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Analysis of a second order discontinuous Galerkin finite element method for the Allen-Cahn equation

by

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ABSTRACT

The paper proposes and analyzes an efficient second-order in time numerical approximation for the Allen-Cahn equation which is a nonlinear singular perturbation of the reaction-diffusion model arising from phase separation in alloys. We firstly present a fully discrete, nonlinear interior penalty discontinuous Galerkin finite element(IPDGFE) method, which is based on the modified Crank-Nicolson scheme and a mid-point approximation of the potential term $f(u)$. We then derive the stability analysis and error estimates for the proposed IPDGFE method under some regularity assumptions on the initial function u_0. There are two key works in our analysis, one is to establish unconditionally energy-stable scheme for the discrete solutions. The other is to use a discrete spectrum estimate to handle the midpoint of the discrete solutions u^m and u^{m+1} in the nonlinear term, instead of using the standard Gronwall inequality technique. We obtain that all our error bounds depend on reciprocal of the perturbation parameter ϵ only in some lower polynomial order, instead of exponential order.

CHAPTER 1. INTRODUCTION

Let $\Omega \subseteq R^d (d = 2, 3)$ be a bounded polygonal or polyhedral domain. Consider the following nonlinear singular perturbation model of the reaction-diffusion equation

$$u_t - \Delta u + \frac{1}{\epsilon^2} f(u) = 0, \qquad \text{in } \Omega_T := \Omega \times (0, T). \tag{1.1}$$

In this paper, we consider the following homogenous Neumann boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} = 0, \qquad \text{in } \partial\Omega_T := \partial\Omega \times (0, T), \tag{1.2}$$

and initial condition

$$u = u_0, \qquad \text{in } \Omega \times \{t = 0\}, \tag{1.3}$$

where, $\mathbf{n}$ denotes the unit outward normal vector to the boundary $\partial\Omega$, and the boundary condition (1.3) means that no mass loss occurs through the boundary walls.

Equation (1.1), which is called the Allen-Cahn equation, was originally introduced by Allen and Cahn in [1] to describe an interface evolving in time in the phase separation process of the crystalline solids. Herein, $\epsilon > 0$ is a parameter related to the interface thickness, which is small compared to the characteristic length of the laboratory scale. u denotes the concentration of one of the two metallic species of the alloy, and $f(u) = F'(u)$ with $F(u)$ being some given energy potential. Several choices of $F(u)$ have been presented in the literature [2-6]. In this paper we focus on the following Ginzburg-Landau double-well potential

$$F(u) = \frac{1}{4}(u^2 - 1)^2 \quad \text{and} \quad f(u) = F'(u) = (u^2 - 1)u. \tag{1.4}$$

Although the potential term (1.4) has been widely used, its quartic growth at infinity leads to a variety of technical difficulties in the numerical approximation for the Allen-Cahn equation. For example, in order to assure that our numerical scheme is second-order in time, we have to employ the modified Crank-Nicolson scheme and a second order in time approximation of the potential term $f(u)$(see (3.4) in section 3.1).

CHAPTER 2. PRELIMINARIES

Let $\mathcal{T}_h$ be a quasi-uniform "triangulation" of Ω such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$. Let h_K denote the diameter of $K \in \mathcal{T}_h$ and $h := \max\{h_K; K \in \mathcal{T}_h\}$. We recall that the standard broken Sobolev space $H^s(\mathcal{T}_h)$ and DG finite element space V_h are defined as

$$H^s(\mathcal{T}_h) := \prod_{K\in\mathcal{T}_h} H^s(K), \qquad V_h := \prod_{K\in\mathcal{T}_h} P_r(K),$$

where $P_r(K)$ denotes the set of all polynomials whose degrees do not exceed a given positive integer r. Let $\mathcal{E}_h^I$ denote the set of all interior faces/edges of $\mathcal{T}_h$, $\mathcal{E}_h^B$ denote the set of all boundary faces/edges of $\mathcal{T}_h$, and $\mathcal{E}_h := \mathcal{E}_h^I \cup \mathcal{E}_h^B$. The L^2-inner product for piecewise functions over the mesh $\mathcal{T}_h$ is naturally defined by

$$(u, v)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_K uv\,dx,$$

and for any set $\mathcal{S}_h \subset \mathcal{E}_h$, the L^2-inner product over $\mathcal{S}_h$ is defined by

$$\big\langle u, v \big\rangle_{\mathcal{S}_h} := \sum_{e\in\mathcal{S}_h} \int_e uv\,ds.$$

Let $K, K' \in \mathcal{T}_h$ and $e = \partial K \cap \partial K'$ and assume global labeling number of K is smaller than that of K'. We choose $n_e := n_K|_e = -n_{K'}|_e$ as the unit normal on e and define the following standard jump and average notations across the face/edge e:

$$[v] := v|_K - v|_{K'} \quad \text{on } e \in \mathcal{E}_h^I, \qquad [v] := v \quad \text{on } e \in \mathcal{E}_h^B,$$
$$\{v\} := \frac{1}{2}\big(v|_K + v|_{K'}\big) \quad \text{on } e \in \mathcal{E}_h^I, \qquad \{v\} := v \quad \text{on } e \in \mathcal{E}_h^B$$

for $v \in V_h$. Let M be a (large) positive integer. Define $\tau := T/M$ and $t_m := m\tau$ for $m = 0, 1, 2, \cdots, M$ be a uniform partition of $[0, T]$. For a sequence of functions $\{v^m\}_{m=0}^M$, we define the (backward) difference operator

$$d_t u^m := \frac{u^m - u^{m-1}}{k}, \qquad m = 1, 2, \cdots, M.$$

First, we introduce the DG elliptic projection operator $P_r^h : H^s(\mathcal{T}_h) \to V_h$ by

$$a_h(v - P_r^h v, w_h) + \big(v - P_r^h v, w_h\big)_{\mathcal{T}_h} = 0 \quad \forall w_h \in V_h \tag{2.1}$$

for any $v \in H^s(\mathcal{T}_h)$.

We start with a well-known fact [18, 27] that the Allen-Cahn equation (1.1) can be interpreted as the L^2-gradient flow for the following Cahn-Hilliard energy functional

$$J_\epsilon(v) := \int_\Omega \Big(\frac{1}{2}|\nabla v|^2 + \frac{1}{\epsilon^2} F(v)\Big)\, dx \tag{2.2}$$

The following assumptions on the initial datum u_0 are made as in [16, 19] to derive a priori solution estimates.

General Assumption (GA)

(1) There exists a nonnegative constant σ_1 such that

$$J_\epsilon(u_0) \le C\epsilon^{-2\sigma_1}. \tag{2.3}$$

(2) There exists a nonnegative constant σ_2 such that

$$\|\Delta u_0 - \epsilon^{-2} f(u_0)\|_{L^2(\Omega)} \le C\epsilon^{-\sigma_2}. \tag{2.4}$$

(3) There exists nonnegative constant σ_3 such that

$$\lim_{s\to 0^+} \|\nabla u_t(s)\|_{L^2(\Omega)} \le C\epsilon^{-\sigma_3}. \tag{2.5}$$

The following solution estimates can be found in [16,19].

Proposition 1 *Suppose that* (2.3) *and* (2.4) *hold. Then the solution u of problem* (1.1)–(1.4) *satisfies the following estimates:*

$$\operatorname*{ess\,sup}_{t\in[0,\infty)} \|u(t)\|_{L^\infty(\Omega)} \le 1, \tag{2.6}$$

$$\operatorname*{ess\,sup}_{t\in[0,\infty)} J_\epsilon(u) + \int_0^\infty \|u_t(s)\|_{L^2(\Omega)}^2\, ds \le C\epsilon^{-2\sigma_1}, \tag{2.7}$$

$$\int_0^T \|\Delta u(s)\|^2\, ds \le C\epsilon^{-2(\sigma_1+1)}, \tag{2.8}$$

$$\operatorname*{ess\,sup}_{t\in[0,\infty)} \Big(\|u_t\|_{L^2(\Omega)}^2 + \|u\|_{H^2(\Omega)}^2\Big) + \int_0^\infty \|\nabla u_t(s)\|_{L^2(\Omega)}^2\, ds \le C\epsilon^{-2\max\{\sigma_1+1,\sigma_2\}}, \tag{2.9}$$

$$\int_0^\infty \Big(\|u_{tt}(s)\|_{H^{-1}(\Omega)}^2 + \|\Delta u_t(s)\|_{H^{-1}(\Omega)}^2\Big)\, ds \le C\epsilon^{-2\max\{\sigma_1+1,\sigma_2\}}. \tag{2.10}$$

In addition to (2.3) *and* (2.4)*, suppose that* (2.5) *holds, then* u *also satisfies*

$$\operatorname*{ess\,sup}_{t\in[0,\infty)} \|\nabla u_t\|^2_{L^2(\Omega)} + \int_0^\infty \|u_{tt}(s)\|^2_{L^2}\,ds \le C\epsilon^{-2\max\{\sigma_1+2,\sigma_3\}}, \tag{2.11}$$

$$\int_0^\infty \|\Delta u_t(s)\|^2_{L^2(\Omega)}\,ds \le C\epsilon^{-2\max\{\sigma_1+2,\sigma_3\}}. \tag{2.12}$$

Next, we quote the following well known error estimate results from [21, 22].

Lemma 1 *Let* $v \in W^{s,\infty}(\mathcal{T}_h)$*, then there hold*

$$\|v - P^h_r v\|_{L^2(\mathcal{T}_h)} + h\|\nabla(v - P^h_r v)\|_{L^2(\mathcal{T}_h)} \le Ch^{\min\{r+1,s\}}\|u\|_{H^s(\mathcal{T}_h)}, \tag{2.13}$$

$$\frac{1}{|\ln h|^{\overline{r}}}\|v - P^h_r v\|_{L^\infty(\mathcal{T}_h)} + h\|\nabla(u - P^h_r u)\|_{L^\infty(\mathcal{T}_h)} \le Ch^{\min\{r+1,s\}}\|u\|_{W^{s,\infty}(\mathcal{T}_h)}. \tag{2.14}$$

where $\overline{r} := \min\{1, r\} - \min\{1, r-1\}$.

