB-HARDY HILBERT SPACES

Recall that the norm of the Hardy space H^2 can be defined three different ways. It can be thought of as

$$\|f\|_{H^2}^2 := \sup_{0<r<1} \int_0^1 |f(re^{2\pi ix})|^2 \, dx, \tag{4.1}$$

or as

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |c_n|^2, \text{ where } f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

or even as

$$\|f\|_{H^2} = \|f^*\|_{L^2([0,1))}, \tag{4.2}$$

where f^* is the Lebesgue boundary function of f.

The Hardy space is intimately tied to Lebesgue measure via Lebesgue measure's use in equations (4.1), and the definition of f^* and (4.2). The idea of using a measure other than Lebesgue measure leads to a main question: What is an appropriate definition of a Hardy space corresponding to a measure μ on $[0, 1)$ other than Lebesgue measure? It would be nice if, taking a cue from (4.1), we could define this space to consist of those holomorphic functions such that

$$\|f\|^2 := \sup_{0<r<1} \int_0^1 |f(re^{2\pi ix})|^2 \, d\mu(x) < \infty.$$

While this certainly yields a set of holomorphic functions, there are several properties of the classical Hardy space that do not go through for general measures. In the Lebesgue case, the integral values $\int_0^1 |f(re^{2\pi ix})|^2 \, dx$ increase monotonically as $r \to 1^-$, allowing an identification of the original Hardy space norm with the $L^2(\mathbb{T})$ norm of the Lebesgue boundary function, and hence by the polarization identity, an identification of the inner products in H^2 and $L^2(\mathbb{T})$ as well. It is easy to see that in the case of a general measure μ, this monotonicity need not hold.

For singular measures μ, we will sidestep these and other difficulties by demoting the supremum definition of norm, instead regarding the boundary integration definition of norm as primary. That is to say, we will start with spaces of holomorphic functions that possess boundary functions that are in $L^2(\mu)$, and we will define the norm of a function in this space to be the $L^2(\mu)$ norm of its boundary function. There are three types of spaces we will obtain from this method. The most special of these spaces will be a subspace of the Hardy space in which the ordinary Hardy space norm and the $L^2(\mu)$-boundary norm actually coincide. The second type of space is a subspace of the Hardy space in which the Hardy space and $L^2(\mu)$-boundary norms differ. The third type of space we will obtain is one whose functions do not belong to H^2 at all.

4.1 Finite-Dimensional μ-Hardy Subspaces

In this short section, we examine the problem of characterizing the finite-dimensional subspaces of H^2 whose kernels reproduce with respect to $L^2(\mu)$-boundary integration. In finite dimensions, we are unburdened by the multitude of convergence issues that will hinder progress, but also facilitate discussion, in the subsequent sections. Because of this ease, it is pointless to restrict our attention to singular measures in this section. In fact, we begin with some observations regarding measures on $[0,1)$ in general:

Definition 19. A function $K : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ will be said to be separable if $K(z,w) = F(z)\cdot G(w)$ for some functions $F : \mathbb{D} \to \mathbb{C}$ and $G : \mathbb{D} \to \mathbb{C}$.

When referring to kernels, we have used the notation $K_z(w)$ in place of $K(z,w)$. For the special case of kernels that are separable, the following observation is immediate:

Proposition 4. *Let μ be a measure on $[0,1)$, and let $K_z(w) \not\equiv 0$ be a separable function that reproduces itself with respect to its $L^2(\mu)$ boundaries. Then $K_z(w) = \overline{F(z)}F(w)$ for some $F : \mathbb{D} \to \mathbb{C}$ such that $\|F^\star\|_{L^2(\mu)} = 1$.*

Conversely, if $F : \mathbb{D} \to \mathbb{C}$ possesses an $L^2(\mu)$-boundary function $F^\star$ of unit norm, then $K_z(w) := \overline{F(z)}F(w)$ reproduces itself with respect to its $L^2(\mu)$ boundaries.

Proof. Write $K_z(w) = K(z,w) = G(z)H(w)$ for some functions $G : \mathbb{D} \to \mathbb{C}$ and $H : \mathbb{D} \to \mathbb{C}$. Since for each fixed $z \in \mathbb{D}$, $K_z(w) = G(z)H(w)$ possesses an $L^2(\mu)$-boundary in the variable w, we must have that $H(w)$ possesses an $L^2(\mu)$-boundary, and

$$K_z^\star = G(z)H^\star.$$

Because $K_z(w)$ reproduces itself with respect to its $L^2(\mu)$-boundary, we have

$$\begin{aligned} K_z(w) &= \int_0^1 K_z^\star(x)\overline{K_w^\star(x)}\, d\mu(x) \\ &= \int_0^1 G(z)H^\star(x)\overline{G(w)H^\star(x)}\, d\mu(x) \\ &= G(z)\overline{G(w)}\|H^\star\|_{L^2(\mu)}^2. \end{aligned}$$

Thus, G is also seen to have an $L^2(\mu)$-boundary, since it is a scalar multiple of $\overline{K_z}$, which has $L^2(\mu)$-boundary. Defining $F : \mathbb{D} \to \mathbb{C}$ by $F(z) = \|H^\star\|_{L^2(\mu)}\overline{G(z)}$, we obtain

$$K_z(w) = \overline{F(z)}F(w).$$

Finally, observe that

$$\begin{aligned} \overline{F(z)}F(w) &= \int_0^1 \overline{F(z)}F^\star(x)F(w)\overline{F^\star(x)}\, d\mu(x) \\ &= \overline{F(z)}F(w)\|F^\star\|_{L^2(\mu)}^2, \end{aligned}$$

so that we must have $\|F^\star\|_{L^2(\mu)} = 1$.

Conversely, suppose $F : \mathbb{D} \to \mathbb{C}$ has an $L^2(\mu)$-boundary function $F^\star$ such that $\|F^\star\|_{L^2(\mu)} = 1$. Define $K_z(w) := \overline{F(z)}F(w)$. For each $z \in \mathbb{D}$, clearly $K_z^\star = \overline{F(z)}F^\star$, and we have

$$\begin{aligned} \int_0^1 K_z^\star(x)\overline{K_w^\star}(x)\, d\mu(x) &= \overline{F(z)}F(w)\int_0^1 F^\star(x)\overline{F^\star(x)}\, d\mu(x) \\ &= \overline{F(z)}F(w) \\ &= K_z(w). \end{aligned}$$

□

Theorem 11. *Let μ be a measure on $[0,1)$. Let S be an n-dimensional reproducing kernel Hilbert space of functions on $\mathbb{D}$ whose inner product is that of $L^2(\mu)$-boundary integration.*

Then the kernel functions $K_z(w)$ are of the form $K_z(w) = \sum_{j=1}^n \overline{F_j(z)}F_j(w)$ for some functions $F_1, \dots, F_n \in S$ whose boundaries $F_1^\star, \dots, F_n^\star$ are orthonormal in $L^2(\mu)$.

Conversely, if $F_1, \dots, F_n : \mathbb{D} \to \mathbb{C}$ have $L^2(\mu)$ boundaries $F_1^\star, \dots, F_n^\star$ that are orthonormal in $L^2(\mu)$, then $K_z(w) := \sum_{j=1}^n \overline{F_j(z)}F_j(w)$ is the reproducing kernel of an n-dimensional Hilbert space S of functions on $\mathbb{D}$ whose inner product is that of $L^2(\mu)$-boundary integration.

Proof. Since S is n-dimensional, we may select an orthonormal basis $F_1, \dots, F_n$ of S. By definition of the inner product on S, $F_1^\star, \dots, F_n^\star$ are orthonormal in $L^2(\mu)$. Then since an orthonormal basis is a Parseval frame, Papadakis's Theorem (see section 1.4) implies

$$K_z(w) = \sum_{i=1}^n \overline{F_i(z)}F_i(w).$$

Conversely, suppose $F_1, \dots, F_n : \mathbb{D} \to \mathbb{C}$ have $L^2(\mu)$ boundaries $F_1^\star, \dots, F_n^\star$ that are orthonormal in $L^2(\mu)$. If $F_1, \dots, F_n$ were linearly dependent, then there would exist scalars $\alpha_1, \dots, \alpha_n$, not all zero, such that $0 \equiv \alpha_1 F_1 + \dots + \alpha_n F_n$ on $\mathbb{D}$. It follows that

$$0 = (\alpha_1 F_1 + \dots + \alpha_n F_n)^\star = \alpha_1 F_1^\star + \dots + \alpha_n F_n^\star.$$

This contradicts the fact that $F_1^\star, \dots, F_n^\star$, being orthonormal, must be linearly independent in $L^2(\mu)$. Hence, $F_1, \dots, F_n$ are linearly independent on $\mathbb{D}$.

Let $S := \operatorname{span}(F_1, \dots, F_n)$, and suppose $f(z) = \sum_{k=1}^n \zeta_k F_k(z)$ and $g(z) = \sum_{k=1}^n \eta_k F_k(z)$ are two typical members of S. By the finite-dimensional Moore-Aronszajn Theorem (see section 1.6), S becomes a reproducing kernel Hilbert space when equipped with the inner product

$$\langle f, g \rangle_S = \sum_{i,j=1}^n \delta_{ij}\zeta_i\overline{\eta_j} = \sum_{i=1}^n \zeta_i\overline{\eta_i},$$

and the kernel is given by

$$K_z(w) = \sum_{i=1}^n \overline{F_i(z)}F_i(w).$$

Since by orthonormality

$$\begin{aligned}\langle f^\star, g^\star \rangle_\mu &= \left\langle \sum_{i=1}^n \zeta_i F_i^\star, \sum_{j=1}^n \eta_j F_j^\star \right\rangle_\mu \\ &= \sum_{i,j=1}^n \zeta_i\overline{\eta_j}\delta_{ij}\end{aligned}$$

$$= \langle f, g \rangle_S,$$

the inner product on S is equally the inner product of $L^2(\mu)$-boundary integration. □

Corollary 7. *Let μ be a measure on $[0,1)$. Let S be an n-dimensional subspace of H^2 whose functions all have $L^2(\mu)$-boundary and is a Hilbert space under the inner product of $L^2(\mu)$-boundary integration. Then the kernel functions $K_z(w)$ of S are of the form $K_z(w) = \sum_{j=1}^n \overline{F_j(z)} F_j(w)$ for some functions $F_1, \ldots, F_n \in S$ whose boundaries $F_1^\star, \ldots F_n^\star$ are orthonormal in $L^2(\mu)$.*

Conversely, if $F_1, \ldots, F_n \in H^2$ have $L^2(\mu)$ boundaries $F_1^\star, \ldots, F_n^\star$ that are orthonormal in $L^2(\mu)$, then $K_z(w) := \sum_{j=1}^n \overline{F_j(z)} F_j(w)$ is the reproducing kernel of an n-dimensional subspace S of H^2 when S is equipped with the norm of $L^2(\mu)$-boundary integration.

Proof. This is immediate from Theorem 11. □

4.2 Infinite-Dimensional μ-Hardy Subspaces

We now turn our attention to the analysis of infinite-dimensional subspaces of H^2 that are RKHS's with respect to $L^2(\mu)$-boundary.

If $\{K_z : \mathbb{D} \to \mathbb{C}\}_{z \in \mathbb{D}}$ is a collection of functions that reproduces itself with respect to $L^2(\mu)$-boundary, then K_z is a positive matrix. This is because if $z_1, \ldots, z_N \in \mathbb{D}$, then

$$(K_{z_i}(z_j))_{ij} = \left(\langle K_{z_i}^\star, K_{z_j}^\star \rangle_\mu \right)_{ij}.$$

Since the right side is the Gram matrix of functions in $L^2(\mu)$, it is positive semidefinite, and so K_z is a positive matrix. By the Moore-Aronszajn Theorem, this implies that there exists a unique reproducing kernel Hilbert space $\mathbb{K}$ for which K_z is the reproducing kernel.

It is clear that on the finite linear span $\operatorname{span}\{K_z : z \in \mathbb{D}\} \subset \mathbb{K}$, the $\mathbb{K}$-norm corresponds to integration with respect to $L^2(\mu)$-boundary. However, $\mathbb{K}$ also contains points adjoined to $\operatorname{span}\{K_z : z \in \mathbb{D}\}$ so as to complete it. These adjoined functions do not *a priori* have $L^2(\mu)$-boundary, and even if they do, the $\mathbb{K}$-norm on them does not *a priori* equal their $L^2(\mu)$-boundary norm. In the results that follow, some of the conditions assumed serve to tame the behavior of these adjoined functions.

Let us summarize the conundra we encounter in the infinite dimensional case. Suppose $K_z(w) \in H^2$ for each $z \in \mathbb{D}$ and reproduces with respect to $L^2(\mu)$-boundary integration. Let $\mathbb{K}$ be the RKHS for which $K_z(w)$ is the reproducing kernel.

1. Decompositions of $K_z(w)$ that converge pointwise do not *a priori* converge in norm. We will provide a condition under which certain decompositions do.

2. $\mathbb{K}$ is not *a priori* a subset of H^2. We provide a condition under which it is.

3. Functions in $\mathbb{K} \setminus \text{span}\{K_z : z \in \mathbb{D}\}$ do not *a priori* have $L^2(\mu)$-boundary. We will provide some conditions under which they must.

4. Even if every function in $\mathbb{K}$ possesses $L^2(\mu)$-boundary, the $\mathbb{K}$-norm on $\mathbb{K}\setminus\text{span}\{K_z : z \in \mathbb{D}\}$ is not *a priori* equal to the $L^2(\mu)$-boundary norm. We will provide some conditions under which it is.

In addition, we will demonstrate other conditions under which $\mathbb{K}$ possesses other properties analogous to those held by H^2 in the classical case.

