Prediction of the incompressible flow over a rearward-facing step

O. Key Kwon

Iowa State University

Follow this and additional works at: http://lib.dr.iastate.edu/rtd

Part of the Mechanical Engineering Commons

Recommended Citation

INFORMATION TO USERS

This was produced from a copy of a document sent to us for microfilming. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the material submitted.

The following explanation of techniques is provided to help you understand markings or notations which may appear on this reproduction.

1. The sign or “target” for pages apparently lacking from the document photographed is “Missing Page(s)”. If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting through an image and duplicating adjacent pages to assure you of complete continuity.

2. When an image on the film is obliterated with a round black mark it is an indication that the film inspector noticed either blurred copy because of movement during exposure, or duplicate copy. Unless we meant to delete copyrighted materials that should not have been filmed, you will find a good image of the page in the adjacent frame. If copyrighted materials were deleted you will find a target note listing the pages in the adjacent frame.

3. When a map, drawing or chart, etc., is part of the material being photographed the photographer has followed a definite method in “sectioning” the material. It is customary to begin filming at the upper left hand corner of a large sheet and to continue from left to right in equal sections with small overlaps. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.

4. For any illustrations that cannot be reproduced satisfactorily by xerography, photographic prints can be purchased at additional cost and tipped into your xerographic copy. Requests can be made to our Dissertations Customer Services Department.

5. Some pages in any document may have indistinct print. In all cases we have filmed the best available copy.
Kwon, O Key

PREDICTION OF THE INCOMPRESSIBLE FLOW OVER A REARWARD-FACING STEP

Iowa State University

Ph.D. 1981

University Microfilms International 300 N. Zeeb Road, Ann Arbor, MI 48106
PLEASE NOTE:

In all cases this material has been filmed in the best possible way from the available copy. Problems encountered with this document have been identified here with a check mark √.

1. Glossy photographs or pages
2. Colored illustrations, paper or print
3. Photographs with dark background
4. Illustrations are poor copy
5. Pages with black marks, not original copy
6. Print shows through as there is text on both sides of page
7. Indistinct, broken or small print on several pages
8. Print exceeds margin requirements
9. Tightly bound copy with print lost in spine
10. Computer printout pages with indistinct print
11. Page(s) _________ lacking when material received, and not available from school or author.
12. Page(s) _________ seem to be missing in numbering only as text follows.
13. Two pages numbered _________. Text follows.
14. Curling and wrinkled pages
15. Other
TABLE OF CONTENTS

NOMENCLATURE xii

I. INTRODUCTION 1

A. The Problem 1

B. Literature Review on Previous Studies 6

1. Laminar flow 6
   a. Experimental work 6
   b. Analytical work 11

2. Turbulent flow 15
   a. Experimental work 15
   b. Analytical work 24

C. Scope of the Present Study 27

II. ANALYSIS 33

A. Viscous Flows 33

1. Geometry and coordinate system 33
2. The continuity and momentum equations 35
   a. The continuity equation 35
   b. The momentum equation 35

3. Laminar flows 36
4. Turbulent flows 37
5. Governing equations 40
6. Boundary conditions 43

B. Turbulence Modeling 45

1. The structure of the turbulent boundary layer 45
2. Introduction to turbulence modeling 52
3. Length scale model 54
   a. Model for the inner region 55
   b. Model for the outer region 56

4. Turbulence kinetic energy model 62

C. Inviscid Flows 66
1. Geometry 66
   a. Governing equation 66
   b. Boundary conditions 66

2. Governing equation and boundary conditions in the physical coordinate system 66
   a. Governing equation 66
   b. Boundary conditions 66

3. Governing equation and boundary conditions in the transformed coordinate system 69
   a. Introduction to the coordinate transformation 69
   b. Governing equation 70
   c. Boundary conditions 72

III. FINITE-DIFFERENCE FORMULATION 74
   A. Nondimensional Form of the Governing Equations 74
      1. Governing viscous flow equations 75
      2. Governing inviscid flow equation 76
      3. Turbulence kinetic energy equation 77
   B. Finite-Difference Representation 78
      1. Governing viscous flow equations 78
      2. Governing inviscid flow equation 83
      3. Turbulence kinetic energy equation 84
   C. Consistency, Stability, and Convergence 85
   D. The Grid Arrangement in the Normal Direction 88

IV. METHOD OF SOLUTION 90
   A. Viscous Flows 90
   B. Inviscid Flows 98
   C. Viscous-Inviscid Interaction Method 104

V. RESULTS AND DISCUSSION 110
   A. Preliminary Study on the New Boundary-Layer Solution Scheme 110
      1. Laminar separation bubble flow without viscous-inviscid interaction 111
      2. Laminar separation bubble flow with viscous-inviscid interaction 118
3. Turbulent separating flows without viscous-inviscid interaction 124
B. Laminar Separating Flows in Symmetric Expansions 133
C. Laminar Separating Flow over a Rearward-Facing Step with Viscous-Inviscid Interaction 150
D. Turbulent Separating Flows over a Rearward-Facing Step with Viscous-Inviscid Interaction 162

VI. CONCLUSIONS AND RECOMMENDATIONS 188
A. Concluding Remarks 188
B. Recommendations for Future Study 192

VII. REFERENCES 194

VIII. ACKNOWLEDGMENTS 210

IX. APPENDIX A: APPROXIMATE RELATIONSHIP BETWEEN $\delta'$ and $\delta$ 211
X. APPENDIX B: DERIVATION OF THE GOVERNING EQUATION FOR INVISCID FLOWS IN THE TRANSFORMED COORDINATE SYSTEM 212
XI. APPENDIX C: DISCUSSION OF SEVERAL LINEARIZING PROCEDURES FOR NONLINEAR TERMS OCCURRING IN THE FINITE-DIFFERENCE REPRESENTATION OF THE MOMENTUM EQUATION 214
XII. APPENDIX D: COEFFICIENTS OF FINITE-DIFFERENCE EQUATIONS FOR VISCOUS FLOWS: THE FULLY IMPLICIT FINITE-DIFFERENCING ALGORITHM WITH NEWTON LINEARIZATION 217
XIII. APPENDIX E: COEFFICIENTS OF THE FINITE-DIFFERENCE EQUATIONS FOR VISCOUS FLOWS: THE FULLY IMPLICIT SCHEME WITH LAGGED COEFFICIENTS AND THE CRANK-NICOLSON SCHEME 219
XIV. APPENDIX F: COEFFICIENTS OF FINITE-DIFFERENCE EQUATIONS FOR INVISCID FLOW 222
XV. APPENDIX G: DISCUSSION ON THE CONSISTENCY OF THE FINITE-DIFFERENCE EQUATIONS 224
XVI. APPENDIX H: DISCUSSION ON THE DIAGONAL DOMINANCE OF THE ALGEBRAIC SYSTEM FOR THE LAPLACE EQUATION 228
XVII. APPENDIX I: COEFFICIENTS IN THE EQUATIONS OBTAINED BY THE MODIFIED THOMAS ALGORITHM 231


APPENDIX L: TABULATION OF SOME TYPICAL TEST CASES 239

APPENDIX M: COMPUTER CODE "KSTEP" 247

APPENDIX N: COMPUTER CODE "KSTEP-2" 304
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table 2.1. Turbulence models used in the present study</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 4.1. Comparison of iterative methods with SOR for a simple test case</td>
<td>61</td>
</tr>
<tr>
<td>Table 5.1. Predicted extent of the recirculation region for laminar flow undergoing 2:1 symmetric expansion in a two-dimensional channel</td>
<td>103</td>
</tr>
<tr>
<td>Table 5.2. Comparison of the present prediction with other predictions for laminar flow undergoing a 2:1 symmetric expansion in a two-dimensional channel</td>
<td>135</td>
</tr>
<tr>
<td>Table 20.1. Predicted velocity and streamfunction profiles in the separated flow region for laminar channel flow in a symmetric sudden expansion ($Re_l = 50$, $h/H_i = 0.5$) at $x/h = 1.0$: fully developed inlet velocity profile</td>
<td>137</td>
</tr>
<tr>
<td>Table 20.2. Predicted velocity and streamfunction profiles in the redeveloping flow region for laminar channel flow in a symmetric sudden expansion ($Re_l = 50$, $h/H_i = 0.5$) at $x/h = 7.5$: fully developed inlet velocity profile</td>
<td>239</td>
</tr>
<tr>
<td>Table 20.3. Prediction of flow development for laminar channel flow downstream of a symmetric sudden expansion ($hu_{i,\text{max}}/v = 56$, $h/H_i = 1.0$)</td>
<td>241</td>
</tr>
<tr>
<td>Table 20.4. Prediction of laminar flow over a rearward-facing step with viscous-inviscid interaction ($hu_{\text{max}}/v = 412$, $h = 0.01016$ m)</td>
<td>242</td>
</tr>
<tr>
<td>Table 20.5. Prediction of turbulent flow over a rearward-facing step with viscous-inviscid interaction (reference flow): the $k-\ell$ turbulence model</td>
<td>243</td>
</tr>
</tbody>
</table>
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 1.1.</td>
<td>Sketches of various types of flow geometries with a sudden expansion</td>
<td>3</td>
</tr>
<tr>
<td>Figure 1.2.</td>
<td>Sketch of rearward-facing step flow field</td>
<td>4</td>
</tr>
<tr>
<td>Figure 1.3.</td>
<td>Computational zones for the flow over a rearward-facing step</td>
<td>29</td>
</tr>
<tr>
<td>Figure 2.1.</td>
<td>Coordinate system used for the analysis</td>
<td>34</td>
</tr>
<tr>
<td>Figure 2.2.</td>
<td>Laminar incompressible separation bubble flow showing reversed flow and reattachment</td>
<td>42</td>
</tr>
<tr>
<td>Figure 2.3.</td>
<td>Semilogarithmic and linear plots of mean velocity distribution across a typical turbulent boundary layer</td>
<td>47</td>
</tr>
<tr>
<td>Figure 2.4.</td>
<td>Inviscid flow computational domain in physical coordinates</td>
<td>67</td>
</tr>
<tr>
<td>Figure 2.5.</td>
<td>Inviscid flow computational domain in transformed coordinates</td>
<td>71</td>
</tr>
<tr>
<td>Figure 3.1.</td>
<td>Finite-difference grid</td>
<td>80</td>
</tr>
<tr>
<td>Figure 4.1.</td>
<td>Skeleton flow chart for the present viscous-inviscid interaction method</td>
<td>107</td>
</tr>
<tr>
<td>Figure 5.1.</td>
<td>Displacement thickness distribution for a laminar separation bubble test case</td>
<td>113</td>
</tr>
<tr>
<td>Figure 5.2.</td>
<td>Comparison of inverse boundary-layer solutions in the reversed flow region: distribution of skin-friction coefficient</td>
<td>114</td>
</tr>
<tr>
<td>Figure 5.3.</td>
<td>Comparison of inverse boundary-layer solutions in the reversed flow region: velocity profiles</td>
<td>117</td>
</tr>
<tr>
<td>Figure 5.4.</td>
<td>Displacement thickness distribution for a laminar separation bubble flow using viscous-inviscid interaction</td>
<td>120</td>
</tr>
<tr>
<td>Figure 5.5.</td>
<td>Edge velocity distribution for a laminar separation bubble flow using viscous-inviscid interaction</td>
<td>121</td>
</tr>
</tbody>
</table>
Figure 5.6. Distribution of skin-friction coefficient for a laminar separation bubble flow using viscous-inviscid interaction

Figure 5.7. Displacement thickness distribution for a turbulent separating flow on a flat plate

Figure 5.8. Edge velocity distribution for a turbulent separating flow on a flat plate

Figure 5.9. Velocity profiles for a turbulent separating flow on a flat plate (1)

Figure 5.10. Velocity profiles for a turbulent separating flow on a flat plate (2)

Figure 5.11. Distribution of skin-friction coefficient for a turbulent separating flow on a flat plate

Figure 5.12. Reynolds stress profiles for a turbulent separating flow on a flat plate

Figure 5.13. Prediction of the reattachment length for laminar flows in a channel with a symmetric sudden expansion, h/H_1 = 0.5

Figure 5.14. Centerline velocity distribution for a laminar flow in a channel with a symmetric sudden expansion, h/H_1 = 0.5

Figure 5.15. Velocity profiles for a laminar flow in a channel with a sudden symmetric expansion, h/H_1 = 0.5

Figure 5.16. Streamline contours for a laminar flow in a channel with a symmetric sudden expansion, Re_h = 50, h/H_1 = 0.5

Figure 5.17. Velocity profiles for a laminar flow in a channel with a symmetric sudden expansion, h/u_{1,max} / √ν = 56, h/H_1 = 1.0

Figure 5.18. Centerline velocity distribution for a laminar flow in a channel with a symmetric sudden expansion, h/u_{1,max} / √ν = 56, h/H_1 = 1.0

Figure 5.19. Distribution of skin-friction coefficient for a laminar flow in a channel with an asymmetric sudden expansion, Re_h = 73
Figure 5.20. Velocity profiles for a laminar flow in a channel with an asymmetric sudden expansion, \( \text{Re}_h = 73 \)

Figure 5.21. Distribution of skin-friction coefficient for a laminar flow in a channel with an asymmetric sudden expansion, \( \text{Re}_h = 229 \)

Figure 5.22. Velocity profiles for a laminar flow in a channel with an asymmetric sudden expansion, \( \text{Re}_h = 229 \)

Figure 5.23. Comparison of the predicted and measured length of the separated region for a laminar flow over a rearward-facing step

Figure 5.24. Displacement thickness distribution for a laminar flow over a rearward-facing step, \( \frac{h u_{\text{max}}}{v} = 412, h = 0.01016 \text{ m} \)

Figure 5.25. Edge velocity distribution for a laminar flow over a rearward-facing step, \( \frac{h u_{\text{max}}}{v} = 412, h = 0.01016 \text{ m} \)

Figure 5.26. Velocity profiles for a laminar flow over a rearward-facing step (1), \( \frac{h u_{\text{max}}}{v} = 412, h = 0.01016 \text{ m} \)

Figure 5.27. Velocity profiles for a laminar flow over a rearward-facing step (2), \( \frac{h u_{\text{max}}}{v} = 412, h = 0.01016 \text{ m} \)

Figure 5.28. Shape factors in a laminar redeveloping boundary layer downstream of a rearward-facing step, \( \frac{h u_{\text{max}}}{v} = 412, h = 0.01016 \text{ m} \)

Figure 5.29. Displacement thickness distribution in a laminar redeveloping boundary layer downstream of a rearward-facing step, \( \frac{h u_{\text{max}}}{v} = 412, h = 0.01016 \text{ m} \)

Figure 5.30. Momentum thickness distribution for a laminar flow over a rearward-facing step, \( \frac{h u_{\text{max}}}{v} = 412, h = 0.01016 \text{ m} \)

Figure 5.31. Distribution of skin-friction coefficient for a laminar flow over a rearward-facing step, \( \frac{h u_{\text{max}}}{v} = 412, h = 0.01016 \text{ m} \)

Figure 5.32. Displacement thickness distribution along the lower (step) wall for a turbulent rearward-facing step flow (reference flow)
Figure 5.33. Displacement thickness distribution for a turbulent rearward-facing step flow (step-1 flow) 165

Figure 5.34. Pressure coefficient distribution along the lower (step) wall for a turbulent rearward-facing step flow (reference flow) 166

Figure 5.35. Pressure coefficient distribution along the upper (no-step) wall for a turbulent rearward-facing step flow (reference flow) 167

Figure 5.36. Pressure coefficient distribution for a turbulent rearward-facing step flow (step-1 flow) 168

Figure 5.37. Velocity profiles downstream of the step for a turbulent rearward-facing step flow (reference flow) 170

Figure 5.38. Velocity profiles downstream of the step for a turbulent rearward-facing step flow (step-1 flow) 171

Figure 5.39. Velocity profiles upstream of the step for a turbulent rearward-facing step flow (step-1 flow) 172

Figure 5.40. Velocity profiles along the upper (no-step) wall for a turbulent rearward-facing step flow (step-1 flow) 173

Figure 5.41. Distribution of maximum Reynolds stress for turbulent rearward-facing step flows 175

Figure 5.42. Reynolds stress profiles for a turbulent rearward-facing step flow (reference flow); the predictions are plotted based on the distance from reattachment 176

Figure 5.43. Reynolds stress profiles for a turbulent rearward-facing step flow (step-1 flow) 177

Figure 5.44. Turbulence kinetic energy profiles for a turbulent rearward-facing step flow (step-1 flow) 178

Figure 5.45. Distribution of skin-friction coefficient for a turbulent rearward-facing step flow (reference flow) 179

Figure 5.46. Distribution of skin-friction coefficient for a turbulent rearward-facing step flow (step-1 flow) 180
Figure 5.47. Displacement thickness distribution for a turbulent rearward-facing step flow (step-2 flow) 183

Figure 5.48. Velocity profiles for a turbulent rearward-facing step flow (step-2 flow) 184

Figure 5.49. Pressure coefficient distribution for a turbulent rearward-facing step flow (step-2 flow) 185

Figure 5.50. Reynolds stress and turbulence kinetic energy profiles for a turbulent rearward-facing step flow (step-2 flow) 186
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>channel cross-sectional area</td>
</tr>
<tr>
<td>$A^+$</td>
<td>damping constant for velocity fluctuations</td>
</tr>
<tr>
<td>$A_j$, $A'_j$</td>
<td>coefficients appearing in the finite-difference expressions for the boundary-layer equations</td>
</tr>
<tr>
<td>$A_\psi$, $A'_\psi$</td>
<td>defined in Equations (14.2) and (14.3)</td>
</tr>
<tr>
<td>B</td>
<td>log law constant</td>
</tr>
<tr>
<td>$B_j$, $B'_j$, $b_j$</td>
<td>coefficients appearing in the finite-difference expressions for the boundary-layer equations</td>
</tr>
<tr>
<td>$B_\psi$, $\bar{B}_\psi$</td>
<td>defined in Equations (14.2) and (14.3)</td>
</tr>
<tr>
<td>$C_1$, $C_2$, $C_3$, $C_4$, $C_5$</td>
<td>empirical constants</td>
</tr>
<tr>
<td>$C_1'$, $C_1''$, $C_\mu$, $C_D$, $c$</td>
<td>defined in Equations (14.2) and (14.3)</td>
</tr>
<tr>
<td>$c_j$, $c'_j$, $c_1$, $c_2$</td>
<td>coefficients appearing in the finite-difference expressions for the boundary-layer equations</td>
</tr>
<tr>
<td>$C_\psi$, $\bar{C}_\psi$</td>
<td>defined in Equations (14.2) and (14.3)</td>
</tr>
<tr>
<td>$C_p$</td>
<td>pressure coefficient</td>
</tr>
<tr>
<td>$C_f$</td>
<td>skin-friction coefficient</td>
</tr>
<tr>
<td>D</td>
<td>damping function</td>
</tr>
<tr>
<td>$D_j$, $D'_j$, $d_j$</td>
<td>coefficients appearing in the finite-difference expressions for the boundary-layer equations</td>
</tr>
<tr>
<td>$E_j$, $E'_j$</td>
<td>coefficients appearing in the finite-difference expressions for the boundary-layer equations</td>
</tr>
<tr>
<td>$F_1$, $F_2$, $F_3$</td>
<td>defined in Equation (18.11) and also appearing in Equation (4.10)</td>
</tr>
<tr>
<td>$f$</td>
<td>body force</td>
</tr>
<tr>
<td>$f_1$, $f_2$</td>
<td>empirical functions</td>
</tr>
<tr>
<td>$g_c$</td>
<td>acceleration due to gravity</td>
</tr>
<tr>
<td>H</td>
<td>channel height; also used for the shape factor</td>
</tr>
</tbody>
</table>
$H_i$  channel inlet height

$H_j, H'_j$  coefficients appearing in the finite-difference expressions for the boundary-layer equations

$H_o$  channel height downstream of step

$h$  step height

$k$  $\Delta Y^+ / \Delta Y^-$

$t$  turbulence kinetic energy

$k_o$  free stream turbulence kinetic energy

$L$  turbulent length scale

$L^*$  relaxation length

$L_e$  extent of interaction region

$L_{REF}$  reference length

$\ell$  mixing length

$\ell_i$  mixing length in the inner region

$\ell_o$  mixing length in the outer region

$\ell_1, \ell_2, \ell_3$  defined in Equation (18.6)

$\ell_e$  distance from the step to the eye of the reversed flow vortex

$\ell_R$  reattachment length

$M$  nondimensional diffusion coefficient

$m_1, m_2, m_3$  defined in Equation (18.6)

$\dot{m}$  mass flow rate

$N$  nondimensional diffusion coefficient

$N_{eq}$  grid point where the unequal grid space in the normal direction terminates

$NJ$  edge of the viscous flow computational domain

$n$  iteration level
P  nondimensional pressure
PrK  turbulence Prandtl number
p  pressure
q1, q2  functions defined in Chapter XVII
Re  Reynolds number
Reh  Reynolds number based on the step height
ReH1  Reynolds number based on the channel inlet height
T  interval of time
T.E.  truncation error
t  time
tB  depth of a trough appearing in Equation (5.2)
to  reference time
U  nondimensional x component of velocity
ΔU  \( U^{i+1} - U^{d+1} \)
u  x component of velocity
ue  edge velocity
ue,0  edge velocity at the interaction starting point
ui  channel inlet velocity
uo  reference velocity
um  mean velocity in the outer region of the turbulent boundary layer, \( (ue + usw)/2 \)
ur  friction velocity
\( \bar{u}_r \)  characteristic turbulent velocity
u+  nondimensional velocity, \( u/u_r \)
V  nondimensional y component of velocity
v  y component of velocity
\[ v_T \] characteristic turbulent velocity

\[ x \] nondimensional x coordinate

\[ \Delta x^+ \]
\[ x^{i+1} - x^i \]

\[ \Delta x^-, \Delta x \]
\[ x^i - x^{i-1} \]

\[ x \] coordinate along the surface

\[ x_e \] interaction end point

\[ x_o \] interaction starting point

\[ x_r \] reattachment point

\[ x_s \] separation point

\[ Y \] nondimensional y coordinate

\[ \Delta Y^+ \]
\[ Y_{j+1} - Y_j \]

\[ \Delta Y^-, \Delta Y \]
\[ Y_j - Y_{j-1} \]

\[ y, y_o \] coordinate normal to the surface

\[ y_D \] dividing streamline

\[ y_{\tau, \text{max}} \] point where the maximum shear stress occurs

\[ y^+ \] nondimensional y, \( yu^*/v \)

\[ \infty \] denotes infinity

**Greek symbols**

\[ \delta \] boundary-layer thickness; also used as the change in variables between two successive iterations

\[ \delta' \] defined in Equation (2.42)

\[ \delta^* \] displacement thickness

\[ \delta_h^* \] displacement thickness at the step

\[ \delta_{UL}^* \]
\[ \delta_U^* - \delta_L^* \]

\[ \varepsilon \] convergence criteria defined in Equation (4.15)

\[ \varepsilon_D \] dissipation rate of turbulence kinetic energy
\( \eta \) transformed y coordinate
\( \Delta \eta_+ \) \( \eta_{j+1} - \eta_j \)
\( \Delta \eta_- \) \( \eta_j - \eta_{j-1} \)
\( \bar{\eta} \) \( y/\delta^* \)

\( \theta \) momentum thickness
\( \theta_h \) momentum thickness at the step
\( \kappa \) von Karman constant

\( \lambda \) second viscosity coefficient
\( \mu \) molecular viscosity
\( \mu_t \) turbulent viscosity
\( \nu \) kinematic viscosity

\( g \) transformed x coordinate
\( \Delta g_+ \) \( g_{i+1} - g_i \)
\( \Delta g_- \) \( g_i - g_{i-1} \)

\( \rho \) density

\( \Sigma \) summation notation

\( \tau \) shear stress

\( \tau_w \) wall shear stress

\( \phi \) surrogate symbol for variables

\( \chi \) pressure gradient

\( \psi, \psi^a \) nondimensional streamfunction

\( \psi_e \) streamfunction at the boundary-layer edge
\( \Delta \psi \) \( \psi_{i+1} - \psi_{i+1} \)

\( \psi \) streamfunction

\( \psi_T \) total volume flow rate per unit width in a channel
perturbation streamfunction defined in Equation (5.1)
relaxation factor

Subscripts
ADI alternating direction implicit method
AVG averaged condition
BL boundary-layer flow
CL centerline
e evaluated at the boundary-layer edge
INV inviscid flow
j mesh index corresponding to Y
L lower wall
max maximum condition
min minimum condition
R reattachment point
step step
SW switch point from inner to outer layer
U upper wall; also denotes the nondimensional x component of velocity
φ surrogate symbol
X pressure gradient
ψ nondimensional streamfunction
ψ streamfunction

Superscripts
i mesh index corresponding to X
( )' fluctuating quantities
( )' time averaged quantities
previous iteration level
nondimensional quantities
I. INTRODUCTION

A. The Problem

The phenomena of flow separation occur in many engineering applications. In many instances, separation of flow is undesirable and leads to adverse influences on the performance of engineering devices, such as large pressure drops and energy losses, etc. In other circumstances, however, separation may be encouraged; for example, in burner flame stabilization or for turbulence promotion leading to enhanced mixing or heat and mass transfer rates. Separation usually occurs when the flow is subjected to a strong adverse pressure gradient on a smooth solid surface. It is also expected to occur near any abrupt geometric change in a solid body submerged in a stream such as a forward or rearward-facing step. In the supersonic flow regime, the interaction of the boundary layer with a shock wave may also cause flow separation from a surface.

Because of the practical importance of flow separation to the performance of engineering equipment, the mechanism of flow separation and reattachment has received a great deal of attention, though it is still far from being fully understood.

In the present study, a two-dimensional, subsonic incompressible, separated flow over a rearward-facing step has been investigated using a numerical prediction scheme. In addition, separated flows occurring on flat surfaces were also studied, although the primary emphasis was on the former.

Among two-dimensional flows, the rearward-facing step geometry
perhaps provides the simplest separating and reattaching flow, since the separation point is generally fixed at the sharp corner. Such step flow has been extensively studied recently for understanding the phenomena of flow reattachment and redevelopment.

Figure 1.1 shows several flow geometries with a rearward-facing step which frequently occur in engineering applications. One example of an application is the flow through an internal passage for fuel assemblies in nuclear reactors. Others are the flow inside a combustion chamber, the film cooling of a wall exposed to a hot gas stream by the injection of a cold fluid, and the cooling gas flow in various parts of a turbogenerator. For external flows, the most common example of such flow geometry is the flow around a building.

Among the flow geometries shown in Figure 1.1, the main emphasis in the present study is on the flow in a two-dimensional duct with single (asymmetric) and double (symmetric) expansions. The flow geometry with a step adjacent to a free stream can be viewed as a special case of the flow in a duct with a single step. That is, the external step case can be obtained from the single expansion by replacing the upper wall of the duct with a streamline.

Although the geometry of a rearward-facing step is simple, the flow field is nevertheless quite complex as shown in Figure 1.2. The boundary layer separates at the edge of the step, forming a new shear layer inside the original shear flow. The separated shear layer grows into the recirculating flow and sharply curves down to the wall in the reattachment zone. Within the reattachment zone, the shear layer is subjected to the effects of strong curvature, adverse pressure gradient, and strong
Figure 1.1. Sketches of various types of flow geometries with a sudden expansion:

(a) Symmetric channel expansion

(b) Asymmetric channel expansion

(c) Axisymmetric pipe expansion

(d) Rearward-facing step in an external flow
Figure 1.2. Sketch of rearward-facing step flow field
interaction with the wall. In this region, part of the incoming flow is deflected upstream into the recirculating zone. Downstream of the reattachment, a new subboundary layer develops on the surface inside the reattached shear layer.

In general, separating and reattaching flows can be divided into three different groups based on whether the flow is fully laminar throughout the flow field including separation and reattachment, or is initially laminar at the separation point but undergoes transition and reattaches as turbulent flow, or is fully turbulent at both separation and reattachment. These groups will be designated as (1) fully laminar flow, (2) transitional, and (3) fully turbulent flow. In the present study, both fully laminar and fully turbulent flows were examined.

A numerical method is developed in this study for the prediction of laminar and turbulent flow over a rearward-facing step. It is generally believed that separated flows have elliptic characteristics such that the fluid inside the flow field is subjected to strong upstream and downstream influences. Consequently, the solutions of such flows are quite different from either the boundary-layer type viscous flow solutions or the inviscid flow solutions. However, several investigators [1, 2, 3, 4, 5] have recently demonstrated that thin separation bubble flows over two-dimensional bodies can be properly predicted by iteratively matching these viscous and inviscid flow solutions so that the upstream-downstream information in the viscous flow solution is obtained through the pressure field established in the outer inviscid flow region. Such a procedure is called a viscous-inviscid interaction method.
In the present study, a viscous-inviscid interaction method has been employed for the solution of the rearward-facing step flows which contain inviscid flow regions. In addition, for the flows which do not have an inviscid flow region, such as channel expansion flows with developed inlet velocity profiles, the boundary layer equations alone were solved in order to examine their applicability to this type of flow.

B. Literature Review on Previous Studies

A considerable amount of research has been done on the flow over a rearward-facing step experimentally and theoretically. A large portion of this research has been for supersonic flow because of the importance of predicting the base pressure of bluff bodies moving at high speeds. However, since the present study deals with low speed flow, the literature review has been restricted to laminar and turbulent flows in the subsonic flow regime.

1. Laminar flow

a. Experimental work A relatively limited number of experimental investigations on the laminar flow over a rearward-facing step have been reported in the literature. The earliest work on the fully laminar separated flow over a rearward-facing step was reported by Moore [6]. The study was carried out in a low-speed wind tunnel. Moore studied laminar reattachment at a Reynolds number based on the boundary-
layer thickness at the step of 338. The measured reattachment length was about 23 step heights. However, the recent analysis by Goldstein, et al. [7] raises questions about Moore's measurement of laminar reattachment. The analysis by Goldstein, et al. shows that data at the Reynolds number of 338 fall in the laminar-turbulent transition zone rather than the fully laminar regime.

Relatively accurate measurements were made by Eriksen [8] and Goldstein, et al. [7] for laminar flow of air over a rearward-facing step with various step heights using a hot wire anemometer. They found that the laminar reattachment length is not a constant number of step heights as for turbulent flow, but varies with Reynolds number and boundary-layer thickness at the step. The shape of the mean velocity profile at reattachment was found to be similar to the shape of a laminar boundary-layer profile at separation. In the redeveloping flow region, the velocity profile was observed to return to the flat plate velocity profile approximately 15 step heights downstream from reattachment.