Let

$$C_1 = \max_{|\xi|\le 2}|f''(\xi)|. \tag{2.15}$$

and $\widehat{P}^h_r$, corresponding to P^h_r, denote the elliptic projection operator on the finite element space $S_h := V_h \cap C^0(\overline{\Omega})$, there holds the following estimate from [12]:

$$\|u - \widehat{P}^h_r u\|_{L^\infty} \le Ch^{2-\frac{d}{2}}||u||_{H^2}. \tag{2.16}$$

We now state our discrete spectrum estimate for the DG approximation.

Proposition 2 *Suppose there exists a positive number* $\gamma > 0$ *such that the solution* u *of problem* (1.1)–(1.4) *satisfies*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_{W^{r+1,\infty}(\Omega)} \le C\epsilon^{-\gamma}. \tag{2.17}$$

Then there exists an ϵ*-independent and* h*-independent constant* $c_0 > 0$ *such that for* $\epsilon \in (0,1)$ *and a.e.* $t \in [0,T]$

$$\lambda^{DG}_h(t) := \inf_{\substack{\psi_h\in V_h\\ \psi_h\ne 0}} \frac{a_h(\psi_h,\psi_h) + \frac{1}{\epsilon^2}\Big(f'\big(P^h_r u(t)\big)\psi_h, \psi_h\Big)_{\mathcal{T}_h}}{\|\psi_h\|^2_{L^2(\mathcal{T}_h)}} \ge -c_0, \tag{2.18}$$

provided that h *satisfies the constraint*

$$h^{2-\frac{d}{2}} \le C_0(C_1C_2)^{-1}\epsilon^{\max\{\sigma_1+3,\sigma_2+2\}}, \tag{2.19}$$

$$h^{\min\{r+1,s\}}|\ln h|^{\overline{r}} \le C_0(C_1C_2)^{-1}\epsilon^{\gamma+2}, \tag{2.20}$$

where C_2 *arises from the following inequality:*

$$\|u - P_r^h u\|_{L^\infty((0,T);L^\infty(\Omega))} \le C_2 h^{\min\{r+1,s\}} |\ln h|^{\overline{r}} \epsilon^{-\gamma}, \tag{2.21}$$

$$\|u - \widehat{P}_r^h u\|_{L^\infty((0,T);L^\infty(\Omega))} \le C_2 h^{2-\frac{d}{2}} \epsilon^{-\max\{\sigma_1+1,\sigma_2\}}. \tag{2.22}$$

Lemma 2 *Let* $\{S_\ell\}_{\ell\ge1}$ *be a positive nondecreasing sequence and* $\{b_\ell\}_{\ell\ge1}$ *and* $\{k_\ell\}_{\ell\ge1}$ *be non-negative sequences, and* $p > 1$ *be a constant. If*

$$S_{\ell+1} - S_\ell \le b_\ell S_\ell + k_\ell S_\ell^p \qquad \text{for } \ell \ge 1, \tag{2.23}$$

$$S_1^{1-p} + (1-p) \sum_{s=1}^{\ell-1} k_s a_{s+1}^{1-p} > 0 \qquad \text{for } \ell \ge 2, \tag{2.24}$$

then

$$S_\ell \le \frac{1}{a_\ell} \left\{ S_1^{1-p} + (1-p) \sum_{s=1}^{\ell-1} k_s a_{s+1}^{1-p} \right\}^{\frac{1}{1-p}} \qquad \text{for } \ell \ge 2, \tag{2.25}$$

where

$$a_\ell := \prod_{s=1}^{\ell-1} \frac{1}{1+b_s} \qquad \text{for } \ell \ge 2. \tag{2.26}$$

CHAPTER 3. FULLY DISCRETE IP-DG APPROXIMATIONS

3.1 Discretized DG scheme

We are now ready to introduce our fully discrete DG finite element methods for problem (1.1)–(1.4). They are defined by seeking $u^m \in V_h$ for $m = 0, 1, 2, \cdots, M$ such that

$$\big(d_t u^{m+1}, v_h\big)_{\mathcal{T}_h} + a_h(u^{m+\frac{1}{2}}, v_h) + \frac{1}{\epsilon^2}\big(f^{m+1}, v_h\big)_{\mathcal{T}_h} = 0 \quad \forall v_h \in V_h, \tag{3.1}$$

where

$$\begin{aligned} a_h(u, v_h) := \big(\nabla u, \nabla v_h\big)_{\mathcal{T}_h} &- \big\langle \{\partial_n u\}, [v_h] \big\rangle_{\mathcal{E}_h^I} \\ &+ \lambda \big\langle [u], \{\partial_n v_h\} \big\rangle_{\mathcal{E}_h^I} + j_h(u, v_h), \end{aligned} \tag{3.2}$$

$$j_h(u, v_h) := \sum_{e \in \mathcal{E}_h^I} \frac{\sigma_e}{h_e} \big\langle [u], [v_h] \big\rangle_e, \tag{3.3}$$

$$\begin{aligned} f^{m+1} &:= \frac{1}{4}\big[(u^{m+1})^3 + (u^{m+1})^2 u^m + u^{m+1}(u^m)^2 + (u^m)^3\big] - u^{m+\frac{1}{2}} \\ &= \frac{F(u^{m+1}) - F(u^m)}{u^{m+1} - u^m}. \end{aligned} \tag{3.4}$$

where $u^{m+\frac{1}{2}} = \frac{u^{m+1} + u^m}{2}$, $\lambda = 0, \pm 1$ and σ_e is a positive piecewise constant function on $\mathcal{E}_h^I$, which will be chosen later (see Lemma 3). In addition, we need to supply u_h^0 to start the time-stepping, whose choice will be clear (and will be specified) below.

Lemma 3 *There exist constants $\sigma_0, \alpha > 0$ such that for $\sigma_e > \sigma_0$ for all $e \in \mathcal{E}_h$ there holds*

$$\Phi^h(v_h) \geq \alpha \|v_h\|_{1,DG}^2 \qquad \forall v_h \in V_h,$$

where

$$\|v_h\|_{1,DG}^2 := \|\nabla v_h\|_{L^2(\mathcal{T}_h)}^2 + j_h(v_h, v_h).$$

Now we introduce three mesh-dependent energy functionals which can be regarded as DG counterparts of the continuous Cahn-Hilliard energy J_ϵ defined in (2.2).

$$\Phi^h(v) := \frac{1}{2}\|\nabla v\|^2_{L^2(\mathcal{T}_h)} - \big\langle\{\partial_n v\},[v]\big\rangle_{\mathcal{E}_h^I} + \frac{1}{2}j_h(v,v) \qquad \forall v\in H^2(\mathcal{T}_h), \tag{3.5}$$

$$J_\epsilon^h(v) := \Phi^h(v) + \frac{1}{\epsilon^2}\big(F(v),1\big)_{\mathcal{T}_h} \qquad \forall v\in H^2(\mathcal{T}_h), \tag{3.6}$$

$$I_\epsilon^h(v) := \Phi^h(v) + \frac{1}{\epsilon^2}\big(F_c^+(v),1\big)_{\mathcal{T}_h} \qquad \forall v\in H^2(\mathcal{T}_h), \tag{3.7}$$

It is easy to check that Φ^h and I_ϵ^h are convex functionals but J_ϵ^h is not because F is not convex. Moreover, we have:

Lemma 4 *Let $\lambda=-1$ in (3.2), then there holds for all $v_h, w_h\in V_h$*

$$\Big(\frac{\delta\Phi^h(v_h)}{\delta v_h}, w_h\Big)_{\mathcal{T}_h} := \lim_{s\to 0}\frac{\Phi^h(v_h+sw_h)-\Phi^h(v_h)}{s} = a_h(v_h,w_h), \tag{3.8}$$

$$\begin{aligned}\Big(\frac{\delta J_\epsilon^h(v_h)}{\delta v_h}, w_h\Big)_{\mathcal{T}_h} &:= \lim_{s\to 0}\frac{J_\epsilon^h(v_h+sw_h)-J_\epsilon^h(v_h)}{s} \\ &= a_h(v_h,w_h) + \frac{1}{\epsilon^2}\big(F'(v_h),w_h\big)_{\mathcal{T}_h},\end{aligned} \tag{3.9}$$

$$\begin{aligned}\Big(\frac{\delta I_\epsilon^h(v_h)}{\delta v_h}, w_h\Big)_{\mathcal{T}_h} &:= \lim_{s\to 0}\frac{I_\epsilon^h(v_h+sw_h)-I_\epsilon^h(v_h)}{s} \\ &= a_h(v_h,w_h) + \frac{1}{\epsilon^2}\big((F_c^+)'(v_h),w_h\big)_{\mathcal{T}_h}.\end{aligned} \tag{3.10}$$

3.2 Stability of the DG scheme

Theorem 1 *The scheme (3.1)–(3.4) is unconditionally stable for all $h,k>0$.*

Proof: We have the DG scheme as below:

$$\big(d_t u^{m+1}, v_h\big) + a_h\big(\frac{u^{m+1}+u^m}{2}, v_h\big) + \frac{1}{\epsilon^2}\big(f^{m+1}, v_h\big) = 0. \tag{3.11}$$

Let $v_h = d_t u^{m+1}$, and we will get:

$$\big(d_t u^{m+1}, d_t u^{n+1}\big) + a_h\big(\frac{u^{m+1}+u^m}{2}, d_t u^{m+1}\big) + \frac{1}{\epsilon^2}\big(\frac{F(u^{m+1})-F(u^m)}{u^{m+1}-u^m}, d_t u^{m+1}\big) = 0. \tag{3.12}$$

Rearrange it to get:

$$\|d_t u^{m+1}\|^2_{L^2} + \frac{1}{2}d_t[a_h(u^{m+1},u^{m+1})] + \frac{1}{\epsilon^2}d_t F(u^{m+1}) = 0, \tag{3.13}$$

$$d_t[\frac{1}{2}a_h(u^{m+1},u^{m+1}) + \frac{1}{\epsilon^2}(F(u^{m+1}),1)] \le 0. \tag{3.14}$$

And the proof is complete.