4.3 Guarantors of Decomposition Norm Convergence

The following result is well-known, but we include a generic proof:

Lemma 7. *Suppose $\mathbb{H}$ is a separable Hilbert space and $\{h_i\}_{i\in I}$ is a collection of elements of $\mathbb{H}$. Then there exists a countable subset $J \subseteq I$ such that $\{h_i\}_{i\in J}$ is complete in $S := \overline{\mathit{span}}\{h_i : i \in I\}$.*

Proof. Since $\mathbb{H}$ is separable, S is separable. Hence, there exists a countable dense subset $\{s_n\}_{n=1}^{\infty}$ of S. For each $n, m \in \mathbb{N}$, there exists a finite linear combination

$$\sum_{p=1}^{P_{n,m}} \alpha_p^{(n,m)} h_{\beta_p^{(n,m)}},$$

where $P_{n,m} \in \mathbb{N}$, $\alpha_1^{(n,m)}, \ldots, \alpha_{P_{n,m}}^{(n,m)} \in \mathbb{C}$ and $\beta_1^{(n,m)}, \ldots, \beta_{P_{n,m}}^{(n,m)} \in I$, such that

$$\left\| s_n - \sum_{p=1}^{P_{n,m}} \alpha_p^{(n,m)} h_{\beta_p^{(n,m)}} \right\| < \frac{1}{m}.$$

Let $C_{n,m} := \left\{ h_{\beta_1^{(n,m)}}, h_{\beta_2^{(n,m)}}, \ldots, h_{\beta_{P_{n,m}}^{(n,m)}} \right\}$. Then by construction, for each $n \in \mathbb{N}$,

$$s_n \in \overline{\text{span}} \left(\bigcup_{m=1}^{\infty} C_{n,m} \right).$$

Thus,

$$\{s_n : n \in \mathbb{N}\} \subseteq \overline{\text{span}} \left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} C_{n,m} \right),$$

and so

$$S = \overline{\text{span}} \left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} C_{n,m} \right).$$

Since $\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} C_{n,m}$ is a countable subset of $\{h_i : i \in I\}$, we need only define

$$J = \left\{ i \in I : h_i \in \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} C_{n,m} \right\},$$

and the proof is complete. □

In the finite-dimensional case, we decomposed every $L^2(\mu)$-boundary-reproducing kernel (non-uniquely) into the form $K_z(w) = \sum_{i=0}^{n} \overline{f_i(z)} f_i(w)$, the f_i possessing orthonormal $L^2(\mu)$ boundaries. Due to the finitude of the sum, convergence occurs in every mode desired. The following result gives a similar decomposition in the infinite-dimensional case, but some modes of convergence come with conditions:

Lemma 8. *Let μ be a Borel measure on $[0, 1)$. Suppose $K_z(w) \in H^2$ for each $z \in \mathbb{D}$ and reproduces itself with respect to $L^2(\mu)$-boundary. Then there exist $f_0, f_1, f_2, \ldots \in H^2$, each possessing $L^2(\mu)$-boundary, such that*

$$K_z(w) = \sum_{i=0}^{\infty} \overline{f_i(z)} f_i(w), \tag{4.3}$$

the convergence occurring pointwise, and such that $\{f_i^\star\}_{i=0}^{\infty}$ is an orthonormal basis of $S := \overline{\text{span}}\{K_z^\star : z \in \mathbb{D}\}$ and

$$K_z^\star = \sum_{i=0}^{\infty} \overline{f_i(z)} f_i^\star. \tag{4.4}$$

Moreover, if $\dim(S) < \infty$ or if

$$\sup_{0<r<1} \int_0^1 \left\| K^\star_{re^{2\pi i x}} \right\|_\mu^2 dx < \infty, \tag{4.5}$$

then the f_i can be chosen such that the convergence in (4.3) occurs in the H^2 norm.

Proof. Because every compact subset of $[0,1)$ is metrizable, the L^2 space of every Radon measure, and hence every Borel measure, on $[0,1)$ is separable. So, $L^2(\mu)$ is separable (see example 7.14.13 of [Bog07]). By Lemma 7, there exists a sequence $\{z_i\}_{i=1}^{\infty}$ of points in $\mathbb{D}$ such that $\{K_{z_i}^{\star}\}_{i=1}^{\infty}$ is complete in $\overline{\text{span}}\{K_z^{\star} : z \in \mathbb{D}\}$.

First, we consider the case where $\overline{\text{span}}\{K_z^{\star} : z \in \mathbb{D}\}$ is infinite-dimensional. Without loss of generality, we may assume the sequence $\{z_i\}_{i=0}^{\infty}$ is such that $K_{z_0}^{\star} \neq 0$ and for each $j \geq 1$, $K_{z_j}^{\star} \notin \text{span}\{K_{z_i}^{\star} : i = 0, 1, \ldots j-1\}$. Otherwise, we may define a subsequence $\{z_{i_j}\}_{j=0}^{\infty}$ by letting i_0 be the least nonnegative integer i such that $K_{z_i} \neq 0$, and then for $k \geq 1$, recursively defining i_k to be the least integer greater than i_{k-1} such that $K_{z_{i_k}}^{\star} \notin \text{span}\{K_{z_{i_j}}^{\star} : j = 0, 1, \ldots, k-1\}$. $\{z_{i_j}\}_{j=0}^{\infty}$ will then be the sequence we want.

Set

$$s_0 = \frac{K_{z_0}^{\star}}{\|K_{z_0}^{\star}\|_{\mu}},$$

and for $i \geq 1$, define

$$s_i = \frac{K_{z_i}^{\star} - \sum_{j=0}^{i-1} \langle K_{z_i}^{\star}, s_j \rangle s_j}{\|K_{z_i}^{\star} - \sum_{j=0}^{i-1} \langle K_{z_i}^{\star}, s_j \rangle s_j\|_{\mu}}.$$

Then $\{s_i\}_{i=0}^{\infty}$ is an orthonormal basis of $\overline{\text{span}}\{K_z^{\star} : z \in \mathbb{D}\}$ constructed by the Gram-Schmidt process.

Now, define functions $f_i : \mathbb{D} \to \mathbb{D}$ by

$$f_i(z) := \langle s_i, K_z^{\star} \rangle_{\mu}. \tag{4.6}$$

We then have

$$\sum_{i=0}^{\infty} \overline{f_i(z)} s_i = K_z^{\star}. \tag{4.7}$$

Observe that for fixed $z \in \mathbb{D}$,

$$\begin{aligned} K_z(w) &= \langle K_z^{\star}, K_w^{\star} \rangle_{\mu} \\ &= \left\langle \sum_{i=0}^{\infty} \overline{f_i(z)} s_i, \sum_{j=0}^{\infty} \overline{f_j(w)} s_j \right\rangle_{\mu} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \overline{f_i(z)} f_j(w) \langle s_i, s_j \rangle_{\mu} \\ &= \sum_{i=0}^{\infty} \overline{f_i(z)} f_i(w) \end{aligned} \tag{4.8}$$

pointwise for all $w \in \mathbb{D}$.

Now, by construction,

$$f_i(z_0) = \begin{cases} \|K^\star_{z_0}\|_\mu & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Hence,

$$K_{z_0}(w) = \overline{f_0(z_0)} f_0(w) = \|K^\star_{z_0}\|_\mu f_0(w).$$

Since $\|K^\star_{z_0}\|_\mu \neq 0$, it follows that

$$f_0(w) = \frac{K_{z_0}(w)}{\|K^\star_{z_0}\|_\mu},$$

and hence that $f_0(w) \in H^2$ and $f_0^\star = \frac{K^\star_{z_0}}{\|K^\star_{z_0}\|_\mu} = s_0$.

Now, assume that for some $j \in \mathbb{N}_0$, $f_i(w) \in H^2$ with $f_i^\star = s_i$ for all $i \leq j$. By construction, $K^\star_{z_{j+1}}$ is in the span of $s_0, s_1, \ldots, s_{j+1}$. Thus by (4.6), $f_i(z_{j+1}) = 0$ for $i > j+1$. Hence, by (4.7) and (4.8), we may write

$$K^\star_{z_{j+1}} = \sum_{i=0}^{j+1} \overline{f_i(z_{j+1})} s_i$$

and

$$K_{z_{j+1}}(w) = \sum_{i=0}^{j+1} \overline{f_i(z_{j+1})} f_i(w),$$

where by the assumption that $K^\star_{z_{j+1}} \notin \operatorname{span}\{K_{z_i} : i = 0, 1, \ldots, j\}$, we have $\overline{f_{j+1}(z_{j+1})} = \langle K^\star_{z_{j+1}}, s_{j+1}\rangle_\mu \neq 0$. Then we may write

$$f_{j+1}(w) = \frac{K_{z_{j+1}}(w) - \sum_{i=0}^{j} \overline{f_i(z_{j+1})} f_i(w)}{\overline{f_{j+1}(z_{j+1})}}.$$

Since all of the terms on the right are in H^2, it follows that $f_{j+1}(w) \in H^2$. By finite linearity of $L^2(\mu)$ boundaries, we have that

$$\begin{aligned} f^\star_{j+1} &= \frac{K^\star_{z_{j+1}} - \sum_{i=0}^{j} \overline{f_i(z_{j+1})} s_i}{\overline{f_{j+1}(z_{j+1})}} \\ &= \frac{\overline{f_{j+1}(z_{j+1})} s_{j+1}}{\overline{f_{j+1}(z_{j+1})}} \\ &= s_{j+1}. \end{aligned}$$

It then follows by induction that for all $i \in \mathbb{N}_0$, $f_i(w) \in H^2$ and $f_i^\star = s_i$. Therefore,

$$K^\star_z = \sum_{i=0}^{\infty} \langle K^\star_z, s_i\rangle s_i = \sum_{i=0}^{\infty} \overline{f_i(z)} f_i^\star.$$

Suppose condition (4.5) holds. Let $C := \sup_{0<r<1} \int_0^1 \left\| K^\star_{re^{2\pi i x}} \right\|_\mu^2 \, dx < \infty$, and fix $z \in \mathbb{D}$. We compute:

$$\begin{aligned}
&\left\| \sum_{j=N}^{M} \overline{f_j(z)} f_j(\cdot) \right\|_{H^2} \\
&= \sup_{0<r<1} \sqrt{\int_0^1 \left| \sum_{j=N}^{M} \overline{f_j(z)} f_j(re^{2\pi i x}) \right|^2 dx} \\
&= \sup_{0<r<1} \sqrt{\int_0^1 \left| \sum_{j=N}^{M} \langle K^\star_z, s_j \rangle_\mu \langle s_j, K^\star_{re^{2\pi i x}} \rangle_\mu \right|^2 dx} \\
&\le \sup_{0<r<1} \sqrt{\int_0^1 \left(\sum_{j=N}^{M} |\langle K^\star_z, s_j \rangle_\mu| \cdot |\langle s_j, K^\star_{re^{2\pi i x}} \rangle_\mu| \right)^2 dx} \\
&\le \sup_{0<r<1} \sqrt{\int_0^1 \left(\sum_{j=N}^{M} |\langle K^\star_z, s_j \rangle_\mu|^2 \right) \cdot \left(\sum_{j=N}^{M} |\langle s_j, K^\star_{re^{2\pi i x}} \rangle_\mu|^2 \right) dx} \\
&\le \sqrt{\sum_{j=N}^{\infty} |\langle K^\star_z, s_j \rangle_\mu|^2} \sup_{0<r<1} \sqrt{\int_0^1 \sum_{j=0}^{\infty} |\langle s_j, K^\star_{re^{2\pi i x}} \rangle_\mu|^2 \, dx} \\
&= \sqrt{\sum_{j=N}^{\infty} |\langle K^\star_z, s_j \rangle_\mu|^2} \sup_{0<r<1} \sqrt{\int_0^1 \left\| K^\star_{re^{2\pi i x}} \right\|_\mu^2 dx} \\
&= \sqrt{C \sum_{j=N}^{\infty} |\langle K^\star_z, s_j \rangle_\mu|^2}.
\end{aligned}$$

Since the sum inside the square root goes to 0 as $N \to \infty$, we have that the partial sums of $\sum_{i=0}^{\infty} \overline{f_i(z)} f_i(w)$ are Cauchy in the H^2 norm, which shows that the convergence in (4.3) occurs in the H^2 norm. Thus the theorem is proved for the infinite-dimensional case.

In the k-dimensional case, where $k < \infty$, it is obvious that there exists a set $\{K^\star_{z_0}, K^\star_{z_1}, \ldots, K^\star_{z_{k-1}}\}$ complete in $\overline{\operatorname{span}}\{K^\star_z : z \in \mathbb{D}\}$. Proceeding in exactly the same manner as before, we obtain an orthonormal basis $\{s_0, \ldots, s_{k-1}\}$ and functions $f_0, \ldots, f_{k-1}$, which by the same arguments satisfy $f_i \in H^2$ and $f_i^\star = s_i$. Taking $f_i(w) \equiv 0$ for $i \ge k$ then proves the theorem. In the finite-dimensional case, the convergence in (4.3) clearly is in the H^2 norm rather than just pointwise, because the sum is finite. This completes the proof. □

The condition (4.5) is sufficient for the existence of f_i that can be selected so that the convergence in the decomposition is in the H^2 norm. It is unknown whether such f_i can always be found in the absence of this condition.