Leal and Acrivos [9] studied the effect of base bleed on the steady separated flow past a rearward-facing step. Based on their experimental observations for Reynolds numbers between 50 and 250 and for bleed coefficients (defined as the ratio of the bleed mass flow rate to the channel mass flow rate at the step) in the range 0 to 0.15, they noted that the streamline pattern near a blunt object is strongly affected by small changes in the rate of bleed; however, the physical dimensions of the near-wake region and the associated streamwise pressure profile qualitatively have the same dependence on Reynolds number as without base bleed. The flow recirculation behind the step was ob-
served to disappear when the bleed coefficient exceeded 0.15. Their flow structure with base bleed was qualitatively similar to that proposed by Bearman [10]. For zero bleed, the recirculation length and downstream distance to the vortex center were found to increase linearly with Reynolds number. The pressure coefficient measured in the recirculation region behind the step was also found to depend qualitatively on Reynolds number. Furthermore, it was not affected by the presence or absence of an open base. A similar result to this for the absence of base bleed was reported by Acrivos, et al. [11].

A study of flow separation and reattachment was also made by O'Leary and Mueller [12] in a water channel. They argued that flow separation occurred below the sharp edge of the step, although in some of their photographs, this was not clearly visible. In their study, they experienced three-dimensional effects when Reynolds number based on the step height was greater than 100. The intensity of the effect increased as Reynolds number increased. The three-dimensionality generally appeared just before reattachment in the form of a helical motion. For further increase in Reynolds number, the recirculation center became increasingly unstable and eventually broke up into several small eddies.

The three-dimensional nature of low Reynolds number flow was also detected recently by Durst, et al. [13] in the region of a plane symmetric sudden expansion using flow visualization and laser-anemometry measurements. They observed asymmetry in the flow downstream of the step for Reynolds number, based on the upstream channel height and the maximum upstream velocity, above about 100. The asymmetric behavior was found to disappear far downstream of the channel expansion. As an
example, at Reynolds number of 114, symmetry in the velocity profile was observed to be restored at 45 channel inlet heights downstream from the expansion. Such an asymmetric nature resulted in an unequal length of the separation region along the upper and lower walls, and at a high Reynolds number it caused a third separation zone to occur on a wall downstream of the smaller of the two separation zones adjacent to the step. At a lower Reynolds number of 56, a symmetric flow pattern was confirmed by them. More recently, Cherdron, et al. [14] also reported a study on asymmetric flows in symmetric duct expansions.

Denham and Patrick [15] studied the laminar flow in a two-dimensional channel with a single expansion using a directionally-sensitive laser anemometer [16] for Reynolds numbers based on the step height from 50 to 250. They observed that the general flow field was similar to other recirculating flows such as the axisymmetric and symmetric two-dimensional duct sudden expansions, but the recirculating zone lengths and the recirculated mass flow-rates were smaller at the Reynolds numbers investigated. However, for Reynolds numbers greater than 140, Denham and Patrick [15] observed longer separated regions than did Goldstein, et al. [7] who also studied flow over a single expansion but at a smaller step height to channel width ratio.

A flow visualization study for the very low Reynolds number regime was carried out by Matsui, et al. [17] in a two-dimensional channel with an asymmetric sudden expansion. Reynolds numbers based on the channel inlet height varied from 4.8 to 45.5. The length of the separated region was found to increase nearly linearly with Reynolds number in the range investigated.
Armaly and Durst [18] reported laser-doppler measurements of the reattachment length behind a single rearward-facing step mounted in a two-dimensional channel. They found additional regions of recirculating flow occurring on both walls in the vicinity of the reattachment point of the main separation region. However, such additional recirculating zones were not reported in the early work by Denham and Patrick [15]. In the laminar flow regime, the separation length was observed to increase with Reynolds number and in the turbulent regime, it was observed to be nearly constant as also noted by other investigators (see Section I.B.2). Transition from laminar to turbulent flow was found to be characterized by an initially strong decrease in the main separation region behind the step and also a decrease in the size of secondary recirculating flow regions developed on the test section walls.

Experimental studies on laminar flow in an axisymmetric expansion were reported by Macagno and Hung [19] and Iribarne [20].

Macagno and Hung [19] investigated flow in an axisymmetric pipe expansion with an expansion ratio of 2:1. By using oil as a working fluid, they could maintain laminar flow up to a Reynolds number based on the pipe inlet diameter of 4500. A flow visualization technique was used to detect the reattachment point of the separation streamline. At a Reynolds number based on the pipe inlet diameter of 4500, they observed that the laminar flow was not axisymmetric; a cellular secondary flow appeared and eventually it resulted in a slow helicoidal motion of the flow.

Iribarne, et al. [20] observed a wavelike instability downstream
of a sudden pipe expansion which was similar to that observed earlier by Macagno and Hung [19] at a Reynolds number of 4500. The instability was detected throughout the entire Reynolds number range from 90 to 1355 when a uniform inlet velocity profile was used. They reported that initially a smooth wave appeared downstream of the expansion. The wave form became increasingly irregular as the flow approached reattachment. For Reynolds numbers greater than 350, tangential motion (varicose wave motion) was detected, resulting in considerable turbulence near the mean position of reattachment. However, they could suppress such a varicose wave behavior up to a Reynolds number of 754 by using a developed laminar starting profile. A study on a wavelike instability in a sudden pipe expansion was also carried out by Back and Roshko [21].

To date, only a few experimental studies of laminar flow over a rearward-facing step with heat transfer have been noted. Experimental measurements of the local heat transfer coefficient for laminar and transitional shear flow behind a rearward-facing step can be found in References [22] and [23].

b. Analytical work Recently the development of high-speed digital computers has made it possible to obtain solutions for complex flows numerically. Numerical predictions of the separated flow over a rearward-facing step started in the mid-1960s. Numerous solutions have been reported to date. The solutions were generally obtained by solving the full Navier-Stokes equations. Most of the Navier-Stokes solutions for the rearward-facing step flow were calculated using streamfunction and vorticity variables rather than primitive (velocity and pressure) variables. Those who used the streamfunction-vorticity scheme include
Hung [24], Macagno and Hung [19], O'Leary and Mueller [12], Durst, et al. [13], Giaquinta [25], Atkins, et al. [26], and Agarwal [27]. Among them, Hung [24] was the earliest investigator. He calculated the solution for laminar separated flow in a two-dimensional and axisymmetric pipe with a sudden expansion by using both the steady and unsteady computational approach. The extent of the separation region was found to increase nearly linearly with Reynolds number for both two-dimensional and axisymmetric cases. The calculated results for axisymmetric cases agreed very well with the measurements, which are also given in [19].

O'Leary and Mueller [12] used an explicit time-dependent finite difference scheme for the prediction of flows over a rearward-facing step.

Durst, et al. [13] predicted one of their measured flows by using a steady streamfunction-vorticity scheme. The solution procedure used was essentially the one developed by Gosman, et al. [28]. The predicted velocity profiles at a Reynolds number of 56 agreed fairly well with the measurements. Measured data were used in the calculation for the upstream boundary conditions. Downstream, a fully developed flow condition was imposed at 20 step heights from the expansion.

Giaquinta [25] obtained the transient and steady solutions for the flow entering a two-dimensional symmetric sudden expansion. He used a uniform velocity distribution at the inlet of the expansion. No direct comparisons were made either with other analytical solutions or with measurements.

Atkins, et al. [26] predicted laminar and turbulent flow in a channel with a single expansion using upwind and central difference
schemes and compared the results with each other and with experiments. At low Reynolds numbers where accurate data are not available, the upwind differences were found to predict a shorter and less intense recirculation zone than the more conventional conditionally stable central difference method. At higher Reynolds numbers, such as Reynolds numbers based on the step height of 73 and 229, the upwind prediction provided good agreement in the comparison with experimental measurements.

A third-order accurate upwind scheme was developed by Agarwal [27] for solution of the steady two-dimensional Navier-Stokes equations in streamfunction and vorticity form. It was found that the scheme was accurate and stable at high Reynolds numbers. He compared the third-order upwind prediction for flow in a channel with symmetric sudden expansion with the available experimental data [13]. He argued that his predictions agreed well with the available experimental data of Durst, et al. [13]. However, the agreement with the measurements turns out to be poor (see Chapter V).

All of the numerical studies mentioned above used streamfunction-vorticity variables. Recently, Morihara [29] developed a method in which the pressure terms were eliminated from the Navier-Stokes equations and the quasilinearization procedure was employed for the nonlinear terms in the resulting momentum equations. He solved the momentum equations and the continuity equation simultaneously so that he could eliminate the need for a relaxation procedure and the danger of divergence due to poor selection of a relaxation factor. He observed no instabilities with this scheme. The major disadvantage of this method is that it requires large computer storage during the execution unless the coeffi-
cient matrix is broken up into small matrices. For the flow in a channel with a double(symmetric) expansion, Morihara's predicted separation zone was found to be shorter than others (see, e.g., [24]) as Reynolds number increased.

Kumar and Yajnik [30] predicted separated flows in a channel with a symmetric expansion for various expansion ratios by solving a set of ordinary differential equations that become progressively decoupled in the downstream direction. Such ordinary differential equations were developed from the two limit equations which were obtained from the governing equation for steady, two-dimensional, laminar motion of an incompressible Newtonian fluid based on a large Reynolds number analysis. In fact, they introduced the eigenfunctions of the Poiseuille flow problem into the limit equations resulting in a set of coupled ordinary differential equations.

Navier-Stokes solutions using primitive variables, u, v, and p were obtained by Leschziner [31] for laminar recirculating flows in a channel with a single expansion and in a pipe with sudden expansion. He examined the performance of three steady-state finite-difference formulations, namely: (1) the hybrid central/upwind differencing scheme, (2) the hybrid central/skew-upwind differencing scheme, and (3) the quadratic, upstream-weighted differencing scheme. For the channel expansion cases, all three schemes were found to yield very similar results except for the largest Reynolds number considered, \( \text{Re}_h = 191 \), where the skew-upwind scheme predicted the longest reattachment length. When they were compared with the measurements of Denham and Patrick [15], it was found that all three schemes yielded excessive reattachment lengths. For the
pipe expansion, in contrast to the previous case, agreement between predictions and experimental data [19] was very good with all three schemes yielding virtually identical reattachment lengths over Reynolds numbers ranging from 60 to 140. Based on the computation, Leschziner [31] concluded that for laminar recirculating flows, skewness errors are relatively small.

Laminar flow in an axisymmetric sudden expansion was predicted by Pollard [32] using primitive variables, namely, u, v, and p. His prediction showed a linear relationship between the reattachment lengths and the Reynolds numbers and a nonlinear relationship between the reattachment lengths and the step heights. He also observed that the reattachment lengths obtained using a parabolic inlet profile were longer than those obtained using a uniform inlet profile.

Recently, several studies for laminar flows over a rearward-facing step using the finite-element method also have been reported (see, e.g., [33, 34]).

2. **Turbulent flow**

   a. **Experimental work** Research on transitional and turbulent separated flows over a plane and axisymmetric rearward-facing step has been actively pursued since the mid-1950s. To date, a large number of experimental studies have been reported. The early studies by Sato [35, 36] and Browand [37] mainly concentrated on the flow instability and transition phenomena associated with a separated shear layer over a plane expansion. More recently, Iribarne, et al. [20], and Back and Roshko [21] also studied the instability in a pipe flow with a
sudden expansion.

Relatively few studies of turbulent separating flows in annular expansions have been noted compared with the studies on flows over planar rearward-facing steps. Chaturavedi [38] and Kangovi and Page [39] measured turbulent characteristics and pressure for annular expansion flows by using hot wire and pitot tube, respectively. Chaturavedi [38] reported also mean velocity profiles downstream of the axisymmetric expansion. Kangovi and Page [39] measured the reattachment point which was in the vicinity of 8 step heights downstream of the annular step. Ha Minh and Chassaing [40] reported hot-wire measurements of mean velocities and turbulent shear stresses for flows in an annular expansion. They found that the "memory effect" plays a fundamental role on the type of flow. Drewry [41] studied the flow in an axisymmetric pipe expansion dump combustor under cold test conditions using a flow visualization technique.

It is well-known that conventional instruments such as the hot wire anemometer are not adequate for measuring turbulence quantities in the reversed flow region. Recently several investigators, including Freeman [42], Moon and Rudinger [43] and Lu [44] studied the separating flow in an annular expansion using laser instruments. Freeman [42] used a laser anemometer employing an electrooptic frequency shifting device for measuring mean flow and axial turbulent velocities. The results were found to agree well with the data measured by Chaturavedi [38] using pitot tubes and hot wire anemometers. Moon and Rudinger [43] used a laser doppler velocimeter (LDV) and measured the reattachment point at 6-9 step heights downstream from the axisymmetric
pipe expansion. The recirculation region was found to be practically independent of Reynolds number based on diameter before the step over the range $10^3$ to $10^6$. A similar result was reported by Lu [44] who also used LDV techniques.

Several experimental studies of turbulent separating flows over planar rearward-facing step have been reported. A detailed discussion of some of the data sets available may be found in Eaton and Johnston [45].

The earliest experimental work on the subsonic flow over a rearward-facing step was done by Hsu [46]. He measured both mean flow and turbulence characteristics. His turbulence data were found not to agree with results reported by later investigators. He observed little unsteadiness in the reattachment line.

Tani, et al. [47] made measurements of the pressure distribution, mean velocity profiles, turbulence intensity and turbulent shear stress for flows with various step heights using hot wires. The measurements are thought to be fairly accurate. The pressure distribution was found to be almost insensitive to changes in step height and boundary layer thickness. They also observed that when transition from laminar to turbulent flow occurred close to the separation point, changes in flow conditions in the approaching boundary layer upstream of the step caused insignificant differences in the separated and redeveloping flows. A relatively rapid decay in turbulence energy was reported downstream of reattachment compared to measurements reported by others.

Abbott and Kline [48] studied flow in a water channel with single and double rearward-facing steps for various step heights. With a flow
visualization technique, they observed that the flow near reattachment was very unsteady. Three zones of flow were found to exist in turbulent separation; namely, a three-dimensional zone of separation, a two-dimensional zone of separation, and a time-dependent reattaching zone. No appreciable change in zone lengths or general stall structure was found for the range of Reynolds numbers investigated and for a variation in inlet turbulence intensity. Turbulence intensity measurements obtained with a hot-film anemometer were also reported.

Mueller, et al. [49] investigated the character of the mean motion and the structure of turbulence for flow over a single step-type roughness element with a hot-wire anemometer. They observed the similarity between separating turbulent shear layers and redeveloping shear layers after reattachment.

Bradshaw and Wong [50] studied the characteristics of the redeveloping boundary layer downstream of the reattachment point in order to demonstrate the complicated nature of the flow and slow, nonmonotonic return of the shear layer to the ordinary boundary layer state. They found that in the redeveloping region, the "law of the wall"-"law of the wake" formulation was not applicable to the mean velocity profiles, and a rapid decay in turbulent shear stress occurred just downstream of reattachment. They suggested that the shear layer split at the reattachment so that the large eddies producing much of the shear stress were torn in two, resulting in a rapid decrease in the turbulent stress and turbulent length scales. Based on that observation, they concluded that the flow just downstream of reattachment bears very little resemblance to a plane mixing layer or any other sort of thin shear layer.
In fact, ordinary turbulent boundary layer characteristics were still not fully recovered at the last data station, 52 step heights downstream of separation.

Narayanan, et al. [51] measured the wall-static-pressure distribution behind steps of various heights. They adjusted the wall opposite to the step to simulate a free stream condition downstream of the step. They found similarities in the pressure distributions and were able to infer the reattachment length from the pressure distribution.

Chandrsuda [52] made extensive measurements for flow over a rearward-facing step using hot wires. Measurements included mean velocity, turbulent stresses, higher-order turbulence quantities, intermittency, and skin friction. With flow visualization, he observed that the large eddies were not torn in two at reattachment [50]; instead, some of the eddies appeared to go downstream and others moved upstream with the recirculating flow in a more or less alternating fashion. He also found that turbulence structure changed rapidly for some distance downstream of reattachment.

Le Balleur and Mirande [53] reported a study on turbulent reattaching flow downstream of a rearward-facing step under varying levels of adverse pressure gradients induced by changing the slope of the lower wall behind the step.

Rothe and Johnston [54] made a flow visualization study for the flow downstream of a step in a channel. The study emphasized the effect of rotation of the system about the spanwise axis on separation and reattachment. They observed gross, quasiperiodic unsteadiness in the reattachment zone.
Moss, et al. [55] measured mean velocity and turbulent normal stresses using a pulsed-wire anemometer, and the turbulent shear stresses using ordinary hot wires. They observed that the hot-wire anemometer indicated a lower turbulence intensity in the separated and reattachment zones than did the pulsed-wire instrument.

Davies and Snell [56] studied the effects of the approaching flow conditions on the flow behind a rearward-facing step using hot wires. Approaching conditions evaluated included a simple duct flow, duct flow with boundary layer suppression and duct flow with imposed shear in a wind tunnel with a rearward-facing step. They observed relatively large variations in turbulence intensity and shear stress distribution brought about by the changes in the nature of the approaching flow.

Davies [57] made measurements of mean velocity and turbulent stresses using hot wires. However, his study was mainly on the development and application of a hot-wire anemometer technique based upon a different method of signal analysis.

Kim, et al. [58] measured mean velocity, turbulent shear stresses, and intermittency in a sudden-expansion channel flow for two different step heights using hot-wire anemometry. The measured maximum values of the shear and normal stresses were found to be substantially lower than the data of other investigators obtained using laser anemometers or pulsed wires. They observed highly unsteady flow behavior in the reattachment region with flow visualization techniques using tufts and oil mixtures. They suggested that spanwise vortices exist at reattachment.
Mehta [59] reported a study on flow in a sudden symmetric expansion. He used pitot tubes and hot wires. The measured mean velocity profiles did not resemble those measured by other workers for two-dimensional turbulent separating flows behind steps. It is believed that his unusual results were caused by the influence of the side walls. His step was on the narrow side of a channel having an aspect ratio of 4:1. He also observed asymmetric flow patterns downstream of the expansion.

Eaton, et al. [60] used a pulsed-wire anemometer to measure mean velocities and turbulence intensities for a low-speed airflow behind a rearward-facing step. The boundary layer in the range of Reynolds numbers based on the momentum thickness at separation from 250 to 1500. They observed three dimensionality in the reattachment region and similarity between the free shear-layer and a plane mixing layer up to the point where the free shear layer begins to interact with the wall. They suggested the existence of a spanwise vortex in the free shear layer when the separating boundary layer is turbulent.

Moss and Baker [61] also used pulsed wires for the measurements of three components of mean velocity and turbulence in a rearward-facing step flow. Measurements of surface pressure and turbulent stresses were also reported.

Eaton and Johnston [62] studied separated flow behind a rearward-facing step in order to understand the reattachment process and the behavior of the large-scale eddies in the flow. Mean velocities, turbulence intensities, and wall shear stress were measured with pulsed wires. Turbulent stress was measured using hot wires. They obtained
relatively large negative skin friction \((-C_f > 1.0 \times 10^{-3}\)) in the recirculating flow region. They also found that the initial boundary layer condition had little effect on the flow at and downstream of reattachment, and that the separated shear layer was very much like a plane mixing layer until about two step heights upstream of reattachment. A spanwise vortex module was observed in the free shear layer when the separating boundary layer was turbulent. They observed flapping of the shear layer in the reattachment zone which caused the impingement point of the shear layer to move up and downstream over a distance of about two step heights.

Recently, several investigators have used the laser anemometer for measurements in planar rearward-facing step flows. Grant, et al. [63] and Denham, et al. [16] are among the earliest workers. However, they used rearward-facing steps mainly to test experimental laser-anemometer arrangements.

Smyth [64] also studied flow behind a symmetric expansion with a laser-doppler anemometer. He observed the Coanda effect in which the fluid takes a preference to one wall surface, resulting in asymmetric velocity profiles. He measured turbulence intensities based on the local streamwise velocities of up to the order of 100% in the recirculation zone. His more recent measurements [65] using a laser anemometer demonstrated the remarkable persistence of the flow structure characteristics of the separated shear layer. In fact, he found that the turbulence profiles were still far from an equilibrium distribution at 48 step heights downstream of separation. He provided measured data of mean velocity, streamwise and transverse turbulence intensities,
turbulent stress and turbulence kinetic energy in a symmetric expansion duct flow.

Restivo and Whitelaw [66, 67] studied flows in symmetric and asymmetric expansions with laser anemometers. For a symmetric expansion channel flow, they found that the lengths of the recirculation regions were dependent upon the shape of the initial mean-velocity profile and not on the expansion ratio. For an asymmetric flow, they investigated the effect of nearly periodic oscillations on the mean flow. They found that the location of reattachment and the centerline velocity decay and shear layer growth rate change with Reynolds number for the flow regime studied.

Etheridge and Kemp [68] used a frequency-shifted laser anemometer to measure a rearward-facing step flow. They reported measurements of mean velocity, turbulence intensities, and turbulent stress. They found that the shear layer split at reattachment with about one-sixth of the mass flow being deflected upstream. They also noticed that the mixing-length ratio $\ell/x$ is larger than that in plane mixing layers, whereas the shear correlation coefficient is roughly the same. The shear correlation coefficient is defined as the ratio of Reynolds stress to the product of the root mean square of the streamwise and normal direction fluctuating mean velocities such as $-\frac{u'v'\sqrt{u'^2\sqrt{v'^2}}}{1/2}$.

Kuehn [69] reported the length of the recirculating region and mean velocity profiles for flow over a rearward-facing step. He also studied the influence of an adverse pressure gradient on boundary-layer reattachment by rotating the opposite wall about a pivot point located directly opposite the step. They observed that channel expans-
sion caused the reattachment to move downstream, and channel contraction caused the reattachment to move upstream.

Armaly and Durst [18] studied the variation of the reattachment point for different Reynolds numbers using a laser-doppler anemometer.

A flow visualization study for flow over a rearward-facing step was reported by Hirata and Kasagi [70].

Several experimental studies also have been conducted for the separated flow past an isolated roughness element or fence. Those include the studies by Plate and Lin [71], Arie and Rouse [72], Durst and Rastogi [73], Agarwall, et al. [74], and Castro [75]. These flows are fundamentally different from the rearward-facing step flow since there is significant streamwise curvature in the separated shear layer.

Experimental research on the heat transfer to flows behind sudden expansions has been more or less continuous since the mid-1950s. Local and average heat transfer data for axisymmetric expansion flows were reported by Hislop and Morris [76], Krall and Sparrow [77], Zemanick and Dougall [78], and Sparrow and O'Brien [79]. For plane expansion flows, such data were obtained by Seban, et al. [80], Seban [81], Filetti and Kays [82], and Seki, et al. [83, 84].

b. Analytical work Early analytical investigations of flows in a sudden expansion were done theoretically, rather than numerically, using the integral approach with limited success. Those include the studies by Mueller, et al. [49], and more recently Teyssandier and Wilson [85].

Numerical studies on turbulent separating flows behind a sudden expansion began relatively recently. Briggs, et al. [86] predicted
the measurements of Abbott and Kline [48] using the streamfunction-vorticity method. Turbulence quantities were evaluated by solving the Reynolds stress equations with additional kinetic energy and length scale equations. Such turbulence quantities were evaluated near the wall just outside of the viscous sublayer based on law of the wall data. They used an alternating direction implicit (ADI) scheme for solving the Poisson equation for the streamfunction. The agreement with the measurements was found to be fair.

Castro [87] used the primitive variables of velocity and pressure for predicting flow behind a normal flat plate. He used a standard two equation eddy viscosity model to obtain the turbulent viscosity. He found that a skew differencing scheme seemed to be better than nonskew differencing schemes.

Ha Minh and Chassaing [88] predicted flows over rearward-facing steps numerically using a streamfunction-vorticity method and a primitive variable method. In the streamfunction-vorticity method, they calculated all the functions at the same grid points. In the primitive variable method, they calculated the velocity components on a staggered grid system. They found that the primitive method provided better results for the pressure field and the turbulence properties than the streamfunction-vorticity scheme.

Gosman, et al. [89] predicted flows in symmetric and axisymmetric expansions by solving two-dimensional, time-averaged conservation equations in elliptic form. They used primitive variables in the computation.

Oliver [90] and Mehta [59] reported predictions using streamfunction-
vorticity transport equations for flows in sudden expansions.

Atkins, et al. [26] predicted turbulent flow in an asymmetric expansion. They obtained better results with a one-equation turbulence model (using a turbulence kinetic energy transport equation) with a prescribed length scale distribution than a standard two-equation model (using turbulence kinetic energy and dissipation rate transport equations) compared to the measurements of Denham, et al. [16]. They suggested, on the basis of their calculation, that the models for ordinary turbulent flows employed in their prediction required some modifications for flows in sudden expansions.

Two studies on the prediction of sudden expansion flows using a viscous-inviscid interaction method have been reported to date. The studies were made by Le Balleur and Mirande [53] and Kim, et al. [58]. They used the so-called zonal method in which the computation domain was divided into several subregions. Kim, et al. [58] solved the momentum integral equation for the viscous flow using slightly different assumptions for the various zones into which the viscous region was divided. For the inviscid solution, they used the Cauchy integral formula. Finally, they iterated until these two (viscous and inviscid) solutions matched within a given tolerance. However, Le Balleur and Mirande [53] calculated the viscous solutions by using direct and inverse "Couche limit" integral methods in the domain except for the reversed flow region where they used empirical correlations. They obtained the inviscid flow solution using conformal mapping. In fact, in the strong interaction region, which included the reversed flow region and a short region downstream of reattachment, they obtained the
inviscid solution directly, whereas, in the weak interaction region, they obtained the solution inversely. They iterated to obtain the flow solution using an empirical interaction correlation. The results predicted by those two groups were found to give good agreement with measured data; however, much empiricism seems to have been involved in both schemes.

The prediction of turbulent flow behind a step using finite element methods was reported by Taylor, et al. [33].

Several numerical studies on turbulent flows over two-dimensional fences also have been reported, including the work by Durst and Rastogi [73].

Only a few numerical predictions of heat transfer to turbulent flows behind a sudden expansion have been noted. These include the works by Oliver [90], and Chieng and Launder [91]. Spalding [92] performed an analytical investigation without using numerical methods.

C. Scope of the Present Study

As discussed above, the prediction of laminar and turbulent flows over rearward-facing steps has generally been carried out by solving the full Navier-Stokes or Reynolds equations. Although significant progress has been made in the development of efficient computational methods for the Navier-Stokes (and Reynolds) equations, very large computation times are still required, especially for turbulent flows. In order to reduce the computation time required with the Navier-Stokes equations, coarse grid systems are generally used. However, large
truncation errors are often associated with the use of such coarse grids. This deterioration in accuracy can only be partially compensated for by the use of higher-order finite-difference formulas without again increasing the computation time. The use of coarse grids for turbulent flows usually requires that additional empiricisms (wall functions) be used to treat the flow very near the wall. Such facts provide motivation for considering the range of applicability of methods based on more approximate forms of the governing equations.

For thin separation bubbles occurring on a smooth two-dimensional body, several investigators have demonstrated that the boundary-layer type models can successfully predict the flow if they take account of the interaction between the boundary layer and external inviscid flow as discussed earlier. Recently, such a viscous-inviscid interaction technique has been successfully used with zonal methods by Kim, et al. [58] and Le Balleur and Mirande [53] for flows over rearward-facing steps. A disadvantage of the zonal methods used by these investigators is the lack of generality of the schemes because of the large amount of empiricism involved.

In the present study, a new viscous-inviscid interaction method was developed based on the scheme developed by Kwon and Fletcher [4] for thin separation bubble flows over two-dimensional smooth bodies. With the new method, the flow domain was divided into three subdomains (see Figure 1.3), namely, two viscous regions and one inviscid core region based on the flow characteristics, whereas in the aforementioned zonal methods, the viscous flow region was further subdivided into several sections based on empirical approximations. With the new method,
Figure 1.3. Computational zones for the flow over a rearward-facing step
the boundary-layer equations are solved inversely in the viscous regions by a finite-difference method using a prescribed displacement thickness, and in the inviscid flow region, the Laplace equation for streamfunction is solved to obtain the velocities at the edge of the boundary layers. The displacement thickness along both the upper and lower walls is updated iteratively based on the difference between the edge velocity obtained from the boundary-layer solutions and that obtained from the inviscid solution. It should be noted that to the author's knowledge, the present viscous-inviscid interaction method is the first interaction method to employ finite-difference procedures in the reversed flow region of a rearward-facing step flow.

A new solution scheme is developed to solve the continuity and momentum equations in a coupled manner. Such coupling was found necessary in order to satisfactorily predict flows having large regions of reversed flow with the boundary-layer equations. Attempts at using conventional fully implicit methods which have worked well for flows with small separated regions were unsuccessful. Solutions obtained without coupling were characterized by unrealistic oscillations in the pressure gradient and skin-friction coefficients. Predictions using the new coupling scheme are compared with results obtained from another coupling scheme for external separation bubble flows [93].

The viscous-inviscid interaction method is evaluated for one laminar and three turbulent flows over a rearward-facing step in two-dimensional channels.

Turbulence modeling appears to be a crucial factor in obtaining accurate predictions for flow details in the separated and reattaching
regions behind rearward-facing steps. None of the simple turbulence models evaluated were found to provide adequate predictions without modifications. The majority of the calculations were performed using a model which employed a solution to a modeled form of the turbulence kinetic energy equation and utilized an algebraic expression for length scale. Some of the calculations also employed a simple length scale transport equation [94].

The present boundary-layer calculation procedure was also applied to predict developed flows undergoing a sudden expansion. No inviscid core can be identified for these flows. Thus, the prediction proceeds without viscous-inviscid interaction. One of the most surprising results of the present study was the extremely good agreement observed between the present predictions based purely on boundary-layer equations and the available experimental data and Navier-Stokes solutions for this type of flow.

This thesis is composed of mainly six chapters. In Chapter II, the complete mathematical model is developed and the assumptions clearly identified. Chapter III is devoted to the development of the finite-difference representation of the mathematical model. The solution procedure for the finite-difference equations as well as the present viscous-inviscid interaction method are discussed in Chapter IV.