3.3 Well-posedness of the DG scheme

We want to get a second order approximation of $f(u^{m+1}, u^m)$, which leads to unconditionally energy stable schemes. We split the function $F(v) = \frac{1}{4}(v^2-1)^2$ into the difference of two convex parts and get the convex decomposition $F(v) = F_c^+(v) - F_c^-(v)$,where $F_c^+(v) := \frac{1}{4}(v^4+1)$and $F_c^-(v) := \frac{1}{2}v^2$.

Now we want to construct a second-order energy-stable scheme to approximate the two convex functions $F_c^+(u)$ and $F_c^-(u)$.

$$f^+(u^{m+1}, u^m) = \frac{F_c^+(u^{m+1}) - F_c^+(u^m)}{u^{m+1} - u^m},$$

$$f^-(u^{m+1}, u^m) = \frac{F_c^-(u^{m+1}) - F_c^-(u^m)}{u^{m+1} - u^m}.$$

Theorem 2 *Under the constraint $k < 2\epsilon^2$, there exists a unique solution of the scheme (3.1)-(3.4).*

Proof: Define the following functional:

$$J(u^{m+1}) = \frac{1}{4}a_h(u^{m+1}, u^{m+1}) + \frac{1}{\epsilon^2}\int_{\mathcal{T}_h} F_+(u^{m+1}, u^m) \tag{3.15}$$
$$+ (\frac{1}{2k} - \frac{1}{4\epsilon^2})\|u^{m+1}\|^2_{L^2(\mathcal{T}_h)} + \frac{1}{2}a_h(u^m, u^{m+1}) + \int_{\mathcal{T}_h}(-\frac{1}{2\epsilon^2} - \frac{1}{k})u^m u^{m+1}.$$

Take the derivative of the functional $J(u^{m+1})$, and will get:

$$\Big(\frac{\delta J(u^{m+1})}{\delta u^{m+1}}, v_h\Big)_{\mathcal{T}_h} = \frac{1}{2}a_h(u^{m+1}, v_h) + \frac{1}{\epsilon^2}\int_{\mathcal{T}_h} f^+(u^{m+1}, u^m) \tag{3.16}$$
$$+ (\frac{1}{2k} - \frac{1}{4\epsilon^2})2(u^{m+1}, v_h)_{\mathcal{T}_h} + \frac{1}{2}a_h(u^m, v_h) + (-\frac{1}{2\epsilon^2} - \frac{1}{k})(u^m, v_h)_{\mathcal{T}_h}.$$

Rearrange it, and we will get:

$$\Big(\frac{\delta J(u^{m+1})}{\delta u^{m+1}}, v_h\Big)_{\mathcal{T}_h} = \big(d_t u^{m+1}, v_h\big)_{\mathcal{T}_h} + a_h(u^{m+\frac{1}{2}}, v_h) + \frac{1}{\epsilon^2}\big(f^{m+1}, v_h\big)_{\mathcal{T}_h} = 0. \tag{3.17}$$

Also we can see from (3.10) the first two terms of $J(u^{m+1})$ are convex, also since the last two terms are linear with respect to u^{m+1}, so they are also convex, so if we restrict the coefficient of third term to be positive, that is, if we restrict $k < 2\epsilon^2$, then the $J(u^{m+1})$ will be a convex functional, and the uniqueness of the solution to this scheme is approved.

3.4 Error estimates analysis

The main result of this subsection is the following error estimate theorem.

Theorem 3 *suppose $\sigma_e > \max\{\sigma_0, \sigma_0'\}$. Let u and $\{u_h^m\}_{m=1}^M$ denote respectively the solutions of problems (1.1)–(1.4) and (3.1)–(3.5). Assume $u \in H^2((0,T);\, L^2(\Omega)) \cap L^2((0,T); W^{s,\infty}(\Omega))$ and suppose (GA) and (2.17) hold. Then, under the following mesh and initial value constraints:*

$$h^{2-\frac{d}{2}} \le C_0(C_1C_2)^{-1}\epsilon^{\max\{\sigma_1+3,\sigma_2+2\}},$$

$$h^{\min\{r+1,s\}}|\ln h|^{\overline{r}} \le C_0(C_1C_2)^{-1}\epsilon^{\gamma+2},$$

$$k < A(\epsilon),$$

$$u_h^0 \in S_h \text{ such that } \quad \|u_0 - u_h^0\|_{L^2(\mathcal{T}_h)} \le Ch^{\min\{r+1,s\}},$$

there hold

$$\max_{0\le m\le M}\|u(t_m) - u_h^m\|_{L^2(\mathcal{T}_h)} \le C(k^2 + h^{\min\{r+1,s\}})\epsilon^{-(\sigma_1+2)}. \tag{3.18}$$

$$\Big(k\sum_{m=1}^{M}\|u(t_m) - u_h^m\|^2_{H^1(\mathcal{T}_h)}\Big)^{\frac{1}{2}} \le C(k^2 + h^{\min\{r+1,s\}-1})\epsilon^{-(\sigma_1+3)}, \tag{3.19}$$

$$\begin{aligned}\max_{0\le m\le M}\|u(t_m) - u_h^m\|_{L^\infty(\mathcal{T}_h)} &\le Ch^{\min\{r+1,s\}}|\ln h|^{\overline{r}}\epsilon^{-\gamma} \\ &\quad + Ch^{-\frac{d}{2}}(k^2 + h^{\min\{r+1,s\}})\epsilon^{-(\sigma_1+2)}.\end{aligned} \tag{3.20}$$

Proof: Since the proof is long, we split the proof into four steps:

Step 1:

We write:

$$u(t_m) - u^m = \eta^m + \xi^m, \quad \eta^m := u(t_m) - P_r^h u(t_m), \quad \xi^m := P_r^h u(t_m) - u^m.$$

Multiply v_h on both sides of the Allen-Cahn equation in (1.1) at the point $u(t_{m+\frac{1}{2}})$

$$\big(u_t(t_{m+\frac{1}{2}}), v_h\big)_{\mathcal{T}_h} + a_h(u(t_{m+\frac{1}{2}}), v_h) + \frac{1}{\epsilon^2}\big(f(u(t_{m+\frac{1}{2}})), v_h\big)_{\mathcal{T}_h} = 0, \tag{3.21}$$

for all $v_h \in V_h$, where $t_{m+\frac{1}{2}} = \frac{t_{m+1}+t_m}{2}$.

Subtract (3.1) from (3.21), we get the following equation:

$$\big(u_t(t_{m+\frac{1}{2}}) - \frac{u^{m+1}-u^m}{k}, v_h\big)_{\mathcal{T}_h} + a_h\big(u(t_{m+\frac{1}{2}}) - \frac{u^{m+1}+u^m}{2}, v_h\big) \tag{3.22}$$
$$+ \frac{1}{\epsilon^2}\big(f(u(t_{m+\frac{1}{2}})) - f^{m+1}, v_h\big)_{\mathcal{T}_h} = 0.$$

From Taylor expansion:

$$u(t_{m+1}) = u(\frac{t_{m+1}+t_m}{2}) + u_t(\frac{t_{m+1}+t_m}{2})(\frac{t_{m+1}-t_m}{2}) + R_1^m,$$

where $R_1^m = u_{tt}(\xi_1)(\frac{t_{m+1}-t_m}{2})^2$.

$$u(t_m) = u(\frac{t_{m+1}+t_m}{2}) - u_t(\frac{t_{m+1}+t_m}{2})(\frac{t_{m+1}-t_m}{2}) + R_2^m,$$

where $R_2^m = u_{tt}(\xi_2)(\frac{t_{m+1}-t_m}{2})^2$. And we will get:

$$u(t_{m+\frac{1}{2}}) = \frac{u(t_{m+1})+u(t_m)}{2} - \frac{\big(R_1^m + R_2^m\big)}{2}, \tag{3.23}$$

$$u_t(t_{m+\frac{1}{2}}) = \frac{u(t_{m+1})-u(t_m)}{k} - \frac{\big(R_1^m - R_2^m\big)}{k}. \tag{3.24}$$

Use (3.23) and (3.24) into (3.22), we will get:

$$\big(\frac{\xi^{m+1}-\xi^m}{k} + \frac{\eta^{m+1}-\eta^m}{k} - \frac{\big(R_1^m - R_2^m\big)}{k}, v_h\big)_{\mathcal{T}_h} \tag{3.25}$$
$$+ a_h\big(\frac{\xi^{m+1}+\xi^m}{2} + \frac{\eta^{m+1}+\eta^m}{2} - \frac{\big(R_1^m + R_2^m\big)}{2}, v_h\big)$$
$$+ \frac{1}{\epsilon^2}\big(f(u(t_{m+\frac{1}{2}})) - f^{m+1}, v_h\big)_{\mathcal{T}_h} = 0.$$

$$\big(d_t\xi^{m+1}, v_h\big)_{\mathcal{T}_h} + a_h\big(\frac{\xi^{m+1}+\xi^m}{2}, v_h\big) \tag{3.26}$$
$$+ \frac{1}{\epsilon^2}\big(f(u(t_{m+\frac{1}{2}})) - f^{m+1}, v_h\big)_{\mathcal{T}_h}$$
$$= \big(\frac{\big(R_1^m - R_2^m\big)}{k}, v_h\big)_{\mathcal{T}_h} - \big(d_t\eta^{m+1}, v_h\big)_{\mathcal{T}_h}$$
$$- a_h\big(\frac{\eta^{m+1}+\eta^m}{2}, v_h\big) + a_h\big(\frac{\big(R_1^m + R_2^m\big)}{2}, v_h\big)$$
$$= \big(\frac{\big(R_1^m - R_2^m\big)}{k}, v_h\big)_{\mathcal{T}_h} - \big(d_t\eta^{m+1}, v_h\big)_{\mathcal{T}_h}$$
$$+ \big(\frac{\eta^{m+1}+\eta^m}{2}, v_h\big)_{\mathcal{T}_h} + a_h\big(\frac{\big(R_1^m + R_2^m\big)}{2}, v_h\big).$$