4.4 Guarantor of Hardy Space Internality

Though all the constituent functions of a positive matrix may lie within the Hardy space, the RKHS $\mathbb{K}$ for which the positive matrix is the kernel could conceivably stray outside the Hardy space if the norm on $\mathbb{K}$ is not equivalent to that of H^2. Condition (4.5) is, more generally speaking, the one that assures us that a kernel does not sprout beyond its Hardy space garden, as the following result shows:

Corollary 8. *Suppose $K_z(w) \in H^2$ for each fixed $z \in \mathbb{D}$ and reproduces itself with respect to $L^2(\mu)$-boundary. Suppose further that*

$$\sup_{0<r<1} \int_0^1 \|K^\star_{re^{2\pi ix}}\|_\mu^2 \, dx := C < \infty.$$

Then the RKHS $\mathbb{K}$ for which K_z is the reproducing kernel is a subset of H^2, and for any $f \in \mathbb{K}$, we have

$$\|f\|_{H^2} \leq \sqrt{C}\|f\|_{\mathbb{K}}.$$

Proof. Let $\{f_i\}_{i=0}^\infty$ be a sequence of functions in H^2 as in Lemma 8. Let $h \in \operatorname{span}\{K_z : z \in \mathbb{D}\}$. For some scalars $\alpha_0, \ldots, \alpha_n \in \mathbb{C}$ and $z_0, \ldots, z_N \in \mathbb{D}$, we have

$$\begin{aligned} h &= \sum_{j=0}^N \alpha_j K_{z_j}(w) \\ &= \sum_{j=0}^N \alpha_j \sum_{i=0}^\infty \overline{f_i(z_j)} f_i(w) \\ &= \sum_{i=0}^\infty \left(\sum_{j=0}^N \alpha_j \overline{f_i(z_j)} \right) f_i(w), \end{aligned}$$

the convergence occurring pointwise (and in fact in the H^2 norm). We therefore have

$$\begin{aligned} \|h\|_{H^2} &= \left\| \sum_{i=0}^\infty \left(\sum_{j=0}^N \alpha_j \overline{f_i(z_j)} \right) f_i(w) \right\|_{H^2} \\ &= \sup_{0<r<1} \sqrt{ \int_0^1 \left| \sum_{i=0}^\infty \left(\sum_{j=0}^N \alpha_j \overline{f_i(z_j)} \right) f_i(re^{2\pi ix}) \right|^2 dx } \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{0<r<1} \sqrt{\int_0^1 \left(\sum_{i=0}^{\infty} \left| \sum_{j=0}^{N} \alpha_j \overline{f_i(z_j)} \right| |f_i(re^{2\pi ix})| \right)^2 dx} \\
&\leq \sup_{0<r<1} \sqrt{\int_0^1 \left(\sum_{i=0}^{\infty} \left| \sum_{j=0}^{N} \alpha_j \overline{f_i(z_j)} \right|^2 \right) \left(\sum_{i=0}^{\infty} |f_i(re^{2\pi ix})|^2 \right) dx} \\
&= \sqrt{\sum_{i=0}^{\infty} \left| \sum_{j=0}^{N} \alpha_j \overline{f_i(z_j)} \right|^2} \sup_{0<r<1} \sqrt{\int_0^1 \sum_{i=0}^{\infty} |\langle s_i, K^{\star}_{re^{2\pi ix}} \rangle_\mu|^2 \, dx} \\
&= \sqrt{\sum_{i=0}^{\infty} \left| \sum_{j=0}^{N} \alpha_j \overline{f_i(z_j)} \right|^2} \sup_{0<r<1} \sqrt{\int_0^1 \|K^{\star}_{re^{2\pi ix}}\|_\mu^2 \, dx} \\
&= \sqrt{C \sum_{i=0}^{\infty} \left| \sum_{j=0}^{N} \alpha_j \overline{f_i(z_j)} \right|^2} \\
&= \sqrt{C} \left\| \sum_{i=0}^{\infty} \left(\sum_{j=0}^{N} \alpha_j \overline{f_i(z_j)} \right) f_i^{\star} \right\|_\mu \\
&= \sqrt{C} \left\| \sum_{j=0}^{N} \alpha_j \sum_{i=0}^{\infty} \overline{f_i(z_j)} f_i^{\star} \right\|_\mu \\
&= \sqrt{C} \left\| \sum_{j=0}^{N} \alpha_j K^{\star}_{z_j} \right\|_\mu \\
&= \sqrt{C} \, \|h^{\star}\|_\mu .
\end{aligned}
$$

Let $\{\varphi_n\}_{n=0}^{\infty} \subset \text{span}\{K_z : z \in \mathbb{D}\} \subseteq \mathbb{K}$ be Cauchy with respect to the $\mathbb{K}$-norm. Since the $\mathbb{K}$-norm and the $L^2(\mu)$-boundary norm coincide on the finite linear span of the K_z, by the preceding computations it follows that $\{\varphi_n\}_{n=0}^{\infty}$ is Cauchy, and hence convergent, in H^2. In a reproducing kernel Hilbert space, a convergent sequence converges to its limit function pointwise. Because H^2 is a RKHS, $\{\varphi_n\}_{n=0}^{\infty}$ converges to its H^2 limit pointwise. Because $\mathbb{K}$ is a RHKS, $\{\varphi_n\}_{n=0}^{\infty}$ also converges to its $\mathbb{K}$-norm limit pointwise. Since there can be only one pointwise limit, this shows that the $\mathbb{K}$-norm limit is in H^2. Thus, every function in the closure of $\text{span}\{K_z : z \in \mathbb{D}\}$ with respect to the $\mathbb{K}$-norm is also in H^2. Since $\text{span}\{K_z : z \in \mathbb{D}\}$ is dense in $\mathbb{K}$, its closure with respect to the $\mathbb{K}$-norm is $\mathbb{K}$, which shows that $\mathbb{K} \subseteq H^2$.

If $f \in \mathbb{K}$, then there exists a sequence of functions $\{h_i\}_{i=0}^{\infty}$ in $\text{span}\{K_z : z \in \mathbb{D}\}$ that converges to f in $\mathbb{K}$ and hence also in H^2. Thus, because the $\mathbb{K}$-norm coincides with the $L^2(\mu)$-boundary norm on $\operatorname{span}\{K_z : z \in \mathbb{D}\}$, we have

$$\begin{aligned}\|f\|_{H^2} &= \left\| \lim_{i\to\infty} h_i \right\|_{H^2} \\ &= \lim_{i\to\infty} \|h_i\|_{H^2} \\ &\le \lim_{i\to\infty} \sqrt{C}\, \|h_i^\star\|_\mu \\ &= \lim_{i\to\infty} \sqrt{C}\, \|h_i\|_{\mathbb{K}} \\ &= \sqrt{C} \left\| \lim_{i\to\infty} h_i \right\|_{\mathbb{K}} \\ &= \sqrt{C}\, \|f\|_{\mathbb{K}} .\end{aligned}$$

□

Our decompositions $K_z(w) = \sum_{j=0}^{\infty} \overline{f_j(z)} f_j(w)$ cannot be any more unique than orthonormal sequences are. In fact, for a given member φ of $L^2(\mu)$, there will be many functions in H^2 that have φ as an $L^2(\mu)$-boundary. However, the decomposition is unique if we specify the desired orthonormal boundaries, the convergence requirements then ensuring us that only one sequence of functions in H^2 with those boundaries will work. More concretely, we have the following result:

Lemma 9. *Let μ be a measure on $[0,1)$. Suppose $\{f_i\}_{i=0}^{\infty}$ is a sequence of functions $f_i : \mathbb{D} \to \mathbb{C}$ satisfying the following properties:*

1. *Each f_i possesses an $L^2(\mu)$-boundary $f_i^\star \neq 0$.*
2. *$\{f_i^\star\}_{i=0}^{\infty}$ is orthogonal in $L^2(\mu)$.*
3. *For each $z \in \mathbb{D}$, $\sum_{i=0}^{\infty} \overline{f_i(z)} f_i(w)$ converges pointwise in w.*
4. *$\left(\sum_{i=0}^{\infty} \overline{f_i(z)} f_i(w)\right)^\star = \sum_{i=0}^{\infty} \overline{f_i(z)} f_i^\star$.*

Suppose $\{g_i\}_{i=0}^{\infty}$ is another sequence of functions $g_i : \mathbb{D} \to \mathbb{C}$ satisfying the above properties. If $g_i^\star = f_i^\star$ for all $i \in \mathbb{N}_0$ and $\sum_{i=0}^{\infty} \overline{g_i(z)} g_i(w) = \sum_{i=0}^{\infty} \overline{f_i(z)} f_i(w)$, then $g_i = f_i$ for all $i \in \mathbb{N}_0$.

Proof. For any fixed $z \in \mathbb{D}$, we have

$$\begin{aligned} 0^\star &= \left(\sum_{i=0}^\infty \overline{f_i(z)} f_i(w) - \sum_{i=0}^\infty \overline{g_i(z)} g_i(w) \right)^\star \\ &= \left(\sum_{i=0}^\infty \overline{f_i(z)} f_i(w) \right)^\star - \left(\sum_{i=0}^\infty \overline{g_i(z)} g_i(w) \right)^\star \\ &= \sum_{i=0}^\infty \overline{f_i(z)} f_i^\star - \sum_{j=0}^\infty \overline{g_i(z)} g_i^\star \\ &= \sum_{i=0}^\infty \left(\overline{f_i(z)} - \overline{g_i(z)} \right) f_i^\star. \end{aligned}$$

Since the $f_i^\star$ are orthogonal and nonzero, this implies $f_i(z) = g_i(z)$ for all $z \in \mathbb{D}$. □

4.5 Guarantors of $L^2(\mu)$-Boundary Existence and Norm Fidelity

While condition (4.5) ensures that $K_z(w)$ does not generate a RKHS $\mathbb{K}$ extending outside of H^2, it does not ensure that functions in $\mathbb{K} \setminus \operatorname{span}\{K_z : z \in \mathbb{D}\}$ all possess $L^2(\mu)$-boundary. To find a condition ensuring this, we are motivated by the fact that in $\mathcal{H}(b)$, where b is the inner function corresponding to a singular Borel probability measure μ, it is the case that every function has $L^2(\mu)$-boundary, and $\mathcal{H}(b)$ is a RKHS under $L^2(\mu)$-boundary integration.

We extend ourselves beyond $\mathcal{H}(b)$ by means of the Wold decomposition. Recall from the previous chapter that

$$H^2 = \bigoplus_{n=0}^\infty T_b^n \mathcal{H}(b),$$

so that every $f \in H^2$ is of the form $f = \sum_{n=0}^\infty \phi_n b^n$. Restricting to finite Wold decompositions is our tactic to ensure $K_z(w)$ generates an RKHS $\mathbb{K}$ where every $f \in \mathbb{K}$ has $L^2(\mu)$-boundary. We exhibit the following result:

Theorem 12. *Let μ be a singular Borel probability measure on $[0,1)$ with corresponding inner function b. Let $K_z(w) \in H^2$ for each fixed $z \in \mathbb{D}$, reproduce itself with respect to $L^2(\mu)$-boundary integration, and also satisfy*

$$\sup_{0<r<1} \int_0^1 \left\| K^\star_{re^{2\pi i x}} \right\|_\mu^2 \, dx < \infty.$$

Let $\mathbb{K}$ denote the RKHS for which K_z is the reproducing kernel. Let $\{v_i\}_{i=0}^\infty$ be an orthonormal basis of $\overline{span}\{K_z^\star : z \in \mathbb{D}\}$. Suppose that at least one of the following two conditions holds:

1. $\{v_i\}_{i=0}^{\infty} \subset span\{K_z^\star : z \in \mathbb{D}\}$.

2. *There exists an* $N \in \mathbb{N}_0$ *such that* $\{K_z : z \in \mathbb{D}\} \subseteq \bigoplus_{n=0}^{N} b^n \mathcal{H}(b)$.

Then there exists a unique sequence $\{h_i\}_{i=0}^{\infty}$ *of functions in* H^2 *satisfying the following properties:*

- *Each* h_i *possesses* $L^2(\mu)$ *boundary* $h_i^\star = v_i$.
- $K_z = \sum_{i=0}^{\infty} \overline{h_i(z)} h_i$ *in the* H^2 *norm.*
- $K_z^\star = \sum_{i=0}^{\infty} \overline{h_i(z)} h_i^\star$ *in the* $L^2(\mu)$ *norm.*

Proof. By Lemma 8, there exists a sequence of functions $\{f_i\}_{i=0}^{\infty}$ in H^2 such that $K_z(w) = \sum_{i=0}^{\infty} \overline{f_i(z)} f_i(w)$ in the H^2 norm, $\{f_i^\star\}_{i=0}^{\infty}$ is an orthonormal basis of $\overline{\text{span}}\{K_z^\star : z \in \mathbb{D}\}$, and $K_z^\star = \sum_{i=0}^{\infty} \overline{f_i(z)} f_i^\star$. For each $i \in \mathbb{N}_0$, define

$$h_i(z) := \langle v_i, K_z^\star \rangle_\mu.$$

First, we claim that each $h_i \in H^2$. To see this, first observe that pointwise we have

$$\begin{aligned} h_i(z) &:= \langle v_i, K_z^\star \rangle_\mu \\ &= \left\langle \sum_{j=0}^{\infty} \left\langle v_i, f_j^\star \right\rangle_\mu f_j^\star, K_z^\star \right\rangle_\mu \\ &= \sum_{j=0}^{\infty} \langle v_i, f_j^\star \rangle_\mu \langle f_j^\star, K_z^\star \rangle_\mu \\ &= \sum_{j=0}^{\infty} \langle v_i, f_j^\star \rangle_\mu f_j(z). \end{aligned}$$

Set $C := \sup_{0<r<1} \int_0^1 \|K_{re^{2\pi i x}}^\star\|_\mu^2 \, dx < \infty$. For fixed $0 \leq M < N$, we compute that

$$\begin{aligned} &\left\| \sum_{j=M}^{N} \langle v_i, f_j^\star \rangle_\mu f_j(z) \right\|_{H^2} \\ &= \sup_{0<r<1} \sqrt{\int_0^1 \left| \sum_{j=M}^{N} \overline{f_i(z)} f_i(re^{2\pi i x}) \right|^2 dx} \\ &= \sup_{0<r<1} \sqrt{\int_0^1 \left| \sum_{j=M}^{N} \langle K_z^\star, f_i^\star \rangle_\mu \langle f_i^\star, K_{re^{2\pi i x}}^\star \rangle_\mu \right|^2 dx} \end{aligned}$$

$$\leq \sup_{0<r<1} \sqrt{\int_0^1 \left(\sum_{j=M}^{N} |\langle K_z^\star, f_j^\star \rangle_\mu| \cdot |\langle f_j^\star, K_{re^{2\pi i x}}^\star \rangle_\mu| \right)^2 dx}$$

$$\leq \sup_{0<r<1} \sqrt{\int_0^1 \left(\sum_{j=M}^{N} |\langle K_z^\star, f_j^\star \rangle_\mu|^2 \right) \cdot \left(\sum_{j=M}^{N} |\langle f_j^\star, K_{re^{2\pi i x}}^\star \rangle_\mu|^2 \right) dx}$$

$$\leq \sqrt{\sum_{j=M}^{\infty} |\langle K_z^\star, f_j^\star \rangle_\mu|^2} \sup_{0<r<1} \sqrt{\int_0^1 \sum_{j=M}^{N} |\langle f_j^\star, K_{re^{2\pi i x}}^\star \rangle_\mu|^2 \, dx}$$

$$\leq \sqrt{\sum_{j=M}^{\infty} |\langle K_z^\star, f_j^\star \rangle_\mu|^2} \sup_{0<r<1} \sqrt{\int_0^1 \|K_{re^{2\pi i x}}^\star\|_\mu^2 \, dx}$$

$$= \sqrt{C \sum_{j=M}^{\infty} |\langle K_z^\star, f_j^\star \rangle_\mu|^2}.$$

Since $\sum_{j=0}^{\infty} |\langle K_z^\star, f_j^\star \rangle_\mu|^2 = \|K_z^\star\|_\mu^2 < \infty$, it follows that the previous sum goes to 0 as $M \to \infty$, and hence that the partial sums of $\sum_{j=0}^{\infty} \langle v_i, f_j^\star \rangle_\mu f_j(z)$ are Cauchy, and therefore convergent, in H^2. Since H^2 norm convergence implies pointwise convergence, and since $h_i(z)$ is the pointwise limit of $\sum_{j=0}^{\infty} \langle v_i, f_j^\star \rangle_\mu f_j(z)$, it follows that h_i is the H^2 limit of this sum, and hence $h_i \in H^2$.