In Chapter V, the validity of the viscous-inviscid interaction method including the turbulence model is tested by comparing the predictions with the available experimental data and the predictions of other numerical schemes. The applicability of the boundary-layer
mathematical model to sudden expansion flows having developed initial velocity profiles is also examined.
II. ANALYSIS

In this chapter, the equations and appropriate boundary conditions for the flow under consideration are presented. Both viscous and inviscid flows are considered. A turbulence model utilizing a simplified formula for the characteristic mixing length scale of the flow is developed. Another model, which employs a turbulence kinetic energy transport equation along with the length scale is also described.

A. Viscous Flows

1. Geometry and coordinate system

The coordinate system chosen for the present analysis is shown, along with the configuration of the flow geometry, in Figure 2.1. It is assumed that the flow passage is a two-dimensional channel with either a single or double step. The horizontal and vertical coordinates are denoted by \( x \) and \( y_o \), respectively. Viscous solutions are obtained for the boundary layers, indicated by regions labeled A in Figure 1.3. It should be pointed out that in some of the cases considered, the viscous flow completely filled the channel.

For convenience, another coordinate \( y \) is introduced such that,

(1) for the lower wall which has a step (see Figure 2.1a):

\[
y = y_o - h \quad \text{if} \quad x < 0
\]

and

\[
y = y_o \quad \text{if} \quad x \geq 0
\]

(2) for the upper wall (see Figure 2.1a):
Figure 2.1. Coordinate system used for the analysis
\[ y = (H_2 + h) - y_0 \]

where \( h \) and \( H_2 \) are the step height and the channel inlet height, respectively.

For a duct with a symmetric double-step expansion as shown in Figure 2.1b, the duct centerline can be considered as the upper boundary for analysis.

2. **The continuity and momentum equations**

Fluid flow obeys the principles of conservation of mass, momentum and energy. Since the present study does not include heat transfer and is incompressible, the energy equation need not be considered. The governing equations derived from the conservation principles for mass and momentum for a two-dimensional flow are discussed below. A Cartesian coordinate system will be used.

a. **The continuity equation** The principle of conservation of mass, which applies to fluids in which no nuclear reactions are taking place, states that the Lagrangian derivative of the mass of fluid contained in an element is zero. This leads to the following partial differential equation, for compressible flows [95]

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k} \left( \rho u_k \right) = 0
\]  

(2.1)

For an incompressible flow, the continuity equation becomes

\[
\frac{\partial u_k}{\partial x_k} = 0
\]  

(2.2)

b. **The momentum equation** The principle of conservation of momentum, which is, in effect, an application of Newton's second law
of motion to an element of the fluid, together with the constitutive relation for a Newtonian fluid yields the well-known Navier-Stokes equations [95]

\[
\rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \lambda \frac{\partial u_i}{\partial x_j} \right) \\
+ \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] + \rho f_j. 
\] (2.3)

If the fluid is assumed to be incompressible and the dynamic viscosity is assumed to be constant, the terms containing the second viscosity coefficient in Equation (2.3) vanish and the viscous-shear term becomes proportional to the Laplacian of the velocity vector.

The Navier-Stokes equations then become [95]

\[
\rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho f_j 
\] (2.4)

3. Laminar flows

Equations (2.2) and (2.4) discussed in the previous section govern the motion of an incompressible Newtonian constant property fluid without heat transfer. For laminar unseparated flows, these equations can be simplified considerably by an order of magnitude analysis, based on the assumption of a thin viscous (shear) layer.

The assumption states that the velocity gradient normal to the principal flow direction is at least an order of magnitude larger than that along the principal direction, and pressure is constant across the principal flow direction for no cross-stream body forces. Under these assumptions, the equations governing two-dimensional steady laminar unseparated flows with constant properties can be written as
4. Turbulent flows

Most flows occurring in nature and in practical applications are turbulent. Turbulence is believed to be the most complicated kind of fluid motion. The scientific study of turbulent flow spans approximately one hundred years and has resulted in significant progress in many directions. However, turbulence is still far from being fully understood.

Turbulence is a three-dimensional time dependent random motion, and is characterized by wide ranges of frequencies and length scales. The size of largest scale is determined by the mean flow, while the size of the smallest is determined by the fluid viscosity. The large-scale motion is believed to carry most of the energy and momentum in the turbulence. The energy is continuously transferred from the largest through the intermediate scales to the smallest, where the energy is dissipated as heat.

The motions comprising turbulence are frequently referred to as eddies of various sizes, although caution has been recommended by
Reynolds [97] in the use of the concept of eddies.

Vortex stretching is generally believed to be the mechanism of energy exchange between eddies of different sizes. Within the smallest eddies, the continuum flow assumption is still applicable since the smallest eddies are several orders of magnitude larger than the length scale of the molecular motion (the mean-free path).

It is generally accepted that turbulence in Newtonian fluids is governed by the full Navier-Stokes equations. Since the equations are highly nonlinear, a numerical method is required to solve the equations. However, they cannot be solved even numerically for turbulent flows by present-day computers because of the extremely large number of grid points which are required to resolve the turbulent motion [98]. A common practice is to solve a more convenient form of equations obtained from Equation (2.4) by using the concept, first introduced by Reynolds [99], that the turbulent motion can be regarded as consisting of the sum of a time mean motion and a fluctuation about the mean. Denoting the time-average of a quantity $\phi$ by $\bar{\phi}$ and its fluctuation by $\phi'$, $\phi$ can be written as

$$\phi = \bar{\phi} + \phi'$$  \hspace{1cm} (2.7)

The time-average is expressed as

$$\bar{\phi} = \frac{1}{T} \int_{t_0}^{t_0+T} \phi(t) dt$$  \hspace{1cm} (2.8)

where $T$ is large compared to the relevant period of the fluctuations.

Using Equation (2.7), the following relations can be written for
velocity components and pressure:

\[ u = \overline{u} + u' \]
\[ v = \overline{v} + v' \]
\[ p = \overline{p} + p' \]  

(2.9)

Substituting these relations into Equations (2.2) and (2.4), using the time-averaging process and the usual boundary layer assumptions results in the following sets of equations for the two-dimensional, incompressible, steady (in the time-mean sense), and constant property turbulent flow [100]:

**Continuity:**

\[ \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \]  

(2.10)

**Momentum:**

\[ \overline{u} \frac{\partial \overline{u}}{\partial x} + \overline{v} \frac{\partial \overline{v}}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial y} (\overline{u'v'}) \]  

(2.11)

The Reynolds stress term \(-\partial(\overline{u'v'}/\partial y)\) in Equation (2.11) must be modeled empirically to close the system of equations. The most common and simplest modeling approach is to follow the suggestion made by Boussinesq in 1877 [2, see e.g. 101, 7] and define a turbulent viscosity, \(\mu_t\),

\[ \mu_t = -\frac{\rho u'v'}{\partial u/\partial y} \]  

(2.12)

The idea is to enforce the laminar formulation upon the total stresses

\[ \tau = \mu \frac{\partial \overline{u}}{\partial y} - \rho u'v' = (\mu + \mu_t) \frac{\partial \overline{u}}{\partial y} \]  

(2.13)

where the turbulent viscosity, \(\mu_t\), is not a property of the fluid but
depends on the particular flow and on the position within it. Details on turbulence modeling will be discussed in Section II.B.

Using Equation (2.13), Equations (2.10) and (2.11) form a parabolic system of equations which can be solved numerically.

5. Governing equations

Equations (2.10) and (2.11), except for the term $-\frac{\partial (u'v')}{\partial y}$ involving fluctuating velocity components, have the same form as Equations (2.5) and (2.6) which apply to laminar flow. Thus, under the assumption given in Equation (2.13), the model equation, both for laminar and turbulent flows, can be written as

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.14)$$

Momentum equation:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial \tau}{\partial y} \quad (2.15)$$

where $\tau$, the mean shear stress, is given as

$$\tau = \mu \frac{\partial u}{\partial y} \text{ for laminar flows}$$

and

$$\tau = (\mu + \mathcal{\mu}_t) \frac{\partial u}{\partial y} \text{ for turbulent flows.}$$

Here, for simplicity, the bars over the mean quantities have been omitted. Equations (2.14) and (2.15) can be solved in a finite-difference form by marching in the streamwise direction for attached flows. In the region of reversed flow, i.e., $u < 0$, it is no longer possible to march.
the solution in the main flow direction (see Figure 2.2); but rather, the correct marching direction is in the negative x direction. Fortunately, the streamwise convective derivative is often negligibly small in regions of reversed flow associated with thin separation bubbles and can be neglected to permit marching the solution in the positive x direction. This idea was suggested first by Reyhner and Flügge-Lotz [102], and is often called the FLARE approximation. Subsequent sample laminar flow calculations by Carter [103] and more recently by Cebeci, et al. [104] where solutions obtained by neglecting this term and by iteratively marching in the correct direction were compared tend to substantiate that the streamwise convective derivative is negligibly small for thin separated regions. This conclusion is also confirmed by the turbulent flow measurements of Simpson, et al. [105]. With this additional approximation added to the analysis, and assuming that the boundary-layer assumptions continue to apply for a thin separated region, the equations can be put into the following form by introducing the streamfunction for convenience:

\[ u = \frac{\partial \psi}{\partial y} \quad (2.16) \]

\[ cu \frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial}{\partial y} \quad (2.17) \]

where \( c = 0 \) when \( u \leq 0 \) and \( c = 1 \) when \( u > 0 \). This modified version of the boundary-layer equations forms the basis of the present analysis for the viscous flows.
Figure 2.2. Laminar incompressible separation bubble flow showing reversed flow and reattachment
6. **Boundary conditions**

To complete the mathematical model for the viscous region, appropriate boundary conditions need to be specified. In this study, two different boundary-layer calculation procedures were used which differed only in the specification of the outer boundary condition. The inner or wall boundary condition is

\[
 u(x, 0) = v(x, 0) = \psi(x, 0) = 0 \tag{2.18}
\]

i.e., no slip, no blowing or suction on the wall. The first of the two procedures is the standard or "direct" method in which, for external flows, the outer edge velocity was prescribed:

As \( y \to \infty \), \( u(x, y) \to u_e(x) \) \hspace{1cm} (2.19)

\[ \psi(x, y) \to \psi_e(x) \tag{2.20} \]

The direct method is suitable for attached flows but becomes singular [106] at the separation point. To overcome the singularity and calculate through regions of separation, an "inverse" procedure was used whereby the displacement thickness was specified:

\[
 \delta^*(x) = \int_0^\infty (1 - \frac{u}{u_e}) dy \tag{2.21-a}
\]

or as \( y \to \infty \),

\[ \psi(x, y) \to u_e(y - \delta^*) \tag{2.21-b} \]

where \( \delta^*(x) \) is a prescribed function. This boundary condition is to be satisfied at each streamwise location. Although the main use of the
inverse procedure was to calculate through regions of reversed flow in the present study, the procedure can also be used for attached flows.

For steady internal flows, Equation (2.21) must be replaced by an equation to meet the requirement that, for no blowing or suction through the walls, overall mass flow should be conserved. The equation is

\[ \dot{m} = \int_A \rho u dA = \text{constant} \quad (2.22) \]

Thus, for a two-dimensional symmetric channel flow, the outer boundary conditions, Equation (2.21), are replaced by

at \( y = \frac{H}{2}, \ \frac{\partial u}{\partial y} = 0 \quad (2.23) \]

\[ \psi(x, \frac{H}{2}) = \frac{\psi_T}{2} \quad (2.24) \]

where \( \psi_T \) is the total volume flow rate per unit width defined by

\[ \psi_T = \int_0^H u dy \]

When an inviscid core exists inside the channel, the outer boundary conditions can be written as the following by using Equation (2.21):

at \( y = \frac{H}{2}, u(x, \frac{H}{2}) = u_e(x) \quad (2.25) \]

\[ \psi(x, \frac{H}{2}) = \frac{\psi_T}{2} = \frac{u_e}{2} (H - 2\delta^k) \quad (2.26) \]

For a two-dimensional asymmetric channel flow, the outer boundary conditions become
at \( y = H \), \( u(x, H) = 0 \) \hspace{1cm} (2.27)

\[ \psi(x, h) = \psi_T \] \hspace{1cm} (2.28)

B. Turbulence Modeling

1. The structure of the turbulent boundary layer

A turbulent boundary layer can be regarded approximately as a composite layer made up of an inner and outer region as shown in Figure 2.3.

The inner region of a turbulent boundary layer is much smaller than the outer region, with a thickness of about 10% to 20% of the total boundary layer thickness. Despite its small extent, the inner region influences the entire flow within the boundary layer, since a significant fraction of the velocity variation occurs within the region. The turbulent energy inside this region is in a state of near local equilibrium [107]. When this case is true, the convective and diffusion of turbulence energy are negligible and the motion is determined by local conditions, particularly by the shear stress in the region \( \tau_w \). Based on the assumption of the local equilibrium, dimensional analysis leads to the following expression known as the "law of the wall:"

\[ u^+ = \frac{u}{u_T} = f(y^+) \] \hspace{1cm} (2.29)

Here \( u_T = (\tau_w/\rho)^{1/2} \) is a parameter having the dimensions of velocity, and is called the friction velocity. The parameter \( y^+ \) is a Reynolds number defined as \( y^+ = yu_T/\nu \). \( u^+ \) and \( y^+ \) are known as law of the wall coordinates.
Figure 2.3. Semilogarithmic and linear plots of mean velocity distribution across a typical turbulent boundary layer
The inner region can be divided into three layers as indicated in Figure 2.3: (1) the viscous sublayer, (2) the buffer layer, and (3) the fully turbulent region.

In the viscous sublayer, the stresses are mainly viscous since turbulent fluctuations, like mean velocities, become zero at the wall. The structure of the sublayer has been studied actively since the early 1950s. The early work, including Kline and Runstadler's [108] observation of the unstable character of the flow in the sublayer using flow visualization, was well-discussed and summarized by Rotta [109]. More recently, several experimenters have studied the eddy structure in the sublayer and identified "streaks" [110], observed first by Kline and Runstadler [108], and "bursts" [111, 112, 113] in which organized bodies of slowly-moving fluid move away from the surface. These observations are well-summarized by Cantwell [114]. Kim, et al. [112] in the early 1970s observed that the bursting phenomenon plays a dominant role in the production of turbulent energy, and conjectured, based on their observations, that the bursting dominates the transfer process between inner and outer regions of the boundary layer and in doing so plays an important role in determining the structure of the entire layer. Somehow this hypothesis contrasts with Bradshaw's argument [115] that most of a boundary layer is strongly affected by the surface or the interface between turbulent and irrotational fluid in a boundary layer.

Corino and Brodkey [116] observed continuous disturbances of the sublayer by small-scale velocity fluctuations and periodic disturbances by fluid elements which penetrated into this region from outside of the
sublayer. However, at any location, a time-mean value of the thickness of the layer may be distinguished which extends to about $y^+ = 5$. In this region, the variation in the mean velocity is determined by the molecular viscosity, and is nearly linear, as in laminar flow. Hence, this region is also called the "linear sublayer." The velocity profile can be written as

$$u^+ = y^+$$

(2.30)

In the region $y^+ > 5$ in Figure 2.3, the effect of the molecular viscosity on the flow decreases gradually with increasing distance from the wall. Finally, a region is reached where the flow is completely turbulent and the effects of molecular viscosity on the shear-stress-producing eddies are negligibly small. The intermediate region, where the viscous and mixing stresses are of comparable magnitude, is called the buffer layer (it is sometimes called the transition region, which should not be confused with the region of true change from laminar to turbulent flow). In general, the thickness of either the viscous sublayer or the buffer layer is quite small in comparison with that of the fully turbulent region. The buffer layer roughly lies in the range $5 < y^+ < 50$ as shown in Figure 2.3.

In the fully turbulent layer ($50 < y^+$ but $y < 0.1-0.25$) the flow is still dominated by the wall, but the turbulence develops sufficiently to render the viscous stress negligible. The mean velocity varies nearly logarithmically in this region, i.e.,

$$u^+ = \frac{1}{\kappa} \ln y^+ + B$$

(2.31)
where $k$ and $B$ are constants. Hence, this region is often called the "logarithmic layer."

The outer region of turbulent boundary layer ($y > 0.1-0.2\delta$) contains 80-90% of the boundary layer thickness (see Figure 2.3). The probability distribution of the interface between turbulent and irrotational (nonturbulent) fluid in a boundary layer is roughly Gaussian with a mean of 0.85 and a standard deviation of 0.15 (so that the interface occasionally extends as far in as 0.45 or as far out as 1.35) [115]. The outer region is dominated by large eddies which transport fluid with low momentum and high turbulent energy from the outer part of the inner layer and deposit it near the outer edge of the boundary layer [115]. These large eddies contribute at least 50% to the turbulent energy associated with the $u$ and $v$ fluctuations and about 80% to the Reynolds stress [117]. Whether the large eddies arise in the outer layer because of some form of interfacial instability [118], or result from the aforementioned bursts observed by Kline, et al. [111] in the viscous sublayer [119], or both, is still an open question.

In this region, the mean velocity can be expressed in terms of the velocity defect $u_e - u$ (see, e.g. [120])

\[
\frac{u_e - u}{u_{\tau}} = f_2 \left( \frac{Y}{\delta} \right)
\]

Equation (2.32) is called the "velocity defect law."

The above discussion is mainly focused on a fully attached turbulent boundary layer. When turbulent flow separates under an adverse streamwise pressure gradient, it is often found that the flow is quite unsteady, sometimes randomly so and sometimes in a quasiperiodic sense.
leading to a vortex street in a wake [120]. Up to now, relatively little information is available about the structure of the turbulence in separating flows.

Recent measurements along a smooth surface by Simpson, et al. [121] show that, upstream of the vicinity of separation, the qualitative turbulence structure is not markedly different from the zero-pressure gradient case. In this region, the wall bursting frequency behavior and the spanwise structure spacing in the viscous sublayer behave similarly to that for the zero-pressure gradient case. As separation is approached, the near wall separating flow appears to be increasingly dominated by the large-scale outer flow, and finally the wall flow is governed by large eddies downstream of the separation point [122].

Simpson [122] argued, based on his measurements [121, 123], that in the reversed flow region, the flow field is strongly dominated by turbulent fluctuations which are greater than or at least comparable to the mean velocities and that the mean reversed-flow appeared to be only large enough to satisfy continuity requirements. The law of the wall type velocity profile and the local equilibrium argument seem not to be valid for the back flow, unless significant turbulent energy production occurs near the wall [122].

When a boundary layer is caused to separate rapidly by an obstacle, the structure of the turbulence is quite different from that of the gradually separating flow over a smooth surface [124]. This is because in the former case, the boundary layer is subjected to a strong perturbation (see, e.g., [125]). However, at the present time, very little reliable data exist to define the structure of the reversed flow as
pointed out by Gosman (see [126]).

In the reattaching and redeveloping region, many questions still remain about the structure of turbulence, despite the large amount of past research. It is not clearly known at present how various parameters affect the reattachment process. Several experimentalists [52, 58, 68, 127] have stated that the separated shear layer is fundamentally different from the plane mixing layer due to the large-scale turbulence in the recirculating flow. However, a recent argument by Eaton and Johnston [62] is contrary to this. They concluded, based on their observation of the flow over a rearward-facing step, that the separated shear layer is very similar to the plane mixing layer upstream of the reattachment zone. In the separated shear layer approaching reattachment, the nature of the large eddy motion is still not clearly understood.

Bradshaw and Wong [50] observed that the large eddies split into two at reattachment, so that one of them deflected upstream into the recirculating flow region, and the other continued to flow downstream. They stated that such phenomena of bifurcation of large eddies results in a rapid decrease in turbulent shear stress and length scale. However, several other investigators disagreed with this; instead, they argued that some of the eddies proceeded downstream and others were swept upstream with the recirculating flow without being split [52, 58, 127]. There is also speculation that both the phenomena of the bifurcation and the alternating distribution of entire eddies, some going upstream and some going downstream, may occur together. Recently, Eaton and Johnston [62] expressed the entirely different opinion that
all of the large eddies flowed downstream.

In the redeveloping boundary layer downstream of reattachment, two shear layers, i.e., the approaching mixing layer and the new wall shear layer, interact and produce an unusual turbulent structure. The mixing layer in the outer part of the boundary layer has been observed to exhibit characteristics of the separation zone far downstream of reattachment. A surprisingly long distance is required for the flow to return to the structure characteristic of an ordinary turbulent boundary layer [50, 127]. Mean velocities, when they are plotted on a semi-logarithmic scale, have a marked "dip" below the log law for the near wall region [50, 52, 58]. In the redeveloping region, the turbulent energy has been observed to decay continuously. However, the reason for this has not been clearly explained.

2. Introduction to turbulence modeling

Time averaging of the Navier-Stokes equations results in the so-called Reynolds equations in which a second-order symmetric tensor for the apparent turbulent stresses appears. These stresses are known as Reynolds stresses. These must be specified before the governing equations can be solved for the mean velocity distribution. While further relationships among these quantities can be developed from the basic conservation laws, these additional constraints inevitably introduce still more unknowns. This is called the closure problem. Therefore, these terms must be closed by empirical assumptions.

Turbulence models can be divided into two main types based on whether or not they employ Boussinesq's suggestion (see, e.g., [101]),
that the stress-strain law for time-averaged turbulent flow is of the same form as that for a Newtonian fluid in laminar motion. Most of the models currently used in engineering calculations, except the so-called Reynolds stress models, are of the type which use the Boussinesq assumption. Experimental evidence indicates that although the Boussinesq concept of a turbulent viscosity [128] appears to be valid in many flow circumstances, exceptions have been noted.

The other common classification of models is according to the number of supplementary partial-differential equations (PDEs) which must be solved in order to supply the modeling parameters [128, 129, 130]. This number ranges from zero for the simplest algebraic mixing length models to twelve for the most complex [131] models. Turbulence models also can be divided into four classes such as mean-flow/field, mean-flow/integral, transport-equation/integral and transport-equation/field methods based on whether turbulence properties are related directly to the mean flow or obtained from transport equations [115]. An alternative open-ended classification is based on the highest order of velocity product for which a transport equation is used [130]. According to this classification, a first-order model evaluates the Reynolds stresses in the momentum equation through functions of the mean velocity and geometry alone. Higher-order models generally employ a modeled form of the transport PDEs for the Reynolds stresses.

Zero and one-half equation models are in common use in the more sophisticated engineering industries, and two-equation models are currently popular in academic research but have not been used extensively for engineering applications. Recently, large eddy simulations [129,
have been developed. The concept of the approach is to calculate the three-dimensional time-dependent large eddy structure by solving numerically the equations deduced from the Navier-Stokes equations using a spatial filtering process. Effects of the small-scale turbulence are evaluated by means of modeling. However, this approach has not been developed to the point of applicability to actual engineering analysis.

For a two-dimensional boundary layer, using Equations (2.10) and (2.11), the problem in modeling reduces to finding an expression for $- \rho' v'$ as discussed in the previous section.

In this study, models employing transport equations for turbulence kinetic energy and an outer region length scale have been employed. The models will be discussed below.

3. **Length scale model**

One of the most successful simple turbulence models was suggested by Prandtl [132] in the 1920s:

$$\nu_t = \rho \tau^2 \frac{\partial u}{\partial y}$$

(2.33)

where $\tau$, a "mixing-length," can be thought of as the transverse distance over which particles retain their original momentum. $\tau \frac{\partial u}{\partial y}$ can be interpreted as a characteristic velocity of turbulence, $v_T$, and $\tau$ a mean free path for the collision or mixing of globules of fluid by drawing an analogy with kinetic theory of gases. Thus,

$$\nu_t = \rho v_T \tau$$

(2.34)

The turbulence model used in this study is composed of two parts.
The first treats the inner part of the flow and the second applies to the outer part.

a. **Model for the inner region** The inner layer of a turbulent boundary layer is close to local equilibrium (a special case of self-preservation [133]) where the only relevant length scale is distance \( y \) from the wall. In this region, the mixing length is believed generally to vary linearly with distance from the wall. Thus,

\[
\ell_i = \kappa y
\]

where \( \kappa \) is von Kármán constant taken as a value of 0.41. Recently, several investigators have pointed out that an assumption of a logarithmic region in the universal law of the wall is incompatible with the mixing length assumption \( \ell = \kappa y \) except where the shear stress is essentially constant through the near wall region [134, 135]; where this is not the case, Reeves [136], and McD Galbraith and Head [137] have suggested that Equation (2.35) be replaced by

\[
\ell_i = \kappa \sqrt{\frac{\tau}{\tau_w}} y
\]

if a logarithmic distribution is to be obtained. Although plausible, this argument deserves further study.

Very close to the wall turbulent fluctuations are damped due to viscous effects that can be taken into account by use of van Driest's hypothesis [138] as

\[
\ell_i = \kappa D y
\]

where damping function \( D \) is given by
This form was suggested by Carter and Wornom [2] and Fletcher [94] for separating flows, and used successfully for separating bubble flows by Kwon and Fletcher [4]. A value of 26 used for the damping constant $A^+$.

The modifications given by Equations (2.36) and (2.37) were combined as

$$
D = 1 - \exp\left[-\left(\frac{1}{\nu} \left| \frac{\partial \mathbf{u}}{\partial y} \right|_{\text{max}} \right)^{1/2} y/A^+\right]
$$

The above evaluation of $\lambda_1$ was used in both the separated and attached flows of the present study. It should be noted that Equation (2.38) reduces to Equation (2.37) for ordinary turbulent flows where the maximum shear stress occurs at the wall.

b. Model for the outer region In the outer region of a turbulent boundary layer, the mixing length remains approximately constant and can be specified as

$$
\lambda_o = C_1 L
$$

where $C_1$ is a constant and $L$ is a characteristic length scale of turbulence. The width of the turbulent region $\delta$ is often used as the characteristic length scale $L$. For that case, the constant $C_1$ varies with the type of turbulent flow (see, e.g., [139]). For a turbulent boundary layer, $C_1$ is approximately 0.085 with $L = \delta$, the boundary layer thickness. With such an approximation, Fletcher [140] and many others have successfully predicted turbulent boundary layers. However,
this model is based on the implicit neglect of transport terms which, in general, cannot be justified.

In the outer region, the length scale transported from upstream is significant since the large eddies that characterize that region travel a streamwise distance of several shear-layer thicknesses in their lifetimes. In this study, the transport equation for the length scale developed by Fletcher [94] has been employed. The general form of this equation is written as

$$\frac{dL}{dx} = \frac{\delta - L}{L^*}$$ (2.40)

Here \(L^*\) is the streamwise distance traversed by the flow during the relaxation time, which is proportional to \(\delta/u_\tau\), and is expressed as

\[L^* = C_2 \bar{u}_\tau \delta/\bar{u}_\tau\] where \(\bar{u}_\tau\) is a characteristic turbulence velocity. By letting \(\bar{u}_\tau = u_\tau L/\delta\), Equation (2.40) can be rewritten as

$$\frac{dL}{dx} = \frac{u_\tau}{C_2 \bar{u}_\tau} L_1 - \left(\frac{L}{\delta}\right)^2$$ (2.41)

The constants \(C_1\) and \(C_2\) are 0.12 and 0.8, respectively.

Equation (2.41) can be identified as a one-dimensional specialization [141] of a more general transport equation for length scale, as given, for example, by Bradshaw [142]. Several investigators [4, 143, 144] have successfully used this model, sometimes with minor modifications, for several different kinds of turbulent flow.

The turbulence structure in step flows differs from that found in ordinary wall boundary layer flows as was pointed out previously. To account for such differences and discontinuities in the geometry, some modifications to Equations (2.39) and (2.41) are in order.
As discussed before, in the length scale transport equation, Equation (2.40), the relaxation time for ordinary wall boundary layer was assumed to be proportional to the time required for a change in turbulence structure to be transferred across the boundary layer δ with the velocity \( \bar{u}_{\tau} \), starting from the wall where the maximum shear stress generally occurs for fully attached flows. In the separated and redeveloping flow regions behind a rearward-facing step, the maximum shear stress is generally found to occur quite some distance from the wall. By assuming (crudely) that the place where the maximum shear stress occurs is the point of origin for the change in structure, and that information about the change is transferred in the normal direction from that point toward the wall and toward the outer edge of the boundary layer, the average time required for the transfer may be expressed as \( \delta'/\bar{u}_{\tau,max} \). Here, \( \delta' \) is a mean value of the distances from the assumed turbulence generation point to the wall \( y_{\tau,max} \) and to the boundary layer edge \( \delta - y_{\tau,max} \), and \( \bar{u}_{\tau,max} \) is the maximum turbulence characteristic velocity. The average distance \( \delta' \) of \( y_{\tau,max} \) and \( \delta - y_{\tau,max} \) can be evaluated in several ways such as by using the arithmetic mean, the root mean square, or the logarithmic mean. In the present study, the root mean square of \( y_{\tau,max} \) and \( \delta - y_{\tau,max} \) was used for \( \delta' \) as

\[
\delta' = \left[ \left\{ y_{\tau,max}^2 + (\delta - y_{\tau,max})^2 \right\}/2 \right]^{1/2}
\]  

(2.42)

Based on these assumptions, the streamwise distance traversed by the fluid in the outer layer during the relaxation time can be written as
The mean velocity of the fluid in the outer part of the boundary layer is evaluated as \((u_e + u_{sw})/2\), where \(u_{sw}\) is the velocity at the switching point from the inner to the outer region. (The evaluation of the switching point will be discussed later.) For fully attached flows, \(u_{sw}\) usually has nearly the same magnitude as \(u_e\), so that \(u_m \approx u_e\). However, for the flows behind the step, \(u_{sw}\) is found to be comparatively small so that the more appropriate form, \((u_e + u_{sw})/2\), was used for \(u_m\).

Now, by letting \(\bar{u}_{\tau,\text{max}} = u_{\tau,\text{max}} L/\delta'\), Equation (2.40) can be written as

\[
\frac{dL}{dx} = \frac{u_{\tau,\text{max}}}{C_3(u_e + u_{sw})} \left(\frac{L}{\delta'}\right)
\]

In the present study, Equation (2.39) was also modified for separated and redeveloping flow regions downstream of a step as

\[
L = C_4 \frac{\delta}{\delta'} L
\]

This modification does not have a solid theoretical basis; however, Equation (2.44) can be developed from Equation (2.39) by using an approximate relationship between \(\delta'\) and \(\delta\) shown in Appendix A.