Let $v_h = \frac{\xi^{m+1}+\xi^m}{2}$, for the first term on the left hand side:

$$\big(d_t\xi^{m+1}, \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h} = \frac{1}{2}d_t\|\xi^{m+1}\|^2_{L^2(\mathcal{T}_h)}. \tag{3.27}$$

We split the third term on the left hand side in (3.26) into two parts and deal with them separately:

$$\begin{aligned}
&\frac{1}{\epsilon^2}\big(f(u(t_{m+\frac{1}{2}})) - f^{m+1}, \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h} \\
&= \frac{1}{\epsilon^2}\big(f(u(t_{m+\frac{1}{2}})) - f(\frac{u(t_{m+1})+u(t_m)}{2}), \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h} \\
&+ \frac{1}{\epsilon^2}\big(f(\frac{u(t_{m+1})+u(t_m)}{2}) - f^{m+1}, \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h}.
\end{aligned} \tag{3.28}$$

Let $\hat{u}(t_{m+\frac{1}{2}}) = \frac{u(t_{m+1})+u(t_m)}{2}$, the we have the following:

$$\begin{aligned}
&f(u(t_{m+\frac{1}{2}})) - f(\hat{u}(t_{m+\frac{1}{2}})) \\
&= f\big(\hat{u}(t_{m+\frac{1}{2}}) - \frac{1}{8}k^2(u''(\xi_1)+u''(\xi_2))\big) - f(\hat{u}(t_{m+\frac{1}{2}})) \\
&= f'(\xi_{12})(-\frac{1}{8})k^2((u''(\xi_1)+u''(\xi_2)) \geq -Ck^2.
\end{aligned} \tag{3.29}$$

Since f' and u'' both are bounded, we will get the following inequality by Cauchy-Schwarz inequality:

$$\begin{aligned}
&\frac{1}{\epsilon^2}\big(f(u(t_{m+\frac{1}{2}})) - f(\hat{u}(t_{m+\frac{1}{2}})), \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h} \\
&\geq -\frac{1}{\epsilon^2}\big(Ck^2, \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h} \\
&\geq -\frac{1}{\epsilon^4}Ck^4 - \|\xi^{m+\frac{1}{2}}\|^2_{L^2(\mathcal{T}_h)}.
\end{aligned} \tag{3.30}$$

For the last term of the right hand side in (3.26):

$$\begin{aligned}
&a_h\big(\frac{(R_1^m+R_2^m)}{2}, \frac{\xi^{m+1}+\xi^m}{2}\big) = a_h\big(\frac{(R_1^m+R_2^m)}{2\epsilon}, \frac{\epsilon(\xi^{m+1}+\xi^m)}{2}\big) \\
&\leq a_h\big(\frac{(R_1^m+R_2^m)}{2\epsilon}, \frac{(R_1^m+R_2^m)}{2\epsilon}\big) + a_h\big(\frac{\epsilon(\xi^{m+1}+\xi^m)}{2}, \frac{\epsilon(\xi^{m+1}+\xi^m)}{2}\big) \\
&\leq Ck^4\epsilon^{-2} + \epsilon^2 a_h\big(\xi^{m+\frac{1}{2}}, \xi^{m+\frac{1}{2}}\big).
\end{aligned} \tag{3.31}$$

Substitute (3.27),(3.30) and (3.31) into (3.26), and we will get:

$$\begin{aligned}
&\frac{1}{2}d_t\|\xi^{m+1}\|^2_{L^2(\mathcal{T}_h)} + a_h(\frac{\xi^{m+1}+\xi^m}{2}, \frac{\xi^{m+1}+\xi^m}{2}) \\
&+ \frac{1}{\epsilon^2}\big(f(\hat{u}(t_{m+\frac{1}{2}})) - f^{m+1}, \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h} \\
&= (\frac{(R_1^m - R_2^m)}{k}, v_h)_{\mathcal{T}_h} - (d_t\eta^{m+1}, v_h)_{\mathcal{T}_h} \\
&+ (\frac{\eta^{m+1}+\eta^m}{2}, v_h)_{\mathcal{T}_h} + a_h(\frac{R_1^m + R_2^m}{2}, v_h). \\
&\leq (\|(\frac{(R_1^m - R_2^m)}{k})\|^2_{L^2(\mathcal{T}_h)} + \|d_t\eta^{m+1}\|^2_{L^2(\mathcal{T}_h)} \\
&+ \|(\frac{\eta^{m+1}+\eta^m}{2})\|^2_{L^2(\mathcal{T}_h)}\big)\big(\frac{\xi^{m+1}+\xi^m}{2})\|^2_{L^2(\mathcal{T}_h)} \\
&+ Ck^4[\epsilon^{-4}+\epsilon^{-2}] + \epsilon^2 a_h\big(\xi^{m+\frac{1}{2}}, \xi^{m+\frac{1}{2}}\big) + \|\xi^{m+\frac{1}{2}}\|^2_{L^2(\mathcal{T}_h)}.
\end{aligned} \tag{3.32}$$

Using the integral form of Taylor formula, we can get:

$$|\frac{R_1^m - R_2^m}{k}| = |\frac{k(u_{tt}(\xi_1) - u_{tt}(\xi_2))}{4} = |\frac{ku_{ttt}(\xi_{11})(\xi_1 - \xi_2)}{4}| \leq Ck^2.$$

Hence

$$\|\frac{R_1^m - R_2^m}{k}\|^2_{L^2(\mathcal{T}_h)} \leq Ck^4. \tag{3.33}$$

Summing in m from 1 to ℓ, using (3.13),(3.32) and (3.33), and we will get the following inequality:

$$\begin{aligned}
&\|\xi^\ell\|^2_{L^2(\mathcal{T}_h)} + 2k\sum_{m=1}^{\ell} a_h(\frac{\xi^m+\xi^{m-1}}{2}, \frac{\xi^m+\xi^{m-1}}{2}) \\
&+ 2k\sum_{m=1}^{\ell}\frac{1}{\epsilon^2}\big(f(\hat{u}(t_{m-\frac{1}{2}})) - f^m, \frac{\xi^m+\xi^{m-1}}{2}\big)_{\mathcal{T}_h} \\
&\leq \|\xi^0\|^2_{L^2(\mathcal{T}_h)} + Ch^{2\min\{r+1,s\}}\|u\|^2_{H^1((0,T);H^s(\Omega))} \\
&+ 2Ck^4[\epsilon^{-4}+\epsilon^{-2}+1] + 2k\sum_{m=1}^{\ell}\epsilon^2 a_h\big(\xi^{m-\frac{1}{2}}, \xi^{m-\frac{1}{2}}\big) + 4k\sum_{m=1}^{\ell}\|\xi^{m-\frac{1}{2}}\|^2_{L^2(\mathcal{T}_h)}.
\end{aligned} \tag{3.34}$$

Step 2: We want to bound the term$\big(f(\hat{u}(t_{m+\frac{1}{2}})) - f^{m+1}, \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h}$ on the left hand side of (3.34):

$$f(\hat{u}(t_{m+\frac{1}{2}})) - f^{m+1} = [f(\hat{u}(t_{m+\frac{1}{2}})) - f(P_r^h\hat{u}(t_{m+\frac{1}{2}}))] + [f(P_r^h\hat{u}(t_{m+\frac{1}{2}})) - f^{m+1}]. \tag{3.35}$$

For the first part on the right hand side of (3.35),we get:

$$|f(u^{m+\frac{1}{2}}) - f(P_r^h \hat{u}(t_{m+\frac{1}{2}}))| = |f'(\xi)|\big|\hat{u}(t_{m+\frac{1}{2}}) - P_r^h \hat{u}(t_{m+\frac{1}{2}})\big| \geq -C|\frac{\eta^{m+1}+\eta^m}{2}|. \quad (3.36)$$

For the second part on the right hand side of (3.35),we get:

$$f(P_r^h \hat{u}(t_{m+\frac{1}{2}})) - f^{m+1} = \big(\frac{P_r^h u(t_{m+1}) + P_r^h u(t_m)}{2}\big)^3 - \big(\frac{P_r^h u(t_{m+1}) + P_r^h u(t_m)}{2}\big) \quad (3.37)$$
$$- \big[\frac{1}{4}[(u^{m+1})^3 + (u^{m+1})^2 u^m + u^{m+1}(u^m)^2 + (u^m)^3] - \frac{u^{m+1}+u^m}{2}\big]$$
$$= \frac{\big(P_r^h u(t_{m+1}) + P_r^h u(t_m)\big)^3}{8} - \frac{2}{8}[(u^{m+1})^3 + (u^{m+1})^2 u^m + u^{m+1}(u^m)^2 + (u^m)^3]$$
$$- \big[\big(\frac{P_r^h u(t_{m+1}) + P_r^h u(t_m)}{2}\big) - \frac{u^{m+1}+u^m}{2}\big]$$
$$= \frac{\big(P_r^h u(t_{m+1}) + P_r^h u(t_m)\big)^3}{8} - \frac{2}{8}[(P_r^h u(t_{m+1}) - \xi^{m+1})^3 + (P_r^h u(t_{m+1}) - \xi^{m+1})^2 (P_r^h u(t_m) - \xi^m)$$
$$+ (P_r^h u(t_{m+1}) - \xi^{m+1})(P_r^h u(t_m) - \xi^m)^2 + (P_r^h u(t_m) - \xi^m)^3] - \frac{(\xi^{(m+1)} + \xi^m)}{2}.$$

We split the above into four terms: constant term with resect to ξ^{m+1} and ξ^m, linear, quadratic and cubic in terms of ξ^{m+1} and ξ^m.