We now claim that h_i possesses $L^2(\mu)$-boundary, and that in fact $h_i^\star = v_i$. By previous computation, we have

$$h_i(w) = \sum_{j=0}^{\infty} \langle v_i, f_j^\star \rangle_\mu f_j(w).$$

Suppose first that condition (1) holds. Then for fixed i, we may write

$$v_i = \sum_{n=0}^{M} c_n K_{z_n}^\star$$

for some scalars $c_0, \ldots, c_M$ and $z_0, \ldots, z_M \in \mathbb{D}$. Hence,

$$\begin{aligned} h_i(w) &= \sum_{j=0}^{\infty} \left\langle \sum_{n=0}^{M} c_n K_{z_n}^\star, f_j^\star \right\rangle_\mu f_j(w) \\ &= \sum_{n=0}^{M} c_n \sum_{j=0}^{\infty} \langle K_{z_n}^\star, f_j^\star \rangle_\mu f_j(w) \\ &= \sum_{n=0}^{M} c_n \sum_{j=0}^{\infty} \overline{f_j(z_n)} f_j(w) \\ &= \sum_{n=0}^{M} c_n K_{z_n}(w). \end{aligned}$$

This implies that

$$h_i^\star = \sum_{n=0}^{M} c_n K_{z_n}^\star = v_i.$$

Now suppose on the other hand that condition (2) holds. Since $v_i \in \overline{\operatorname{span}}\{K_z^\star : z \in \mathbb{D}\}$, there exist functions $\{q_{j,i}\}_{j=0}^\infty \subseteq \operatorname{span}\{K_z : z \in \mathbb{D}\}$ with corresponding $L^2(\mu)$ boundaries $\{q_{j,i}^\star\}_{j=0}^\infty \subseteq \operatorname{span}\{K_z^\star : z \in \mathbb{D}\}$ such that

$$\lim_{j\to\infty} q_{j,i}^\star = v_i$$

in the $L^2(\mu)$ norm. The functions $\{q_{j,i}\}_{j=0}^\infty$ are thus Cauchy in $\mathbb{K}$, and so converge in $\mathbb{K}$ and hence pointwise to a function in $\mathbb{K}$. The function to which they converge must be h_i, because on $\operatorname{span}\{K_z : z \in \mathbb{D}\} \subset \mathbb{K}$ the norm of $\mathbb{K}$ is the $L^2(\mu)$-boundary norm, and so

$$\lim_{j\to\infty} q_{j,i}(w) = \lim_{j\to\infty} \langle q_{j,i}^\star, K_w^\star\rangle_\mu = \left\langle \lim_{j\to\infty} q_{j,i}^\star, K_w^\star \right\rangle_\mu = \langle v_i, K_w^\star\rangle_\mu =: h_i(w).$$

Observe that this shows that $h_i \in \mathbb{K}$. By Corollary 8, $\lim_{j\to\infty} q_{j,i} = h_i$ in the H^2 norm as well. Since by assumption $\{q_{j,i}\}_{j=0}^\infty \subseteq \operatorname{span}\{K_z : z \in \mathbb{D}\} \subseteq \bigoplus_{n=0}^N b^n \mathcal{H}(b)$, and the latter space is closed in H^2, we have that

$$h_i \in \bigoplus_{n=0}^{N} b^n \mathcal{H}(b).$$

Thus we may write

$$h_i = \sum_{n=0}^{N} \psi_{n,i} b^n,$$

where each $\psi_n \in \mathcal{H}(b)$, implying that h_i possesses $L^2(\mu)$-boundary

$$h_i^\star = \sum_{n=0}^{N} \psi_{n,i}^\star \in L^2(\mu).$$

Now, for each j, we may write

$$q_{j,i} = \sum_{n=0}^{N} \phi_{n,i}^{(j)} b^n.$$

Since $\lim_{j\to\infty} q_{j,i} = h_i$ in H^2, by orthogonality of the spaces $\{b^n\mathcal{H}(b) : n \in \mathbb{N}_0\}$ in H^2, it follows that for each $n = 0, 1, \ldots, N$, we have

$$\lim_{j\to\infty} \phi_{n,i}^{(j)} = \psi_{n,i}$$

in the H^2 norm. Recall that in each of the spaces $b^n\mathcal{H}(b)$, the H^2 norm and the $L^2(\mu)$-boundary norm are equal. It follows immediately that

$$\lim_{j\to\infty}\left(\phi_{n,i}^{(j)}\right)^\star = \psi_{n,i}^\star$$

in the $L^2(\mu)$ norm. Therefore,

$$\begin{aligned}
\|h_i^\star - v_i\|_\mu &= \|h_i^\star - \lim_{j\to\infty} q_{j,i}^\star\|_\mu \\
&= \lim_{j\to\infty}\|h_i^\star - q_{j,i}^\star\|_\mu \\
&= \lim_{j\to\infty}\left\|\sum_{n=0}^{N}\psi_{n,i}^\star - \sum_{n=0}^{N}\left(\phi_{n,i}^{(j)}\right)^\star\right\|_\mu \\
&\le \lim_{j\to\infty}\sum_{n=0}^{N}\left\|\psi_{n,i}^\star - \left(\phi_{n,i}^{(j)}\right)^\star\right\|_\mu \\
&= 0.
\end{aligned}$$

This shows that $h_i^\star = v_i$.

We have

$$K_z^\star = \sum_{i=0}^{\infty}\langle K_z^\star, v_i\rangle v_i = \sum_{i=0}^{\infty}\overline{h_i(z)}h_i^\star.$$

Thus, it remains only to show that

$$K_z = \sum_{i=0}^{\infty}\overline{h_i(z)}h_i$$

in the H^2 norm. However, this follows immediately from Corollary 8, because

$$\begin{aligned}
\lim_{N\to\infty}\left\|K_z - \sum_{i=0}^{N}\overline{h_i(z)}h_i\right\|_{H^2} &\le \sqrt{C}\lim_{N\to\infty}\left\|K_z^\star - \sum_{i=0}^{N}\overline{h_i(z)}h_i^\star\right\|_\mu \\
&= \sqrt{C}\lim_{N\to\infty}\left\|K_z^\star - \sum_{i=0}^{N}\langle K_z^\star, v_i\rangle v_i\right\|_\mu \\
&= 0.
\end{aligned}$$

Uniqueness of the decomposition follows from Lemma 9. □

Remark 5. By using the technique in the proof showing that $h_i^\star = v_i$, it is easily seen that, under the hypotheses of the theorem along with condition (2), the inner product of $\mathbb{K}$ is given by

$$\langle f, g\rangle_{\mathbb{K}} = \langle f^\star, g^\star\rangle_\mu \tag{4.9}$$

for every $f, g \in \mathbb{K}$. That is to say, the inner product of $\mathbb{K}$ is given by $L^2(\mu)$-boundary integration even for the functions in $\mathbb{K}\backslash\text{span}\{K_z : z \in \mathbb{D}\}$. In the absence of condition (2) this is not *a priori* true, even if every function in $\mathbb{K}$ possesses $L^2(\mu)$-boundary. For this reason, if (4.9) is satisfied for all $f, g \in \mathbb{K}$, we say that the inner product of $\mathbb{K}$ is **universally that of** $L^2(\mu)$-boundary.

4.6 Guarantors of Classical Properties

One of the properties of the classical Hardy space is that for any $f \in H^2$, the norms $\|f(rz)\|_{H^2}$ increase monotonically as $r \to 1^-$. As a result, $\|f(rz)\|_{H^2} \leq \|f\|_{H^2}$ for $0 < r < 1$. We seek to emulate this property in the RKHS's $\mathbb{K}$ coming from $L^2(\mu)$-boundary-reproducing kernels. The following result provides a sufficient condition, and the result that follows it shows that the class of finite sums of Dirac measures has this property.

Lemma 10. *Let μ be a Borel measure on $[0, 1)$. Suppose $K_z \in H^2$ reproduces itself with respect to $L^2(\mu)$-boundary integration, and let $\mathbb{K}$ be the RKHS for which K_z is the reproducing kernel. Set*

$$C := \sup_{0<r<1} \int_0^1 \left\|K^\star_{re^{2\pi ix}}\right\|_\mu^2 \, d\mu(x).$$

Then $C \geq 1$, and every $f \in \mathbb{K}$ satisfies

$$\left\|f(re^{2\pi ix})\right\|_\mu \leq \sqrt{C}\, \|f\|_{\mathbb{K}}$$

for all $0 < r < 1$. Moreover, if the inner product of $\mathbb{K}$ is universally that of $L^2(\mu)$-boundary integration and $C = 1$, then we have

$$\|f^\star\|_\mu = \sup_{0<r<1} \sqrt{\int_0^1 |f(re^{2\pi ix})|^2 \, d\mu(x)}$$

for every $f \in \mathbb{K}$.

Proof. Let $f \in \mathbb{K}$. For $0 < r < 1$, we have

$$\|f(re^{2\pi ix})\|_\mu^2 = \int_0^1 |f(re^{2\pi ix})|^2 \, d\mu(x)$$

$$= \int_0^1 |\langle f, K_{re^{2\pi ix}}\rangle_{\mathbb{K}}|^2 \, d\mu(x)$$
$$\leq \int_0^1 \|f\|_{\mathbb{K}}^2 \|K_{re^{2\pi ix}}\|_{\mathbb{K}}^2 \, d\mu(x)$$
$$= \|f\|_{\mathbb{K}}^2 \int_0^1 \|K^\star_{re^{2\pi ix}}\|_\mu^2 \, d\mu(x)$$
$$\leq C\|f\|_{\mathbb{K}}^2,$$

and hence

$$\|f(re^{2\pi ix})\|_\mu \leq \sqrt{C}\|f\|_{\mathbb{K}}.$$

Select a $z \in \mathbb{D}$ such that $K_z \neq 0$. On $\operatorname{span}\{K_z : z \in \mathbb{D}\}$ the norm of $\mathbb{K}$ is the $L^2(\mu)$-boundary norm, so that the above implies

$$\|K_z(re^{2\pi ix})\|_\mu \leq \sqrt{C}\|K_z^\star\|_\mu.$$

Hence,

$$\|K_z^\star\|_\mu = \lim_{r\to 1^-} \|K_z(re^{2\pi ix})\|_\mu \leq \sqrt{C}\|K_z^\star\|_\mu,$$

whence $C \geq 1$.

Now suppose $C = 1$ and that the inner product of $\mathbb{K}$ is universally that of $L^2(\mu)$-boundary integration. For any $f \in \mathbb{K}$, the above shows that

$$\|f(re^{2\pi ix})\|_\mu \leq \|f^\star\|_\mu.$$

Hence,

$$\sup_{0<r<1} \|f(re^{2\pi ix})\|_\mu \leq \|f^\star\|_\mu.$$

Since

$$\lim_{r\to 1^-} \|f(re^{2\pi ix})\|_\mu = \|f^\star\|_\mu,$$

the reverse inequality also follows, and we obtain

$$\|f^\star\|_\mu = \sup_{0<r<1} \sqrt{\int_0^1 |f(re^{2\pi ix})|^2 \, d\mu(x)}.$$

□

Lemma 11. *If μ is a finite sum of weighted Dirac measures with associated inner function b, then*

$$\sup_{0<r<1} \int_0^1 \left\| (k^b_{re^{2\pi ix}})^\star \right\|^2_\mu d\mu(x) < \infty. \tag{4.10}$$

Proof. We have that

$$\begin{aligned}
\left\| (k^b_{re^{2\pi ix}})^\star \right\|^2_\mu &= \left\| k^b_{re^{2\pi ix}} \right\|^2_{H^2} \\
&= \left\langle k^b_{re^{2\pi ix}}, k^b_{re^{2\pi ix}} \right\rangle_{H^2} \\
&= k^b_{re^{2\pi ix}}(re^{2\pi ix}) \\
&= \frac{1 - \overline{b(re^{2\pi ix})} b(re^{2\pi ix})}{1 - \overline{re^{2\pi ix}} re^{2\pi ix}} \\
&= \frac{1 - \left| b(re^{2\pi ix}) \right|^2}{1 - r^2}.
\end{aligned}$$