It should be noted that these modified equations, Equations (2.43) and (2.44), reduce to the original form of the equations, Equations (2.39) and (2.41), for an ordinary turbulent boundary layer. In the present analysis, the constants \(C_3\) and \(C_4\) were set to 0.4 and 0.12, respectively.

The modified turbulence model seems to provide reasonably good
prediction of flows over a rearward-facing step. However, flow reattachment was found to be delayed somewhat when the model was used with the present viscous-inviscid interaction scheme. Although it is possible that the prediction could be improved if the model was used with a better mathematical model, for example, the full Reynolds equations, a simple algebraic model has been found to give somewhat better predictions in the separating and redeveloping flow regions after the step. In those regions, Equation (2.44) was mainly used with the replacement of L by δ such as

$$L_o = 0.1(\frac{\delta}{\delta'})\delta$$

(2.45)

When this equation was applied immediately after the step, a sudden jump in the mixing length was observed. Thus, the following formulation was employed to get a smooth variation in the mixing length distribution for a short distance downstream of the step:

$$L_o = 0.08(1 + C_5 \frac{\delta}{\delta'})(\delta - y_D)$$

(2.46)

In fact, in the present analysis, the smaller value of $L_o$ obtained from the above two equations, Equations (2.45) and (2.46), was taken as the mixing length in the outer layer for the separated and redeveloping flow regions downstream of the step. Equation (2.46) was found to work reasonably well with a value of $C_5$ of about 0.3. However, for the calculations of the present study, where most of the flow configurations have a relatively small channel inlet height to step height ratio, $C_5$ was evaluated using the following correlation:
This equation gives about 0.3 of $C_5$ for the present turbulent flow configurations.

For an ordinary boundary layer, Equation (2.46) will also reduce to the form $\beta = C_6$.

This algebraic model seems to work reasonably well for the step flows considered in the present study. However, the generality of the model is still uncertain.

Except for the separating and redeveloping regions downstream of the step, Fletcher's L-equation model, Equations (2.39) and (2.41), has been used as shown in Table 2.1. The switch between the inner and outer region model is made whenever $\lambda_i \geq \lambda_o$.

Table 2.1. Turbulence models used in the present study

<table>
<thead>
<tr>
<th>Turbulence models</th>
<th>$\beta$-model</th>
<th>$k-\beta$ model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flow on the wall with a rearward-facing step</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inner layer</td>
<td>Equation (2.38)</td>
<td>Equation (2.38)</td>
</tr>
<tr>
<td>Outer layer</td>
<td>Equations (2.39) and (2.41)</td>
<td>Equations (2.39), (2.41), and (2.51)</td>
</tr>
<tr>
<td>Before step</td>
<td>Equations (2.45) and (2.46)</td>
<td>Equations (2.45), (2.46), and (2.51)</td>
</tr>
<tr>
<td>After step</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Flow on the wall without a step</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inner layer</td>
<td>Equation (2.38)</td>
<td>Equation (2.38)</td>
</tr>
<tr>
<td>Outer layer</td>
<td>Equations (2.39) and (2.41)</td>
<td>Equations (2.39), (2.41), and (2.51)</td>
</tr>
</tbody>
</table>
4. **Turbulence kinetic energy model**

The suggestion of Prandtl and Kolmogorov in the 1940s (see, e.g., [145]) was to let $\nu_t$ be proportional to the square root of the turbulent kinetic energy, $k$. Thus, Equation (2.33) becomes

$$\nu_t = C_p \frac{k^{1/2}}{\delta}$$  \hspace{1cm} (2.47)

where the turbulent kinetic energy $k$ is defined as

$$k = \frac{1}{2} (u'^2 + v'^2 + w'^2)$$

A transport equation for turbulent kinetic energy can be derived from the Navier-Stokes equations. For steady, incompressible thin shear layer flows, the equation is of the following form [141]

$$\rho \frac{\partial k}{\partial t} + \rho v \frac{\partial k}{\partial y} = \frac{\partial}{\partial y} \left( \mu \frac{\partial k}{\partial y} \right) - \frac{\partial}{\partial y} \left( \rho v'k' + v'\rho \delta \right)$$

$$- \rho u'v' \frac{\partial u}{\partial y} - \rho \varepsilon$$  \hspace{1cm} (2.48)

In the above equation, the diffusion and dissipation terms must be modeled in order to avoid introducing additional unknowns.

By assuming that $k$ diffuses down its gradient, the diffusion flux can be written as

$$\rho v'k' + v'\rho = \mu_t \frac{\partial k}{\partial y}$$  \hspace{1cm} (2.49)

where $\text{Pr}_k$ is known as the turbulent Prandtl number. This expression was suggested by Kolmogorov, Prandtl, and others (see, e.g., [139]).

When the local Reynolds number of turbulence defined as $k^{1/2}/\nu$ is large, the dissipation of turbulence energy by viscous action occurs...
mainly in the smallest eddies. However, the rate at which energy is fed into these eddies is governed by the large eddies which contain most of the energy. At large Reynolds numbers, the rate of dissipation is, therefore, independent of the molecular viscosity, and Kolmogorov in 1941 (see, e.g., [145]) suggested the following expression

$$e_D = C_D \kappa^{3/2}/\ell$$  \hspace{1cm} (2.50)

Introduction of Equations (2.12), (2.49) and (2.50) into Equation (2.48) leads to the following modeled form of the turbulence kinetic energy equation,

$$\rho u \frac{\partial k}{\partial x} + \rho \nu \frac{\partial k}{\partial y} = \frac{\partial}{\partial y} \left[ \left( \mu + \frac{\mu_t}{\text{Pr}_k} \right) \frac{\partial k}{\partial y} \right] + \mu_t \left( \frac{\partial u}{\partial y} \right)^2 - C_D \rho \kappa^{3/2}/\ell$$  \hspace{1cm} (2.51)

The constants $C_D$ and $C_\mu$ in Equation (2.51) must be determined.

In the inner region where the turbulence kinetic energy usually is in a near local equilibrium state (as discussed earlier), the convective and diffusion terms can be neglected and Equation (2.51) reduces to

$$\mu_t \left( \frac{\partial u}{\partial y} \right)^2 = C_D \rho \kappa^{3/2}/\ell$$  \hspace{1cm} (2.52)

From Equations (2.12) and (2.47)

$$\mu_t \left( \frac{\partial u}{\partial y} \right)^2 = \frac{(-\rho u'v')^2}{\mu_t} = \frac{(-\rho u'v')^2}{C_\mu \rho \kappa^{1/2}/\ell}$$  \hspace{1cm} (2.53)

Therefore, the following equation is obtained

$$\frac{-u'v'}{k} = (C_\mu C_D)^{1/2}$$  \hspace{1cm} (2.54)

It has been found experimentally that $-u'v'/k$ lies between 0.25 for
the flat plate boundary layer and 0.3 for fully developed pipe flow [146]. By taking 0.3 for \(-\frac{u'v'}{k}\)

\[ C_{\mu D} = 0.09 \]

It can be assumed as usual that the Prandtl's mixing length formulation, Equation (2.33), holds in the inner region. Thus, from Equations (2.12) and (2.33), \(-\frac{u'v'}{k}\) can be written as

\[ -\frac{u'v'}{k} = \rho \frac{l^2}{\partial y} \frac{\partial u}{\partial y} \]

By introducing Equation (2.53) into this equation

\[ \mu_t^2 = \rho \frac{l^2}{(-\frac{u'v'}{k})} \]

or

\[ \mu_t = \rho \frac{l}{(-\frac{u'v'}{k})}^{1/2} \]

The substitution of this equation into Equation (2.47) leads to

\[ C_{\mu} = \left(-\frac{u'v'}{k}\right)^2 \]

Since 0.3 is used for \(-\frac{u'v'}{k}\)

\[ C_{\mu} = (0.3)^{1/2} = 0.548 \]

Therefore,

\[ C_D = 0.164 \]

The turbulent Prandtl number \(Pr_k\) was assigned a value of 1.0. At the present time, several different combinations of values for the constants,
\( C_{\mu}, C_D \) and \( Pr_k \), can be found in the literature (see, e.g., [146]). The values used in the study appear to be among the most widely used set.

Equation (2.51) requires boundary conditions at inner and outer boundaries. The inner boundary condition is not specified at the wall where flow is largely influenced by laminar viscosity. Instead, it is specified at some distance from the wall, i.e., at the switching point from the inner to the outer layer as

\[
k(x, y_{sw}) = -\rho u'v'(x, y_{sw})/C_D^{2/3}
\]

The above boundary conditions follow from the usual assumptions that in the fully turbulent region near the wall, the generation and dissipation terms balance one another and that Prandtl's mixing length formula, Equation (2.33), also holds in that region.

The outer boundary condition is specified as

\[
k(x, \infty) = k_0
\]

where \( k_0 \) is the free stream value of turbulence kinetic energy.

An initial distribution of \( k \) is provided by assuming that the turbulent viscosity predicted by both Equations (2.33) and (2.47) are equal at some distance downstream of the inlet of a duct where the calculation of \( k \) begins. The equation of \( k \) at that location is thus written as

\[
k(\text{initial}) = \rho^2 \frac{\partial u}{\partial y}^2 / \kappa^2
\]

The length scale needed in Equation (2.57) is provided by using the model described in Section II.B.3.
C. Inviscid Flows

1. Geometry

Figure 1.3 shows an inviscid core region, region B, which is surrounded by displacement surfaces on the upper and lower sides and the vertical inlet and outlet sections. Region B is often called an effective flow channel (EFC) [58], since a displacement thickness $\delta^*$ is subtracted from the channel walls to provide a new effective passage for the inviscid flow. The velocity is assumed constant across the inlet. No flow is permitted across the upper and lower $\delta^*$ lines, which are thus approximated as streamlines, as shown in Figure 2.4. The Laplace equation for a streamfunction $\psi$ is solved in this region.

2. Governing equation and boundary conditions in the physical coordinate system

a. Governing equation By assuming that flow is two-dimensional, incompressible and irrotational in the domain defined by EFC, the governing equation can be written as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (2.58)$$

where the streamfunction $\psi$ is defined as usual:

$$u = \frac{\partial \psi}{\partial y} \quad (2.59-a)$$

$$v = -\frac{\partial \psi}{\partial x} \quad (2.59-b)$$

b. Boundary conditions The differential equation, Equation (2.58), can be solved numerically subject to the following boundary conditions along the boundaries of the calculation domain as shown in
Figure 2.4. Inviscid flow computational domain in physical coordinates
Figure 2.4.

At the inlet, 1-4 in Figure 2.4, a constant, uniform velocity, $u_{e,o}$, is assumed. In terms of the streamfunction, this boundary condition is expressed as

$$\psi(x_0, y) = u_{e,o} \left( y - \delta^L(x_0) \right) \tag{2.60}$$

Since the lower and upper boundaries, 1-2 and 4-3 in Figure 2.4, are approximated as streamlines, the boundary conditions at these boundaries become

$$\psi(x, \delta^L_U) = 0 \tag{2.61}$$

$$\psi(x, \delta^*_{U, L}) = u_{e,o} \left( \delta^*_{U}(x_0) - \delta^*_{L}(x_0) \right) = \psi_T \tag{2.62}$$

The boundary condition at the exit of the computation region, 2-3 in Figure 2.4, requires special consideration. Since the flow is developing as it moves along downstream and the outflow condition is not known a priori, specifying a particular velocity profile at the outflow condition is too restrictive. Up to now, several different techniques have been suggested for the treatment of the downstream boundary conditions (see, e.g., [147]). The least restrictive outflow boundary conditions among them is

$$\frac{\partial \psi}{\partial x} = \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{2.63}$$

Equation (2.63) was first developed by Thoman and Szewczyk [148] and has been used successfully by several investigators (see, e.g., [149, 150]).
In the present study, instead of using Equation (2.63) directly, the discretized ordinary differential equation deduced from Equation (2.58), following Roache [151] and Briley [152], is used as the outlet boundary condition. That is,

\[ \frac{d^2 \tilde{y}}{dy^2} = 0 \]  

(2.64)

3. Governing equation and boundary conditions in the transformed coordinate system

a. Introduction to the coordinate transformation. The development of the displacement surfaces along the upper and lower walls of the channel causes the inviscid calculation domain to be irregular in the physical coordinates (see Figure 2.4). In the numerical analysis, such an irregular domain is not preferred since the use of irregular nodes on the boundary is generally believed to cause a deterioration of accuracy in the solution. The irregular grid may also cause the order of the formal truncation error of an otherwise second-order accurate scheme to be reduced to first order or even to zero order [147, 153]. The numerical stability characteristics of the method may also be adversely affected by such an irregular mesh because some of the intervals next to the boundary may become excessively small [154]. Consequently, a coordinate transformation is often used to generate a regular geometry in transformed coordinates to replace the irregular domain in the physical Cartesian coordinates for computational purposes.

In recent years, much attention has been given to ways of numerically generating transformed curvilinear coordinate systems such that the new grid lines are coincident with the surface of the bodies
and/or the boundaries of the integration domain (see, e.g., [155, 156, 157]). Significant progress has been made for arbitrarily shaped two-dimensional bodies or regions [158, 159, 160, 161]. These grid generation techniques generally require that two additional elliptic partial differential equations be solved in order to generate the curvilinear coordinates. Such coordinate transforms are not only used to allow accurate implementation of the boundary conditions at irregular nodes, but also used to increase resolution by refining the grid size in a particular part of the computational plane where large gradients occur [160, 161].

However, by using the coordinate transformation techniques, the governing differential equations in the new coordinates become more complicated than in the original physical plane. Therefore, a problem having simple equations but complex geometry, has been exchanged for a problem having complex equations and simple geometry. This is a main drawback of using coordinate transformations.

b. Governing equation In the present study, the following equations have been introduced to transform the original computation domain in Figure 2.4 to a rectangular domain on the new ξ, η coordinates as shown in Figure 2.5. The equations are

$$\xi = \frac{x - x_0}{L_e}$$

(2.65-a)

and

$$\eta = \frac{y - \delta^*_y}{\delta^*_U - \delta^*_L}$$

(2.65-b)

where \(L_e\) is a reference length taken as \(x_e - x_0\). Equation (2.58) can
Figure 2.5. Inviscid flow computational domain in transformed coordinates.
be rewritten for the transformed coordinates $\xi$ and $\eta$, defined in Equations (2.65), as

\[
\frac{\partial^2 \psi}{\partial \xi^2} - \frac{2}{\delta^*_{UL}} \left( \frac{d\delta^*}{d\xi} + \eta \frac{d\delta^*_{UL}}{d\xi} \right) \frac{\partial^2 \psi}{\partial \eta^2} + \eta \left( \frac{d\delta^*_{UL}}{d\xi} \right)^2 + \left( \frac{2}{\delta^*_{UL}} \right) \left( \frac{d\delta^*_{UL}}{d\xi} \right) \frac{\partial^2 \psi}{\partial \eta \partial \xi} + \frac{1}{\delta^*_{UL}} \left( \frac{d\delta^*_{UL}}{d\xi} \right) \frac{\partial \psi}{\partial \xi} + \eta \frac{d\delta^*_{UL}}{d\xi} = 0
\]  

(2.66)

where $\delta^*_{UL}$ is defined as $\delta^*_{UL} - \delta^*_{L}$. Details of the derivation of Equation (2.66) are discussed in Appendix B.

c. Boundary conditions By using Equations (2.65), the boundary conditions, Equations (2.60)-(2.62) and (2.64), can be rewritten for the transformed plane as shown in Figure 2.5, as the following:

- at the inlet, $1'-4'$ in Figure 2.5,

\[
\psi(0, \eta) = \psi_T \eta
\]  

(2.67-a)

- at the lower boundary, $1'-2'$ in Figure 2.5,

\[
\psi(\xi, 0) = 0
\]  

(2.67-b)

- at the upper boundary, $3'-4'$ in Figure 2.5,

\[
\psi(\xi, 1) = \psi_T
\]  

(2.67-c)

and at the outlet, $2'-3'$ in Figure 2.5,

\[
\frac{d^2 \psi}{d\eta^2} = 0
\]  

(2.67-d)
With these boundary conditions, Equation (2.66) was solved for the inviscid flow in a two-dimensional channel expansion in the present study.
III. FINITE-DIFFERENCE FORMULATION

This chapter describes the finite-difference method used to solve the governing partial differential equations. Consistency and stability of the finite-difference formulations are discussed. Finally, the grid system used in the normal direction for the boundary-layer solution is discussed.

A. Nondimensional Form of the Governing Equations

It is generally useful to write the governing equations in dimensionless form. In the present study, the variables are nondimensionalized as follows:

\[ U = \frac{u}{u_o}, \quad V = \frac{v}{u_o}, \quad P = \frac{P}{\rho u_o^2}, \]

\[ X = \frac{\rho u x}{\mu}, \quad Y = \frac{\rho u y}{\mu}, \quad \hat{L}_e = \frac{\rho u L}{\mu}, \]

\[ \hat{\theta}^* = \frac{\rho u \delta^*}{\mu}, \quad \hat{H} = \frac{\rho u H}{\mu}, \quad \hat{L} = \frac{\rho u L}{\mu}, \]

\[ \hat{\delta} = \frac{\rho u \delta}{\mu}, \quad \hat{\xi} = \frac{\rho u \xi}{\mu}, \quad \hat{\mu}_t = \frac{\mu_t}{\mu}, \quad k = \frac{k}{u_o^2}, \]

\[ \psi = \frac{\rho u}{\mu} \text{ (used for the viscous flow), } \Psi = \frac{\psi}{\psi_T} \text{ (used for the inviscid flow)} \]

Here, since the fluid density and viscosity were earlier assumed to be constant for the flows being considered, those quantities, \( \rho \) and \( \mu \), and an arbitrary reference velocity, \( u_o \), were used as reference values for the nondimensionalization.
1. Governing viscous flow equations

With the nondimensionalized variables introduced in Equation (3.1), the governing equations for viscous flow, Equations (2.16) and (2.17), become

\[ U = \frac{\partial \psi}{\partial Y} \]  
\[ cU \frac{\partial U}{\partial X} - \frac{\partial \psi}{\partial X} \frac{\partial U}{\partial Y} = - \frac{\partial p}{\partial X} + \frac{\partial}{\partial Y} \left[ (1 + \mu_t) \frac{\partial U}{\partial Y} \right] \]  

The nondimensional boundary conditions are as follows:

For external boundary layers:

\[ U(X, 0) = \psi(X, 0) = 0 \]  
\[ U(X, Y) = U_e(X) \quad \text{and} \quad \psi(X, Y) = \psi_e(X), \text{as } Y \to \infty \]  

for the direct method. For the inverse method, the second boundary condition is replaced by

\[ U(X, Y) = U_e(X) \]  

and

\[ \psi(X, Y) = U_e(Y - \delta_e), \text{as } Y \to \infty \]  

For a two-dimensional channel:

\[ U(X, 0) = \psi(X, 0) = 0 \]  

At \( Y = \frac{H}{2} \),

\[ \frac{\partial U}{\partial Y} = 0 \]  

and
\[ \psi(X, \frac{\hat{H}^2}{2}) = \frac{\psi_T}{2} \]  

(3.8-b)

for a symmetric channel flow. For a developing flow in a symmetric two-dimensional channel, containing an inviscid core, Equation (3.8) can be replaced by

\[ U(X, \frac{\hat{H}}{2}) = U_e(X) \]  

(3.9-a)

and

\[ \psi_T = U_e(\hat{H} - 2\hat{X}) \]  

(3.9-b)

For an asymmetric channel, the second boundary condition is replaced by

\[ U(X, \hat{H}) = 0 \]  

(3.10-a)

\[ \psi(X, \hat{H}) = \psi_T \]  

(3.10-b)

2. Governing inviscid flow equation

For the governing inviscid flow equation, the streamfunction \( \psi \) was nondimensionalized differently from that for the boundary-layer equations as shown in Equation (3.1),

\[ \hat{\psi} = \frac{\psi}{\psi_T} \]

where \( \psi_T \) was defined in Equation (2.26).

Using Equation (3.1), the governing inviscid flow equation written for the transformed coordinates, Equation (2.66), becomes
The boundary conditions become

at the upstream boundary, 1'-4' in Figure 2.5,

\[ \hat{\psi}(0, \eta) = \eta \]  

(3.12-a)

at the lower boundary, 1'-2' in Figure 2.5,

\[ \hat{\psi}(\xi, 0) = 0 \]  

(3.12-b)

at the upper boundary, 3'-4' in Figure 2.5,

\[ \hat{\psi}(\xi, 1) = 1 \]  

(3.12-c)

and at the downstream boundary, 2'-3' in Figure 2.5,

\[ \frac{\delta^2 \hat{\psi}}{\delta \eta^2} = 0 \]  

(3.12-d)

3. **Turbulence kinetic energy equation**

The turbulence kinetic energy equation, Equation (2.51) can also be nondimensionalized as follows

\[
U \frac{\partial \hat{k}}{\partial x} + V \frac{\partial \hat{k}}{\partial y} = \frac{\partial}{\partial y} \left[ \left(1 + \frac{\hat{u}'}{Pr_k} \right) \frac{\partial \hat{k}}{\partial y} + \hat{u}' \left( \frac{\partial u}{\partial y} \right)^2 - \frac{C_{Dk}^{3/2}}{\hat{k}} \right]
\]  

(3.13)
The boundary conditions for turbulence kinetic energy, Equations (2.55) and (2.56), are expressed in nondimensional variables as

\[ \hat{k}(X, \frac{Y}{Y_s}) = - \frac{U'V'(X, \frac{Y}{Y_s})}{c_D^{2/3}} \]  
\[ \text{and} \]
\[ \hat{k}(X, \infty) = \hat{k}_o \]

B. Finite-Difference Representation

The set of nondimensionalized equations discussed in Section III.A is to be solved over the region of interest using a finite-difference method. In this section, the finite-difference scheme employed to solve the governing partial differential equations is described.

1. Governing viscous flow equations

As discussed before, the boundary layer equations are parabolic in nature so they can be solved by an implicit finite-difference scheme which computes the entire viscous flow from the wall outward. The momentum and continuity equations are most commonly solved separately in an uncoupled manner in conventional implicit difference schemes. However, in the present study, this procedure proved unsatisfactory for flows containing large separated regions due to the appearance of wiggles in the velocity profiles and oscillations in the pressure gradient in the reversed flow region. A new procedure has been developed in the present study to solve the momentum and continuity equations simultaneously in a coupled manner. Coupling was first suggested by Davis for the boundary-layer equations (see [162]) and has been used...
by several authors \([3, 93, 96, 104, 163, 164]\). The present scheme differs somewhat from those used previously. Further details of the schemes will be discussed in Chapter IV. Figure 3.1 shows the finite-difference grid utilized the present calculation method.

The mesh size generally varies throughout the flow. The finite-difference representation of Equations (3.2) and (3.3) for a variable grid can be written as

\[
\frac{U_{j+1}^{i+1} + U_{j-1}^{i+1}}{2} - \frac{\psi_{j+1}^{i+1} - \psi_{j-1}^{i+1}}{\Delta Y} = 0
\]  

(3.16)

\[
cU_{j}^{i+1} \left( \frac{U_{1}^{i+1} - U_{1}^{i}}{\Delta X} \right) - \left( \frac{\psi_{1}^{i+1} - \psi_{1}^{i}}{\Delta X} \right) \frac{U_{1}^{i+1} - U_{1}^{i-1}}{\Delta Y_{+} + \Delta Y_{-}} \right)
\]

\[
= \chi + \frac{2}{(\Delta Y_{+} + \Delta Y_{-})} \left[ M_{j+1/2}^{i} \frac{U_{j+1}^{i+1} - U_{j}^{i+1}}{\Delta Y_{+}} \right] - M_{j-1/2}^{i} \left( \frac{U_{j}^{i+1} - U_{j-1}^{i+1}}{\Delta Y_{-}} \right)
\]

(3.17)

where the pressure gradient \(\chi\) is evaluated at the \(i+1\) station. It should be noted that Equations (3.16) and (3.17) are used for both internal and external flows. For internal flows, the pressure gradient \(\chi\) is set equal to \(dP/dX\) and carried in the algebraic formulation as an unknown. However, for external boundary-layer flows, \(\chi\) was evaluated by using the edge velocity such as \(\chi = -U_{e} \frac{dU_{e}}{dX}\). For external flow solutions proceeding in the direct mode, \(U_{e}\) is specified as a boundary condition. When the inverse mode is used, \(U_{e}\) is determined as part of the solution in such a manner that the displacement thickness is matched.

In the above equation, when \(U_{j}^{i+1} > 0\), \(c = 1.0\) and when \(U_{j}^{i+1} < 0\), \(c = 0\). The nondimensional diffusion coefficient \(M\) at \(j + 1/2\) and \(j - 1/2\)
Figure 3.1. Finite-difference grid
was evaluated as the arithmetic averages of these quantities at neighboring integer grid points:

\[
M_{j+1/2} = \frac{M_{j+1} + M_{j}}{2} \\
M_{j-1/2} = \frac{M_{j} + M_{j-1}}{2}
\]  

(3.18)

The momentum equation, Equation (3.17), is algebraically nonlinear in the unknowns due to the appearance of unknowns at the \( i + 1 \) level in the coefficients. The simplest and most common linearizing procedure is to evaluate the coefficients at the \( i \) level. This is known as "lagging" the coefficients [94]. For separating flows, such a simple lagging technique was found to cause unrealistic oscillations in the solution for the pressure gradient and wall shear stress distributions. Other procedures for linearizing the coefficients which can and have been followed are extrapolating coefficients [165], the simple iterative updating of coefficients [162], iterative update by the use of Newton linearization and Newton linearization with coupling. Several of these linearization schemes were evaluated during the course of the present study. Generally, they all worked satisfactorily for attached flows. However, for separated flows, the Newton linearization with coupling proved much superior to the other procedures and was the only scheme which resulted in a well-behaved solution when large separation regions occurred in the flow field. Details of these schemes are discussed in Appendix C.

In the present study, Newton linearization with coupling has been used to linearize the coefficients of the convective terms in the
momentum equation. The coefficient of the diffusion term was evaluated by lagging. With such approximations, Equations (3.16) and (3.17) can be rewritten as the following for the unknown quantities \( U \) and \( \Psi \) at the \( i + 1 \) station:

\[
b_j U_{j-1}^{i+1} + d_j U_j^{i+1} - \frac{\Psi_{j-1}^{i+1} + \Psi_j^{i+1}}{2} = 0
\] (3.19)

\[
B_j U_j^{i+1} + D_j U_{j+1}^{i+1} + A_j U_j^{i+1} + E_j \Psi_j^{i+1} = H_j \chi + C_j
\] (3.20)

The values of \( U \) and \( \Psi \) above in Equations (3.19) and (3.20) are unknowns. Thus, the algebraic problem to be solved at each streamwise step requires the simultaneous solution of \( 2NJ - 2 \) equations with \( 2NJ - 2 \) unknowns where \( NJ \) is the number of grid points across the flows including points on the boundaries.

The coefficients in Equations (3.19) and (3.20), which are assumed known from the previous station and previous iteration, are given in Appendix D. It should be noted that Equations (3.19) and (3.20) represent a general form for the numerical coupling algorithm for the boundary-layer equations, if the finite-difference expressions for the boundary-layer equations are written for the variables themselves, for example, \( U \) and \( \Psi \) as in the present case, and not for the variation of the variables such as \( \delta U \) and \( \delta \Psi \). Here, the variations \( \delta U \) and \( \delta \Psi \) are the changes in \( U \) and \( \Psi \), respectively, between two successive iterations as defined in Equation (11.5) for a general variable \( \phi \) (see Appendix C). Thus, different finite-difference schemes for coupled boundary-layer solutions from the scheme discussed above may result in the same equations as Equations (3.19) and (3.20); however, the coefficients are different.
for each scheme. Appendix E provides the coefficients in Equations (3.19) and (3.20) for a few different finite-difference schemes such as, a fully implicit scheme with lagged coefficients, and the Crank-Nicolson scheme with Newton linearization.

2. **Governing inviscid flow equation**

The governing equation for the inviscid flow, Equation (2.66), is elliptic in nature. An elliptic problem is often called a boundary-value or jury problem, since the boundary conditions have a strong influence on the solutions inside the calculation domain. Thus, its solution requires a different numerical method from the method used for the boundary layer equations which are parabolic in nature. For the finite-difference representation of an elliptic equation, a central differencing scheme is generally used.

Utilizing a central differencing scheme in Equation (3.11) and rearrangement of the resulting finite-difference representation leads to the following equation (see Appendix F)

\[- \psi_{j+1}^{i+1} + \psi_j^{i+1} + \psi_j^{i+1} + (B_\psi + C_\psi) \psi_{j+1}^i - 2(1 + B_\psi) \psi_j^i + (B_\psi - C_\psi) \psi_j^{i-1} + A_\psi \psi_{j+1}^i + \psi_j^{i-1} - A_\psi \psi_j^{i-1} = 0 \]  \hspace{1cm} (3.21)

The coefficients $A_\psi$, $B_\psi$, and $C_\psi$ in Equation (3.21) are given in Appendix F. Equation (3.21) was obtained for constant grid spacing, that is, $\Delta_\psi^+ = \Delta_\psi^-$ and $\Delta_\eta^+ = \Delta_\eta^-$ (see Figure 3.1 where X and Y are replaced by $\xi$ and $\eta$, respectively).
Equation (3.21) is written for each position \((i, j)\) for which \(\hat{v}\) is unknown. This gives rise to a system of simultaneous linear algebraic equations to be solved throughout the inviscid flow field. The system was solved by a form of the alternating direction implicit (ADI) method with successive over-relaxation. The procedure used to solve the algebraic equations will be discussed in more detail in Chapter IV.