For constant term, we have

$$\big(\frac{1}{8}(P_r^h u(t_{m+1}) - P_r^h u(t_m))^2 (P_r^h u(t_{m+1}) + P_r^h u(t_m)), \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h} \quad (3.38)$$
$$\geq -C(h^4 + k^2)\big(1, \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h}$$
$$\geq -C(h^8 + k^4) - C\|\xi^{m+\frac{1}{2}}\|^2_{L^2(\mathcal{T}_h)}.$$

By the boundness of $P_r^h u^m$ and $|P_r^h u(t_{m+1}) - P_r^h u(t_m)| \leq h^2 + k$.

For the linear term, we have the following:

$$l = \frac{1}{4}\big\{\xi^{m+1}[3(P_r^h u(t_{m+1}))^2 + P_r^h u(t_{m+1}) P_r^h u(t_m) + (P_r^h u(t_m))^2] \quad (3.39)$$
$$+ \xi^m [3(P_r^h u(t_m))^2 + P_r^h u(t_{m+1}) P_r^h u(t_m) + (P_r^h u(t_{m+1}))^2]\big\} - \frac{(\xi^{m+1}+\xi^m)}{2}$$
$$= \frac{1}{4}(\xi^{m+1}+\xi^m)(P_r^h u(t_{m+1}) + P_r^h u(t_m))^2$$
$$+ \frac{1}{2}[\xi^{m+1}(P_r^h u(t_{m+1}))^2 + \xi^m (P_r^h u(t_m))^2] - \frac{(\xi^{m+1}+\xi^m)}{2}.$$

And we have:

$$
\begin{aligned}
&\big(\frac{1}{4}(\xi^{m+1}+\xi^{m})(P_r^h u(t_{m+1})+P_r^h u(t_m))^2 \\
&+\frac{1}{2}[\xi^{m+1}(P_r^h u(t_{m+1}))^2+\xi^{m}(P_r^h u(t_m))^2], \frac{\xi^{m+1}+\xi^{m}}{2}\big)_{\mathcal{T}_h} \\
&= \big(\frac{1}{2}(P_r^h u(t_{m+1})+P_r^h u(t_m))^2, (\frac{\xi^{m+1}+\xi^{m}}{2})^2\big)_{\mathcal{T}_h} \\
&+\big(\frac{1}{2}[\xi^{m+1}(P_r^h u(t_{m+1}))^2+\xi^{m}(P_r^h u(t_m))^2], \frac{\xi^{m+1}+\xi^{m}}{2}\big)_{\mathcal{T}_h}.
\end{aligned} \tag{3.40}
$$

By using the Schwarz Inequality and $|P_r^h u(t_{m+1})-P_r^h u(t_m)| \le C(h^2+k)$, we get the following inequalities for the first and second terms of the right hand side of (3.40):

$$
\begin{aligned}
&\big(\frac{1}{2}(P_r^h u(t_{m+1})+P_r^h u(t_m))^2, (\frac{\xi^{m+1}+\xi^{m}}{2})^2\big)_{\mathcal{T}_h} \\
&\ge \big(2(P_r^h u(t_m))^2, (\frac{\xi^{m+1}+\xi^{m}}{2})^2\big)_{\mathcal{T}_h} - C(h^2+k)\|\xi^{m+\frac{1}{2}}\|^2_{L^2(\mathcal{T}_h)}
\end{aligned} \tag{3.41}
$$

$$
\begin{aligned}
&\big(\frac{1}{2}[\xi^{m+1}(P_r^h u(t_{m+1}))^2+\xi^{m}(P_r^h u(t_m))^2], \frac{\xi^{m+1}+\xi^{m}}{2}\big)_{\mathcal{T}_h} \\
&\ge \big((P_r^h u(t_m))^2, (\frac{\xi^{m+1}+\xi^{m}}{2})^2\big)_{\mathcal{T}_h} - C(h^2+k)\|\xi^{m+\frac{1}{2}}\|^2_{L^2(\mathcal{T}_h)}.
\end{aligned} \tag{3.42}
$$

$$
\begin{aligned}
&\big(l, \frac{\xi^{m+1}+\xi^{m}}{2}\big)_{\mathcal{T}_h} \\
&\ge \big(3(P_r^h u(t_m))^2-1, (\frac{\xi^{m+1}+\xi^{m}}{2})^2\big)_{\mathcal{T}_h} - C(h^2+k)\|\xi^{m+\frac{1}{2}}\|^2_{L^2(\mathcal{T}_h)} \\
&= \big((f'(P_r^h u(t_m)), (\frac{\xi^{m+1}+\xi^{m}}{2})^2\big)_{\mathcal{T}_h} - C(h^2+k)\|\xi^{m+\frac{1}{2}}\|^2_{L^2(\mathcal{T}_h)}.
\end{aligned} \tag{3.43}
$$

For the quadratic term, we get the inequality below;

$$
\begin{aligned}
q &= 3(\xi^{m+1})^2 P_r^h u(t_{m+1}) + (\xi^{m+1})^2 P_r^h u(t_m) + 2\xi^{m+1}\xi^{m} P_r^h u(t_{m+1}) \\
&+ (\xi^{m})^2 P_r^h u(t_{m+1}) + 2\xi^{m+1}\xi^{m} P_r^h u(t_m) + 3(\xi^{m})^2 P_r^h u(t_m) \\
&\ge -C_1[(\xi^{m+1})^2+(\xi^{m})^2].
\end{aligned} \tag{3.44}
$$

So we get:

$$
\begin{aligned}
&\big(q, \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h} \\
&\geq -C_1\big((\xi^{m+1})^2+(\xi^m)^2, \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h} \\
&\geq -C\|\xi^{m+\frac{1}{2}}\|^3_{L^3(\mathcal{T}_h)}.
\end{aligned} \tag{3.45}
$$

For cubic term, we have:

$$
\begin{aligned}
c &= \frac{1}{4}\big[(\xi^{m+1})^3+(\xi^{m+1})^2\xi^m+\xi^{m+1}(\xi^m)^2+(\xi^m)^3\big] \\
&= \frac{1}{4}\big[(\xi^{m+1})^2+(\xi^m)^2\big](\xi^{m+1}+\xi^m),
\end{aligned} \tag{3.46}
$$

the we have

$$
\big(c, \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h} = \big((\xi^{m+1})^2+(\xi^m)^2, (\xi^{m+1}+\xi^m)^2\big)_{\mathcal{T}_h} \geq 0. \tag{3.47}
$$

Combine all above together, we will get:

$$
\begin{aligned}
&\big(f(P_r^h\hat{u}(t_{m+\frac{1}{2}}))-f^{m+1}, \frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h} \\
&\geq -C|(\eta^{m+\frac{1}{2}}, \xi^{m+\frac{1}{2}})|_{\mathcal{T}_h} - C(h^8+k^4) - C\|\xi^{m+\frac{1}{2}}\|_{L^2(\mathcal{T}_h)} \\
&\big((f'(P_r^h u(t_m))), (\frac{\xi^{m+1}+\xi^m}{2})^2\big)_{\mathcal{T}_h} - C(h^2+k)\|\xi^{m+\frac{1}{2}}\|^2_{L^2(\mathcal{T}_h)} - C\|\xi^{m+\frac{1}{2}}\|^3_{L^3(\mathcal{T}_h)} \\
&+\frac{4k}{\epsilon^2}\big((\xi^{m+1})^2+(\xi^m)^2, (\xi^{m+1}+\xi^m)^2\big)_{\mathcal{T}_h}.
\end{aligned} \tag{3.48}
$$

Summing in m a from 1 to ℓ and we will get the following:

$$\frac{2k}{\epsilon^2}\sum_{m=1}^{\ell}\big(f(P_r^h\hat{u}(t_{m+\frac{1}{2}}))-f^{m+1},\frac{\xi^{m+1}+\xi^m}{2}\big)_{\mathcal{T}_h} \tag{3.49}$$
$$\geq -\frac{Ck}{\epsilon^2}\sum_{m=1}^{\ell}\|\eta^{m+\frac{1}{2}}\|_{\mathcal{T}_h}\|\xi^{m+\frac{1}{2}}\|_{\mathcal{T}_h}-C\frac{1}{\epsilon^2}(h^8+k^4)-\frac{Ck}{\epsilon^2}\sum_{m=1}^{\ell}\|\xi^m\|^2_{L^2(\mathcal{T}_h)}$$
$$+\frac{2k}{\epsilon^2}\sum_{m=1}^{\ell}\big(f'(P_r^hu(t_m)),(\frac{\xi^{m+1}+\xi^m}{2})^2\big)_{\mathcal{T}_h}-C\frac{k}{\epsilon^2}(h^2+k)\sum_{m=1}^{\ell}\|\xi^m\|^2_{L^2(\mathcal{T}_h)}$$
$$-C\frac{k}{\epsilon^2}\sum_{m=1}^{\ell}\|\xi^{m+\frac{1}{2}}\|^3_{L^3(\mathcal{T}_h)}+\sum_{m=1}^{\ell}\big((\xi^m)^2+(\xi^{m+1})^2,(\xi^m+\xi^{m+1})^2\big)_{\mathcal{T}_h},$$
$$\geq -Ch^{2\min\{r+1,s\}}\epsilon^{-4}\|u\|^2_{L^2((0,T);H^s(\Omega)}-C\frac{1}{\epsilon^2}(h^8+k^4)+\frac{2k}{\epsilon^2}\sum_{m=1}^{\ell}\big(f'(P_r^hu(t_m)),(\frac{\xi^{m+1}+\xi^m}{2})^2\big)_{\mathcal{T}_h}$$
$$+\frac{2k}{\epsilon^2}\sum_{m=1}^{\ell}\big((\xi^m)^2+(\xi^{m+1})^2,(\xi^m+\xi^{m+1})^2\big)_{\mathcal{T}_h}-C\frac{k}{\epsilon^2}\sum_{m=1}^{\ell}\|\xi^{m+\frac{1}{2}}\|^3_{L^3(\mathcal{T}_h)}$$
$$-C\frac{k}{\epsilon^2}(h^2+k+1)\sum_{m=1}^{\ell}\|\xi^m\|^2_{L^2(\mathcal{T}_h)}-k^2\sum_{m=1}^{\ell}\|\xi^m\|^2_{L^2(\mathcal{T}_h)}.$$