Thus condition (4.10) is equivalent to the existence of a constant C such that

$$\int_0^1 1 - \left| b(re^{2\pi ix}) \right|^2 d\mu(x) \leq C(1 - r^2).$$

It is easily verified that for $0 < r < 1$,

$$\frac{2(r^2 + 1 - 2r\cos(2\pi r))}{(1 - r)(1 - r^2)} \leq 1 + 4\pi^2.$$

Since μ is a finite sum of weighted Dirac measures, there exists among those weights one that is smallest. Call it w. So for $x \in \text{supp}(\mu)$, we have $\mu(\{x\}) \geq w$. Let $M := \frac{1+4\pi^2}{w}$. Then for $x \in \text{supp}(\mu)$ and $\frac{1}{2} < r < 1$,

$$\begin{aligned}
\frac{1}{M} \frac{2(r^2 + 1 - 2r\cos(2\pi r))}{(1 - r)(1 - r^2)} &\leq \frac{1 + 4\pi^2}{M} \\
&= w \\
&\leq \mu(\{x\}) \\
&\leq \mu((x - (1 - r), x + (1 - r))),
\end{aligned}$$

which implies

$$\frac{2}{M(1 - r)} \leq \frac{1 - r^2}{r^2 + 1 - 2r\cos(2\pi r)} \mu((x - (1 - r), x + (1 - r)))$$

$$\begin{aligned}
&\leq \int_0^1 \frac{1-|re^{2\pi i x}|^2}{|re^{2\pi i x}-e^{2\pi i t}|^2}\,d\mu(t) \text{ [by inequality (A.1) from the appendix]}\\
&= \operatorname{Re}\left(\frac{1+b(re^{2\pi i x})}{1-b(re^{2\pi i x})}\right)\\
&= \frac{1-\left|b(re^{2\pi i x})\right|^2}{|1-b(re^{2\pi i x})|^2}\\
&\leq \frac{1-\left|b(re^{2\pi i x})\right|^2}{(1-|b(re^{2\pi i x})|)^2}\\
&= \frac{1+\left|b(re^{2\pi i x})\right|}{1-|b(re^{2\pi i x})|}\\
&\leq \frac{2}{1-|b(re^{2\pi i x})|}
\end{aligned}$$

for x in the support of μ. Thus,

$$1-\left|b(re^{2\pi i x})\right| \leq M(1-r).$$

It follows that

$$\begin{aligned}
1-\left|b(re^{2\pi i x})\right|^2 &= (1+\left|b(re^{2\pi i x})\right|)(1-\left|b(re^{2\pi i x})\right|)\\
&\leq (1+r)M(1-r)\\
&= M(1-r^2)
\end{aligned}$$

for x in the support of μ. Consequently,

$$\int_0^1 1-\left|b(re^{2\pi i x})\right|^2\,d\mu(x) \leq \|\mu\| M(1-r^2).$$

Thus setting $C=\|\mu\|M$, the theorem is proved. □

APPENDIX: PROOFS OF SELECTED THEOREMS

Radial Limits

Theorem 13. *Let μ be a positive singular Borel measure on $[0,1)$, and b the inner function corresponding to μ. Then for μ-almost-every x, $\lim_{r\to 1^-} b(re^{2\pi i x}) = 1$.*

Proof. Recall that since μ is a singular measure, for μ-almost-every $x \in [0,1)$, we have

$$\lim_{h\to 0^+} \frac{\mu((x-h,x+h))}{2h} = \infty.$$

(For a proof, see Rudin's *Real and Complex Analysis*, Theorem 7.15 page 143. The above limit is the symmetric derivative of μ at x.)

Fix such an x that satisfies the above property. For any $\frac{1}{2} < r < 1$, we have

$$\begin{aligned}
\int_0^1 \frac{1-|re^{2\pi i x}|^2}{|re^{2\pi i x} - e^{2\pi i t}|^2}\, d\mu(t) &= (1-r^2)\int_0^1 \frac{1}{|re^{2\pi i x} - e^{2\pi i t}|^2}\, d\mu(t) \\
&\geq (1-r^2)\int_{(x-(1-r))}^{x+(1-r)} \frac{1}{|re^{2\pi i x} - e^{2\pi i t}|^2}\, d\mu(t) \\
&\geq (1-r^2)\int_{(x-(1-r))}^{x+(1-r)} \frac{1}{|re^{2\pi i x} - e^{2\pi i (x+(1-r))}|^2}\, d\mu(t) \\
&= \frac{1-r^2}{|re^{2\pi i x} - e^{2\pi i (x+(1-r))}|^2}\mu((x-(1-r), x+(1-r))) \qquad \text{(A.1)} \\
&= \frac{1-r^2}{|r - e^{2\pi i (1-r)}|^2}\mu((x-(1-r), x+(1-r))) \\
&= \frac{1-r^2}{r^2+1-2r\cos(2\pi r)}\mu((x-(1-r), x+(1-r))) \\
&= \frac{2(1-r)(1-r^2)}{r^2+1-2r\cos(2\pi r)}\frac{\mu((x-(1-r), x+(1-r)))}{2(1-r)}.
\end{aligned}$$

By L'Hôpital's Rule,

$$\begin{aligned}
\lim_{r\to 1^-} \frac{2(1-r)(1-r^2)}{r^2+1-2r\cos(2\pi r)} &= \lim_{r\to 1^-} \frac{6r^2-4r-2}{2r-2\cos(2\pi r)+4\pi r\sin(2\pi r)} \\
&= \lim_{r\to 1^-} \frac{12r-4}{2+8\pi\sin(2\pi r)+8\pi^2 r\cos(2\pi r)}
\end{aligned}$$

$$= \frac{8}{2+8\pi^2}.$$

By assumption,

$$\lim_{r\to 1^-} \frac{\mu((x-(1-r), x+(1-r)))}{2(1-r)} = \infty.$$

Hence,

$$\lim_{r\to 1^-} \int_0^1 \frac{1-|re^{2\pi ix}|^2}{|re^{2\pi ix} - e^{2\pi it}|^2}\, d\mu(t) = \infty.$$

Recall that

$$\mathrm{Re}\left(\frac{1+b(z)}{1-b(z)}\right) = \int_0^1 \frac{1-|z|^2}{|z-\xi|^2}\, d\mu(\xi).$$

So, we must have

$$\lim_{r\to 1^-} \mathrm{Re}\left(\frac{1+b(re^{2\pi ix})}{1-b(re^{2\pi ix})}\right) = \lim_{r\to 1^-} \frac{1-|b(re^{2\pi ix})|^2}{|1-b(re^{2\pi ix})|^2} = \infty.$$

Since the numerator in the above limit is bounded, it follows that $\lim_{r\to 1^-} b(re^{2\pi ix}) = 1$. Since the above is true for μ-almost-every x, the proof is complete. □

Boundary Sameness

Proposition 5. *Let b be an inner function with corresponding singular measure μ. Suppose $f \in H^2$ has $L^2(\mu)$-boundary $f^\star$. Then $T_b f = bf$ has $L^2(\mu)$-boundary $f^\star$ as well.*

Proof. Let $f \in H^2$ have $L^2(\mu)$ boundary $f^\star$. For brevity of notation, for $0 < r < 1$, let $f_r(x) := f(re^{2\pi ix})$ and $b_r(x) = b(re^{2\pi ix})$. Observe that

$$\begin{aligned}
|f_r - f_r b_r|^2 &= |f_r(1-b_r)|^2 \\
&= |f_r(1-b_r) - f^\star(1-b_r) + f^\star(1-b_r)|^2 \\
&= |(f_r - f^\star)(1-b_r) + f^\star(1-b_r)|^2 \\
&\le (|(f_r - f^\star)(1-b_r)| + |f^\star(1-b_r)|)^2 \\
&= |(f_r - f^\star)(1-b_r)|^2 + 2|(f_r - f^\star)(1-b_r)| \cdot |f^\star(1-b_r)| + |f^\star(1-b_r)|^2.
\end{aligned}$$

By assumption,

$$\lim_{r\to 1^-} \int_0^1 |(f_r - f^\star)(1-b_r)|^2\, d\mu(x) \le \lim_{r\to 1^-} 4 \int_0^1 |f_r - f^\star|^2\, d\mu(x) = 0.$$

For all $0 < r < 1$, $|f^\star(1-b_r)|^2 \le 4|f^\star|^2$, and

$$\int_0^1 4|f^\star|^2\, d\mu(x) = 4\|f^\star\|_\mu^2 < \infty.$$

Thus, by the Dominated Convergence Theorem,

$$\lim_{r\to 1^-} \int_0^1 |f^\star(1-b_r)|^2\, d\mu(x) = \int_0^1 \lim_{r\to 1^-} |f^\star(1-b_r)|^2\, d\mu(x) = \int_0^1 |f^\star \cdot 0|^2\, d\mu(x) = 0,$$

where we recalled that $b_r(x) \to 1$ for μ-almost-every $x \in [0,1)$. Finally, by the Cauchy-Schwarz Inequality, we have that

$$\begin{aligned}
&\lim_{r\to 1^-} \int_0^1 2|(f_r - f^\star)(1-b_r)| \cdot |f^\star(1-b_r)|\, d\mu(x) \\
&\le \lim_{r\to 1^-} 2\sqrt{\int_0^1 |(f_r - f^\star)(1-b_r)|^2\, d\mu(x)}\sqrt{\int_0^1 |f^\star(1-b_r)|^2\, d\mu(x)} \\
&= 0
\end{aligned}$$

by previous computation. We thus have that

$$\lim_{r\to 1^-} \|f_r - f_r b_r\|_\mu^2 = \lim_{r\to 1^-} \int_0^1 |f_r - f_r b_r|^2\, d\mu(x) = 0.$$

To conclude the proof, we note that

$$\lim_{r\to 1^-} \|f^\star - f_r b_r\|_\mu \le \lim_{r\to 1^-} \left(\|f^\star - f_r\|_\mu + \|f_r - f_r b_r\|_\mu\right) = 0.$$

□

Corollary 9. *The function b has $L^2(\mu)$-boundary 1.*

Proof. $b = 1 \cdot b$, and 1 has $L^2(\mu)$-boundary 1. □

We now generalize the previous theorem.

Proposition 6. *For any $\varphi \in H^\infty$ possessing an $L^2(\mu)$ boundary satisfying $\lim_{r\to 1^-} \varphi(re^{2\pi i x}) = \varphi^\star(x)$ for μ-almost every x, we have that $(f\varphi)^\star = f^\star\varphi^\star$.*

Proof. Let $f \in H^2$ have $L^2(\mu)$ boundary $f^\star$. For brevity of notation, for $0 < r < 1$, let $f_r(x) := f(re^{2\pi i x})$ and $\varphi_r(x) = \varphi(re^{2\pi i x})$. Observe that

$$|f_r\varphi^\star - f_r\varphi_r|^2 = |f_r(\varphi^\star - \varphi_r)|^2$$

$$= |f_r(\varphi^\star - \varphi_r) - f^\star(\varphi^\star - \varphi_r) + f^\star(\varphi^\star - \varphi_r)|^2$$
$$= |(f_r - f^\star)(\varphi^\star - \varphi_r) + f^\star(\varphi^\star - \varphi_r)|^2$$
$$\leq (|(f_r - f^\star)(\varphi^\star - \varphi_r)| + |f^\star(\varphi^\star - \varphi_r)|)^2$$
$$= |(f_r - f^\star)(\varphi^\star - \varphi_r)|^2 + 2|(f_r - f^\star)(\varphi^\star - \varphi_r)| \cdot |f^\star(\varphi^\star - \varphi_r)| + |f^\star(\varphi^\star - \varphi_r)|^2.$$

Now, φ is bounded on $\mathbb{D}$ by $\|\varphi\|_{H^\infty}$. This implies that $|\varphi^\star(x)| = |\lim_{r\to 1^-} \varphi(re^{2\pi i x})| \leq \|\varphi\|_{H^\infty}$ for μ-almost-every x. Hence,

$$\lim_{r\to 1^-} \int_0^1 |(f_r - f^\star)(\varphi^\star - \varphi_r)|^2 \, d\mu(x) \leq \lim_{r\to 1^-} 4\|\varphi\|_{H^\infty}^2 \int_0^1 |f_r - f^\star|^2 \, d\mu(x) = 0.$$

For all $0 < r < 1$, $|f^\star(\varphi^\star - \varphi_r)|^2 \leq 4\|\varphi\|_{H^\infty}^2 |f^\star|^2$, and

$$\int_0^1 4\|\varphi\|_{H^\infty}^2 |f^\star|^2 \, d\mu(x) = 4\|\varphi\|_{H^\infty}^2 \|f^\star\|_\mu^2 < \infty.$$

Thus, by the Dominated Convergence Theorem,

$$\lim_{r\to 1^-} \int_0^1 |f^\star(\varphi^\star - \varphi_r)|^2 \, d\mu(x) = \int_0^1 \lim_{r\to 1^-} |f^\star(\varphi^\star - \varphi_r)|^2 \, d\mu(x) = \int_0^1 |f^\star \cdot 0|^2 \, d\mu(x) = 0,$$

where we used the assumption that $\varphi_r(x) \to \varphi^\star(x)$ for μ-almost-every $x \in [0, 1)$. Finally, by the Cauchy-Schwarz Inequality, we have that

$$\lim_{r\to 1^-} \int_0^1 2|(f_r - f^\star)(1 - \varphi_r)| \cdot |f^\star(1 - \varphi_r)| \, d\mu(x)$$
$$\leq \lim_{r\to 1^-} 2\sqrt{\int_0^1 |(f_r - f^\star)(\varphi^\star - \varphi_r)|^2 \, d\mu(x)} \sqrt{\int_0^1 |f^\star(\varphi^\star - \varphi_r)|^2 \, d\mu(x)}$$
$$= 0$$

by previous computation. We thus have that

$$\lim_{r\to 1^-} \|f_r\varphi^\star - f_r\varphi_r\|_\mu^2 = \lim_{r\to 1^-} \int_0^1 |f_r\varphi^\star - f_r\varphi_r|^2 \, d\mu(x) = 0.$$

To conclude the proof, we note that

$$\lim_{r\to 1^-} \|f^\star\varphi^\star - f_r\varphi_r\|_\mu \leq \lim_{r\to 1^-} (\|f^\star\varphi^\star - f_r\varphi^\star\|_\mu + \|f_r\varphi^\star - f_r\varphi_r\|_\mu)$$
$$\leq \lim_{r\to 1^-} (\|\varphi\|_{H^\infty}\|f^\star - f_r\|_\mu + \|f_r\varphi^\star - f_r\varphi_r\|_\mu)$$
$$= 0.$$

□

Cesàro and Abel Summability in General Normed Linear Spaces

Theorem: Let X be a normed linear space and $\{c_n\}_{n=0}^{\infty}$ a sequence in X such that $S_N := \sum_{n=0}^{N} c_n$ converges to s in norm as $N \to \infty$. Then the Cesàro mean $\frac{1}{N}\sum_{n=0}^{N-1} S_n$ converges in norm to s as $N \to \infty$.