3. Turbulence kinetic energy equation

The turbulence kinetic energy equation was solved in an uncoupled manner after \(U_{j}^{i+1}, V_{j}^{i+1}\) had been determined by solving the momentum and continuity equations. The coefficients of the convective terms in the turbulence kinetic energy equation were thus known and no linearization of those coefficients was required. In the present study, a fully implicit finite-difference method was used to solve the turbulence kinetic energy equation. The finite-difference representation of Equation (3.13) for a variable grid shown in Figure 3.1 can be written as

\[
\begin{align*}
U_{j}^{i+1} & = \frac{k_{i+1}^{j} - k_{i}^{j}}{\Delta X_{+}} + V_{j}^{i+1} \left( \frac{k_{i+1}^{j} - k_{i-1}^{j}}{\Delta Y_{+}} + \frac{k_{i+1}^{j} - k_{i-1}^{j}}{\Delta Y_{-}} \right) \\
& + \frac{k_{i}^{j}}{\Delta X_{+} + \Delta Y_{-}} \left( \frac{U_{i+1}^{j} - U_{i-1}^{j}}{\Delta X_{+}} \right)^{2} - \frac{C_{D}}{k_{i}^{j}} \frac{k_{i+1}^{j}}{k_{i}^{j}}
\end{align*}
\]

In the above equation, the nondimensional diffusion coefficient \(N\) at \(j + 1/2\) and \(j - 1/2\) was evaluated in the same manner as \(M\) in Equation (3.17). That is, the arithmetic average of \(N_{s}\) at neighboring grid points
was taken as the value of \( N \) at \( j + 1/2 \) and \( j - 1/2 \) (see Equation (3.18), which is written for \( M \)).

The central-differenced term distinguished by an asterisk in Equation (3.22) was replaced by an upwind difference whenever the mesh Reynolds number defined by

\[
\left| v_{j}^{i+1} \right| \times \left\{ \text{Max of } \frac{\Delta y_{+}}{M_{j+1}^{i}}, \frac{\Delta y_{-}}{M_{j-1}^{i}} \right\}
\]

exceeded two in Equation (3.22) in order to prevent unrealistic "wiggles" in the solution [147] and to ensure diagonal dominance in the solution of the algebraic equations by the Thomas algorithm. The upwind difference was,

\[
v_{j}^{i+1} = \begin{cases} 
\frac{\hat{v}_{j}^{i+1} - v_{j}^{i+1 - 1}}{\Delta y_{-}}, & \text{when } v_{j}^{i+1} > 0 \\
\frac{\hat{v}_{j}^{i+1} - v_{j}^{i+1 + 1}}{\Delta y_{+}}, & \text{when } v_{j}^{i+1} < 0
\end{cases}
\]

The dissipation term in Equation (3.22), \( C_{D}(k_{j}^{i})^{1/2}v_{j}^{i+1} + \frac{\hat{v}_{j}^{i+1}}{\hat{v}_{j}} \), has been linearized by following Patankar's suggestion [166].

For separating flow where \( U_{j}^{i+1} < 0 \), the FLARE approximation was used to permit marching the solution in the positive X direction for Equation (3.22) as for the momentum equation discussed earlier.

C. Consistency, Stability, and Convergence

Consistency and stability are major concerns in the use of a finite-difference method. It is found that when finite-difference
expressions (at least for linear partial differential equations) meet the consistency and stability requirements, then a converged solution can be obtained, according to Lax's equivalency theorem (see, e.g., [147]). Convergence here means that the solution to the finite-difference equation approaches the true solution to the partial differential equations having the same boundary and initial conditions as the mesh is refined.

Consistency deals with the extent to which the finite-difference equations approximate the partial differential equations. A finite-difference scheme is said to be consistent if the truncation error, which is the difference between a partial differential equation and the difference representation of it, vanishes as the mesh is refined. It is shown in Appendix G that the finite-difference representations, Equations (3.16), (3.17), (3.21) and (3.22), of the model equations, Equations (3.2), (3.3), (3.11) and (3.13), respectively, are mathematically consistent.

Stability is also an important consideration, since even a highly accurate finite-difference scheme in terms of truncation error can be unstable so that converged solutions cannot be obtained. The essence of stability is that errors from any source, such as, for example, round-off errors, do not grow in the sequence of numerical procedures as the calculation proceeds from one marching step to the next. No general theory exists for the stability analysis of nonlinear partial differential equations. For linear partial differential equations, the theory of von Neumann (see, e.g., [167]) can generally be applied in order to obtain stability restriction.
It is well-known that implicit schemes are unconditionally stable in the von Neumann sense. However, in general, there is a very real constraint on the use of implicit schemes for boundary layer flows when the continuity and momentum equations are solved in an uncoupled manner. Though not detected by the von Neumann stability analysis, a behavior very much characteristic of numerical instability can occur if the choice of grid spacing permits the convective transport to dominate the viscous transport. Such instability problems can be eliminated when the diagonal term in the system of algebraic equations dominates. Diagonal dominance was maintained in the present algebraic system for those equations such as the turbulence kinetic energy equation which are solved in an uncoupled manner by shifting from a central difference to an upwind representation (see Section III.B.3) whenever

\[ \frac{|V^i| \Delta Y}{\nu} > 2 \]

In the coupled procedure for the continuity and momentum equations, the problem related to the diagonal dominance has not been observed. The fact that \( V \) in \( \nabla \psi / \partial Y \) is treated as an unknown rather than a coefficient in the coupled algebraic formulation is thought to eliminate the need to constrain the mesh Reynolds number in order to eliminate wiggles.

The Laplace equation for the streamfunction was solved in an implicit manner. Thus, as discussed above, the diagonal dominance of the algebraic system is the only requirement for convergence.
in Appendix H shows that the diagonal dominance of the algebraic system is maintained.

D. The Grid Arrangement in the Normal Direction

For the boundary layer solution, the $\Delta Y$-grid should be carefully specified. For laminar flows, an equal grid has been found to be satisfactory in most cases. However, for flows where a steep velocity gradient is expected, a variable grid will be desirable so that a fine mesh size can be obtained in the region where the velocity changes rapidly.

For turbulent flows, an unequal grid is highly recommended since steep velocity gradients occur near the wall except near the separation and reattachment points. The accurate solution of the boundary layer equations for such flows requires that at least one grid point be located within the viscous sublayer unless the turbulence quantities at the wall are evaluated by using special functions known as wall functions (see, e.g., [168]). In the present study, the turbulence quantities were evaluated directly by using models discussed in Section II.B. Therefore, a very small value $\Delta Y$ was specified at the wall in order to locate at least one grid point within the viscous sublayer. The grid scheme used in the present study was such that the cross-stream spacing increased in a geometric progression away from the wall,

$$\Delta Y_{j+1} = K \Delta Y_j \quad \text{for } j = 1, 2, 3, \ldots, N_{eq}$$

where $N_{eq}$ denotes the grid point at which the unequal grid spacing
terminates. From \( j = N_{e\text{q}} \) to \( j = NJ \), which locates at the outer edge of the computation domain, an equal grid was used. \( K \) is usually a number between 1.0 and 2.0. A value of \( K = 1.15 \) was used in the present study.

For the inviscid flow, an equal grid was used in both the streamwise and normal directions in the transformed coordinates.
IV. METHOD OF SOLUTION

In this chapter, the general method of solution is presented. In the first section, the newly developed boundary-layer solution scheme is discussed. Next, the solution method used for the inviscid solution is described. Finally, the viscous-inviscid interaction method is presented.

A. Viscous Flows

As discussed in Section III.B.1, the continuity and momentum equations were solved in a coupled manner for internal and external flows. The solution procedures for both internal and external flows are the same except for the evaluation of the pressure gradient. The pressure gradient is determined by the boundary conditions specified for the flows being considered. In the present study, the pressure gradient for internal flows is obtained from the global mass flow rate constraint, whereas for external boundary-layer flows, it is evaluated by using Euler's equation, $\chi = -\frac{dP}{dX} = U_e \frac{dU_e}{dX}$, either directly with the specified edge velocity (the direct method) or indirectly through matching the specified displacement thickness (the inverse method). Since there is no major difference in the solution algorithms for such flows, only the inverse solution scheme for external boundary-layer flows is discussed in this section. It should be noted that the present solution scheme generally requires iterations for obtaining solutions at each streamwise station because of the Newton linearization procedure employed for linearizing the nonlinear terms in the
finite-difference representations of the governing equations as discussed in Chapter IV.

Equations (3.19) and (3.20) can be rewritten as

\[
\begin{bmatrix}
B_j & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
U_{i+1}^{j-1}
\end{bmatrix}
+ \begin{bmatrix}
D_j & E_j \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
U_i^{j} \\

\psi_i^{j+1}
\end{bmatrix}
+ \begin{bmatrix}
A_j & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
U_{i+1}^{j+1} \\
\psi_{i+1}^{j+1}
\end{bmatrix}
= \begin{bmatrix}
H_j x + C_j \\
0
\end{bmatrix}
\tag{4.1}
\]

Equation (4.1) results in a system of block tridiagonal linear equations of the form

\[
\begin{bmatrix}
D_1 & E_1 & A_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
B_2 & 0 & D_2 & E_2 & A_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
B_3 & 0 & D_3 & E_3 & A_3 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
U_1 \\
\psi_1 \\
U_2 \\
\psi_2 \\
U_3 \\
\psi_3 \\
\vdots \\
\vdots \\
U_{NJ} \\
\psi_{NJ}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\vdots \\
\vdots \\
0 \\
0
\end{bmatrix}
\]
where for convenience, the superscript \( i + 1 \) has been deleted.

Equation (4.2) can be solved by using a block elimination scheme. Since the block elimination procedure for this problem is similar to the widely used Thomas algorithm, it is also known as the modified Thomas algorithm [162]. Elimination of the lower diagonal elements in the coefficient matrix and rearrangement of the resulting equations provides

\[
\begin{align*}
\{H_1U + C_1\} \\
\{0\} \\
\{H_2U + C_2\} \\
\{0\} \\
\{H_3U + C_3\} \\
\{0\} \\
\vdots \\
\{H_{NJ}U + C_{NJ}\} \\
\{0\}
\end{align*}
\]  
\tag{4.2}
\]

for \( j = 1, NJ \) where \( NJ \) denotes the outer edge of the boundary layer.

Since in Equations (4.3) and (4.4), \( U \) and \( \psi \) at the \( j \) level are written in terms of only \( U \) at the \( j + 1 \) level and the pressure gradient \( \chi \), these unknown quantities \( U_j \) and \( \psi_j \) can be calculated if and only if \( U_{j+1} \) and \( \chi \) are known a priori. In the inverse solution method for ex-
ternal boundary-layer flows, the pressure gradient and, thus, the edge velocity \( U_e \) or \( U \) at \( j = NJ \), can be determined from the displacement thickness specified as the boundary condition, which will be discussed below. Consequently, the solution procedure for all the \( U \)'s and \( \psi \)'s across the boundary layer starts from the outer edge of the boundary layer (\( j = NJ \)) and continues down to the wall (\( j = 1 \)). Such a calculation procedure is often called the back-substitution procedure since the calculation starts from the equation for the last point and proceeds continuously to that for the first point.

The coefficients \( A_j', B_j', C_j', D_j', E_j' \) and \( H_j' \) in Equations (4.3) and (4.4) are given in Appendix I. Since the inner boundary conditions are all homogeneous, that is, \( U_1 = 0 \) and \( \psi_1 = 0 \), the coefficients are initialized at \( j = 1 \) by letting

\[
A_1' = B_1' = C_1' = D_1' = E_1' = H_1' = 0
\]

Therefore, the coefficients \( A_j', B_j', C_j', D_j', E_j' \) and \( H_j' \) can be computed from the wall throughout to the outer edge of the computation domain as discussed in Appendix I.

The pressure gradient \( \chi \) is obtained by solving simultaneously the equations obtained from Equations (4.3) and (4.4) by replacing \( j \) by \( NJ - 1 \) and the boundary conditions given in Equation (3.6) as follows:

At \( j = NJ - 1 \), Equations (4.3) and (4.4) become

\[
\begin{align*}
U_{NJ-1}^{i+1} &= A_{NJ-1}' U_{NJ}^{i+1} + H_{NJ-1}' \chi_{NJ-1}^{i+1} + C_{NJ-1}' \\
\psi_{NJ-1}^{i+1} &= B_{NJ-1}' U_{NJ}^{i+1} + D_{NJ-1}' \chi_{NJ-1}^{i+1} + E_{NJ-1}' 
\end{align*}
\] (4.5, 4.6)
The boundary conditions, Equation (3.6), are written as

$$v_{NJ}^{i+1} = u_{NJ}^{i+1}(y_{NJ} - y_{x+j+1})$$

(4.7)

and

$$\chi^{i+1} = \frac{1}{\Delta x_+} \left\{ (2u_{NJ}^{i+1} - u_{NJ}^i)u_{NJ}^{i+1} - (\bar{u}_{NJ}^{i+1})^2 \right\}$$

(4.8)

Equation (3.16) is written at $NJ - 1/2$ as

$$v_{NJ}^{i+1} = v_{NJ-1}^{i+1} + \frac{\Delta y}{2} (u_{NJ}^{i+1} + u_{NJ-1}^{i+1})$$

(4.9)

Solving Equations (4.5)-(4.9) for the pressure gradient $\chi^{i+1}$ gives

$$\chi^{i+1} = \frac{\frac{F_3}{F_1}(2u_{NJ}^{i+1} - u_{NJ}^i) - (\bar{u}_{NJ}^{i+1})^2}{\Delta x_+ - \frac{F_2}{F_1}(2u_{NJ}^{i+1} - u_{NJ}^i)}$$

(4.10)

where

$$F_1 = y_{NJ} - y_{x+j+1} - B_{NJ-1}^i - \frac{\Delta y}{2} (1.0 + A_{NJ-1}^i)$$

$$F_2 = D_{NJ-1}^i + \frac{\Delta y}{2} H_{NJ-1}^i$$

and

$$F_3 = E_{NJ-1}^i + \frac{\Delta y}{2} C_{NJ-1}^i$$

Once the pressure gradient $\chi$ is obtained, the edge velocity $u_{NJ}$ and the edge streamfunction $v_{NJ}$ can be calculated as follows:

From Equations (4.5)-(4.9), the edge velocity $u_{NJ}$ is written in terms of the pressure gradient $\chi$ as

$$u_{NJ}^{i+1} = \left( \frac{F_2}{F_1} \right) \chi^{i+1} + \left( \frac{F_3}{F_1} \right)$$

(4.11)

where $\chi^{i+1}$ is given in Equation (4.10). With the edge velocity obtained
from Equation (4.11), the edge streamfunction \( \psi_{NJ}^{i+1} \) can be obtained directly from Equation (4.7). Now the back-substitution process is initiated, and the process continues down to the wall by using Equations (4.3) and (4.4). Since the present solution method requires iterations as discussed above, the solutions obtained from Equations (4.3) and (4.4), \( U's \) and \( \psi's \), must be compared with those obtained at the previous iteration. The iteration at each streamwise location is continued until the maximum change in \( U's \) and \( \psi's \) between two successive iterations, i.e.,

\[
\text{Max. of } (|\Delta U_j^i|, |\Delta \psi_j^i|)
\]

is less than or equal to the convergence criteria which was set equal to \( 5.0 \times 10^{-4} \) in all the present calculations. The iterative calculation is initiated with the solutions obtained at the previous streamwise station as the initial assumed values for the present station. The number of iterations typically required for each streamwise station is 2 or 3.

The calculation procedure discussed above can be summarized as follows:

1) Assume the solutions of \( U \) and \( \psi \) across the viscous flow region.

2) Calculate the coefficients \( A_j^i, B_j^i, C_j^i, D_j^i, E_j^i \) and \( H_j^i \) in Equations (4.3) and (4.4) for \( j = 2, NJ \) using the previously calculated values for \( U's \) and \( \psi's \) at the previous and present stations.

3) Calculate the pressure gradient \( \chi_{i+1} \) using Equation (4.10).

4) Calculate the edge velocity \( U_{NJ} \) using Equation (4.11).
5) Calculate the edge streamfunction $\psi_{NJ}$ using Equation (4.7).

6) Calculate the solutions for $U$ and $\psi$ across the computation domain using Equations (4.3) and (4.4) by means of the back-substitution process.

7) Evaluate the convergence of the solutions. If the solutions meet the convergence criteria, proceed to the next streamwise station. Otherwise, return to step 2.

Appendix J provides the calculation procedure for the edge velocity, the edge streamfunction, and the pressure gradient for the direct method for the external boundary-layer flows, as well as the procedure for evaluating the pressure gradient for internal flows.

In the course of the present study, the Crank-Nicolson scheme with Newton linearization, and the fully implicit method with lagged coefficients were also examined. These methods were found to give equivalently good results for fully attached boundary-layer flows compared to the fully implicit scheme with Newton linearization discussed above. For separated flows, the Crank-Nicolson scheme with Newton linearization still provides almost the same solutions as those for the present fully implicit scheme with Newton linearization; however, the fully implicit scheme with lagged coefficients results in unfavorable solutions as discussed previously. Appendix K provides the calculation procedure for the edge velocity, the edge streamfunction, and the pressure gradient for the inverse solution method with the Crank-Nicolson scheme with Newton linearization, and the fully implicit scheme with lagged coefficients. It is interesting to note that, when the present calculation procedure for the pressure gradient was applied to the fully implicit
scheme with lagged coefficients, no iterative procedures were necessary for obtaining solutions at a given streamwise station even in the separated flow regions.

All viscous flow calculations were started by using the direct calculation mode. The switch to the inverse procedure was easily implemented since the numerics for the two procedures are identical except for the calculation of the solutions at the outer edge (see Appendix J). For both procedures, as discussed above, the solutions for the velocities, U's, and the streamfunctions, \(\psi\)'s, were obtained for each columnwise iteration such that they satisfy the specified boundary conditions, i.e., the edge velocity for the direct method and the displacement thickness for the inverse method. The distinction between the two methods lies in the fact that for the inverse procedure, use of the specified displacement thickness permits the pressure gradient, \(\chi\), to be obtained as part of the solution, as discussed above. For the direct method, the value of the edge velocity also fixes the pressure gradient \(\chi\).

The inverse procedure of the new coupling scheme with Newton linearization differs from that of the conventional fully implicit scheme with lagged coefficients given by Pletcher [94] mainly in the evaluation of the pressure gradient as well as the evaluation of the coefficients in the momentum equation. The conventional fully implicit method utilizes the variable secant procedure to obtain the pressure gradient. In general, with the variable secant procedure, the solution has been found to converge reasonably fast; however, the convergence rate seemed to be highly dependent upon the initial two guesses which
must be supplied to start the procedure. The difference in the evalua-
tion of the coefficients of the convective terms in the momentum equation
between the two methods is discussed in Section III.B.1.

B. Inviscid Flows

A system of simultaneous linear algebraic equations results when
the discretized form of the Laplace equation for the streamfunction is
applied at each grid point inside the computation domain. There are
several methods available for solving such a system of equations. The
methods can be in general classified as either direct or iterative.
Direct methods are those which give the solution exactly (if roundoff
error is neglected) in a finite and predeterminable number of operations
using an algorithm which is often quite complicated. Iterative methods
consist of a repeated application of an algorithm which is usually
simple. They yield the exact solution only as a limit of sequence.
However, with a permissible range of convergence criteria, the solution
can be obtained in a finite but usually not predeterminable number of
operations.

The most elementary methods for solving such a system of linear
algebraic equations in the direct manner are Cramer's rule and the
various forms of Gaussian elimination (see [169]). However, these
methods are not adequate for the problems of interest which require
a solution to a large number of equations because such methods either
involve an unacceptably large number of operations or large roundoff
errors [147].
In recent years, highly efficient direct methods have been developed. Such methods include the cyclic reduction method [170, 171, 172], Fourier series method [173, 174, 175] utilizing the fast Fourier transformation [176], odd-even reduction (or double cyclic reduction) methods [177, 178], Green's integral methods [179], and the error vector propagation (EVP) method [180]. One of the most efficient methods among such advanced direct methods is the odd-even reduction method of Buneman [178]. It is generally believed that the method is the fastest method for solving the Poisson equation among all the solution schemes available at present and it is not essentially limited due to accumulation of roundoff errors. The fast Fourier transformation method of Hockney [174] is comparable to the Buneman's method as far as computation time and accuracy are concerned. Hockney [174] argued that his fast Fourier transformation algorithm is even faster than the Buneman's odd-even reduction algorithm. These two methods were found to be on the order of 10 to 20 times faster than the best iterative procedures when they were used for the Poisson equation arising from the solution of the two-dimensional Navier-Stokes equations [181]. The EVP method developed for the Poisson equation by Roache [180] is also among the simplest and fastest advanced direct methods.

Unfortunately, most of the advanced direct methods at present cannot handle irregular geometries; that is, they are limited to rectangular domains like L- or T-shapes. Although several investigators [182, also see 183] suggested algorithms recently for solving problems for general bounded regions using the capacitance matrix method of Proskurowski and Widlund [184], the problem related to the irregular
boundaries still remains. Other problems associated with the direct methods as a general class include field size limitations due to accumulation of roundoff errors, the restrictions on the number of grid points along the horizontal and/or vertical coordinates and large storage requirements. Boundary conditions also provide restrictions to some of the methods in their range of application, that is, some of the methods cannot handle a certain type of boundary condition. Algorithm complexity is another disadvantage of most of the fast solvers. Furthermore, most of these advanced direct methods are applicable only for a Poisson equation in which the coefficients of the second-order derivative terms are constants but not functions of independent variables.

As discussed in Section II.C.1, the present inviscid computation domain is irregular and, furthermore, at every global iteration of the viscous-inviscid interaction scheme (see Section IV.C), the upper and lower boundaries vary. In the present study, much consideration was given to these advanced direct solution methods discussed above for their applicability to the present inviscid flow problem. Because of the irregularity of the domain in the physical coordinates, most of the methods were found to be inapplicable to the present inviscid flow problem. The EVP method was found to be applicable to such an irregular geometry if and only if the upper boundary did not cross any grid lines parallel to the horizontal coordinate. However, even if this is the case, the variation of the upper and lower boundaries at every global iteration raised serious doubt about the superiority of the EVP method to other iterative methods because of the need for evaluating the coefficient matrix, which is the most time-consuming procedure in the
method, at every global iteration. In the transformed coordinates, the prospects for applying the direct methods were found to be even worse than in the physical coordinates because the coefficients of the second-order terms in the transformed governing equation are functions of the independent variables $\xi$ and $\eta$, as shown in Appendix B. Thus, an iterative scheme was used for the present inviscid flow solution.

Iterative methods can be classified into explicit (or point) iterative methods and implicit (or block) iterative methods. For point iterative methods, the same algorithm is applied to each point inside the computation domain where the unknown function is to be determined in successive iterative sweeps. In block iterative methods, subgroups of points are singled out for solution by elimination schemes in an overall iterative procedure. All of the iterative methods are restricted by the constraints for convergence. The sufficient condition for convergence of the explicit methods is that the magnitude of the coefficient on the diagonal term in each equation must be greater than or equal to the sum of the magnitude of the other coefficients with the greater than holding for at least one equation. The implicit methods require that diagonal dominance be maintained in the subgroups of each equation.

The iterative procedures can be, in general, accelerated by using the successive over-relaxation (SOR) technique which gives an arbitrary correction to the intermediate values of the unknowns from any iterative procedure according to the form

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n} + \omega(\phi_{i,j}^{n+1} - \phi_{i,j}^{n})$$

Here $n$ denotes iterative level and $\phi_{i,j}^{n+1}$ is the most recent value...
of $\phi_{i,j}$ calculated from an iterative procedure. $\phi_{i,j}^{n+1}$ is the newly adjusted or over-relaxed value of $\phi_{i,j}$ at the $n$ iteration level. The value of the over-relaxation factor $\omega$ ranges from 1.0 to 2.0. When $\omega$ is less than 1.0 but greater than zero, the procedure is called under-relaxation.

Commonly used block iterative methods with SOR are the Gauss-Seidel method with SOR, SOR by lines, and alternating direction implicit (ADI) methods. For the Gauss-Seidel method, SOR is applied immediately at each point after $\phi_{i,j}^{n+1}$ has been obtained and $\phi_{i,j}^{n+1}$, obtained from Equation (4.12) replaces $\phi_{i,j}^{n+1}$ in all subsequent calculations in the cycle. For the SOR by lines and ADI methods, SOR can be applied in the same manner as discussed above before moving on to the next row or column. For the SOR by lines, one iteration cycle is completed when all the rows or columns have been calculated, while for the ADI, a complete iteration cycle consists of a sweep over all rows followed by a sweep over all columns. These latter two methods generally require fewer iterations than point SOR. For example, for a simple problem with Dirichlet boundary conditions, the SOR by lines requires $1/\sqrt{2}$ as many iterations as for Gauss-Seidel iterations with SOR to reduce the initial errors by the same amount [185]. However, as far as computation time is concerned, the advantage of these two methods over the Gauss-Seidel method with optimum over-relaxation factor is still questionable.

For the present problem, the three iterative methods with SOR were tested for a typical inviscid flow domain. Table 4.1 shows the comparison

---

1 Fletcher, R. H. Dept. of Mechanical Engineering, Iowa State University, Ames, private communication, 1981.
of the three methods. The calculation was performed for 49 x 40 uniform grids with an over-relaxation factor of 1.5. The convergence criteria used were $\varepsilon = 0.5 \times 10^{-4}$. It is clear that for this one test case, the ADI method was the most efficient scheme among them. However, the generality of the result is uncertain.

Based on this comparison and favorable reports in the literature, the ADI method was adopted for the inviscid solution. The ADI method has been found typically to provide converged solutions in 10-15 iterations at the start of the viscous-inviscid interaction calculation. From the second iteration of the interaction calculation (or the global iteration), the number of the iterations is usually reduced to less than half the number of iterations used for the first global iteration.

This improvement in convergence was made possible by using the inviscid solution at the previous global iteration level as the initially assumed values. By doing this, as the solution of the interaction method converged, the number of iterations required for the inviscid solution was reduced. Finally only 1-2 iterations were needed when the interaction solution approached convergence.
C. Viscous-Inviscid Interaction Method

For the problem of separating flow over a rearward-facing step, the viscous and inviscid solutions can be obtained by using Equations (3.16), (3.17), and (3.21) with the methods discussed above, if the displacement thickness $\delta^*(x)$ and appropriate flow conditions are provided. If the specified displacement thickness distribution is correct, the viscous solution for $u_e, BL(x)$ and the inviscid solution for $u_e, INV(x)$, will match on the displaced surface. However, in general, the displacement thickness is not known a priori but must be determined by the iterative matching of the viscous and inviscid solutions. In the present study, the viscous-inviscid matching procedure developed by Carter [3] and Kwon and Fletcher [4] was used. This method has been successfully used for thin separation bubble flows over a two-dimensional body [3, 4, 5].

In this method, the viscous solutions, $u_{e, BL}(x)$, along the upper and lower walls are obtained by solving the boundary-layer equations inversely with the specified $\delta^*$'s. The inviscid solution $u_{e, INV}(x)$ is also obtained for the domain bounded by the $\delta^*$'s and vertical inlet and outlet sections. The $u_e(x)$ from the two calculations, inviscid and boundary-layer, will not agree until convergence has been achieved. The difference between the $u_e(x)$ calculated both ways was used as a potential to calculate an improved value for $\delta^*(x)$. Based on a local continuity concept in the boundary layer [4], the following equation was obtained:
where \( n \) denotes the global iteration level. This equation was used for updating \( \delta^*(x) \) to provide the new input to the boundary layer and inviscid calculation.

In the present study, the interaction calculation was initiated by prescribing a guessed distribution of \( \delta* \) along the upper and lower solid walls in the interaction region. After the calculation of the boundary layer solutions along both walls and the inviscid solution with the assumed \( \delta^* \), a better displacement thickness is obtained from Equation (13). However, because of the discontinuity in the geometry at the step, \( \delta^* \) before the step along the lower wall must be added by the step height \( (\delta^* + h) \). This procedure accounts for the translation of the reference axis for the inviscid flow before the step to that after the step. Such an adjustment is necessary since the displacement thickness before the step is smaller than that after the step by \( h \), so that, without such an adjustment, the updating using Equation (4.13) will cause a sudden jump in the displacement thickness distribution at the step by approximately \( h \left( \frac{u_{e, BL}}{u_{e, INV}} \right) \). Although such an adjustment of adding to \( \delta^* \)'s by \( h \) before the step results in a smooth distribution in the displacement thickness, the variation of the displacement thickness before and after updating is often unreasonably large. Such large variations often caused the displacement thicknesses before the step to be even smaller than the step height. In order to avoid such unrealistic results and to permit the calculations to continue, under-relaxation was used for updating \( \delta^* \)'s along the
lower wall for some problems:

\[ \delta_{n+1}^* = \omega [ \delta_n^* \frac{u_{e, BL}}{u_{e, INV}} ] + (1 - \omega) \delta_n^* \]  
(4.14)

The required value of the under-relaxation factor seemed to be a strong function of the ratio between \( \delta^* \) before the step and that after the step. When \( \delta^* \) at the step was larger than the step height \( h \), under-relaxation appeared not to be necessary. The numerical value of the under-relaxation factor will be given in Chapter V.

The calculation proceeds as shown in Figure 4.1. The initial velocity distribution required for the boundary layer calculation at each iterative pass is the same and is obtained from the boundary-layer finite-difference calculation for a two-dimensional channel from the channel inlet to the start of the interaction region \( x^* \).

Convergence was said to have occurred when

\[ \frac{|u_{e, BL} - u_{e, INV}|}{u_{e, INV}} \leq \varepsilon \]  
(4.15)

at all calculation points in the interaction zone.

The numerical value of \( \varepsilon \) will be given in Chapter V.

The calculation procedure of the viscous-inviscid interaction method is summarized as follows:

1) Assume a distribution for \( \delta^* \) along the upper and lower walls.

2) Using the \( \delta^* \) distribution, calculate the viscous flow solution for \( u_{e, BL} \) for region A in Figure 1.3 using the inverse solution procedure discussed in Section IV.A.

3) Using the same \( \delta^* \) as in step 2, calculate the inviscid flow solution for \( u_{e, INV} \) for region B in Figure 1.3 using the method dis-
Figure 4.1. Skeleton flow chart for the present viscous-inviscid interaction method
108

START

INITIALLY PRESCRIBED $\delta^*$ ALONG UPPER AND LOWER WALLS

IS ONLY BOUNDARY-LAYER SOLUTION REQUIRED?

YES

BOUNDARY-LAYER SOLUTION FOR LOWER WALL

IS CALCULATION OF BOUNDARY-LAYER SOLUTION FOR UPPER WALL REQUIRED?

NO

INVISCID SOLUTION

IS INTERACTION SOLUTION REQUIRED?

NO

BOUNDARY-LAYER SOLUTION FOR UPPER WALL

IS INTERACTION SOLUTION REQUIRED?