Substitute the inequality above into (3.34), and we get:

$$\|\xi^\ell\|^2_{L^2(\mathcal{T}_h)}+\frac{2k}{\epsilon^2}\sum_{m=1}^{\ell}\big((\xi^m)^2+(\xi^{m-1})^2,(\xi^m+\xi^{m-1})^2\big) \tag{3.50}$$
$$+2k(1-\epsilon^2)\sum_{m=1}^{\ell}\Big(a_h(\xi^{m-\frac{1}{2}},\xi^{m-\frac{1}{2}})+\frac{1}{\epsilon^2}\Big(f'\big(P_r^hu(t_{m-1})\big),(\xi^{m-\frac{1}{2}})^2\Big)_{\mathcal{T}_h}\Big)$$
$$+2k\sum_{m=1}^{\ell}\Big(f'\big(P_r^hu(t_{m-1})\big),(\xi^{m-\frac{1}{2}})^2\Big)_{\mathcal{T}_h}$$
$$\leq \|\xi^0\|^2_{L^2(\mathcal{T}_h)}+Ch^{2\min\{r+1,s\}}\big(\|u\|^2_{H^1((0,T);H^s(\Omega))}+\epsilon^{-4}\|u\|^2_{L^2((0,T);H^s(\Omega)}\big)+\frac{C}{\epsilon^2}(h^8+k^4)$$
$$+Ck^4[\epsilon^{-4}+\epsilon^{-2}+1]+Ck(1+\frac{k\epsilon^2}{\epsilon^2}+\frac{h^2+k}{\epsilon^2})\sum_{m=1}^{\ell}\|\xi^m\|^2_{L^2(\mathcal{T}_h)}+C\frac{k}{\epsilon^2}\sum_{m=1}^{\ell}\|\xi^{m+\frac{1}{2}}\|^3_{L^3(\mathcal{T}_h)}.$$

Step 3: In order to control the last two terms on the right-hand side of (3.49) we use the following Gagliardo-Nirenberg inequality [23]:

$$\|v\|^3_{L^3(K)}\leq C\Big(\|\nabla v\|^{\frac{d}{2}}_{L^2(K)}\big\|v\big\|^{\frac{6-d}{2}}_{L^2(K)}+\|v\|^3_{L^2(K)}\Big)\qquad \forall K\in\mathcal{T}_h,$$

to get

$$\frac{Ck}{\epsilon^2}\sum_{m=1}^{\ell}\|\xi^m\|^3_{L^3(\mathcal{T}_h)} \le \epsilon^2\alpha k\sum_{m=1}^{\ell}\|\nabla\xi^m\|^2_{L^2(\mathcal{T}_h)} + \epsilon^2 k\sum_{m=1}^{\ell}\|\xi^m\|^2_{L^2(\mathcal{T}_h)} \tag{3.51}$$
$$+ C\epsilon^{-\frac{2(4+d)}{4-d}} k\sum_{m=1}^{\ell}\sum_{K\in\mathcal{T}_h}\big\|\xi^m\big\|^{\frac{2(6-d)}{4-d}}_{L^2(K)}$$
$$\le \epsilon^2\alpha k\sum_{m=1}^{\ell}\|\nabla\xi^m\|^2_{L^2(\mathcal{T}_h)}$$
$$+ C\epsilon^{-\frac{2(4+d)}{4-d}} k\sum_{m=1}^{\ell}\big\|\xi^m\big\|^{\frac{2(6-d)}{4-d}}_{L^2(\mathcal{T}_h)}.$$

Finally, for the third term on the left-hand side of the above inequality, we utilize the discrete spectrum estimate (2.18) to bound it from below as follows:

$$2k(1-\epsilon^2)\sum_{m=1}^{\ell}\Big(a_h(\xi^{m-\frac12},\xi^{m-\frac12}) + \frac{1}{\epsilon^2}\Big(f'\big(P_r^h u(t_{m-1})\big),(\xi^{m-\frac12})^2\Big)_{\mathcal{T}_h}\Big) \tag{3.52}$$
$$+ 4k\sum_{m=1}^{\ell}\Big(f'\big(P_r^h u(t_{m-1})\big),(\xi^{m-\frac12})^2\Big)_{\mathcal{T}_h}$$
$$= 2k(1-2\epsilon^2)\sum_{m=1}^{\ell}\Big(a_h(\xi^{m-\frac12},\xi^{m-\frac12}) + \frac{1}{\epsilon^2}\Big(f'\big(P_r^h u(t_{m-1})\big),(\xi^{m-\frac12})^2\Big)_{\mathcal{T}_h}\Big)$$
$$+ 2k\epsilon^2 a_h(\xi^{m-\frac12},\xi^{m-\frac12}) + 4k\sum_{m=1}^{\ell}\Big(f'\big(P_r^h u(t_{m-1})\big),(\xi^{m-\frac12})^2\Big)_{\mathcal{T}_h}$$
$$\ge -2(1-2\epsilon^2)c_0 k\sum_{m=1}^{\ell}\|\xi^m\|^2_{L^2(\mathcal{T}_h)} + 4\epsilon^2\alpha k\sum_{m=1}^{\ell}\|\xi^{m-\frac12}\|^2_{1,\mathrm{DG}} - Ck\sum_{m=1}^{\ell}\|\xi^m\|^2_{L^2(\mathcal{T}_h)}.$$

Step 4: Substitute (3.51) and (3.52) into (3.50), and we get the following:

$$\|\xi^\ell\|^2_{L^2(\mathcal{T}_h)} + 3\epsilon^2\alpha k\sum_{m=1}^{\ell}\|\xi^m\|^2_{1,\mathrm{DG}} + \frac{2k}{\epsilon^2}\sum_{m=1}^{\ell}\big((\xi^m)^2 + (\xi^{m-1})^2, (\xi^m + \xi^{m-1})\big) \tag{3.53}$$
$$\le Ck\Big(1 + \frac{k\epsilon^2}{\epsilon^2} + \frac{h^2+k}{\epsilon^2}\Big)\sum_{m=1}^{\ell}\|\xi^m\|^2_{L^2(\mathcal{T}_h)} + C\epsilon^{-\frac{2(4+d)}{4-d}} k\sum_{m=1}^{\ell}\big\|\xi^m\big\|^{\frac{2(6-d)}{4-d}}_{L^2(\mathcal{T}_h)}$$
$$+ \|\xi^0\|^2_{L^2(\mathcal{T}_h)} + Ch^{2\min\{r+1,s\}}\big(\|u\|^2_{H^1((0,T);H^s(\Omega))} + \epsilon^{-4}\|u\|^2_{L^2((0,T);H^s(\Omega))}\big)$$
$$+ \frac{C}{\epsilon^2}(h^8 + k^4) + Ck^4[\epsilon^{-4} + \epsilon^{-2} + 1].$$

Notice that on the right hand side, we need to choose the appropriate initial value u_h^0, so that $\|\xi^0\|_{L^2(\mathcal{T}_h)} = O(h^{\min\{r+1,s\}})$ to maintain the optimal rate of convergence in h. Clearly, both the

L^2 and the elliptic projection of u_0 work. and in the latter case, we get $\xi^0 = 0$.

It then follows from (2.7), (2.9), (2.12) and (3.53) that

$$\|\xi^\ell\|^2_{L^2(\mathcal{T}_h)} + 3\epsilon^2\alpha k \sum_{m=1}^{\ell} \|\xi^m\|^2_{1,\mathrm{DG}} + \frac{2k}{\epsilon^2} \sum_{m=1}^{\ell} \big((\xi^m)^2 + (\xi^{m-1})^2, (\xi^m + \xi^{m-1})\big) \tag{3.54}$$
$$\le Ck(1 + \frac{k\epsilon^2}{\epsilon^2} + \frac{h^2+k}{\epsilon^2}) \sum_{m=1}^{\ell} \|\xi^m\|^2_{L^2(\mathcal{T}_h)} + C\epsilon^{-\frac{2(4+d)}{4-d}} k \sum_{m=1}^{\ell} \big\|\xi^m\big\|^{\frac{2(6-d)}{4-d}}_{L^2(\mathcal{T}_h)}$$
$$+ Ch^{2\min\{r+1,s\}}\epsilon^{-2(\sigma_1+2)} + \frac{C}{\epsilon^2}(h^8 + k^4) + Ck^4[\epsilon^{-4} + \epsilon^{-2} + 1].$$

since u^ℓ can be written as

$$u^\ell = k\sum_{m=1}^{\ell} d_t u^m + u^0, \tag{3.55}$$

then by (2.3) and (3.11), we get

$$\|u^\ell\|_{L^2(\mathcal{T}_h)} \le k\sum_{m=1}^{\ell} \|d_t u^m\|_{L^2(\mathcal{T}_h)} + \|u^0\|_{L^2(\mathcal{T}_h)} \le C\epsilon^{-2\sigma_1}. \tag{3.56}$$

By the boundedness of the projection, we have

$$\|\xi^\ell\|^2_{L^2(\mathcal{T}_h)} \le C\epsilon^{-2\sigma_1}. \tag{3.57}$$

Then the above inequality is equivalent to the form below:

$$\|\xi^\ell\|^2_{L^2(\mathcal{T}_h)} + k\sum_{m=1}^{\ell} 3\epsilon^2\alpha \|\xi^m\|^2_{1,\mathrm{DG}} \le H_1 + H_2, \tag{3.58}$$

where

$$H_1 := Ck(1 + \frac{k\epsilon^2}{\epsilon^2} + \frac{h^2+k}{\epsilon^2}) \sum_{m=1}^{\ell-1} \|\xi^m\|^2_{L^2(\mathcal{T}_h)} + C\epsilon^{-\frac{2(4+d)}{4-d}} k \sum_{m=1}^{\ell-1} \big\|\xi^m\big\|^{\frac{2(6-d)}{4-d}}_{L^2(\mathcal{T}_h)} \tag{3.59}$$
$$+ Ch^{2\min\{r+1,s\}}\epsilon^{-2(\sigma_1+2)} + \frac{C}{\epsilon^2}(h^8 + k^4) + Ck^4[\epsilon^{-4} + \epsilon^{-2} + 1],$$

$$H_2 := Ck(1 + \frac{k\epsilon^2}{\epsilon^2} + \frac{h^2+k}{\epsilon^2})\|\xi^\ell\|^2_{L^2(\mathcal{T}_h)} + C\epsilon^{-\frac{2(4+d)}{4-d}} k \big\|\xi^\ell\big\|^{\frac{2(6-d)}{4-d}}_{L^2(\mathcal{T}_h)}. \tag{3.60}$$