Proof: We will prove even more. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence in X such that $a_n \to L \in X$ in norm. For each $N \in \mathbb{N}$, define $\sigma_N = \frac{1}{N}\sum_{n=0}^{N-1} a_n$. Let $\epsilon > 0$. Let N' be so large that for $n \geq N'$, $\|a_n - L\| < \epsilon$. Then for $N > N'$, we have

$$\begin{aligned}
\|\sigma_N - L\| &= \left\| \left(\frac{1}{N} \sum_{n=0}^{N-1} a_n \right) - L \right\| \\
&= \left\| \frac{1}{N} \sum_{n=0}^{N-1} (a_n - L) \right\| \\
&= \frac{1}{N} \left\| \sum_{n=0}^{N-1} (a_n - L) \right\| \\
&\leq \frac{1}{N} \left(\left\| \sum_{n=0}^{N'-1} (a_n - L) \right\| + \sum_{n=N'}^{N-1} \|(a_n - L)\| \right) \\
&\leq \frac{1}{N} \left(\left\| \sum_{n=0}^{N'-1} (a_n - L) \right\| + (N - N')\epsilon \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\limsup_{N\to\infty} \|\sigma_N - L\| &\leq \limsup_{N\to\infty} \frac{1}{N} \left(\left\| \sum_{n=0}^{N'-1} (a_n - L) \right\| + (N - N')\epsilon \right) \\
&\leq \limsup_{N\to\infty} \frac{1}{N} \left\| \sum_{n=0}^{N'-1} (a_n - L) \right\| + \limsup_{N\to\infty} \frac{N - N'}{N}\epsilon \\
&= 0 + \epsilon \\
&= \epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$, we have that $0 \leq \limsup_{N\to\infty} \|\sigma_N - L\| \leq 0$, which implies $\lim_{n\to\infty} \|\sigma_N - L\| = 0$. By applying this result to $a_n = S_n$ and $L = s$, we obtain the desired result. $\square$

Theorem: Let X be a normed linear space and $\{c_n\}_{n=0}^{\infty}$ a sequence in X. If $\sum_{n=0}^{\infty} r^n c_n$ converges in norm for all $0 < r \leq 1$, then

$$\lim_{r\to 1^-} \left\| \left(\sum_{n=0}^{\infty} c_n \right) - \left(\sum_{n=0}^{\infty} r^n c_n \right) \right\| = 0.$$

Proof: It is obvious that the series

$$c_0 + rc_1 + r^2c_2 + r^3c_3 + \cdots$$

converges in norm for all $0 < r \leq 1$ and converges in norm to $s \in X$ when $r = 1$ if and only if the series

$$(c_0 - s) + rc_1 + r^2c_2 + r^3c_3 + \cdots$$

converges in norm for all $0 < r \leq 1$ and converges in norm to 0 when $r = 1$. Thus, we may assume without loss of generality that $\sum_{n=0}^{\infty} c_n = 0$. We have that for all $0 < r \leq 1$,

$$\begin{aligned}
\sum_{n=0}^{N} r^n c_n =& c_0 + rc_1 + r^2c_2 + \ldots + r^N c_N \\
=& S_0 \\
& + rS_1 - rc_0 \\
& + r^2S_2 - r^2c_1 - r^2c_0 \\
& + r^3S_3 - r^3c_2 - r^3c_1 - r^3c_0 \\
& \vdots \\
& + r^NS_N - r^Nc_{N-1} - \ldots - r^Nc_0 \\
=& S_0 \\
& + rS_1 - rS_0 \\
& + r^2S_2 - r^2S_1 \\
& + r^3S_3 - r^3S_2 \\
& \vdots \\
& + r^NS_N - r^NS_{N-1} \\
& = \sum_{n=0}^{N} r^n S_n - \sum_{n=0}^{N-1} r^{n+1} S_n \\
& = \sum_{n=0}^{N} r^n S_n - r\sum_{n=0}^{N-1} r^n S_n \\
& = \sum_{n=0}^{N} r^n S_n - r\sum_{n=0}^{N} r^n S_n + r^{N+1} S_N
\end{aligned}$$

$$= (1-r)\sum_{n=0}^{N} r^n S_n + r^{N+1} S_N.$$

Let $\epsilon > 0$. Let N' be so large that $\|S_n\| \leq \epsilon$ for all $n \geq N'$. Then for $N > N'$, we have

$$\begin{aligned}
\left\|\sum_{n=0}^{N} r^n c_n\right\| &= \left\|(1-r)\sum_{n=0}^{N} r^n S_n + r^{N+1} S_N\right\| \\
&\leq (1-r)\left\|\sum_{n=0}^{N'-1} r^n S_n\right\| + (1-r)\sum_{n=N'}^{N} r^n \|S_n\| + r^{N+1}\|S_N\| \\
&\leq (1-r)\left\|\sum_{n=0}^{N'-1} r^n S_n\right\| + (1-r)\epsilon \sum_{n=N'}^{N+1} r^n \\
&\leq (1-r)\left\|\sum_{n=0}^{N'-1} r^n S_n\right\| + \frac{1-r}{1-r}\epsilon \\
&= (1-r)\left\|\sum_{n=0}^{N'-1} r^n S_n\right\| + \epsilon.
\end{aligned}$$

Letting $N \to \infty$, we thus obtain

$$\left\|\sum_{n=0}^{\infty} r^n c_n\right\| \leq (1-r)\left\|\sum_{n=0}^{N'-1} r^n S_n\right\| + \epsilon.$$

Hence,

$$\lim_{r\to 1^-}\left\|\sum_{n=0}^{\infty} r^n c_n\right\| \leq \epsilon.$$

Letting $\epsilon \to 0$, we see that

$$\lim_{r\to 1^-}\sum_{n=0}^{\infty} r^n c_n = 0,$$

the convergence in the norm of X. This completes the proof. □

Note: If $\{c_n\}_{n=0}^{\infty}$ is a sequence of complex numbers such that $\sum_{n=0}^{\infty} c_n$ converges, then we have that the power series $\sum_{n=0}^{\infty} c_n z^n$ converges at $z = 1$ and hence has radius of convergence at least 1. This implies that $\sum_{n=0}^{\infty} r^n c_n$ converges for all $0 < r \leq 1$. However, it is not necessarily the case that if $\{c_n\}_{n=0}^{\infty}$ is a sequence in a general linear space X, then the convergence of $\sum_{n=0}^{\infty} c_n$ implies the convergence of $\sum_{n=0}^{\infty} r^n c_n$ in X for all $0 < r \leq 1$. Thus, we made this an extra assumption.

Inner Function Interpolation

Proposition 7. *Suppose μ and ν are finite nonnegative singular Borel measures on $\mathbb{T}$ with corresponding inner functions G and H, respectively. If J is the inner function corresponding* *to $\mu+\nu$, we have*

$$J = \frac{1+G+H-3GH}{3-G-H-GH}.$$

Proof. Let J be the inner function corresponding to $\mu+\nu$. Because J, G, and H are inner, they are bounded by 1 on $\mathbb{D}$. (See [SNF70].) Hence, by the maximum modulus principle they do not attain the value 1 on $\mathbb{D}$, and so the functions

$$\frac{1+J}{1-J}, \frac{1+G}{1-G}, \text{ and } \frac{1+H}{1-H}$$

are holomorphic on $\mathbb{D}$. Now, we must have

$$\begin{aligned}\operatorname{Re}\left(\frac{1+J}{1-J}\right) &= \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi-z|^2}\, d[\mu+\nu](\xi)\\ &= \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi-z|^2}\, d\mu(\xi) + \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi-z|^2}\, d\nu(\xi)\\ &= \operatorname{Re}\left(\frac{1+G}{1-G}\right) + \operatorname{Re}\left(\frac{1+H}{1-H}\right).\end{aligned}$$

Hence,

$$\operatorname{Re}\left(\frac{1+J}{1-J} - \frac{1+G}{1-G} - \frac{1+H}{1-H}\right) = 0.$$

It follows from the Cauchy-Riemann Equations that

$$\frac{1+J}{1-J} - \frac{1+G}{1-G} - \frac{1+H}{1-H} = 0$$

on $\mathbb{D}$, and solving for J we obtain

$$J = \frac{1+G+H-3GH}{3-G-H-GH}.$$

as desired. □

Proposition 8. *Let μ be a finite positive singular Borel measure on $\mathbb{T}$ with corresponding inner function G. Then for any $\alpha > 0$, the inner function corresponding to $\alpha\, d\mu$ is*

$$J = \frac{(\alpha-1)+(\alpha+1)G}{(\alpha+1)+(\alpha-1)G}.$$

Proof. Let J be the inner function corresponding to $\alpha\, d\mu$. As in the previous theorem, we have that $\frac{J+1}{J-1}$ and $\frac{G+1}{G-1}$ are holomorphic on $\mathbb{D}$. We have

$$\operatorname{Re}\left(\frac{1+J}{1-J}\right) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi-z|^2}\, \alpha\, d\mu(\xi)$$

$$= \alpha \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2}\, d\mu(\xi)$$
$$= \alpha \mathrm{Re}\left(\frac{1+G}{1-G}\right).$$

Therefore, by the Cauchy-Riemann Equations,

$$\frac{1+J}{1-J} = \alpha \frac{1+G}{1-G}.$$

Solving for J, we obtain the desired result. □

Corollary 10. *Let μ, ν be singular finite positive Borel measures on $\mathbb{T}$ with corresponding inner functions G and H, respectively, and let $0 < \eta \leq 1$. Then the inner function of $\eta\mu + (1-\eta)\nu$ is*

$$\frac{H(1-G) + \eta(G-H)}{1 - G + \eta(G-H)}.$$

Proof. Combine the previous two theorems. □

Proposition 9. *Let μ be a singular finite Borel measure on $\mathbb{T} = [0,1)$. Let ν be the measure obtained by translating μ right by $m \in \mathbb{R}$. That is to say,*

$$\nu(E) = \int_0^1 \chi_E(x)\, d\nu(x) := \int_0^1 \chi_E(x)\, d\mu(x-m) = \int_0^1 \chi_E(x+m)\, d\mu(x).$$

(Note: Recall that in the integrals above we are identifying points modulo 1.) If G is the inner function corresponding to μ, then the inner function corresponding to ν is

$$J(z) := G(e^{-2\pi i m} z).$$

Proof. Let J be the inner function corresponding to ν. For all $z \in \mathbb{D}$, we have

$$\mathrm{Re}\left(\frac{1+J(z)}{1-J(z)}\right) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2}\, d\nu(\xi)$$
$$= \int_0^1 \frac{1-|z|^2}{|e^{2\pi i x} - z|^2}\, d\nu(x)$$
$$= \int_0^1 \frac{1-|z|^2}{|e^{2\pi i x} - z|^2}\, d\mu(x-m)$$
$$= \int_0^1 \frac{1-|z|^2}{|e^{2\pi i (x+m)} - z|^2}\, d\mu(x)$$
$$= \int_0^1 \frac{1-|e^{-2\pi i m} z|^2}{|e^{2\pi i x} - e^{-2\pi i m} z|^2}\, d\mu(x)$$

$$= \text{Re}\left(\frac{1 + G(e^{-2\pi i m}z)}{1 - G(e^{-2\pi i m}z)}\right).$$

Since $G(e^{-2\pi i m}z)$ is obviously an inner function, it follows that $J(z) = G(e^{-2\pi i m}z)$, as desired.

□

Reproduction of Sarason's Kernel via Boundary Integration

In II-1 and II-3 of [Sar94], Sarason notes that the kernel function of $\mathcal{H}(b)$ is

$$k_z^b(w) = \frac{1 - \overline{b(z)}b(w)}{1 - \overline{z}w},$$

a consequence that follows from I-3. We paraphrase his argument as follows:

Let $A = (I - T_b T_{\overline{b}})^{1/2}$. As mentioned in Chapter 1, the inner product $\langle \cdot, \cdot \rangle_b$ is defined on $\mathcal{H}(b)$ so as to make A a coisometry from H^2 onto $\mathcal{H}(b)$. At the same time, since $\mathcal{H}(b)$ is a subset of H^2, we can also put its elements into the usual inner product $\langle \cdot, \cdot \rangle_2$ on H^2. Now, H^2 has the Szegő Kernel:

$$k_z(w) = \frac{1}{1 - \overline{z}w}.$$

Any element $f(w) \in H^2$ is reproduced via

$$f(z) = \langle f, k_z \rangle_2.$$

In particular, if $f \in \mathcal{H}(b)$, it is likewise reproduced via the above formula.

Since $\mathcal{H}(b)$ is the range of A, there exists an element $x \in H^2$ such that $f = Ax$. Then by the fact that A is a coisometry, we have

$$f(z) = \langle f, k_z \rangle_2 = \langle Ax, k_z \rangle_2 = \langle x, A^* k_z \rangle_2 = \langle Ax, AA^* k_z \rangle_b.$$

This proves that $AA^* k_z$ is the kernel in $\mathcal{H}(b)$. We have $AA^* = I - T_b T_{\overline{b}}$, so the kernel in $\mathcal{H}(b)$ equipped with the inner product $\langle \cdot, \cdot \rangle_b$, which we will denote k_z^b, is

$$k_z^b = (I - T_b T_{\overline{b}}) k_z.$$

Since $b \in H^\infty$ and $k_z \in H^2$, clearly $bk_z \in H^2$ and hence $T_b k_z = bk_z$. It is a basic property of Toeplitz operators that $T_b^* = T_{\overline{b}}$. Therefore,

$$T_{\overline{b}} k_z(w) = \langle T_{\overline{b}} k_z, k_w \rangle_2$$

$$= \langle k_z, T_b k_w \rangle_2$$
$$= \langle k_z, b k_w \rangle_2$$
$$= \overline{\langle b k_w, k_z \rangle_2}$$
$$= \overline{b(z) k_w(z)}$$
$$= \overline{b(z)} k_z(w).$$

Then

$$k_z^b = k_z - T_b \overline{b(z)} k_z = k_z - \overline{b(z)} b k_z = (1 - \overline{b(z)} b) k_z.$$

Therefore,

$$k_z^b(w) = \frac{1 - \overline{b(z)} b(w)}{1 - \overline{z} w},$$

as we wished to show.