YES

$\left| \frac{U_{e, BL} - U_{e, INV}}{U_e, INV} \right| \leq \varepsilon$

YES

$\delta^*_\text{NEW} = \frac{U_{e, BL}}{U_e, INV}$

NO

$\delta^*_\text{NEW} = \omega \delta^*_\text{NEW} + (1 - \omega) \delta^*_\text{OLD}$

STOP
cussed in Section IV.B.

4) Examine convergence using Equation (4.15): If the solutions meet the convergence criteria throughout the computation domain, terminate the calculation. Otherwise, proceed to step 5.

5) Adjust $\delta^*$ before the step along the step-side wall by adding $h$.

6) Update $\delta^*$ using Equation (4.13). Under-relaxation may be required for the displacement thicknesses upstream of the step along the step-side wall.

7) Readjust $\delta^*$ before the step along the step-side wall from

$$\delta^* = \delta^* - h$$

8) Return to step 2.
V. RESULTS AND DISCUSSION

The results of the present study have been divided into three categories. First, the capabilities of the new boundary-layer prediction procedure for separated flows are evaluated by comparisons with other results, numerical and experimental, for separated flows over smooth, continuous surfaces. Next, predictions are presented for developed laminar flows through channels containing abrupt symmetric expansion in flow cross-sectional area. This flow configuration is especially interesting as it tests the applicability of the boundary-layer equations to this type of flow. Finally, computational results for one laminar and three turbulent flows over rearward-facing steps are presented to establish the capabilities of the present viscous-inviscid interaction calculation procedure.

A. Preliminary Study on the New Boundary-Layer Solution Scheme

In order to evaluate the capabilities of the new boundary-layer solution scheme for separating flows, the following flows were predicted and compared with other numerical predictions and experimental data:

1) one laminar separation bubble flow without viscous-inviscid interaction,

2) one laminar separation bubble flow with viscous-inviscid interaction, and

3) one turbulent separating flow without viscous-inviscid interaction.

The results are discussed below in order.
1. **Laminar separation bubble flow without viscous-inviscid interaction**

Since there are no experimental data for the laminar separation bubble flow available for simple external flows, the present solution scheme is compared with results from other numerical solution methods. There have been several finite-difference solution methods for laminar separating boundary layers presented in the literature to date. One of the most successful is the coupling method developed by Carter [93] based on the Crank-Nicolson numerical algorithm. In the method, Carter introduced a perturbation stream function defined as

\[ \tilde{\psi} = \left[ \psi - u(y - \delta^y) \right] / \sqrt{2x} \]  

(5.1)

and used a transformed coordinate \( \tilde{\eta} \) defined as \( \tilde{\eta} = y / \delta^y \). By using \( \delta^\phi + \delta^\psi \) for all the unknown quantities, \( u \), \( \tilde{\psi} \), and \( \chi \) (pressure gradient) in the finite-difference expressions of the resulting governing equations for \( u \) and \( \tilde{\psi} \) and employing Newton linearization, he obtained linear equations for the perturbations of the dependent variables such as \( \delta^u \), \( \delta^\psi \), and \( \delta^\chi \). (These are defined in Appendix C for a general function \( \phi \).) He solved the linear equations for the perturbations \( \delta^u \), \( \delta^\psi \), and \( \delta^\chi \), iteratively.

However, in the present solution method, neither the perturbation stream function \( \tilde{\psi} \) nor the transformed coordinate \( \tilde{\eta} \) was used. Furthermore, solutions were obtained directly for the variables themselves such as \( u \), \( \psi \), and \( \chi \) as discussed before. The fully implicit scheme with Newton linearization was used in the present method. It should be noted that Newton linearization in the present method was only used for linearizing the nonlinear terms in the finite-difference equations.
Thus, it can be said that the present scheme is algebraically simpler than the Carter method.

Carter [93] also used the FLARE approximation for the reversed flow. Recently, it was found that the solution obtained with the Carter's coupling scheme for a separation bubble flow is very close to that obtained with an upstream-downstream iteration procedure [104] which does not employ the FLARE approximation. Thus, the Carter's coupling method was chosen for the comparison with the newly developed method. Predictions of the present method are also compared with those obtained from a fully implicit method with lagged coefficients in [94]. This fully implicit scheme with lagged coefficients utilizes the secant method for evaluating the pressure gradient, as discussed briefly in Chapter IV, when the boundary-equations are solved inversely. The method has been used for fully attached flows and thin separation bubble flows by Pletcher [94] and Kwon and Pletcher [4].

In this first comparison, the boundary-layer equations were solved inversely using an arbitrarily assumed displacement thickness distribution as shown in Figure 5.1. The inverse solution started at $x = 0.251m$ where the displacement thickness and Reynolds number based on the displacement thickness were 0.004712m and 184, respectively. At the leading edge, the free stream velocity was set equal to 0.597 m/sec.

The skin-friction coefficients obtained by the three methods are shown in Figure 5.2. The present coupling method with Newton linearization gives a very smooth skin-friction coefficient distribution which agrees reasonably well with that obtained by using Carter's coupling
Figure 5.1. Displacement thickness distribution for a laminar separation bubble test case
Figure 5.2. Comparison of inverse boundary-layer solutions in the reversed flow region: distribution of skin-friction coefficient
method. The difference between the two solutions seems to be mainly caused by the difference in the grid sizes near the wall used for the two methods. In the present solution scheme, the y-grid size, $\Delta y$, near the wall remains constant along x-direction, while in Carter's scheme, it varies since the y-coordinate in the method was normalized by the local displacement thickness. The displacement thickness generally grows very rapidly under the adverse pressure gradient characteristics of separating flows. Consequently, it is expected that the displacement thickness for an adverse pressure gradient flow will be larger than for a zero or favorable pressure gradient flow for the same initial conditions. For the comparison, the same y-grid size for both schemes was used at the start of the inverse solution procedure. Thus, the y-grid size near the wall used for the present method in the reversed flow region is much finer than that for Carter's method.

Differences in the numerical algorithms may have also contributed something to the discrepancies in the two solutions. The present scheme used a fully implicit method as discussed above whereas Carter's method utilized the Crank-Nicolson algorithm.

As discussed before, the uncoupled fully implicit method with lagged coefficients resulted in oscillations in the skin-friction coefficient distribution in the reversed flow region and it predicted early reattachment as shown in Figure 5.2. At first it was thought that the wiggles might have been caused by the use of the secant method to determine the edge velocity required to satisfy the displacement thickness boundary condition. However, extensive evaluation finally determined that they are not caused by the choice of difference schemes.
but by the choice of the linearization procedure for evaluating the coefficients of the convective terms in the momentum equation. It was found that both the simple lagging and extrapolation procedures resulted in such oscillations in the separation flow region. It is interesting to note that when the pressure gradient was not evaluated in a coupled manner as discussed in Chapter IV, a coupling scheme for $u$ and $v$ with Newton linearization failed to provide converged solutions in the separated flow region. This latter observation suggests that coupling of the pressure gradient as well as $u$ and $v$ (or $\psi$) are essential for obtaining oscillation free converged solutions in the separated flow region.

Velocity profiles predicted by the three methods near separation and near reattachment points are shown in Figure 5.3. At the separation point, all three methods are in good agreement; however, near reattachment, the uncoupled fully implicit method with lagged coefficients predicts profiles which are noticeably different from those obtained by the coupling methods. The velocity profiles predicted by the two coupling methods agree reasonably well as can be seen in Figure 5.3.

The present coupling scheme with Newton linearization was found to be the fastest among the three methods evaluated, as far as the number of iterations required to provide converged solutions at each streamwise location was concerned. In the present inverse calculation, the present solution scheme gave converged solutions in two to three iterations (in most of the calculation region two iterations were found to be enough), whereas Carter's coupling method and the fully implicit method with lagged coefficients generally required
Figure 5.3. Comparison of inverse boundary-layer solutions in the reversed flow region: velocity profiles
3 (not less than 3) and 4-6 iterations, respectively. In the calculations, the convergence criteria were set equal to 0.0005 as mentioned in Section IV.A.

2. **Laminar separation bubble flow with viscous-inviscid interaction**

One of the flows studied previously by Carter and Wornom [2] was calculated in order to demonstrate the capability of the present boundary-layer solution scheme for predicting a laminar separation bubble flow, when viscous-inviscid interaction is included. Carter and Wornom simulated a two-dimensional, incompressible, laminar separation bubble flow by assuming a two-dimensional body whose surface is prescribed by

\[ y = t_B \text{sech} 4(x - 2.5) \]  

(5.2)

This equation provides a trough, whose maximum depth is \( t_B \), on a smooth surface which is essentially flat far upstream and downstream of \( x = 2.5 \) m. The trough is expected to cause flow separation. They calculated two cases, for \( t_B = -0.015 \) m and \(-0.03 \) m, using a viscous-inviscid interaction scheme. More details of the flow geometry can be found elsewhere [2, 186]. In the calculation, they solved the boundary-layer equations inversely in terms of vorticity and streamfunction for the viscous flow solutions, and the inverse Cauchy integral formulation with the small-perturbation approximation for the inviscid flow solution. The interaction region was assumed to extend from \( x = 1.0 \) m to \( 4.0 \) m.

In the present study, only the \( t_B = -0.03 \) m case was investigated.
This flow (t_B = - 0.03 m) was also predicted earlier by Kwon and Fletcher [4] using a viscous-inviscid interaction method. However, in that earlier calculation, the boundary-layer equations were solved inversely in terms of the velocities u and v by the fully implicit finite-difference scheme with lagged coefficients. The viscous solutions obtained from the boundary-layer equations were matched with the inviscid flow solution which was computed by numerically solving a direct Cauchy integral formulation for which the source was simplified by use of a small perturbation approximation.

In the present prediction, the fully implicit boundary-layer solution scheme with lagged coefficients in the viscous-inviscid interaction method developed by Kwon and Fletcher [4] was replaced by the new boundary-layer solution scheme. The same inverse Cauchy integral formulation for the inviscid solution was used [4].

The present interaction calculation was started by using the solution for the displacement thicknesses obtained by Kwon and Fletcher [4] as the initial assumed displacement thickness distribution. The solution converged in 11 iterations to ε = 5.0 x 10^{-4} (where ε is the convergence criteria defined in Equation (4.15)).

The results predicted with the new boundary-layer solution scheme are compared with the predictions of Kwon and Fletcher [4] and Carter and Wornom [2] in Figures 5.4-5.6. The predicted displacement thickness and edge velocities are found to agree very well with the previous predictions by Kwon and Fletcher. In fact, no noticeable difference between the present prediction and the earlier Kwon and Fletcher prediction can be seen in Figures 5.4 and 5.5. However, near the reat-
Figure 5.4. Displacement thickness distribution for a laminar separation bubble flow using viscous-inviscid interaction.
Figure 5.5. Edge velocity distribution for a laminar separation bubble flow using viscous-inviscid interaction.
Figure 5.6. Distribution of skin-friction coefficient for a laminar separation bubble flow using viscous-inviscid interaction.
attachment point, the two methods predict somewhat different values of skin-friction coefficient as can be seen in Figure 5.6. The skin-friction coefficients predicted by the new method agree well with the predictions of Carter and Wornom.

Differences in the prediction of the present method and the solution obtained by Carter and Wornom are at least partially explained by the fact that Carter and Wornom added the displacement thickness which would have been generated by a zero pressure gradient laminar boundary layer to the input for the inverse boundary-layer calculation. This was done to be consistent with second-order boundary-layer theory. However, no such adjustment was made in the present prediction, or in the prediction by Kwon and Pletcher [4], because there is no obvious method for extending this procedure to general complex transitional or turbulent flows which are of interest in applications.

In both the present prediction and the Kwon and Pletcher prediction, the displacement thicknesses were found to return to that of the Blasius exact solution at the end of the interaction region. However, the protuberance in the displacement line seems to cause the apparent leading edge of the downstream Blasius flow to move upstream of the actual leading edge. Further discussion of differences between the predictions for this flow can be found elsewhere [4, 186].

In the calculation, over-relaxation was used with a relaxation factor of 1.5. The present calculation required typically 8.92 sec of CPU time per iteration with single precision on an NAS AS/6 digital computer, while the calculation performed by Kwon and Pletcher [4] required 67 sec per iteration on an ITEL AS/5. However, these computa-
tion times cannot be directly compared with each other because the computing machines are different. The present calculation used the same grid system as did the Kwon and Pletcher calculation.

3. **Turbulent separating flows without viscous-inviscid interaction**

The present coupling scheme for the boundary-layer equations was also applied for predicting the turbulent separating flow measured by Simpson, et al. [187]. They obtained the data for an incompressible turbulent separating boundary layer in a 4.9 m long converging-diverging channel using hot-wires and a laser anemometer. The channel has a flat bottom wall of a constant width and variable height. The flow separation occurred on the bottom wall. Two-dimensionality in mean flow was promoted by suction and tangential injection control of the side and top wall boundary layers. The test section entrance was 0.91 m wide by 0.38 m high and the velocity there was 16.5 m/sec. Free-stream turbulence intensity was 0.1%. In this experiment, Simpson, et al. [187] measured a very large and thick separation region.

In the prediction, the boundary-layer equations were solved inversely between \( x = 0.607 - 4.35 \) m using the measured displacement thickness (see Figure 5.7) as a boundary condition. From the leading edge to \( x = 0.607 \) m, the equations were solved directly using the measured edge velocities as the outer boundary condition.

The predicted edge velocities are plotted in Figure 5.8 and compared with the measurements. The present solution for the edge velocity was slightly overpredicted near the intersection of the converging-diverging sections and in the separated flow region.
Figure 5.7. Displacement thickness distribution for a turbulent separating flow on a flat plate.
Figure 5.8. Edge velocity distribution for a turbulent separating flow on a flat plate.
(x > 3.472 m). However, the predicted velocity profiles are overall in good agreement with the measured data as can be seen in Figures 5.9 and 5.10. In the reversed flow region, the negative velocities are somewhat smaller than the measurements. In the prediction, the standard turbulence kinetic energy equation, Equation (2.51), with Fletcher's length scale transport equation, Equation (2.41), was used for the turbulence modeling in the outer region, and in the inner region, Equation (2.38) was used.

Recently, Cebeci, et al. [188] predicted a flow similar to this using both the full Reynolds averaged Navier-Stokes equations and the boundary-layer equations in an inverse mode. The flow was measured by Simpson, et al. [121] in the same wind tunnel with almost the same flow geometry as the one discussed above. With the Reynolds equations, they used the elliptic form of the kinetic energy and dissipation rate transport equations for the turbulence modeling. In the inverse boundary-layer solutions, they used the algebraic eddy viscosity formulation developed by Cebeci and Smith (see [96]). They found that, when the Reynolds equations were solved, the predicted turbulence kinetic energy profiles agreed reasonably well with the measurements, but the mean velocity profiles provided only fair agreement with the measurements. The mean velocity profiles obtained from the boundary-layer equations by Cebeci, et al. [188] were in poor agreement with the measured data. Murphy and Rubesin [189] also predicted a flow similar to these two flows (Strickland and Simpson [190]) using the full Reynolds averaged Navier-Stokes equations. In the prediction by Murphy and Rubesin, the flow did not even separate. It is believed that the poor results
Figure 5.9. Velocity profiles for a turbulent separating flow on a flat plate (1)
Figure 5.10. Velocity profiles for a turbulent separating flow on a flat plate (2)
observed by these investigators were caused by the turbulence models they used.

The predicted skin-friction coefficients and turbulent stress (Reynolds stress) are plotted in Figures 5.11 and 5.12, respectively. The Reynolds stresses predicted by the present turbulence modeling do not agree well with the measurements. The predicted Reynolds stresses are observed to spread and decrease in magnitude from one streamwise station to the next as shown in Figure 5.12. However, in contrast to this, the Reynolds stresses measured by Simpson, et al. [121] are observed to grow. The measured time averaged fluctuation of the streamwise mean velocity were very large even fairly far away from the wall; at \( x = 4.43 \text{ m} \), the maximum fluctuation which occurred at \( y \approx 0.6 \delta \) was on the order of 23% of the outer edge velocity. Thus, some question might be raised as to whether the data measured by Simpson, et al. [121] for the fluctuating quantities were completely isolated from the unsteadiness. It is believed that the inclusion of elliptic type terms in the turbulence modeling equations would be helpful in obtaining improved predictions of turbulence quantities.

It was not possible to compare the skin-friction coefficients predicted in the separated flow region with the measurements since these data were not reported by Simpson, et al. [121]. In the attached flow region upstream of the step, the agreement between the predictions and measurements is reasonably good as can be seen in Figure 5.11.

In the calculation, the solutions generally converged in two iterations within the tolerance of 0.0005 at each streamwise station throughout the inverse solution region.
Figure 5.11. Distribution of skin-friction coefficient for a turbulent separating flow on a flat plate.
Figure 5.12. Reynolds stress profiles for a turbulent separating flow on a flat plate
In concluding this section, it can be argued based on these comparisons that the present boundary-layer solution method behaves very well and predicts fast and accurate solutions for separated flows compared to other finite-difference solution methods for the boundary-layer equations and the measured data. However, further studies seem to be in order on the turbulence modeling for predicting the flows with large separation regions.

B. Laminar Separating Flows in Symmetric Expansions

The separating flow in a channel with a sudden expansion is generally thought to be elliptic in nature. That is, it is generally believed that an elliptic form of the governing equations must be solved to accurately predict this type of flow. As discussed in Chapter I, until now most of the analytical studies on the type of flow have been performed using full Navier-Stokes equations (or Reynolds equations for turbulent flow) which are elliptic equations. In the prediction of the viscous-inviscid interaction method presented in Section A and to be presented below in Sections C and D, the elliptic effects were accounted for in the solution for the inviscid region. For flows which are nearly fully developed at the expansion so that no inviscid region can be identified, it is natural to question the importance of the elliptic effects despite the fact that the previous numerical predictions of the flows have invariably employed the full Navier-Stokes equations. Such considerations motivated the present evaluation of the use of the boundary-layer mathematical model for these flows. It is important to
know if the boundary-layer model will provide a good approximate solution in this case since the computer time required for the boundary-layer calculation is an order of magnitude less than required for the solution of the full Navier-Stokes equations.

In the present study, the boundary-layer equations were evaluated for predicting laminar separated flows in two-dimensional channel with sudden expansions. For a symmetric sudden expansion with expansion ratio of 2:1, predictions were obtained for Reynolds numbers based on the channel inlet height ranging from 1 to 320 (see Table 5.1) using fully developed inlet velocity profiles. The calculation for the present prediction was carried out throughout the channel by using the mass flow rate conservation constraint as a boundary condition (see Chapter II).

The predicted reattachment lengths are plotted in Figure 5.13 and are compared with the Navier-Stokes predictions by Hung [24] and Morihara [29]. The predictions are in excellent agreement except for very low Reynolds numbers (below $Re_H = 20$). It is noted that Morihara's prediction lies below Hung's prediction and the difference between the two predictions increases as Reynolds number increases. The reason for the difference is not clear. However, recent calculations by Kumar and Yajnik [30] and Agarwal [27] for the same flow problem at $Re_H = 46.6$, which is equivalent to $Re_H = 93.2$, provided good agreement with Hung's prediction. Kumar and Yajnik used two so-called limit equations developed from the governing equation for a two-dimensional, steady, laminar motion of an incompressible Newtonian fluid based on a large Reynolds number analysis. In fact, they solved a set of ordinary dif-
Table 5.1. Predicted extent of the recirculation region for laminar flow undergoing 2:1 symmetric expansion in a two-dimensional channel

<table>
<thead>
<tr>
<th>Reynolds number (\frac{u_{1,AVG}H}{\nu})</th>
<th>Extent of recirculation region (\frac{R}{H_1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
</tr>
<tr>
<td>5</td>
<td>0.175</td>
</tr>
<tr>
<td>10</td>
<td>0.350</td>
</tr>
<tr>
<td>20</td>
<td>0.700</td>
</tr>
<tr>
<td>30</td>
<td>0.105</td>
</tr>
<tr>
<td>40</td>
<td>1.350</td>
</tr>
<tr>
<td>100</td>
<td>3.450</td>
</tr>
<tr>
<td>320</td>
<td>10.850</td>
</tr>
</tbody>
</table>

Differential equations deduced from the two limit equations by expressing the limit equations with eigenfunctions for a Poiseuille flow problem. Agarwal solved the Navier-Stokes equations for primitive variables \(u, v, p\) with a third-order accurate scheme.

More recently, Madavan\(^1\) also predicted the same flow using the partially parabolized Navier-Stokes equations resulting in good agreement with these results as well as the present solution. The present prediction at \(Re_\text{h} = 50\) (\(Re_\text{h,1} = 100\)) is compared in detail with the results obtained by other investigators (see Table 5.2). All the

\(^1\)Madavan, N. Dept. of Mechanical Engineering, Iowa State University, Ames, private communication, 1981.
Figure 5.13. Prediction of the reattachment length for laminar flows in a channel with a symmetric sudden expansion, $h/H_1 = 0.5$
Table 5.2. Comparison of the present prediction with other predictions for laminar flow undergoing a 2:1 symmetric expansion in a two-dimensional channel

<table>
<thead>
<tr>
<th>Investigator</th>
<th>Equations solved</th>
<th>Predictions</th>
<th>Reynolds number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present calculations</td>
<td>Boundary-layer equations</td>
<td>0.069</td>
<td>50.0</td>
</tr>
<tr>
<td>Hung</td>
<td>Navier-Stokes equations</td>
<td>0.066</td>
<td>46.6</td>
</tr>
<tr>
<td>Agarwal</td>
<td>Navier-Stokes equations</td>
<td>0.068</td>
<td>46.6</td>
</tr>
<tr>
<td>Kumar and Yajnik</td>
<td>Two principal limits of the governing equation</td>
<td>0.064</td>
<td>46.6</td>
</tr>
</tbody>
</table>
solutions are found to agree very well for the normalized reattachment length, $\frac{\lambda_R}{Re_h}$, the normalized distance between the step and the eye of the vortex, $\frac{\lambda_e}{Re_h}$, and the value of streamfunction at the eye of the vortex, $\psi_{\text{min}}$.

The centerline velocity distribution obtained by the present boundary-layer solution is compared with the predictions by Kumar and Yajnik [30], and Agarwal [27] in Figure 5.14. Although the present predictions lie slightly above the latter, overall agreement between them is good. In the plotting, the streamwise coordinate was normalized by $Re_h$. Mean velocity profiles obtained from the boundary-layer solutions also provide good agreement with those obtained from the Navier-Stokes solutions as shown in Figure 5.15.

A typical streamline pattern predicted using the boundary-layer equations is shown in Figure 5.16. In the near step region, the dividing streamline predicted by using boundary-layer equations appears not to be tangential to the wall upstream of the step, and the negative streamlines are more densely populated than the solutions generally obtained by using the Navier-Stokes equations.

At the present time, there is only one set of experimental data available for symmetric laminar channel expansion flows. The data were measured by Durst, et al. [13] using a laser anemometer. The Reynolds number based on the step height and the maximum inlet velocity for the measurements is 56, and the channel expansion ratio is 3:1. The velocity profile at $x/h = -0.25$ was in the nearly developed state. Far downstream from the step ($x/h = 40$), they observed that the flow was fully developed. The boundary-layer equations were solved for the
Figure 5.14. Centerline velocity distribution for a laminar flow in a channel with a symmetric sudden expansion, $h/H_i = 0.5$
Figure 5.15. Velocity profiles for a laminar flow in a channel with a sudden symmetric expansion, $h/H_1 = 0.5$.
Figure 5.16. Streamline contours for a laminar flow in a channel with a symmetric sudden expansion, $Re_h = 50$, $h/H_1 = 0.5$
flow using the measured inlet velocity profile and the mass flow rate conservation constraint as a boundary condition. The predicted mean velocity profiles and centerline velocity distribution are in good agreement with the measurements and with the predictions by Durst, et al. [13] using the Navier-Stokes equations (see Figures 5.17 and 5.18). They used the streamfunction-vorticity scheme for the solution. However, the boundary-layer solutions are found to provide slightly earlier development downstream from the reattachment point. The reasons for the differences in the fully developed velocity profiles obtained from the Navier-Stokes equations and the boundary-layer equations at x/h = 40 shown in Figure 5.17 are not clear. However, it might be caused by the author's misinterpretation of the results plotted by Durst, et al. [13].

As mentioned above, Agarwal [27] and Kumar and Yajnik [30] also predicted the same flow. They used a fully developed inlet velocity profile rather than the measured profile in their predictions. Actually, the measured profile is not much different from the fully developed profile. Consequently, it is expected that their results would be quite close to the measurements. Surprisingly, their predicted centerline velocity distribution and the reattachment length were found to be very different from the measurements. The centerline velocity distributions are compared in Figure 5.18. The reason for the poor predictions by Agarwal [27] and Kumar and Yajnik [30] is not clear. Those poor results raise questions about the adequacy of the solution methods used by these investigators.

The boundary-layer equations were also applied to a laminar asym-
Figure 5.17. Velocity profiles for a laminar flow in a channel with a symmetric sudden expansion, $\frac{h u_{i,\text{max}}}{\nu} = 56$, $\frac{h}{H_i} = 1.0$
Figure 5.18. Centerline velocity distribution for a laminar flow in a channel with a symmetric sudden expansion, $\frac{h u_{i,\text{max}}}{\gamma} = 56$, $h/H_1 = 1.0$
metric channel flow problem. Denham and Patrick [15] reported measured data of developed flows in an asymmetric channel expansion flow for four different Reynolds numbers ranging from 73 to 229. Two of the cases measured by them, for $Re_h = 73$ and 229, were calculated by assuming fully developed inlet velocity profile.

In the calculation, a separation region along the wall opposite to the step was detected. As the solution approached the opposite wall separation point, the present coupling scheme failed to provide converged solutions probably because of the large pressure gradient which is expected to exist at the location. It should be noted that in the boundary-layer equations, the normal component of the pressure gradient is assumed to be negligible. However, the fully implicit scheme with lagged coefficients was found not to diverge but to provide solutions with a slightly oscillatory pressure gradient along the streamwise direction. Such oscillation of the pressure gradient seems to help the numerical solutions adjust to the large variation of the pressure gradient. Therefore, by replacing the present coupling scheme by the fully implicit scheme with lagged coefficients near the separation region on the opposite wall, it was possible to carry out the calculation in the entire flow domain. The oscillation of pressure gradient in the solution obtained by the replaced fully implicit scheme seemed to cause early reattachment as discussed before. Figures 5.19 and 5.20 show the calculated skin-friction coefficient distribution and velocity profiles for $Re_h = 73$ and Figures 5.21 and 5.22 show those for $Re_h = 229$. In Figures 5.19 and 5.21, the regions where the fully implicit scheme with lagged coefficients was used are shown. The predicted velocity profiles
Along the lower wall
Along the upper wall

Figure 5.19. Distribution of skin-friction coefficient for a laminar flow in a channel with an asymmetric sudden expansion.

Re_f = 73

\[ \text{X} \]

FULLY IMPLICIT SCHEME WITH LAGGED COEFFICIENTS WAS USED

\[ \text{C_f} \]

along the lower wall
along the upper wall
Figure 5.20. Velocity profiles for a laminar flow in a channel with an asymmetric sudden expansion, Re_h = 73
ALONG THE LOWER WALL
ALONG THE UPPER WALL

FULLY IMPLICIT SCHEME WITH LAGGED COEFFICIENTS WAS USED

Figure 5.21. Distribution of skin-friction coefficient for a laminar flow in a channel with an asymmetric sudden expansion, $Re_h = 229$
Figure 5.22. Velocity profiles for a laminar flow in a channel with an asymmetric sudden expansion, $Re_h = 229$
are plotted in Figures 5.20 and 5.22 resulting in reasonable agreement with the measured data.

The recirculation region occurring on the opposite wall is found to increase as Reynolds number $Re_h$ increases. Downstream of the main separation region attached to the step, another small recirculation region is detected at higher Reynolds number, $Re_h = 229$. Such findings agree qualitatively with the recent laser anemometer measurements of Armaly and Durst [18] for asymmetric channel flows.

C. Laminar Separating Flow over a Rearward-Facing Step with Viscous-Inviscid Interaction

The viscous-inviscid interaction method was applied for predicting laminar flows over rearward-facing steps to evaluate its capabilities for predicting such flows. At the present time, only a few reliable measurements are available for the laminar rearward-facing step flow which has inviscid core region inside the flow field. Eriksen [8] measured such laminar flows over rearward-facing steps (the results were also reported by Goldstein, et al. [7]). The measurements were performed in a small wind tunnel with a rectangular test section of 0.102 m width and 0.153 m height upstream of the rearward-facing step. The step was located on the flat top wall. The test section was 0.203 m long downstream of the step with a movable top to provide for various step heights. The range of the experimental conditions included 0.00356-0.01016 m in step height, 0.6096-2.4384 m in free stream velocity at the step, and 0.00163-0.005 m in the displacement thickness at the
step. Data were taken by using hot wires. The measured turbulence intensity in the test section was less than 0.05%. In the measurements, Eriksen observed that the boundary layers on both the step side and no-step side walls were fairly thin so that a large inviscid flow region existed inside the channel throughout the test section.

O'Leary and Mueller [12] and Leal and Acrivos [9] also reported measurements for rearward-facing step flows which have inviscid flow region in the flow field. However, they did not report clearly the flow conditions, especially on the wall opposite to the step. The measured reattachment lengths by O'Leary and Mueller [12] and Leal and Acrivos [9] are much larger than those measured by Eriksen [8] for the range of Reynolds numbers investigated by them (see Figure 5.23).

In the present study, the laminar flow over rearward-facing step measured by Eriksen [8] (reported also by Goldstein, et al. [7]) was predicted using the present viscous-inviscid interaction method.