It is easy to check that

$$H_2 < \frac{1}{2}\|\xi^\ell\|^2_{L^2(\mathcal{T}_h)} \qquad \text{provided that} \quad k < A(\epsilon). \tag{3.61}$$

By (3.58) we have

$$\|\xi^\ell\|^2_{L^2(\mathcal{T}_h)} + k\sum_{m=1}^{\ell} 3\epsilon^2\alpha\|\xi^m\|^2_{1,\mathrm{DG}} \le 2H_1 \tag{3.62}$$
$$\le 2Ck(1+\frac{k\epsilon^2}{\epsilon^2}+2\frac{h^2+k}{\epsilon^2})\sum_{m=1}^{\ell-1}\|\xi^m\|^2_{L^2(\mathcal{T}_h)} + 2C\epsilon^{-\frac{2(4+d)}{4-d}}k\sum_{m=1}^{\ell-1}\|\xi^m\|^{\frac{2(6-d)}{4-d}}_{L^2(\mathcal{T}_h)}$$
$$+2Ch^{2\min\{r+1,s\}}\epsilon^{-2(\sigma_1+2)} + 2\frac{C}{\epsilon^2}(h^8+k^4) + 2Ck^4[\epsilon^{-4}+\epsilon^{-2}+1]$$
$$\le Ck(1+\frac{k\epsilon^2}{\epsilon^2}+\frac{h^2+k}{\epsilon^2})\sum_{m=1}^{\ell-1}\|\xi^m\|^2_{L^2(\mathcal{T}_h)} + C\epsilon^{-\frac{2(4+d)}{4-d}}k\sum_{m=1}^{\ell-1}\|\xi^m\|^{\frac{2(6-d)}{4-d}}_{L^2(\mathcal{T}_h)}$$
$$+Ch^{2\min\{r+1,s\}}\epsilon^{-2(\sigma_1+2)} + \frac{C}{\epsilon^2}(h^8+k^4) + Ck^4[\epsilon^{-4}+\epsilon^{-2}+1].$$

Let $d_\ell \ge 0$ be the slack variable such that

$$\|\xi^\ell\|^2_{L^2(\mathcal{T}_h)} + k\sum_{m=1}^{\ell} 3\epsilon^2\alpha\|\xi^m\|^2_{1,\mathrm{DG}} + d_\ell$$
$$= Ck(1+\frac{k\epsilon^2}{\epsilon^2}+\frac{h^2+k}{\epsilon^2})\sum_{m=1}^{\ell-1}\|\xi^m\|^2_{L^2(\mathcal{T}_h)} + C\epsilon^{-\frac{2(4+d)}{4-d}}k\sum_{m=1}^{\ell-1}\|\xi^m\|^{\frac{2(6-d)}{4-d}}_{L^2(\mathcal{T}_h)} \tag{3.63}$$
$$+Ch^{2\min\{r+1,s\}}\epsilon^{-2(\sigma_1+2)} + \frac{C}{\epsilon^2}(h^8+k^4) + Ck^4[\epsilon^{-4}+\epsilon^{-2}+1].$$

and define for $\ell \ge 1$

$$S_{\ell+1} := \|\xi^\ell\|^2_{L^2(\mathcal{T}_h)} + k\sum_{m=1}^{\ell} 3\epsilon^2\alpha\|\xi^m\|^2_{1,\mathrm{DG}} + d_\ell, \tag{3.64}$$

$$S_1 := Ch^{2\min\{r+1,s\}}\epsilon^{-2(\sigma_1+2)} + \frac{C}{\epsilon^2}(h^8+k^4) + Ck^4[\epsilon^{-4}+\epsilon^{-2}+1]. \tag{3.65}$$

then we have

$$S_{\ell+1} - S_\ell \le C(1+\frac{k\epsilon^2}{\epsilon^2}+\frac{h^2+k}{\epsilon^2})kS_\ell + C\epsilon^{-\frac{2(4+d)}{4-d}}kS_\ell^{\frac{6-d}{4-d}} \qquad \text{for } \ell \ge 1. \tag{3.66}$$

Applying Lemma 2 to $\{S_\ell\}_{\ell\ge1}$ defined above, we obtain for $\ell \ge 1$

$$S_\ell \le a_\ell^{-1}\left\{S_1^{-\frac{2}{4-d}} - \frac{2Ck}{4-d}\sum_{s=1}^{\ell-1}\epsilon^{-\frac{2(4+d)}{4-d}}a_{s+1}^{-\frac{2}{4-d}}\right\}^{-\frac{4-d}{2}} \tag{3.67}$$

provided that

$$\frac{1}{2}S_1^{-\frac{2}{4-d}} - \frac{2Ck}{4-d}\sum_{s=1}^{\ell-1}\epsilon^{-\frac{2(4+d)}{4-d}}a_{s+1}^{-\frac{2}{4-d}} > 0. \tag{3.68}$$

We note that $a_s\,(1 \leq s \leq \ell)$ are all bounded as $k \to 0$, therefore, (3.68) holds under the mesh constraint stated in the theorem. It follows from (3.66) and (3.67) that

$$S_\ell \leq 2a_\ell^{-1} S_1 \leq Ck^4 \epsilon^{-2(\sigma_1+2)} + Ch^{2\min\{r+1,s\}} \epsilon^{-2(\sigma_1+2)}. \tag{3.69}$$

Finally, using the above estimate and the properties of the operator P_r^h we obtain (3.18) and (3.19). The estimate (3.20) follows from (3.19) and the inverse inequality bounding the L^∞-norm by the L^2-norm and (2.21). The proof is complete.

CHAPTER 4. CONVERGENCE OF THE NUMERICAL INTERFACE TO THE MEAN CURVATURE FLOW

In this section, we prove the rate of convergence of the numerical interface to its limit geometric interface of the Allen-Cahn equation. This convergence theory is based on the maximum norm error estimates, which is proven above. The rate of convergence can be proven by the sharper error estimates, which is the negative polynomial function of the interaction length ϵ. It can't be proven if the coarse error estimate, which is the exponential function of ϵ, is used.

For all the DG problem, the the zero-level set of u_h^n may not be well defined since the zero-level set may not be continuous. Therefore, we introduce the finite element approximation $\widehat{u}_h^m$ of the DG solution u_h^m It is defined by using the averaged degrees of freedom of u_h^n as the degrees of freedom for determining $\widehat{u}_h^m$ (cf. [24]). We get the following results [24].

Theorem 4 *Let $\mathcal{T}_h$ be a conforming mesh consisting of triangles when $d = 2$, and tetrahedra when $d = 3$. For $v_h \in V_h$, let $\widehat{v}_h$ be the finite element approximation of v_h as defined above. Then for any $v_h \in V_h$ and $i = 0, 1$ there holds*

$$\sum_{K\in\mathcal{T}_h} \|v_h - \widehat{v}_h\|^2_{H^i(K)} \leq C \sum_{e\in\mathcal{E}_h^I} h_e^{1-2i} \|[v_h]\|^2_{L^2(e)}, \tag{4.1}$$

where $C > 0$ is a constant independent of h and v_h but may depend on r and the minimal angle θ_0 of the triangles in $\mathcal{T}_h$.

Using the above approximation result we can show that the error estimates of Theorem 3 also hold for $\widehat{u}_h^n$.

Theorem 5 *Let u_h^m denote the solution of the DG scheme (3.1)–(3.4) and $\widehat{u}_h^m$ denote its finite element approximation as defined above. Then under the assumptions of Theorem 3 the error* *estimates for u_h^m given in Theorem 3 are still valid for $\widehat{u}_h^m$, in particular, there holds*

$$\max_{0\le m\le M} \|u(t_m) - \widehat{u}_h^m\|_{L^\infty(\mathcal{T}_h)} \le Ch^{\min\{r+1,s\}}|\ln h|^{\overline{r}}\epsilon^{-\gamma} + Ch^{-\frac{d}{2}}(k^2 + h^{\min\{r+1,s\}})\epsilon^{-(\sigma_1+2)}. \tag{4.2}$$

Proof: We only give a proof for (4.2) because other estimates can be proved likewise. By the triangle inequality we have

$$\|u(t_m) - \widehat{u}_h^m\|_{L^\infty(\mathcal{T}_h)} \le \|u(t_m) - u_h^m\|_{L^\infty(\mathcal{T}_h)} + \|u_h^m - \widehat{u}_h^m\|_{L^\infty(\mathcal{T}_h)}. \tag{4.3}$$

Hence, it suffices to show that the second term on the right-hand side is an equal or higher order term compared to the first one.