By construction, this kernel reproduces all functions in $\mathcal{H}(b)$, including itself, with respect to the inner product $\langle \cdot, \cdot \rangle_b$. Without appealing to the $\{g_n\}$ sequence of the Kaczmarz algorithm, we will show that in also reproduces itself with respect to integration over $\mathbb{T}$ with respect to the measure μ on $\mathbb{T}$ whose inner function is b. By [Pol93], for μ-almost every x, we have

$$\lim_{r \to 1^-} b(re^{2\pi i x}) = 1.$$

So for every $z \in \mathbb{D}$, we may define a boundary function

$$k_z^{b\star}(x) = \frac{1 - \overline{b(z)}}{1 - \overline{z} e^{2\pi i x}}$$

that is defined for all $x \in [0, 1)$ and is the radial limit of the k_z^b for μ-almost-every x. We have $|b| \leq 1$ on $\mathbb{D}$. Moreover,

$$|1 - \overline{z} e^{2\pi i x}| \geq |1 - |z e^{2\pi i x}|| = 1 - |z|.$$

Thus, we have

$$|k_z^{b\star}(x)| \leq \frac{2}{1 - |z|}$$

on $[0, 1)$. By a similar argument, we have

$$|k_z^b(re^{2\pi i x})| \leq \frac{2}{1 - |z|}$$

for all $0 < r < 1$ and $x \in [0,1)$. Then by the Bounded Convergence Theorem,

$$\lim_{r\to 1^-} \int_{\mathbb{T}} \left| k_z^{b\star}(x) - k_z^b(re^{2\pi i x}) \right|^2 d\mu(x) = \int_0^1 \lim_{r\to 1^-} |k_z^{b\star}(x) - k_z^b(re^{2\pi i x})|^2 \, d\mu(x)$$
$$= \int_0^1 0 \, d\mu(x) = 0.$$

This shows that $k_z^{b\star}$ is an $L^2(\mu)$-boundary function of k_z^b.

Now observe that

$$\langle k_z^{b\star}, k_w^{b\star} \rangle_\mu = \int_0^1 \frac{(1-\overline{b(z)})(1-b(w))}{(1-\overline{z}e^{2\pi i x})(1-ze^{-2\pi i x})} \, d\mu(x)$$
$$= \langle k_z^b, k_w^b \rangle_b$$
$$= k_z^b(w),$$

as Sarason computes on page 18 of [Sar94]. Thus, we have shown without appeal to the Kaczmarz algorithm that k_z^b reproduces itself via a boundary integral with respect to μ.

Clark Measure Equivalences

Recall that given an inner function θ, Clark [Cla72] defines a family of positive measures $\{\sigma_\alpha : \alpha \in \mathbb{T}\}$ given by

$$\operatorname{Re}\left(\frac{\alpha+\theta(z)}{\alpha-\theta(z)}\right) = P\sigma_\alpha := \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\xi|^2} \, d\sigma_\alpha(\xi).$$

Observe that since

$$\operatorname{Re}\left(\frac{\alpha+\theta(z)}{\alpha-\theta(z)}\right) = \operatorname{Re}\left(\frac{1+\alpha^{-1}\theta(z)}{1-\alpha^{-1}\theta(z)}\right),$$

$\alpha^{-1}\theta(z)$ is the unique inner function associated to σ_α. Let θ_μ denote the unique inner function associated to a singular measure μ. It is obvious from the previous observation that the relation $\sim$ given by $\mu \sim \nu$ if $\theta_\mu = \alpha\theta_\nu$ for some $\alpha \in \mathbb{T}$ is an equivalence relation, partitioning the singular Borel measures on $\mathbb{T}$ into equivalence classes.

We give some necessary conditions for $\mu \sim \nu$:

Proposition 10. *Let ν, μ be singular Borel measures on $\mathbb{T}$. Then $\nu \sim \mu$ if and only if there exists an $\alpha \in \mathbb{T}$ such that*

$$\widehat{\nu}(n) = \frac{1}{n!}\left(\frac{\mu_+(z)}{(1-\alpha)\mu_+(z)+\alpha}\right)^{(n)}\Bigg|_{z=0} \quad \textit{for all } n \geq 0.$$

In particular, if μ is a singular Borel probability measure and $\nu \sim \mu$, then

$$|\widehat{\mu}(1)| = |\widehat{\nu}(1)|.$$

Proof. Let μ_+ denote the Cauchy integral of μ. Suppose μ is a singular Borel probability measure and that $\nu \sim \mu$. Then we have

$$\theta_\mu(z) = 1 - \frac{1}{\mu_+(z)}$$

and hence

$$\mu_+(z) = \frac{1}{1 - \theta_\mu(z)}.$$

Since $\mu \sim \nu$, we must have that $\theta_\nu = \alpha\theta_\mu$ for some $\alpha \in \mathbb{T}$. Therefore,

$$\begin{aligned}
\nu_+(z) &= \frac{1}{1 - \alpha\theta_\mu(z)} \\
&= \frac{1}{1 - \alpha\left(1 - \frac{1}{\mu_+(z)}\right)} \\
&= \frac{\mu_+(z)}{\mu_+(z) - \alpha\left(\mu_+(z) - 1\right)} \\
&= \frac{\mu_+(z)}{(1-\alpha)\mu_+(z) + \alpha}.
\end{aligned}$$

Recall that

$$\nu_+(z) = \sum_{n=0}^{\infty} \widehat{\nu}(n) z^n.$$

Hence, equating coefficients of power series,

$$\widehat{\nu}(n) = \frac{1}{n!}\left(\frac{\mu_+(z)}{(1-\alpha)\mu_+(z) + \alpha}\right)^{(n)}\Bigg|_{z=0} \tag{A.2}$$

for all $n \geq 0$.

Conversely, given a singular Borel measure μ, if the above equation holds for some $\alpha \in \mathbb{T}$, and all $n \geq 0$, then reversing the preceding calculations implies $\nu_+(z) = \frac{1}{1-\alpha\theta_\mu(z)}$, and therefore that $\theta_\nu = \alpha\theta_\mu$. Thus, $\nu \sim \mu$. □

In particular, if μ is a singular Borel probability measure, then $\widehat{\mu}(0) = 1$, and so

$$\nu_+(z) = \frac{\sum_{n=0}^{\infty} \widehat{\mu}(n) z^n}{1 + \sum_{n=1}^{\infty} (1-\alpha)\widehat{\mu}(n) z^n}.$$

By division of power series, the reader may verify that

$$\nu_+(z) = 1 + \alpha\widehat{\mu}(1)z + \left(\alpha\widehat{\mu}(2) - (\alpha - \alpha^2)\widehat{\mu}(1)^2\right) z^2 + \cdots .$$

Thus, the following identities hold:

$$\widehat{\nu}(0) = 1$$

$$\widehat{\nu}(1) = \alpha\widehat{\mu}(1)$$

$$\widehat{\nu}(2) = \alpha\widehat{\mu}(2) - (\alpha - \alpha^2)\widehat{\mu}(1)^2.$$

Therefore, $|\widehat{\nu}(1)| = |\widehat{\mu}(1)|$.

Corollary 11. $\mu_4 \not\sim \mu_3$.

Proof. $x = 1$ is a zero of $\widehat{\mu_4}(x)$, but not of $\widehat{\mu_3}(x)$. □

BIBLIOGRAPHY

[Ale89] A. B. Aleksandrov, *Inner functions and related spaces of pseudocontinuable functions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **170** (1989), no. Issled. Linein. Oper. Teorii Funktsii. 17, 7–33, 321. MR 1039571

[Aro50] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404. MR 0051437

[Bar05] Anton Baranov, *Stability of bases and frames of reproducing kernels in model spaces*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 7, 2399–2422. MR 2207388

[BBtH11] Joseph A. Ball, Vladimir Bolotnikov, and Sanne ter Horst, *Interpolation in de Branges-Rovnyak spaces*, Proc. Amer. Math. Soc. **139** (2011), no. 2, 609–618. MR 2736342

[BCMO95] Larry Baggett, Alan Carey, William Moran, and Peter Ohring, *General existence theorems for orthonormal wavelets, an abstract approach*, Publ. Res. Inst. Math. Sci. **31** (1995), no. 1, 95–111. MR 1317525

[Bes15] R. V. Bessonov, *Duality theorems for coinvariant subspaces of H^1*, Adv. Math. **271** (2015), 62–90. MR 3291857

[Beu48] Arne Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1948), 17. MR 0027954

[BFG$^+$13] Alain Blandignères, Emmanuel Fricain, Frédéric Gaunard, Andreas Hartmann, and William Ross, *Reverse Carleson embeddings for model spaces*, J. Lond. Math. Soc. (2) **88** (2013), no. 2, 437–464. MR 3106730

[BFM10] Anton Baranov, Emmanuel Fricain, and Javad Mashreghi, *Weighted norm inequalities for de Branges-Rovnyak spaces and their applications*, Amer. J. Math. **132** (2010), no. 1, 125–155. MR 2597508

[Bog07] V. I. Bogachev, *Measure theory. Vol. I, II*, Springer-Verlag, Berlin, 2007. MR 2267655

[BR04] V. Bolotnikov and L. Rodman, *Finite dimensional backward shift invariant subspaces of a class of reproducing kernel Hilbert spaces*, Linear Multilinear Algebra **52** (2004), no. 5, 321–334. MR 2075032

[Car66] Lennart Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135–157. MR 0199631

[CFP06] I. Chalendar, E. Fricain, and J. R. Partington, *Overcompleteness of sequences of reproducing kernels in model spaces*, Integral Equations Operator Theory **56** (2006), no. 1, 45–56. MR 2256996

[CFT09] Nicolas Chevrot, Emmanuel Fricain, and Dan Timotin, *On certain Riesz families in vector-valued de Branges-Rovnyak spaces*, J. Math. Anal. Appl. **355** (2009), no. 1, 110–125. MR 2514455

[CGR10] Nicolas Chevrot, Dominique Guillot, and Thomas Ransford, *De Branges-Rovnyak spaces and Dirichlet spaces*, J. Funct. Anal. **259** (2010), no. 9, 2366–2383. MR 2674117

[Chu15] Cheng Chu, *Compact product of Hankel and Toeplitz operators on the Hardy space*, Indiana Univ. Math. J. **64** (2015), no. 4, 973–982. MR 3385782

[Cla72] Douglas N. Clark, *One dimensional perturbations of restricted shifts*, J. Analyse Math. **25** (1972), 169–191. MR 0301534 (46 #692)

[CMR04] Joseph A. Cima, Alec Matheson, and William T. Ross, *The backward shift on the space of Cauchy transforms*, Proc. Amer. Math. Soc. **132** (2004), no. 3, 745–754 (electronic). MR 2019951

[CMR06] Joseph A. Cima, Alec L. Matheson, and William T. Ross, *The Cauchy transform*, Mathematical Surveys and Monographs, vol. 125, American Mathematical Society, Providence, RI, 2006. MR 2215991

[CR00] Joseph A. Cima and William T. Ross, *The backward shift on the Hardy space*, Mathematical Surveys and Monographs, vol. 79, American Mathematical Society, Providence, RI, 2000. MR 1761913

[CR13] Constantin Costara and Thomas Ransford, *Which de Branges-Rovnyak spaces are Dirichlet spaces (and vice versa)?*, J. Funct. Anal. **265** (2013), no. 12, 3204–3218. MR 3110499

[Cro94] R. Bruce Crofoot, *Multipliers between invariant subspaces of the backward shift*, Pacific J. Math. **166** (1994), no. 2, 225–246. MR 1313454

[CS97] Kit C. Chan and Steven M. Seubert, *Reducing subspaces of compressed analytic Toeplitz operators on the Hardy space*, Integral Equations Operator Theory **28** (1997), no. 2, 147–157. MR 1451498

[CS14] Kit Chan and Irina Seceleanu, *Cyclicity of vectors with orbital limit points for backward shifts*, Integral Equations Operator Theory **78** (2014), no. 2, 225–232. MR 3157979

[CT13] Wojciech Czaja and James H. Tanis, *Kaczmarz algorithm and frames*, Int. J. Wavelets Multiresolut. Inf. Process. **11** (2013), no. 5, 1350036, 13. MR 3117886

[DHS09] Dorin Ervin Dutkay, Deguang Han, and Qiyu Sun, *On the spectra of a Cantor measure*, Adv. Math. **221** (2009), no. 1, 251–276. MR 2509326

[DHS14] ———, *Divergence of the mock and scrambled Fourier series on fractal measures*, Trans. Amer. Math. Soc. **366** (2014), no. 4, 2191–2208. MR 3152727

[DHSW11] Dorin Ervin Dutkay, Deguang Han, Qiyu Sun, and Eric Weber, *On the Beurling dimension of exponential frames*, Adv. Math. **226** (2011), no. 1, 285–297. MR 2735759 (2012a:42058)

[DHW11] Dorin Ervin Dutkay, Deguang Han, and Eric Weber, *Bessel sequences of exponentials on fractal measures*, J. Funct. Anal. **261** (2011), no. 9, 2529–2539. MR 2826404

[DHW12] ______, *Iterative approximations of exponential bases on fractal measures*, Numer. Funct. Anal. Optim. **33** (2012), no. 7-9, 928–950. MR 2966138

[DHW14] ______, *Continuous and discrete Fourier frames for fractal measures*, Trans. Amer. Math. Soc. **366** (2014), no. 3, 1213–1235. MR 3145729

[DJ11] Dorin Ervin Dutkay and Palle E. T. Jorgensen, *Affine fractals as boundaries and their harmonic analysis*, Proc. Amer. Math. Soc. **139** (2011), no. 9, 3291–3305. MR 2811284 (2012e:28008)

[DS52] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366. MR 0047179 (13,839a)

[DSS67] R. G. Douglas, H. S. Shapiro, and A. L. Shields, *On cyclic vectors of the backward shift*, Bull. Amer. Math. Soc. **73** (1967), 156–159. MR 0203465

[DSS70] ______, *Cyclic vectors and invariant subspaces for the backward shift operator.*, Ann. Inst. Fourier (Grenoble) **20** (1970), no. fasc. 1, 37–76. MR 0270196

[Fao82] Nazih S. Faour, *Toeplitz operators on Bergman and Hardy spaces*, Houston J. Math. **8** (1982), no. 2, 185–191. MR 674036

[FM08a] Emmanuel Fricain and Javad Mashreghi, *Boundary behavior of functions in the de Branges-Rovnyak spaces*, Complex Anal. Oper. Theory **2** (2008), no. 1, 87–97. MR 2390675

[FM08b] ______, *Integral representation of the n-th derivative in de Branges-Rovnyak spaces and the norm convergence of its reproducing kernel*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 6, 2113–2135. MR 2473631