In the present predictions, the step height was set at 0.01016 m and the free stream velocity at the step was 0.645 m/sec. The interaction zone ranged from $x/h = -13.5$ to $x/h = 36$. The solutions obtained after 12 global iterations are shown in Figures 5.23-5.31 along with the measurements by Eriksen [8]. The solutions converged to $\varepsilon = 0.0007$ ($\varepsilon$ is defined in Equation (4.15)). The agreement of the present results with the measured mean velocity profiles is very good as can be seen in Figures 5.26 and 5.27. The reattachment velocity profile obtained from the present prediction is shown to be slightly different from the Karman-Pholhausen separation profile (see Figure 5.27). The present predicted velocity profile at 14 step heights downstream of
Figure 5.23. Comparison of the predicted and measured length of the separated region for a laminar flow over a rearward-facing step.
Figure 5.24. Displacement thickness distribution for a laminar flow over a rearward-facing step, $h \omega_{max}/\nu = 412$, $h = 0.01016$ m
Figure 5.25. Edge velocity distribution for a laminar flow over a rearward-facing step, $\frac{h_u}{v \max} = 412$, $h = 0.01016$ m
Figure 5.26. Velocity profiles for a laminar flow over a rearward-facing step (1), $h u_{max}/\nu = 412$, $h = 0.01016$ m
Figure 5.27. Velocity profiles for a laminar flow over a rearward-facing step (2), $\frac{hu_{max}}{\nu} = 412$, $h = 0.01016$ m
Figure 5.28. Shape factors in a laminar redeveloping boundary layer downstream of a rearward-facing step, $\text{Re}_{\text{u}_{\text{max}}} = 412$, $h = 0.01016 \text{ m}$
Figure 5.29. Displacement thickness distribution in a laminar redeveloping boundary layer downstream of a rearward-facing step, $h_u/\nu = 412$, $h = 0.1016$ m
Figure 5.30. Momentum thickness distribution for a laminar flow over a rearward-facing step, $h u_{max}/v = 412$, $h = 0.01016$ m
Figure 5.31. Distribution of skin-friction coefficient for a laminar flow over a rearward-facing step, $h u_{max} / \nu = 412$, $h = 0.01016$ m.
reattachment agrees very well with the Blasius profile. The shape factor at that point is about 2.59 which is almost the same value as for the exact solution. However, the shape factor was found to keep slowly decreasing downstream in the same fashion as for the measurements as shown in Figure 5.28. The agreement of the predicted shape factor with the measurements is reasonably good except for near the reattachment point where the predicted shape factor is larger than the measurements.

In the redeveloping flow region, the displacement thickness distribution is slightly under predicted compared with the measured data (see Figure 5.29). The momentum thickness in this region agrees very well with the measured data as can be seen in Figure 5.30. In the reversed flow region, the predicted momentum thickness is lower than the measurements. However, the measured data in this region may involve large uncertainties since the negative mean velocities were not measured.

The predicted reattachment length is about 13.5 step heights which is longer than the Goldstein, et al.'s measurement [7] by 23% as shown in Figure 5.23. However, the predicted reattachment point is located below the points extrapolated from the measurements by O'Leary and Mueller [12] and Leal and Acrivos [9] as can be seen in Figure 5.23. It is interesting to note that the present predicted point is not far from the point extrapolated from Denham and Patrick's measurements [15] for asymmetric expansion with developed velocity profiles as discussed in Section B above.

Relatively large negative skin-friction coefficients were obtained from the present prediction in the separating flow region (see Figure
5.31). Unfortunately, no data are available for comparison. In the prediction, neither under-relaxation nor over-relaxation was used. One iteration of the present interaction calculation required typically 17 sec of CPU time on an NAS AS/6 digital computer.

D. Turbulent Separating Flows over a Rearward-Facing Step with Viscous-Inviscid Interaction

The measurements by Kim, et al. [58] for turbulent separating flows over rearward-facing steps were taken as the primary test cases for this type of flow. The data were obtained in a wind tunnel using hot wires. The wind tunnel had a rearward-facing step on the lower wall. The tunnel inlet section was 0.6096 m in height, 0.0762 m in width, and 0.3048 m in length. The test section of the tunnel behind the step was 2.3368 m long. The step height could be varied from 0 to 0.203 m; however, the measurements were performed for the step height of 0.0381 m (reference flow) and 0.0254 m (step-1 flow). Here, the reference and step-1 flows are named following Kim, et al. [58]. The typical reference velocity was 18.2 m/sec with a variation of 0.15 m/sec throughout the experiment. Kim, et al. [58] identified an inviscid flow region inside the flow field. They also predicted the flows using a zonal method with a viscous-inviscid interaction procedure as discussed in Chapter I. Their results agreed very well with the measurements.

The present predictions were made for the region ranging from \(x/h = -4\) to \(x/h = 16\) for the reference flow and from \(x/h = -5\) to \(x/h = 20\) for the step-1 flow. In the predictions, the kinetic energy-mixing
length (k-ε) model was generally used for evaluating the turbulence viscosity.

Figures 5.32 and 5.33 show the predicted displacement thickness distributions for the reference and step-1 flows. The solutions converged to $\delta = 0.0125$ in 45 global iterations for the reference flow and to $\delta = 0.0185$ in 35 global iterations for the step-1 flow. For both cases, the initially assumed displacement thickness distributions along the step-side walls were similar to the Kim, et al.'s predictions. Small under-relaxation factors were used for both flow problems, 0.1 for the reference flow and 0.2 for the step-1 flow. As shown in Figures 5.32 and 5.33, the present viscous-inviscid interaction method overpredicted the displacement thickness distributions along the step side walls. The predicted displacement thicknesses along the no-step side wall for the step-1 flow are also presented in Figure 5.33; however, no comparisons were made either with measurements or with predictions, since they are not reported by Kim, et al. [58].

The pressure coefficient distributions plotted in Figures 5.34-5.36 provide reasonable agreement with the measurement except very near the step. However, the present viscous-inviscid interaction method did not predict a strong favorable pressure gradient very close to the step along the step side wall, but it gave only a relatively mild favorable pressure gradient as shown in Figures 5.34 and 5.36. In the short region downstream of the step, the predicted pressure coefficients were found to be interestingly nearly constant. Kim, et al. [58] assumed a constant pressure coefficient in more or less the same region in their predictions with the zonal method. The magnitude of the negative pressure coeffi-
Figure 5.32. Displacement thickness distribution along the lower (step) wall for a turbulent rearward-facing step flow (reference flow)
Figure 5.33. Displacement thickness distribution for a turbulent rearward-facing step flow (step-1 flow)
Figure 5.34. Pressure coefficient distribution along the lower (step) wall for a turbulent rearward-facing step flow (reference flow)
Figure 5.35. Pressure coefficient distribution along the upper (no-step) wall for a turbulent rearward-facing step flow (reference flow)
Figure 5.36. Pressure coefficient distribution for a turbulent rearward-facing step flow (step-1 flow)
coefficients predicted by the present method and the zonal method in the region are almost the same. In the present prediction, the flows behind the step generally turned down slightly late compared to the measurements and predictions by Kim, et al. [58]. The pressure coefficients along the no-step side wall for the reference flow are shown to be in very good agreement with the measurements (see Figure 5.35). For the step-1 flow, the pressure coefficients were not reported by Kim, et al. so that in the present study the comparison of the predicted pressure coefficients for the step-1 flow was not possible.

The predicted solutions were found to be strongly dependent upon the turbulence modeling used especially in the separated and redeveloping flow regions. In general, the simple algebraic mixing length ($\ell$) model, Equations (2.38), (2.45), and (2.46), predicted a better reattachment length and mean velocity profiles than did the turbulence kinetic energy-mixing length ($k-\ell$) equation model, Equations (2.38), (2.45), (2.46), and (2.51), as can be seen in Figures 5.37 and 5.38. However, the $k-\ell$ model was found to predict better turbulence quantities such as turbulence kinetic energies and turbulent stresses than the $\ell$ model compared to the measured data, although these data may not be reliably accurate. Upstream of the step, the solutions for the mean flow quantities and turbulent stresses obtained using the $k-\ell$ and the $\ell$ models were indistinguishable (see Figures 5.37 and 5.39 for mean velocity profiles).

The velocity profiles along the no-step side wall for the step-1 flow are plotted in Figure 5.40. It can be seen that close to the step
Figure 5.37. Velocity profiles downstream of the step for a turbulent rearward-facing step flow (reference flow)
Figure 5.38. Velocity profiles downstream of the step for a turbulent rearward-facing step flow (step-1 flow)
Figure 5.39. Velocity profiles upstream of the step for a turbulent rearward-facing step flow (step-1 flow)
Figure 5.40. Velocity profiles along the upper (no-step) wall for a turbulent rearward-facing step flow (step-1 flow)
and far downstream of the step, the profiles follow the logarithmic law. In the mid-interaction region, the velocity profiles exhibit characteristics of an adverse pressure gradient. This shows up as the larger slope of the velocity profile plotted on a semi-log scale than given by the logarithmic law [191]. However, such phenomena were not observed in the measurements by Kim, et al. [58].

The predicted maximum turbulent stresses in the reattachment region for both the reference and step-1 flows predicted by using the k-¿ model were larger than the measured data as shown in Figure 5.41. However, upstream and downstream of the zone, they agreed fairly well with the measured data. Such results also can be seen in the turbulent stress profiles plotted in Figures 5.42 and 5.43 and in the turbulence kinetic energy profiles in Figure 5.44. These turbulent stress and kinetic energy profiles plotted based on the distance from the reattachment point generally provide a better agreement with the measured profiles near the reattachment point than those plotted based on the distance from the step as shown in Figures 5.42-5.44.

The predicted skin-friction coefficients are shown in Figures 5.45 and 5.46. In the reversed flow region, they have large negative values (the minimum values ranged from - 0.0027 to - 0.0029) which are even larger than the value \( C_f > - 0.001 \) measured by Eaton and Johnston [62] in the reversed flow region of a rearward-facing step flow (see Chapter I). The largest negative skin-friction coefficient occurred a short distance upstream of reattachment. Near reattachment, relatively unstable flow solutions were found as shown in Figures 5.45 and 5.46. These might indicate that the present turbulence models do not provide
Figure 5.41. Distribution of maximum Reynolds stress for turbulent rearward-facing step flows.
Figure 5.42. Reynolds stress profiles for a turbulent rearward-facing step flow (reference flow); the predictions are plotted based on the distance from reattachment.
Figure 5.43. Reynolds stress profiles for a turbulent rearward-facing step flow (step-1 flow)
Figure 5.44. Turbulence kinetic energy profiles for a turbulent rearward-facing step flow (step-1 flow)
Figure 5.45. Distribution of skin-friction coefficient for a turbulent rearward-facing step flow (reference flow)
Figure 5.46. Distribution of skin-friction coefficient for a turbulent rearward-facing step flow (step-1 flow)
large enough turbulent stresses near the wall in the reattachment region.

The reattachment lengths calculated were $x/h = 7.95$ for the reference flow and 7.65 for the step-1 flow by using the $k-\varepsilon$ model, and $x/h = 7.32$ for the reference flow and 7.28 for the step-1 flow by using the $\lambda$ model. As discussed above, the $\lambda$ model provided better reattachment lengths compared to the measurements than did the $k-\varepsilon$ model. However, all of these are longer than the measured value of 7.0.

In the calculation, typically 9.7 sec of CPU time was required for one global iteration of the interaction calculation on an NAS AS/6 digital computer.

The present viscous-inviscid interaction method was tested for predicting turbulent flow in a two-dimensional symmetric channel expansion (step-2 flow) in which an inviscid core region existed. At the present time, no accurate measured data are available for such a flow.

Mehta [59] did report measurements for this flow, but because of an extremely small channel aspect ratio (1/4), two-dimensionality could not be maintained in the flow. This can be clearly seen by comparing his measured mean velocity profiles with other two-dimensional flow velocity profiles.

The symmetric expansion flows measured by Abbott and Kline [48] were not fully developed at the step so that an inviscid flow region could be identified there. However, the boundary layers on both the walls merged downstream of the step so that the inviscid core region soon disappeared behind the symmetric expansion. The viscous-inviscid
interaction procedure would not appear to be applicable for this case.

The solutions for the turbulent symmetric expansion flow similar to the flow measured by Mehta [59] are shown in Figures 5.47-5.50. The channel inlet height and the step height were 0.1 m and 0.0125 m, respectively. The channel inlet velocity was 11.66 m/sec. The interaction region ranged from x/h = -5.76 to 17.76. The solutions converged to ε = 0.00095 in 17 iterations with an under-relaxation factor of 0.5. The displacement thicknesses by a one-through solution of the boundary-layer equations were used as the initial distribution for the interaction calculation (see Figure 5.47). The one-through boundary-layer solutions were obtained using the same procedure as used for the symmetric expansion with fully developed profiles as discussed in Section V.B.

A fair amount of difference between the one-through boundary-layer solution and the viscous-inviscid interaction solution can be seen in Figure 5.47. Consequently, for this type of flow which has inviscid core inside the flow field, the boundary-layer equations alone appear not to provide a sufficiently accurate mathematical model.

Velocity profiles predicted are compared with Mehta's measured profiles in Figure 5.48. The agreement is poor as expected due to the three-dimensionality of the flow in his small aspect ratio channel. The predicted pressure coefficients along the wall and centerline, and turbulence kinetic energy and turbulent stress profiles are also plotted in Figures 5.49 and 5.50.

The calculated reattachment length is 4.32 step heights. This
Figure 5.47. Displacement thickness distribution for a turbulent rearward-facing step flow (step-2 flow)
Figure 5.48. Velocity profiles for a turbulent rearward-facing step flow (step-2 flow)
Figure 5.49. Pressure coefficient distribution for a turbulent rearward-facing step flow (step-2 flow)
Figure 5.50. Reynolds stress and turbulence kinetic energy profiles for a turbulent rearward-facing step flow (step-2 flow)
agrees reasonably well with the prediction by Gosman, et al. [89]
for the somewhat similar flow. They predicted flows in a symmetric
channel with low expansion ratio using the Reynolds equations and
compared the predicted reattachment length with the measured data by
Abbott and Kline [48] resulting in a reasonable agreement. The direct
comparison of the present results with that predicted by Gosman, et al.
[89] was not made by means of figures or tables in the present study,
since the mean velocity profiles predicted by Gosman, et al. were re­
ported on extremely small-scaled figures in [89] so that an accurate
interpretation of their results was not possible.

The calculation time required for this test case was 5.53 sec of
CPU for one iteration of viscous-inviscid interaction calculation
on an NAS AS/6 digital computer.
VI. CONCLUSIONS AND RECOMMENDATIONS

A. Concluding Remarks

In the present study, an improved boundary-layer solution method has been developed and applied to predict the two-dimensional symmetric channel flows with developed inlet velocity profiles. A viscous-inviscid interaction method has also been developed for rearward-facing step flows where both the viscous and inviscid flow regions exist throughout the computational domain. The main results are summarized below:

1. On the development of the boundary-layer solution method:

The new coupling scheme was found to be well-behaved and to provide nonoscillatory solutions in the reversed flow region for both laminar and turbulent separation bubble flows occurring on a smooth surface. The results compared quite favorably with other numerical predictions and experimental data. As far as the number of iterations required for an inverse boundary-layer solution at a streamwise location is concerned, the present solution method is the fastest among the calculation schemes being considered. For the inverse solutions, the present method provides converged solutions generally in two iterations, even in a large separation region, which are about 2/3 of those required for the Carter's coupling method [93], and approximately 1/2 of those for the fully implicit method with lagged coefficients employing the secant solution procedure for the evaluation of the pressure gradient [94].
2. On the prediction of laminar flows in a two-dimensional channel expansion with developed inlet velocity profiles using the boundary-layer equations:

For laminar symmetric channel expansion flows with developed inlet velocity profiles, the prediction for mean velocity profiles and channel centerline velocities predicted using the boundary-layer equations were found to agree very well with the full Navier-Stokes solutions and measurements for Reynolds number based on the channel inlet height and inlet mean velocity greater than 20. The predicted reattachment length for such flows was found to vary nearly linear with the Reynolds number. Such characteristics of the linear variation of the reattachment length with respect to the Reynolds number have been observed with the full Navier-Stokes solutions. However, inconsistency between the present boundary-layer solutions and the general Navier-Stokes solutions was observed in the streamline pattern very near the step. That is, in the boundary-layer solutions, the dividing streamline was found not to leave the step tangentially and the negative streamlines were densely populated near the step, whereas in the Navier-Stokes solutions noted in the literatures, the dividing streamline departs tangentially from the step and the negative streamlines near the step are populated relatively sparsely.

For asymmetric channel expansion flows, the mean velocity profiles and reattachment lengths predicted using the present boundary-layer solution method agreed reasonably well with the measured data. In the present prediction using the boundary-layer equations, secondary type small separation bubbles were found on the upper and/or lower walls
downstream of the step. This finding agrees qualitatively with recent laser anemometer measurements [18].

When the boundary-layer equations were applied for a turbulent symmetric channel expansion flow which included viscous and inviscid core regions in the flow field, the predictions were not as successful as for the laminar flows without an inviscid flow region. For this type of flow, an elliptic mathematical model seems to be required.

3. On the prediction of laminar and turbulent flows over a rearward-facing step using a viscous-inviscid interaction method:

A major part of this study was devoted to the development and evaluation of a viscous-inviscid interaction method and a turbulence length-scale model for flows over rearward-facing steps. The predictions using the viscous-inviscid interaction method were found to agree reasonably well with the measurements for both laminar and turbulent flows. For laminar flows, the predicted mean flow quantities and redeveloping boundary-layer characteristics are in very good agreement with the measurements. However, the reattachment was overpredicted by 23% compared with the measured data. The solutions for turbulent flows seem to be quite dependent upon the turbulence model used. For example, the mixing length model in the present study predicted mean velocity profiles and reattachment lengths more accurately than did the turbulence kinetic energy–mixing length model, but overpredicted the magnitude of turbulence quantities near the reattachment point. The predicted reattachment lengths with the mixing length model were longer than the measurements by about 4-5%. The turbulence kinetic energy/mixing length model was found to overpredict the reattachment
lengths for the two asymmetric channel expansion cases being investigated by about 10-15% compared with the measured data, but the turbulence kinetic energies and turbulent stresses were predicted generally better than the mixing length model compared to the measurements.

The number of viscous-inviscid interaction iterations required for convergence was found to be very much dependent upon the ratio of the displacement thickness at the step to the step height. When the ratio is small, that is, the step height was much higher than the displacement thickness, a very small under-relaxation factor is required for convergence (consequently, a large amount of time is needed). When the ratio was large, relatively fewer iterations are required since a large under-relaxation factor or even over-relaxation can be used. A typical time required for one iteration of the viscous-inviscid interaction solution for a turbulent asymmetric channel expansion flow was 9.7 seconds on the NAS AS/6 computer.

Based on the main results summarized above, the following major conclusions are drawn:

1. The present boundary-layer solution method is a relatively accurate and efficient solution procedure.

2. The boundary-layer equations are found to be a reasonably good approximate mathematical model for symmetric channel expansion flows with fully developed inlet velocity profiles for Reynolds number based on channel inlet height and inlet mean velocity greater than 20. For asymmetric channel expansion flows, the boundary-layer solutions are reasonably accurate except for near the separation region on the wall opposite to the step. It is found that for symmetric channel
expansion flows which have inviscid flow region inside, the boundary-layer solutions do not agree well with the viscous-inviscid interaction solutions.

3. The newly developed viscous-inviscid interaction method provides reasonably accurate solutions for both laminar and turbulent rearward-facing step flows which contain both viscous and inviscid flow regions, although the solutions for turbulent flows seem to be very dependent upon the turbulent model used. Of the two models evaluated, the newly developed simple algebraic mixing length model was found to provide better reattachment lengths and mean velocity profiles. However, as far as the predicted turbulence quantities are concerned, the kinetic energy-mixing length model appears to be better than the mixing length model.

4. The only empiricism in the present viscous-inviscid interaction method is contained in the turbulence modeling. For this reason, the present interaction method is believed to represent a more general calculation procedure than the existing zonal methods.

B. Recommendations for Future Study

Although the present viscous-inviscid interaction method was found to provide reasonably good solutions, further study on this method seems to be required for further refinement of the procedure. In the near future, however, a better and more complete mathematical model, such as the full Reynolds averaged Navier-Stokes equations, should be investigated for predicting complex turbulent flows. Recently, the
Navier-Stokes equations have been widely used and significant progress has been made in reducing the computation time required. However, further studies are still required for improvement and development of an advanced algorithm for numerical solutions of the equations. For example, most of the numerical methods using primitive variables for Navier-Stokes solutions employ the "lagging" procedure for evaluation of the coefficients in the momentum equations. This approximation may be improved by introducing a coupling procedure. Pioneering work on such an approach was done by Morihara [29] as discussed earlier.

As far as turbulence modeling is concerned, it is believed that a simple mixing length model is not sufficient unless it takes account properly of the elliptic nature of the flow. A higher-order model retaining elliptic terms seems to be more promising. Recently, studies have been made on such a model, but further continuous studies with such a model seem to be necessary.

It is also recommended that heat transfer to separated flows be investigated either with a viscous-inviscid interaction method or with the Navier-Stokes equations.

As regards the need for future experimental studies, continuous accurate measurements are needed to understand more clearly the mean flow and turbulence characteristics in the separated and redeveloping flow regions behind the step, and, thus, to provide a physical basis for improving the turbulence model used to analyze the flow.
VII. REFERENCES


VIII. ACKNOWLEDGMENTS

The author wishes to express his heartfelt gratitude to Dr. Richard H. Fletcher, his thesis advisor and committee chairman, for his invaluable instructions, advice, suggestions and his continuous encouragement.

The author is indebted to his doctoral committee, Dr. Arthur E. Bergles, Dr. William J. Cook, Dr. Bruce R. Munson, and Dr. John C. Tennehill, for their interest and cooperation during the course of the study.

The author is thankful to the National Science Foundation for sponsoring the research under Grant ENG-7812901.

The author is grateful to his parents, brothers and sisters for their unfailing encouragement and support.

Finally, the author wishes to thank his wife, Young-Mee, for her patience and understanding throughout these recent trying years.
IX. APPENDIX A: APPROXIMATE RELATIONSHIP BETWEEN $\delta'$ AND $\delta$

Equation (2.42) can be expanded in a series form as

$$\delta' = \frac{1}{\sqrt{2}} \left[ y_{\tau,\text{max}}^2 + (\delta - y_{\tau,\text{max}})^2 \right]^{1/2}$$

$$= \frac{1}{\sqrt{2}} y_{\tau,\text{max}} \left[ 1 + \frac{1}{2} \left( \frac{\delta - y_{\tau,\text{max}}}{y_{\tau,\text{max}}} \right)^2 + \frac{1}{8} \left( \frac{\delta - y_{\tau,\text{max}}}{y_{\tau,\text{max}}} \right)^4 + \ldots \right],$$

if $\frac{1}{2} \delta < y_{\tau,\text{max}} \leq \delta$

$$= \frac{1}{\sqrt{2}} (\delta - y_{\tau,\text{max}}) \left[ 1 + \frac{1}{2} \left( \frac{y_{\tau,\text{max}}}{\delta - y_{\tau,\text{max}}} \right)^2 + \frac{1}{8} \left( \frac{y_{\tau,\text{max}}}{\delta - y_{\tau,\text{max}}} \right)^4 + \ldots \right],$$

if $0 < y_{\tau,\text{max}} \leq \frac{1}{2} \delta$

Considering first for $\frac{1}{2} \delta < y_{\tau,\text{max}} \leq \delta$, since $\frac{\delta - y_{\tau,\text{max}}}{y_{\tau,\text{max}}} < 1$, the terms higher than second order can be neglected as

$$\delta' \approx \frac{1}{\sqrt{2}} y_{\tau,\text{max}} \left[ \frac{1}{2} + \frac{\delta}{y_{\tau,\text{max}}} + \frac{1}{2} \left( \frac{\delta}{y_{\tau,\text{max}}} \right)^2 \right]$$

$$= \frac{1}{\sqrt{2}} \left( y_{\tau,\text{max}} + 2\delta + \delta^2 / y_{\tau,\text{max}} \right)$$

Since $y_{\tau,\text{max}}$ is of the same order of magnitude as $\delta$, $\delta'$ can be approximated as a weak linear function of $\delta$, i.e.,

$$\delta' \approx C' \delta$$

Similarly, the same result as this can be drawn for the case of $0 < y_{\tau,\text{max}} \leq \frac{1}{2} \delta$. 
X. APPENDIX B: DERIVATION OF THE GOVERNING EQUATION FOR INVIScid
FLOWS IN THE TRANSFORMED COORDINATE SYSTEM

The equations for transforming the original inviscid flow domain into a rectangular domain on the $\xi$, $\eta$ coordinates are given in Equations (2.65) as,

\[
\xi = \frac{x - x_0}{L_e} \quad (2.65-a)
\]

\[
\eta = \frac{y - \delta^*}{\delta^* - \delta^*} \quad (2.65-b)
\]

Differentiation of these equations with respect to $x$ and $y$ leads to the following:

\[
\frac{d\xi}{dx} = \frac{1}{L_e} \quad (10.1)
\]

\[
\frac{d\xi}{dy} = 0 \quad (10.2)
\]

\[
\frac{d\eta}{dx} = -\frac{1}{\delta^* - \delta^*} \left\{ \eta \frac{d\delta^*}{dx} + \frac{d\delta^*}{dx} \right\} \quad (10.3)
\]

\[
\frac{d\eta}{dy} = \frac{1}{\delta^* - \delta^*} \quad (10.4)
\]

\[
\frac{d^2\xi}{dx^2} = 0 \quad (10.5)
\]

\[
\frac{d^2\xi}{dy^2} = \frac{1}{L_e} \left\{ \eta \left( \frac{d\delta^*}{dx} \frac{d\delta^*}{dx} - \frac{\eta}{\delta^* - \delta^*} \frac{d^2\delta^*}{dx^2} \right) + \frac{2(1 - \eta)}{\delta^* - \delta^*} \frac{d\delta^*}{dx} \frac{d\delta^*}{dx} - \frac{1 - \eta}{\delta^* - \delta^*} \frac{d^2\delta^*}{dx^2} \right\} \quad (10.6)
\]

\[
\frac{d^2\xi}{dy^2} = 0 \quad (10.7)
\]
\[
\frac{d^2 \psi}{dy^2} = 0 \quad (10.8)
\]

with \[\delta^*_L - \delta^*_U\]

Using Equations (10.1)-(10.8), \[\frac{\partial^2 \psi}{\partial x^2}\] and \[\frac{\partial^2 \psi}{\partial y^2}\] can be written as

\[
\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{d \xi}{dx}\right)^2 \frac{\partial^2 \psi}{\partial \xi^2} + 2\left(\frac{d \xi}{dx}\right)\frac{d \eta}{dx} \frac{\partial^2 \psi}{\partial \xi \partial \eta} + \left(\frac{d \eta}{dx}\right)^2 \frac{\partial^2 \psi}{\partial \eta^2}
\]

\[
+ \left(\frac{d^2 \xi}{dx^2}\right) \frac{\partial \psi}{\partial \xi} + \left(\frac{d^2 \eta}{dx^2}\right) \frac{\partial \psi}{\partial \eta}
\]

\[
= \frac{1}{\delta^*_{UL}} \frac{\partial^2 \psi}{\partial \xi^2} - \frac{2}{\delta^*_{UL}} \left\{ \frac{d \delta^*_L}{d \xi} + \eta \frac{d \delta^*_UL}{d \xi} \right\} \frac{\partial^2 \psi}{\partial \xi \partial \eta}
\]

\[
+ \frac{1}{\delta^*_{UL}} \frac{\partial \psi}{\partial \xi} \left\{ \frac{d \delta^*_L}{d \xi} + \eta \frac{d \delta^*_UL}{d \xi} \right\} \frac{\partial^2 \psi}{\partial \eta^2}
\]

\[
+ \frac{1}{\delta^*_{UL}} \frac{\partial \psi}{\partial \xi} \left\{ \frac{2}{\delta^*_{UL}} \frac{d \delta^*_L}{d \xi} + \eta \frac{d \delta^*_UL}{d \xi} \right\}
\]

\[
- \frac{1}{\delta^*_{UL}} \left( \frac{d^2 \delta^*_L}{d \xi^2} + \eta \frac{d^2 \delta^*_UL}{d \xi^2} \right) \frac{\partial \psi}{\partial \eta}
\]

(10.9)

and

\[
\frac{\partial^2 \psi}{\partial y^2} = \left(\frac{d \xi}{dy}\right)^2 \frac{\partial^2 \psi}{\partial \xi^2} + 2\left(\frac{d \xi}{dy}\right)\frac{d \eta}{dy} \frac{\partial^2 \psi}{\partial \xi \partial \eta} + \left(\frac{d \eta}{dy}\right)^2 \frac{\partial^2 \psi}{\partial \eta^2}
\]

\[
+ \left(\frac{d^2 \xi}{dy^2}\right) \frac{\partial \psi}{\partial \xi} + \left(\frac{d^2 \eta}{dy^2}\right) \frac{\partial \psi}{\partial \eta}
\]

\[
= \frac{1}{\delta^*_{UL}} \frac{\partial^2 \psi}{\partial \eta^2}
\]

(10.10)

Substitution of Equations (10.9) and (10.10) into Equation (2.58) results in Equation (2.66).
XI. APPENDIX C: DISCUSSION OF SEVERAL LINEARIZING PROCEDURES FOR NONLINEAR TERMS OCCURRING IN THE FINITE-DIFFERENCE REPRESENTATION OF THE MOMENTUM EQUATION

Procedures for linearizing the nonlinear terms occurring in the finite-difference formulations of the momentum equation are as follows:

1. **Lagging coefficients**: This procedure evaluates coefficients at the $i$ level (see Figure 3.1), such as, for a general function $\phi(x, y)$,

$$
(\phi_j^{i+1})^2 \approx \phi_j^i \phi_j^{i+1} \tag{11.1}
$$

This representation is consistent since $\phi(x_0 + \Delta x, y_0) = \phi(x_0, y_0) + O(\Delta x)$, but does ensure that the scheme is formally no better than first order accurate in the marching direction.

2. **Extrapolating coefficients**: Coefficients are evaluated at the $i+1$ level by extrapolation based on values previously obtained at the $i$ and/or $i-1$ level. This procedure can provide formally higher order accuracy as far as the truncation error is concerned.

For example, a second order accurate representation of a coefficient $\phi_j^{i+1}$ can be attained as follows:

$$
\phi_j^{i+1} = \phi_j^i + \frac{\partial \phi}{\partial x}_j \Delta x + O(\Delta x)^2 \tag{11.2}
$$

With a first order approximation of $\frac{\partial \phi}{\partial x}_j$, this equation becomes

$$
\phi_j^{i+1} = \phi_j^i + \frac{1}{\Delta x} (\phi_j^i - \phi_{j-1}^i) \Delta x + O(\Delta x)^2 \tag{11.3}
$$
which has a truncation error of $O(\Delta x)^2$.

The procedure of lagging coefficients is in a sense a first order accurate representation of the procedure of extrapolating coefficients.