Let $u^I(t)$ denote the finite element interpolation of $u(t)$ into S_h. It follows from (4.1) and the trace inequality that

$$\begin{aligned}
\|u_h^m - \widehat{u}_h^m\|_{L^2(\mathcal{T}_h)}^2 &\le C\sum_{e\in\mathcal{E}_h^I} h_e\|[u_h^m]\|_{L^2(e)}^2 \\
&= C\sum_{e\in\mathcal{E}_h^I} h_e\|[u_h^m - u^I(t_m)]\|_{L^2(e)}^2 \\
&\le C\sum_{K\in\mathcal{T}_h} h_e h_K^{-1}\|u_h^m - u^I(t_m)\|_{L^2(K)}^2 \\
&\le C\big(\|u_h^m - u(t_m)\|_{L^2(\mathcal{T}_h)}^2 + \|u(t_m) - u^I(t_m)\|_{L^2(\mathcal{T}_h)}^2\big).
\end{aligned} \tag{4.4}$$

Substituting (4.4) into (4.3) after using the inverse inequality yields

$$\begin{aligned}
\|u(t_m) - \widehat{u}_h^m\|_{L^\infty(\mathcal{T}_h)} &\le \|u(t_m) - u_h^m\|_{L^\infty(\mathcal{T}_h)} + Ch^{-\frac{d}{2}}\|u_h^m - \widehat{u}_h^m\|_{L^2(\mathcal{T}_h)} \\
&\le \|u(t_m) - u_h^m\|_{L^\infty(\mathcal{T}_h)} \\
&\quad + Ch^{-\frac{d}{2}}\big(\|u_h^m - u(t_m)\|_{L^2(\mathcal{T}_h)} + \|u(t_m) - u^I(t_m)\|_{L^2(\mathcal{T}_h)}\big),
\end{aligned}$$

which together with (3.18) implies the desired estimate (4.2). The proof is complete.

We are now ready to state the main theorem of this section.

Theorem 6 *Let* $\{\Gamma_t\}$ *denote the (generalized) mean curvature flow defined in* [25]*, that is,* Γ_t *is the zero-level set of the solution* w *of the following initial value problem:*

$$w_t = \Delta w - \frac{D^2 w Dw \cdot Dw}{|Dw|^2} \qquad \text{in } \mathbf{R}^d \times (0,\infty), \tag{4.5}$$

$$w(\cdot,0) = w_0(\cdot) \qquad \text{in } \mathbf{R}^d. \tag{4.6}$$

Let $u^{\epsilon,h,k}$ *denote the piecewise linear interpolation in time of the numerical solution* $\{\widehat{u}_h^m\}$ *defined by*

$$u^{\epsilon,h,k}(x,t) := \frac{t-t_m}{k}\widehat{u}_h^{m+1}(x) + \frac{t_{m+1}-t}{k}\widehat{u}_h^m(x), \quad t_m \le t \le t_{m+1} \tag{4.7}$$

for $0 \le m \le M-1$*. Let* $\{\Gamma_t^{\epsilon,h,k}\}$ *denote the zero-level set of* $u^{\epsilon,h,k}$*, namely,*

$$\Gamma_t^{\epsilon,h,k} = \{x \in \Omega;\, u^{\epsilon,h,k}(x,t) = 0\}. \tag{4.8}$$

Suppose $\Gamma_0 = \{x \in \overline{\Omega}; u_0(x) = 0\}$ *is a smooth hypersurface compactly contained in* Ω*, and* $k = O(h^2)$*. Let* t_* *be the first time at which the mean curvature flow develops a singularity, then there exists a constant* $\epsilon_1 > 0$ *such that for all* $\epsilon \in (0,\epsilon_1)$ *and* $0 < t < t_*$ *there holds*

$$\sup_{x \in \Gamma_t^{\epsilon,h,k}} \{\mathrm{dist}(x,\Gamma_t)\} \le C\epsilon^2 |\ln \epsilon|^2.$$

Proof: We note that since $u^{\epsilon,h,k}(x,t)$ is continuous in both t and x, then $\Gamma_t^{\epsilon,h,k}$ is well defined. Let I_t and O_t denote the inside and the outside of Γ_t defined by

$$I_t := \{x \in \mathbf{R}^d;\, w(x,t) > 0\}, \qquad O_t := \{x \in \mathbf{R}^d;\, w(x,t) < 0\}. \tag{4.9}$$

Let $d(x,t)$ denote the signed distance function to Γ_t which is positive in I_t and negative in O_t. By Theorem 6.1 of [26], there exist $\widehat{\epsilon}_1 > 0$ and $\widehat{C}_1 > 0$ such that for all $t \ge 0$ and $\epsilon \in (0,\widehat{\epsilon}_1)$ there hold

$$u_\epsilon(x,t) \ge 1-\epsilon \qquad \forall x \in \{x \in \overline{\Omega};\, d(x,t) \ge \widehat{C}_1 \epsilon^2 |\ln \epsilon|^2\}, \tag{4.10}$$

$$u_\epsilon(x,t) \le -1+\epsilon \qquad \forall x \in \{x \in \overline{\Omega};\, d(x,t) \le -\widehat{C}_1 \epsilon^2 |\ln \epsilon|^2\}. \tag{4.11}$$

Since for any fixed $x \in \Gamma_t^{\epsilon,h,k}$, $u^{\epsilon,h,k}(x,t) = 0$, by (4.2) with $k = O(h^2)$, we have

$$\begin{aligned} |u^\epsilon(x,t)| &= |u^\epsilon(x,t) - u^{\epsilon,h,k}(x,t)| \\ &\le \tilde{C}\Big(h^{\min\{r+1,s\}}|\ln h|^{\overline{r}}\epsilon^{-\gamma} + h^{-\frac{d}{2}}(k + h^{\min\{r+1,s\}})\epsilon^{-(\sigma_1+2)}\Big). \end{aligned}$$

Then there exists $\widetilde{\epsilon}_1 > 0$ such that for $\epsilon \in (0, \widetilde{\epsilon}_1)$ there holds

$$|u^\epsilon(x,t)| < 1 - \epsilon. \tag{4.12}$$

Therefore, the assertion follows from setting $\epsilon_1 = \min\{\widehat{\epsilon}_1, \widetilde{\epsilon}_1\}$. The proof is complete.

CHAPTER 5. NUMERICAL EXPERIMENTS

In this section, we provide a two-dimensional numerical experiment to gauge the accuracy and reliability of the fully discrete IPDGFE method developed in the previous sections. We use a square domain $\Omega = [-1,1] \times [-1,1] \subset \mathbf{R}^2$, and $u_0(x) = \tanh(\frac{d_0(x)}{\sqrt{2}\epsilon})$, where $d_0(x)$ stands for the signed distance from x to the initial curve Γ_0.

The test uses the smooth initial curves Γ_0, hence the requirements for u_0 are satisfied. Consequently, the results established in this paper apply to the test example. In the test we first verify the spatial rate of convergence given in (3.18) and (3.20). We then compute the evolution of the zero-level set of the solution of the Allen-Cahn problem with $\epsilon = 0.025$ and at various time instances.

Test . Consider the Allen-Cahn problem with the following initial condition:

$$u_0(x) = \begin{cases} \tanh(\frac{d(x)}{\sqrt{2}\epsilon}), & \text{if } x_1^2 + x_2^2 \geq 0.25^2, \\ \tanh(\frac{-d(x)}{\sqrt{2}\epsilon}), & \text{if } x_1^2 + x_2^2 < 0.25^2, \end{cases}$$

here $d(x)$ stands for the distance function to the circle $x_1^2 + x_2^2 = 0.25^2$.

Table 5.1. Spatial errors and convergence rates

h	$L^\infty(L^2)$ error	$L^\infty(L^2)$ order	$L^2(H^1)$ error	$L^2(H^1)$ order
$\sqrt{2}/10$	0.02451		0.34216	
$\sqrt{2}/20$	0.00539	2.1850	0.17258	0.9874
$\sqrt{2}/40$	0.00142	1.9244	0.08394	1.0398
$\sqrt{2}/80$	0.00036	1.9798	0.04172	1.0086

Table 5.1 shows the spatial L^2 and H^1-norm errors and convergence rates, which are consistent with what are proved for the linear element in the convergence theorem.

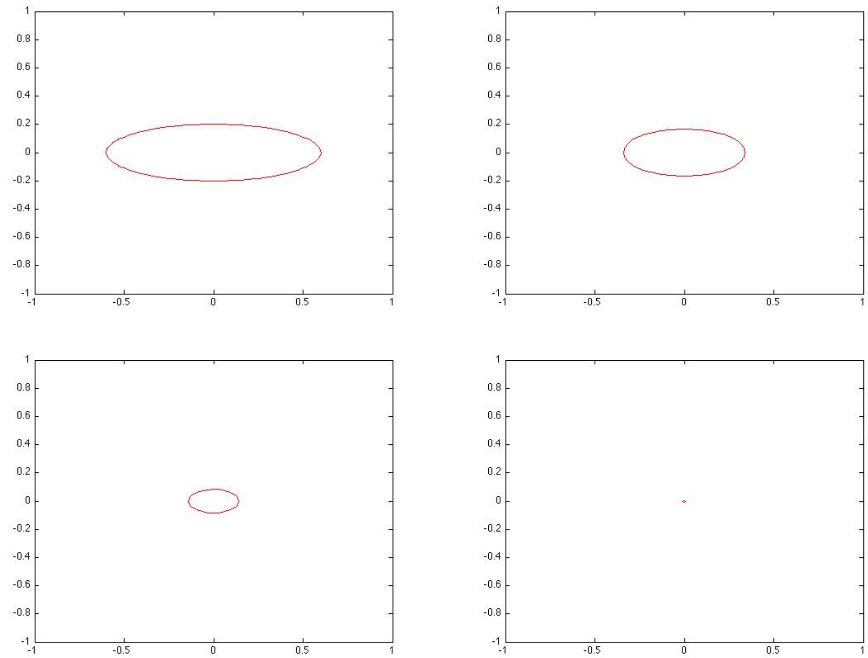

Figure 5.1 Test 1: Snapshots of the zero-level set of $u^{\epsilon,h,k}$ at time $t = 0, 1 \times 10^{-2}, 3 \times 10^{-2}, 4 \times 10^{-2}$ and $\epsilon = 0.025$.

Figure 5.1 displays four snapshots at four fixed time points of the zero-level set of the numerical solution $u^{\epsilon,h,k}$ with four different ϵ. Once again, we observe that as ϵ is small enough the zero-level set converges to the mean curvature flow Γ_t as time goes on.

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