[FMS15] Emmanuel Fricain, Javad Mashreghi, and Daniel Seco, *Cyclicity in non-extreme de Branges–Rovnyak spaces*, Invariant subspaces of the shift operator, Contemp. Math., vol. 638, Amer. Math. Soc., Providence, RI, 2015, pp. 131–136. MR 3309352

[Fri01] Emmanuel Fricain, *Bases of reproducing kernels in model spaces*, J. Operator Theory **46** (2001), no. 3, suppl., 517–543. MR 1897152

[Fri05] E. Fricain, *Bases of reproducing kernels in de Branges spaces*, J. Funct. Anal. **226** (2005), no. 2, 373–405. MR 2159461

[FY89] N. Faour and S. Yousef, *Toeplitz operators associated with Bergman and Hardy spaces*, Riv. Mat. Univ. Parma (4) **15** (1989), 51–62. MR 1064243

[Gar03] Stephan Ramon Garcia, *Conjugation, the backward shift, and Toeplitz kernels*, ProQuest LLC, Ann Arbor, MI, 2003, Thesis (Ph.D.)–University of California, Berkeley. MR 2705182

[Gar05] ———, *Conjugation, the backward shift, and Toeplitz kernels*, J. Operator Theory **54** (2005), no. 2, 239–250. MR 2186351

[GR15] Stephan Ramon Garcia and William T. Ross, *Model spaces: a survey*, Invariant subspaces of the shift operator, Contemp. Math., vol. 638, Amer. Math. Soc., Providence, RI, 2015, pp. 197–245. MR 3309355

[Guy91] James Guyker, *The de Branges-Rovnyak model*, Proc. Amer. Math. Soc. **111** (1991), no. 1, 95–99. MR 1031663

[Har11] Andreas Hartmann, *Some remarks on analytic continuation in weighted backward shift invariant subspaces*, Arch. Math. (Basel) **96** (2011), no. 1, 59–75. MR 2755070

[Hay90] Eric Hayashi, *Classification of nearly invariant subspaces of the backward shift*, Proc. Amer. Math. Soc. **110** (1990), no. 2, 441–448. MR 1019277

[Her11] G. Herglotz, *Über Potenzreihen mit positiven reellen Teil im Einheitskreis*, Ber. Verh. Sächs, Ges. Wiss. Leipzig Math. Phys. Kl. **63** (1911), 501–511.

[HKLW07] Deguang Han, Keri Kornelson, David Larson, and Eric Weber, *Frames for undergraduates*, Student Mathematical Library, vol. 40, American Mathematical Society, Providence, RI, 2007. MR 2367342

[HS01] Nina N. Huang and Robert S. Strichartz, *Sampling theory for functions with fractal spectrum*, Experiment. Math. **10** (2001), no. 4, 619–638. MR 1881762 (2003g:94017)

[HS05] Rainis Haller and Ryszard Szwarc, *Kaczmarz algorithm in Hilbert space*, Studia Math. **169** (2005), no. 2, 123–132. MR 2140451 (2006b:41049)

[HW15] John Herr and Eric Weber, *Fourier Series for Singular Measures.*

[JP98] Palle E. T. Jorgensen and Steen Pedersen, *Dense analytic subspaces in fractal L^2-spaces*, J. Anal. Math. **75** (1998), 185–228. MR 1655831 (2000a:46045)

[Jur07] Michael T. Jury, *Reproducing kernels, de Branges-Rovnyak spaces, and norms of weighted composition operators*, Proc. Amer. Math. Soc. **135** (2007), no. 11, 3669–3675 (electronic). MR 2336583

[Kac37] Stefan Kaczmarz, *Angenäherte auflösung von systemen linearer gleichungen*, Bulletin International de l'Académie Plonaise des Sciences et des Lettres. Classe des Sciences Mathématiques et Naturelles. Série A. Sciences Mathématiques **35** (1937), 355–357.

[KM01] Stanisław Kwapień and Jan Mycielski, *On the Kaczmarz algorithm of approximation in infinite-dimensional spaces*, Studia Math. **148** (2001), no. 1, 75–86. MR 1881441 (2003a:60102)

[KM06] ______, *Erratum to the paper: "On the Kaczmarz algorithm of approximation in infinite-dimensional spaces" [Studia Math.* **148** *(2001), no. 1, 75–86; mr1881441]*, Studia Math. **176** (2006), no. 1, 93. MR 2263965

[Koo98] Paul Koosis, *Introduction to H_p spaces*, second ed., Cambridge Tracts in Mathematics, vol. 115, Cambridge University Press, Cambridge, 1998, With two appendices by V. P. Havin [Viktor Petrovich Khavin]. MR 1669574

[KZ15] Karim Kellay and Mohamed Zarrabi, *Two-isometries and de Branges–Rovnyak spaces*, Complex Anal. Oper. Theory **9** (2015), no. 6, 1325–1335. MR 3390195

[Lai12] Chun-Kit Lai, *Spectral analysis on fractal measures and tiles*, Ph.d. dissertation, The Chinese University of Hong Kong, 2012, Published by ProQuest, LLC: UMI Dissertation Publishing, Number 3535691 (2013).

[Lax02] Peter D. Lax, *Functional analysis*, Pure and Applied Mathematics (New York), Wiley-Interscience [John Wiley & Sons], New York, 2002. MR 1892228

[Lee14] Hong Youl Lee, *Invariant subspaces for the backward shift on the Hardy space*, Honam Math. J. **36** (2014), no. 4, 901–906. MR 3288829

[Lee15] Young Joo Lee, *Zero sums of products of Toeplitz and Hankel operators on the Hardy space*, Studia Math. **227** (2015), no. 1, 41–53. MR 3359956

[LO01] Shidong Li and Hidemitsu Ogawa, *Pseudo-duals of frames with applications*, Appl. Comput. Harmon. Anal. **11** (2001), no. 2, 289–304. MR 1848709 (2002f:46014)

[Lot92] B. A. Lotto, *Toeplitz operators on weighted Hardy spaces*, Function spaces (Edwardsville, IL, 1990), Lecture Notes in Pure and Appl. Math., vol. 136, Dekker, New York, 1992, pp. 295–300. MR 1152354

[LS93] B. A. Lotto and Donald Sarason, *Multipliers of de Branges-Rovnyak spaces*, Indiana Univ. Math. J. **42** (1993), no. 3, 907–920. MR 1254125

[LS97] T. L. Lance and M. I. Stessin, *Multiplication invariant subspaces of Hardy spaces*, Canad. J. Math. **49** (1997), no. 1, 100–118. MR 1437202 (97m:30048)

[LS98] Benjamin A. Lotto and Donald Sarason, *Multipliers of de Branges-Rovnyak spaces. II*, Harmonic analysis and hypergroups (Delhi, 1995), Trends Math., Birkhäuser Boston, Boston, MA, 1998, pp. 51–58. MR 1614726

[MT14] Javad Mashreghi and Dan Timotin, *Nonextreme de Branges–Rovnyak spaces as models for contractions*, Integral Equations Operator Theory **80** (2014), no. 1, 137–152. MR 3248478

[Nak08] Takahiko Nakazi, *Invariant subspaces of Toeplitz operators and uniform algebras*, Bull. Belg. Math. Soc. Simon Stevin **15** (2008), no. 1, 1–8. MR 2406082

[Nik02] Nikolai K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 1*, Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002, Hardy, Hankel, and Toeplitz, Translated from the French by Andreas Hartmann. MR 1864396

[NT93] Takahiko Nakazi and Katsutoshi Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338** (1993), no. 2, 753–767. MR 1162103

[Pau08] Vern I. Paulsen, *An introduction to the theory of reproducing kernel Hilbert spaces*, 2008.

[Pav14] Miroslav Pavlović, *Function classes on the unit disc*, De Gruyter Studies in Mathematics, vol. 52, De Gruyter, Berlin, 2014, An introduction. MR 3154590

[Pol93] A. G. Poltoratskiĭ, *Boundary behavior of pseudocontinuable functions*, Algebra i Analiz **5** (1993), no. 2, 189–210, English translation in St. Petersburg Math. 5:2 (1994): 389–406. MR 1223178 (94k:30090)

[RF10] Halsey L. Royden and Patrick M. Fitzpatrick, *Real Analysis, Fourth Edition*, Prentice Hall, 2010.

[Ros05] William T. Ross, *The backward shift on H^p*, Selected topics in complex analysis, Oper. Theory Adv. Appl., vol. 158, Birkhäuser, Basel, 2005, pp. 191–211. MR 2147597

[RR16] Frigyes Riesz and Marcel Riesz, *Über die randwerte einer analytischen funktion*, Quatrième Congrès des Mathématiciens Scandinaves (1916), 27–44, See Rudin, *Real and Complex Analysis, 3rd Edition*: pg. 341.

[Rud90] Walter Rudin, *Fourier analysis on groups*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1990, Reprint of the 1962 original, A Wiley-Interscience Publication. MR 1038803

[Sar88] Donald Sarason, *Nearly invariant subspaces of the backward shift*, Contributions to operator theory and its applications (Mesa, AZ, 1987), Oper. Theory Adv. Appl., vol. 35, Birkhäuser, Basel, 1988, pp. 481–493. MR 1017680

[Sar90] ______, *Function theory and de Branges's spaces*, Operator theory: operator algebras and applications, Part 1 (Durham, NH, 1988), Proc. Sympos. Pure Math., vol. 51, Amer. Math. Soc., Providence, RI, 1990, pp. 495–502. MR 1077404

[Sar94] ______, *Sub-Hardy Hilbert spaces in the unit disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, 10, John Wiley & Sons, Inc., New York, 1994, A Wiley-Interscience Publication. MR 1289670 (96k:46039)

[Sar97] ______, *Local Dirichlet spaces as de Branges-Rovnyak spaces*, Proc. Amer. Math. Soc. **125** (1997), no. 7, 2133–2139. MR 1396993

[Sar10] ______, *Unbounded operators commuting with the commutant of a restricted backward shift*, Oper. Matrices **4** (2010), no. 2, 293–300. MR 2667340

[Sar12] ______, *Self-adjoint truncated Toeplitz operators on finite-dimensional model spaces*, Indag. Math. (N.S.) **23** (2012), no. 4, 650–662. MR 2991916

[SNF70] Béla Sz.-Nagy and Ciprian Foiaş, *Harmonic analysis of operators on Hilbert space*, Translated from the French and revised, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest, 1970. MR 0275190

[SS03] Elias M. Stein and Rami Shakarchi, *Fourier analysis*, Princeton Lectures in Analysis, vol. 1, Princeton University Press, Princeton, NJ, 2003, An introduction. MR 1970295

[SŚ13] Ryszard Szwarc and Grzegorz Świderski, *Kaczmarz algorithm with relaxation in Hilbert space*, Studia Math. **216** (2013), no. 3, 237–243. MR 3092425

[Ste99] M. I. Stessin, *Wold decomposition of the Hardy space and Blaschke products similar to a contraction*, Colloq. Math. **81** (1999), no. 2, 271–284. MR 1715351 (2000g:47036)

[Str90] Robert S. Strichartz, *Fourier asymptotics of fractal measures*, J. Funct. Anal. **89** (1990), no. 1, 154–187. MR 1040961

[Str98] ———, *Remarks on: "Dense analytic subspaces in fractal L^2-spaces" [J. Anal. Math.* **75** *(1998), 185–228; MR1655831 (2000a:46045)] by P. E. T. Jorgensen and S. Pedersen*, J. Anal. Math. **75** (1998), 229–231. MR 1655832

[Str99] Karel Stroethoff, *Algebraic properties of Toeplitz operators on the Hardy space via the Berezin transform*, Function spaces (Edwardsville, IL, 1998), Contemp. Math., vol. 232, Amer. Math. Soc., Providence, RI, 1999, pp. 313–319. MR 1678343

[Str00] Robert S. Strichartz, *Mock Fourier series and transforms associated with certain Cantor measures*, J. Anal. Math. **81** (2000), 209–238. MR 1785282 (2001i:42009)

[Str06] ———, *Convergence of mock Fourier series*, J. Anal. Math. **99** (2006), 333–353. MR 2279556

[Suá95] Fernando Daniel Suárez, *Multipliers of de Branges-Rovnyak spaces in H^2*, Rev. Mat. Iberoamericana **11** (1995), no. 2, 375–415. MR 1344898

[Suá97] Daniel Suárez, *Backward shift invariant spaces in H^2*, Indiana Univ. Math. J. **46** (1997), no. 2, 593–619. MR 1481605

[SZ03a] Michael Stessin and Kehe Zhu, *Generalized factorization in Hardy spaces and the commutant of Toeplitz operators*, Canad. J. Math. **55** (2003), no. 2, 379–400. MR 1969797

[SZ03b] ———, *Generalized factorization in Hardy spaces and the commutant of Toeplitz operators*, Canad. J. Math. **55** (2003), no. 2, 379–400. MR 1969797

[Szw07] Ryszard Szwarc, *Kaczmarz algorithm in Hilbert space and tight frames*, Appl. Comput. Harmon. Anal. **22** (2007), no. 3, 382–385. MR 2311862 (2008b:42065)

[Tim15] Dan Timotin, *A short introduction to de Branges–Rovnyak spaces*, Invariant subspaces of the shift operator, Contemp. Math., vol. 638, Amer. Math. Soc., Providence, RI, 2015, pp. 21–38. MR 3309347

[TKA11] Hiroyuki Takagi, Hironao Koshimizu, and Hiroaki Ariizumi, *Backward shifts on function algebras*, J. Math. Anal. Appl. **377** (2011), no. 1, 135–144. MR 2754814

[Vuk03] Dragan Vukotić, *Analytic Toeplitz operators on the Hardy space H^p: a survey*, Bull. Belg. Math. Soc. Simon Stevin **10** (2003), no. 1, 101–113. MR 2032329

[YFD07] B. Yousefi, A. Farrokhinia, and J. Doroodgar, *Cyclic vectors of the backward shift operator on Hilbert spaces of analytic functions*, Southeast Asian Bull. Math. **31** (2007), no. 4, 827–834. MR 2363035

[Zhu96] Kehe Zhu, *A class of Toeplitz operators on the Hardy space*, Functional analysis in China, Math. Appl., vol. 356, Kluwer Acad. Publ., Dordrecht, 1996, pp. 366–373. MR 1379621