3. **Simple iterative update of coefficients:** Coefficients are evaluated at the $i + 1$ level by updating iteratively. The "updating" proceeds by utilizing the solution just obtained at the $i + 1$ level as the coefficient such as

$$\phi_j^{i+1} \simeq \phi_j^{d+1}$$  \hspace{1cm} (11.4)

where $\phi_j^{d+1}$ is the value obtained at the previous $n$th iteration level.

The calculation is repeated until the changes of the solution between two successive $n$ and $(n + 1)$ iteration levels are small.

4. **Iteratively updating coefficients by the use of Newton linearization:** The Newton linearization (also called quasilinearization) technique is used to linearize the nonlinear terms as follows:

$$\phi_j^{i+1} \text{ at } (n + 1) \text{ iteration level can be written as}$$

$$\phi_j^{i+1} = \phi_j^{d+1} + \delta \phi$$  \hspace{1cm} (11.5)

where $\delta \phi$ denotes the changes in $\phi$ between two successive iterations, and $\phi_j^{d+1}$ is the value obtained at the $n$th iteration level as defined earlier.

Thus, $(\phi_j^{i+1})^2$ can be evaluated as

$$(\phi_j^{i+1})^2 = (\phi_j^{d+1} + \delta \phi)^2 = (\phi_j^{d+1})^2 + 2\phi_j^{d+1} \delta \phi + \delta \phi^2$$  \hspace{1cm} (11.6)

Here $\delta \phi^2$ can be dropped by assuming $\delta \phi$ is small, so that $(\phi_j^{i+1})^2$
is linearized. Now replacing $\phi$ in the linearized expression of $(\phi_j^{i+1})^2$ by $\phi_j^{i+1} - \phi_j^{i+1}$, the following equation is obtained

\[(\phi_j^{i+1})^2 = 2\phi_j^{i+1}\phi_j^{i+1} - (\phi_j^{i+1})^2 \tag{11.7}\]

Use of Newton linearization is generally known to enhance the convergence rate in the iterative updating procedure for the coefficients.

The governing continuity and momentum equations can be solved in an uncoupled manner by employing the Newton linearization procedure to evaluate x-convective term only. Other nonlinear terms, such as y-convective and diffusion terms, must be evaluated by the simple updating procedure discussed earlier.

5. Newton linearization with coupling: The Newton linearization procedure discussed previously can be used to linearize not only x-convective terms but also the y-convective and diffusion terms. If this is done, the continuity and momentum equations must be solved in coupled manner. The convergence rate of the solution was found to be accelerated greatly by this procedure.
The finite-difference representations, Equations (3.16) and (3.17), were obtained by differencing the streamfunction and momentum equations. A fully implicit scheme was used. By applying the Newton linearization procedure to Equations (3.16) and (3.17), Equations (3.19) and (3.20) were obtained. The coefficients appearing in these equations, Equations (3.19) and (3.20), are evaluated as:

\[
A_j = - \frac{\psi_{j+1}^1 - \psi_j^1}{\Delta x \Delta y (K + 1)} + \frac{2M_{j+1/2}^1}{(\Delta y)^2(K + 1)} \\
B_j = \frac{\psi_{j+1}^1 - \psi_j^1}{\Delta x \Delta y (K + 1)} - \frac{2M_{j-1/2}^1}{(\Delta y)^2(K + 1)} \\
C_j = c \frac{(U_{j+1}^1)^2}{\Delta x} - \frac{\psi_{j+1}^1(U_{j+1}^1 - U_{j+1}^1)}{\Delta x \Delta y (K + 1)} \\
D_j = c \frac{2U_{j+1}^1 - U_j^1}{\Delta x} + \frac{2}{(\Delta y)^2(K + 1)} \left( \frac{M_{j+1/2}^1}{K} + M_{j-1/2}^1 \right) \\
E_j = - \frac{U_{j+1}^1 - U_{j-1}^1}{\Delta x \Delta y (K + 1)} \\
H_j = 1 \\
b_j = \frac{\Delta y}{2} \\
d_j = \frac{\Delta y}{2}
\]
and 

\[ K = \frac{\Delta Y_+}{\Delta Y_-} \]
XIII. APPENDIX E: COEFFICIENTS OF THE FINITE-DIFFERENCE EQUATIONS FOR VISCOUS FLOWS: THE FULLY IMPLICIT SCHEME WITH LAGGED COEFFICIENTS AND THE CRANK-NICOLSON SCHEME

Newton linearization can also be applied to the Crank-Nicolson implicit form of the difference equations for solving the streamfunction and momentum equations in a coupled manner. The resulting difference equations can be written in the same form as Equations (2.19) and (2.20); however, the coefficients in the resulting equations differ from those for the fully implicit scheme with Newton linearization. The following are the coefficients evaluated for fully implicit scheme with lagged coefficients and Crank-Nicolson scheme for solving the streamfunction and momentum equations simultaneously:

1. **Fully implicit scheme with lagged coefficients**: The coefficients of the convective and diffusion terms are evaluated at the $i - 1$ level.

\[
\begin{align*}
A_j &= -\frac{\psi_i^1 - \psi_i^{1-1}}{\Delta X \Delta Y (K + 1)} - \frac{2M_i^{i+1/2}}{(\Delta Y_j)^2(K + 1)} \\
B_j &= \frac{\psi_i^1 - \psi_i^{1-1}}{\Delta X \Delta Y (K + 1)} - \frac{2M_i^{i-1/2}}{(\Delta Y_j)^2(K + 1)} \\
C_j &= c \frac{(U_i^1)^2}{\Delta X} \\
D_j &= c \frac{U_i^1}{\Delta X} + \frac{2}{(\Delta Y_j)^2(K + 1)} \left( M_i^{i+1/2} + \frac{M_j^{i-1/2}}{K} \right) \\
E_j &= 0
\end{align*}
\]
and

\[ K = \frac{\Delta Y_+}{\Delta Y_-} \]

2. **Crank-Nicolson scheme with Newton linearization**: When the Crank-Nicolson scheme with Newton linearization is applied to the streamfunction and \( x \)-momentum equations, Equations (3.19) and (3.20), the following coefficients result:

\[
A_j = -\frac{\psi^i_{j+1} - \psi^i_j}{2\Delta X_+ \Delta Y_-(K+1)} - \frac{M^i_{j+1/2}}{(\Delta Y_-)^2(K+1)}
\]

\[
B_j = \frac{\psi^i_{j+1} - \psi^i_j}{2\Delta X_+ \Delta Y_-(K+1)} - \frac{M^i_{j-1/2}}{(\Delta Y_-)^2(K+1)}
\]

\[
C_j = c \frac{\left(\frac{U^{i+1}}{j+1} - \frac{U^i}{j-1}\right)^2 + \left(\frac{U^i}{j}\right)^2}{2\Delta X_+ \Delta Y_-(K+1)} - \frac{1}{2\Delta X_+ \Delta Y_-(K+1)} \left\{ \left(\frac{U^{i+1}}{j+1} - \frac{U^{i+1}}{j-1}\right)\psi^{i+1}_{j+1} - \psi^i_j\left(\frac{U^{i+1}}{j+1} - \frac{U^{i+1}}{j-1}\right) \right\} + \frac{1}{(\Delta Y_-)^2(K+1)} \left\{ \frac{M^i_{j+1/2}U^{i+1}}{K} - \left(\frac{M^i_{j-1/2}}{K} + M^i_{j-1/2}\right)U^i_j + M^i_{j-1/2}U^{i+1}_j \right\}
\]

\[
D_j = c \frac{\frac{U^{i+1}}{j+1}}{\Delta X_+} + \frac{1}{(\Delta Y_-)^2(K+1)} \left( \frac{M^i_{j+1/2}}{K} + M^i_{j-1/2} \right)
\]

\[
E_j = -\frac{\frac{U^{i+1}}{j+1} - \frac{U^{i+1}}{j-1} + U^i_j}{2\Delta X_+ \Delta Y_-(K+1)} - \frac{U^i_j}{2\Delta X_+ \Delta Y_-(K+1)}
\]

\[ H_j = 1 \]
\[ b_j = \frac{\Delta Y}{2} \]
\[ d_j = \frac{\Delta Y}{2} \]

and

\[ K = \frac{\Delta Y_+}{\Delta Y_-} \]
XIV. APPENDIX F: COEFFICIENTS OF FINITE-DIFFERENCE EQUATIONS FOR INVISCID FLOW

Applying the central differencing scheme to the governing inviscid flow equation, Equation (3.11), the following finite-difference representation can be obtained for constant grid spacing, $\Delta x_+ = \Delta x_- = \Delta x$ and $\Delta \eta_+ = \Delta \eta_- = \Delta \eta$:

$$
\frac{\psi^{i+1}}{\Delta x^2} - \frac{\psi^i}{\Delta x^2} - \frac{\psi^{i+1} - \psi^i}{4 \Delta x \Delta \eta} + \frac{\psi^{i+1} - 2 \psi^i + \psi^{i-1}}{\Delta \eta^2} + \frac{\psi^i}{2 \Delta \eta} = 0
$$

(14.1)

where the coefficients $\bar{A}_\psi$, $\bar{B}_\psi$, and $\bar{C}_\psi$ are evaluated by using the central differencing scheme, as

$$
\bar{A}_\psi = \frac{2}{\delta \psi} \left\{ \frac{\delta^2 \psi^{i+1} - \delta^2 \psi^i}{2 \Delta x} + \eta_j \frac{\delta^2 \psi^{i+1} - \delta^2 \psi^i}{2 \Delta x} \right\}
$$

(14.2-a)

$$
\bar{B}_\psi = \frac{1}{\delta \psi^2} \left[ \left( \delta \psi^2 \right)^2 + \left\{ \frac{\delta^2 \psi^{i+1} - \delta^2 \psi^i}{2 \Delta x} + \eta_j \frac{\delta^2 \psi^{i+1} - \delta^2 \psi^i}{2 \Delta x} \right\}^2 \right]
$$

(14.2-b)

$$
\bar{C}_\psi = \frac{2}{(\delta \psi)^2} \left\{ \frac{\delta^2 \psi^{i+1} - \delta^2 \psi^i}{2 \Delta x} \right\} \left\{ \frac{\delta^2 \psi^{i+1} - \delta^2 \psi^i}{2 \Delta x} \right\} + \eta_j \frac{\delta^2 \psi^{i+1} - \delta^2 \psi^i}{2 \Delta x} - \frac{i}{\delta \psi} \left\{ \frac{\delta^2 \psi^{i+1} - 2 \delta \psi^i + \delta \psi^i}{\Delta x^2} \right\} + \eta_j \frac{\delta^2 \psi^{i+1} - 2 \delta \psi^i + \delta \psi^i}{\Delta x^2}
$$

(14.2-c)

In order to simply Equation (14.1), $A_\psi$, $B_\psi$, and $C_\psi$ are introduced:
\[ A_\psi = \frac{\Delta \xi}{4 \Delta \eta} A_\psi \]
\[ = \frac{1}{4 \Delta \eta \hat{\delta}_{UL}^2} \left( \hat{\delta}_L^{i+1} - \hat{\delta}_L^{i-1} + \eta_j (\hat{\delta}_{UL}^{i+1} - \hat{\delta}_{UL}^{i-1}) \right) \]  
\[ (14.3-a) \]

\[ B_\psi = \left( \frac{\Delta \xi}{\Delta \eta} \right)^2 B_\psi \]
\[ = \frac{1}{(2 \Delta \eta \hat{\delta}_{UL}^2)} \left[ (2 \Delta \xi_{UL})^2 + \left( \hat{\delta}_L^{i+1} - \hat{\delta}_L^{i-1} + \eta_j (\hat{\delta}_{UL}^{i+1} - \hat{\delta}_{UL}^{i-1}) \right)^2 \right] \]  
\[ (14.3-b) \]

\[ C_\psi = \frac{\Delta \xi^2}{2 \Delta \eta} C_\psi \]
\[ = \frac{1}{2 \Delta \eta} \left[ \frac{\hat{\delta}_L^{i+1} - \hat{\delta}_L^{i-1}}{2 (\hat{\delta}_{UL}^2)} \left( \hat{\delta}_L^{i+1} - \hat{\delta}_L^{i-1} + \eta_j (\hat{\delta}_{UL}^{i+1} - \hat{\delta}_{UL}^{i-1}) \right) \right. \]
\[ - \frac{1}{\hat{\delta}_{UL}^2} \left( \hat{\delta}_L^{i+1} - \hat{\delta}_L^{i-1} \right) \]
\[ (14.3-c) \]

Equation (14.1) thus simplified to Equation (3.21).
In order to show that Equations (3.16), (3.17), (3.21) and (3.22) are consistent representations of the model equations, Equations (3.2), (3.3), (3.11) and (3.13), respectively, the following Taylor series expansions for a general function $\phi$ can be written:

About the point $(i, j)$ in Figure 2.5

\[
\phi_{j+1} = \phi_j + \left( \frac{\partial \phi}{\partial x} \right)_j \Delta x^+ + \left( \frac{\partial^2 \phi}{\partial x^2} \right)_j \Delta x^2 + \ldots \tag{15.1}
\]

\[
\phi_{j-1} = \phi_j - \left( \frac{\partial \phi}{\partial x} \right)_j \Delta x^- + \left( \frac{\partial^2 \phi}{\partial x^2} \right)_j \Delta x^2 - \ldots \tag{15.2}
\]

\[
\phi_{j+1} = \phi_j + \left( \frac{\partial \phi}{\partial y} \right)_j \Delta y^+ + \left( \frac{\partial^2 \phi}{\partial y^2} \right)_j \Delta y^2 + \ldots \tag{15.3}
\]

\[
\phi_{j-1} = \phi_j - \left( \frac{\partial \phi}{\partial y} \right)_j \Delta y^- + \left( \frac{\partial^2 \phi}{\partial y^2} \right)_j \Delta y^2 - \ldots \tag{15.4}
\]

and about the point $(i + 1, j)$ in Figure 2.5

\[
\phi_{j+1} = \phi_{j+1} + \left( \frac{\partial \phi}{\partial y} \right)_j \Delta y^+ + \left( \frac{\partial^2 \phi}{\partial y^2} \right)_j \Delta y^2 + \ldots \tag{15.5}
\]

\[
\phi_{j-1} = \phi_{j-1} - \left( \frac{\partial \phi}{\partial y} \right)_j \Delta y^- + \left( \frac{\partial^2 \phi}{\partial y^2} \right)_j \Delta y^2 - \ldots \tag{15.6}
\]

Equation (15.1) can be rearranged to obtain an expression for

\[
\left( \frac{\partial \phi}{\partial x} \right)_j
\]

as

\[
\left( \frac{\partial \phi}{\partial x} \right)_j = \frac{\phi_{j+1} - \phi_j}{\Delta x^+} - \frac{\partial^2 \phi_j}{\partial x^2} \Delta x^2 - \ldots \tag{15.7}
\]

where the first term on the right-hand side is the finite-difference
representation of $\partial \phi / \partial x$ used. Thus, the truncation error which is the difference between the derivative and the finite-difference representation, is

$$T.E. = \frac{\partial^2 \phi}{\partial x^2} \Delta x + \ldots$$  \hspace{1cm} (15.8)$$
or

$$T.E. = O(\Delta x)$$

Similarly, it can be shown that the one-way differencing representation for $\partial \phi / \partial y$ is first order accurate by using Equation (15.6). The central-differencing representation for $\partial \phi / \partial y$ about $(i, j)$ can be obtained by using Equations (15.3) and (15.4) as

$$\frac{\partial \phi}{\partial y} \bigg|_{j} = \frac{\phi_{i+1}^j - \phi_{i-1}^j}{\Delta y} - \frac{\partial^2 \phi}{\partial y^2} \Delta y + \ldots$$  \hspace{1cm} (15.9)$$

and

$$T.E. = - \frac{\partial^2 \phi}{\partial y^2} \Delta y + \Delta y - \frac{\partial^3 \phi}{\partial y^3} \frac{\Delta y^3}{2} + \ldots$$  \hspace{1cm} (15.10)$$

The representation for $\partial \phi / \partial y$ about $(i + 1, j)$ can also be obtained by using Equations (15.5) and (15.6). The truncation error for the case is the same as shown in Equation (15.10). It is clear that for equal grid spacing, the truncation error shown in Equation (15.10) is $O((\Delta x)^2)$ and for an unequal grid, it is $O(\Delta x^+ \Delta x^-)$.

Using Equations (15.1) and (15.2), $\partial^2 \phi / \partial x^2$ can be written as
It can be seen that the truncation error for an unequal grid is \( O(\Delta X_+, \Delta X_-) \) and for an equal grid, \( O[(\Delta X)^2] \).

In a similar fashion, it can be shown that the representations for \( \frac{\partial^2 \phi}{\partial Y^2} \) about \((i + 1, j)\) and \((i, j)\) are first order accurate for an unequal grid, while they become second order accurate for an equal grid. Recently, Blottner [162] argued that such representations for \( \frac{\partial^2 \phi}{\partial Y^2} \) with a geometric grid which was used for turbulent flows in the present study, are actually equivalent to second order in accuracy.

The finite-difference representation of \( \frac{\partial^2 \phi}{\partial X \partial Y} \) can be obtained by a double Taylor series expansion in \( X \) and \( Y \). The procedure provides second-order accuracy, with truncation error \( T.E. = O[(\Delta X)^2 + (\Delta Y)^2] \), which can be demonstrated as follows:

A double Taylor series expansion for \( \phi \) about \((i, j)\) can be written for equal grids as

\[
\frac{\partial^2 \phi}{\partial x^2} = \phi_{i+1}^j + \frac{1}{2!} \left[ \left( \frac{\partial^2 \phi}{\partial x^2} \right)_{i+1}^j \Delta x^2 + 2 \left( \frac{\partial^2 \phi}{\partial x \partial y} \right)_{i+1}^j \Delta x \Delta y + \left( \frac{\partial^2 \phi}{\partial y^2} \right)_{i+1}^j \Delta y^2 \right] + \frac{1}{3!} \left[ \left( \frac{\partial^3 \phi}{\partial x^3} \right)_{i+1}^j \Delta x^3 + 3 \left( \frac{\partial^3 \phi}{\partial x^2 \partial y} \right)_{i+1}^j \Delta x^2 \Delta y + \left( \frac{\partial^3 \phi}{\partial y^3} \right)_{i+1}^j \Delta y^3 \right] + \ldots
\]

(15.11)
\[ + \frac{1}{4!} \left[ \left( \frac{\partial^4 \phi}{\partial x^4} \right)_j \Delta x^4 + 4 \left( \frac{\partial^4 \phi}{\partial x^3 \partial y} \right)_j \Delta x^3 \Delta y + 6 \left( \frac{\partial^4 \phi}{\partial x^2 \partial y^2} \right)_j \Delta x^2 \Delta y^2 + \frac{1}{3} \left( \frac{\partial^4 \phi}{\partial x \partial y^3} \right)_j \Delta x \Delta y^3 + \frac{1}{3} \left( \frac{\partial^4 \phi}{\partial y^4} \right)_j \Delta y^4 \right] + \ldots \]

(15.12)

Similarly, expressions for \( \phi_{j+1} \), \( \phi_{j-1} \) and \( \phi_{-1} \) can be obtained. Solving these for \( \frac{\partial^2 \phi}{\partial x \partial y} \), gives

\[ \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\phi_{j+1} - \phi_{j-1} - \phi_{j+1} + \phi_{j-1}}{4 \Delta x \Delta y} - \frac{2}{3} \left( \frac{\partial^4 \phi}{\partial x^3 \partial y} \right)_j \Delta x^2 + \left( \frac{\partial^4 \phi}{\partial y^4} \right) \Delta y^2 \] + \ldots \]

(15.13)

Since all the truncation errors discussed above vanish when \( \Delta x_+ \to 0 \) and \( \Delta y_+ \to 0 \), it is concluded that the finite-difference representations of Equations (3.16), (3.17), (3.21) and (3.22) are consistent.
XVI. APPENDIX H: DISCUSSION ON THE DIAGONAL DOMINANCE OF THE ALGEBRAIC SYSTEM FOR THE LAPLACE EQUATION

Since the alternating direction implicit (ADI) scheme was used for the inviscid solution, diagonal dominance must be maintained in both the streamwise and normal directions. In the streamwise direction: In Equation (3.21), the absolute value of the diagonal term is $2(1 + B^\psi)$, where $B^\psi$ is defined in Appendix F and is always positive. The sum of the absolute value of the off-diagonal terms is 2. Thus,

$$|\text{the diagonal term}| > \Sigma|\text{the off-diagonal term}|$$

In the normal direction: Similarly, from Equation (3.21),

$$|\text{the diagonal term}| = 2(1 + B^\psi)$$

and

$$\Sigma|\text{the off-diagonal term}| = |B^\psi + C^\psi| + |B^\psi - C^\psi|$$

$$= \begin{cases} 
2|B^\psi|, & \text{if } |B^\psi| \geq |C^\psi| \\
2|C^\psi|, & \text{if } |B^\psi| < |C^\psi| 
\end{cases}$$

Since $B^\psi$ is always positive, $|1 + B^\psi|$ is always greater than $|B^\psi|$. Thus, a comparison between $|1 + B^\psi|$ and $C^\psi$ must be made. Using Equation (14.3)

$$(1 + B^\psi)^2 = 1 + 2B^\psi + B^\psi^2$$

$$= 1 + \frac{2}{(2\Delta^n_{UL})^2} (2\Delta^n_{e})^2 + \delta^i_{L} - \delta^{i-1}_{L} + 1$$
Comparing the fifth term on the right-hand side of Equation (16.1) with the first term on the right-hand side of Equation (16.2) gives

\[ \frac{1}{(2\Delta\eta)^2 \Delta^4} \left[ \left( \frac{\delta_{\text{UL}}^{*i+1} - \delta_{\text{UL}}^{*i-1}}{\delta_{\text{UL}}^{*i}} \right)^2 \left( \delta_{L}^{*i+1} - \delta_{L}^{*i-1} + \gamma_j (\delta_{\text{UL}}^{*i+1} - \delta_{\text{UL}}^{*i-1}) \right)^2 \right. \\
- \left. \frac{1}{(\Delta\xi)^2} \left( \delta_{\text{UL}}^{*i+1} - \delta_{\text{UL}}^{*i-1} \right) \left( \delta_{L}^{*i+1} - \delta_{L}^{*i-1} + \gamma_j (\delta_{\text{UL}}^{*i+1} - \delta_{\text{UL}}^{*i-1}) \right) \left( \delta_{L}^{*i+1} - \delta_{L}^{*i-1} \right) \right] \\
+ \frac{1}{(\Delta\xi)^2} \left( \delta_{L}^{*i+1} - 2\delta_{L}^{*i} + \delta_{L}^{*i-1} + \gamma_j (\delta_{\text{UL}}^{*i+1} - 2\delta_{\text{UL}}^{*i} + \delta_{\text{UL}}^{*i-1}) \right)^2 \]

(16.2)

Comparing the fifth term on the right-hand side of Equation (16.1) with the first term on the right-hand side of Equation (16.2) gives

\[ \frac{1}{8(\Delta\eta)^2 \Delta^4} \left( \delta_{\text{UL}}^{*i+1} - \delta_{\text{UL}}^{*i-1} \right)^2 \left( \delta_{L}^{*i+1} - \delta_{L}^{*i-1} + \gamma_j (\delta_{\text{UL}}^{*i+1} - \delta_{\text{UL}}^{*i-1}) \right)^2 \\
- \frac{2\Delta^4 \xi e}{\Delta} \left( \delta_{\text{UL}}^{*i+1} - \delta_{\text{UL}}^{*i-1} \right)^2 \]

(16.3)
Here, $\Delta g$ and $\Delta L$ are of the same order of magnitude. $(\Delta^*_{UL} - \Delta^*_{UL-1})$ is more or less of the same order of magnitude as $\Delta g$, which is very small compared to $L_e$. Thus, it can be said that Equation (16.3) remains positive; in other words, the fifth term on the right-hand side of Equation (16.1) is greater than the first term on the same side of Equation (16.2). Similarly, the comparison of the second and third terms on the same side of Equation (16.1), respectively, results in the conclusion that the former is larger than the latter. Consequently, it can be said that

$$|1 + B_\psi| > |C_\psi|$$

The inequality means that diagonal dominance is satisfied also in the normal direction.
XVII. APPENDIX I: COEFFICIENTS IN THE EQUATIONS OBTAINED
BY THE MODIFIED THOMAS ALGORITHM

The finite-difference equations for the boundary-layer equations, Equations (3.19) and (3.20), result in a system of block tridiagonal linear equations, Equation (4.2). The system of equations are solved by using the modified Thomas algorithm [162]. The following are the coefficients in the resulting equations, Equations (4.3) and (4.4):

\[ A'_j = - \frac{A_j}{q_1} \]
\[ B'_j = A'_j q_2 \]
\[ C'_j = \frac{1}{q_1} \left[ C_j - B'_j C'_{j-1} - E_j (b_j C'_{j-1} + E'_j) \right] \]
\[ D'_j = b_j H'_j + D'_{j-1} + q_2 H'_j \]
\[ E'_j = b_j C'_j + E'_j + C'_j q_2 \]
\[ H'_j = \frac{1}{q_1} \left[ H_j - B'_j H'_{j-1} - E_j (b'_j H'_{j-1} + D'_{j-1}) \right] \]
\[ q_1 = D_j + A'_{j-1} (b_j E_j + B_j) + E_j (b'_j - 1 + d_j) \]

and
\[ q_2 = d_j + b_j A'_{j-1} + B'_j \]

These coefficients are computed starting from the point nearest the wall \( j = 2 \) and continuing to the edge of the computation domain \( j = NJ \).
XVIII. APPENDIX J: EVALUATION OF THE VELOCITY AND
STREAMFUNCTION AT THE OUTER BOUNDARY, AND
THE PRESSURE GRADIENT FOR VISCOUS FLOWS

For the different outer boundary conditions discussed in
Section III.A.1, the velocity and streamfunction at the outer boundary,
and the pressure gradient are obtained as follows:

1. For the external boundary-layer flow. Since an inverse
procedure was discussed in the text (see Section IV.A), a direct
solution procedure is discussed here.
   a. The direct solution procedure. When the pressure gradient
or the edge velocity is specified, the streamfunction at the outer
edge must be obtained before starting the back-substitution process.

   With the given boundary condition, $U_{NJ-1}$ and $\psi_{NJ-1}$ can be ob­
tained by using Equations (4.5) and (4.6). $\psi_{NJ}$ is thus obtained from
Equation (4.9).

2. For two-dimensional internal flows.
   a. A symmetric channel flow. When Equation (3.8) is used as
the outer boundary condition, the centerline velocity, $U_{NJ}$, and the
pressure gradient, $\chi$, are obtained as the following.

   Since $\partial U/\partial Y$ can be represented as

\[
\frac{\partial U}{\partial Y}_{NJ} \simeq \frac{U_{i+1}^{NJ} - U_{i-1}^{NJ}}{2} - \frac{1}{\Delta Y} \left( \frac{\Delta Y}{\Delta Y} \right)^{-2} + \frac{U_{i+1}^{NJ-2}}{2\Delta Y} - \frac{U_{i+1}^{NJ-1}}{2\Delta Y} \tag{18.1}
\]

   where $\Delta Y = Y_{NJ} - Y_{NJ-1}$ and $\Delta Y = Y_{NJ-1} - Y_{NJ-2}$.

Equation (3.8) becomes
Equations (18.2) and (18.3) are to be solved with Equations (4.5), (4.6) and (4.9). However, one additional equation is required, since the number of unknowns is 6, while the number of equation is 5. The additional equation can be obtained from Equation (4.3) by writing it for \( j = NJ - 2 \):

\[
U_{NJ-2} = A'_{NJ-2} U_{NJ-1} + H'_{NJ-2} X + C'_{NJ-2}
\]  

(18.4)

Solving Equations (18.2)-(18.4) with Equations (4.5), (4.6) and (4.9) for \( X \) gives

\[
X^{i+1} = \frac{\kappa_{i+1} m_3 - \kappa_{i+1} m_1}{\kappa_{i+1} m_1 - \kappa_{i+1} m_2}
\]  

(18.5)

where

\[
\kappa_1 = 1 - A'_{NJ-1}(c_1 - c_2 A'_{NJ-2})
\]

\[
\kappa_2 = (c_1 - c_2 A'_{NJ-2}) H'_{NJ-1} - c_2 H'_{NJ-2}
\]

\[
\kappa_3 = (c_1 - c_2 A'_{NJ-2}) C'_{NJ-1} - c_2 C'_{NJ-2}
\]

\[
m_1 = 1 + \frac{2}{\Delta Y} - B'_{NJ-1} + A'_{NJ-1}
\]

\[
m_2 = - (H'_{NJ-1} + \frac{2}{\Delta Y} - D'_{NJ-1})
\]

and

\[
m_3 = \frac{\psi_T}{\Delta Y} - \frac{2}{\Delta Y} - E'_{NJ-1} - C'_{NJ-1}
\]
Thus, $U_{NJ}$ can be calculated from

$$U_{NJ}^{i+1} = \frac{\delta}{2} \left( \frac{U_{1m3} - U_{3m1}}{\delta_{2m1} - \delta_{1m2}} + \frac{\delta}{\delta^2} \right)$$  \hfill (18.7)

When Equation (3.9) is used as the outer boundary condition, the centerline velocity, $U_{NJ}$, and the pressure gradient, $\chi$, are obtained by solving

$$U_{NJ}^{i+1} \left( \frac{H}{2} + \delta_{k1} \right) = \frac{T}{2}$$  \hfill (18.8)

with Equations (4.5), (4.6), and (4.8). The solutions are

$$\chi^{i+1} = \frac{(2U_{NJ}^{i+1} - U_{NJ}^{i})F_3 - (U_{NJ}^{i+1})^2}{\Delta x + (2U_{NJ}^{i+1} - U_{NJ}^{i})F_2}$$  \hfill (18.9)

and

$$U_{NJ}^{i+1} = \left( \frac{F_2}{F_1} \right) \chi^{i+1} + \left( \frac{F_3}{F_1} \right)$$  \hfill (18.10)

Here,

$$F_1 = \frac{H}{2} - \frac{V_{NJ}}{\Delta Y} + \frac{\Delta Y}{2} (1 + A'_{NJ-1})$$

$$F_2 = - \frac{D'_{NJ-1} - \frac{\Delta Y}{2}}{H_{NJ-1}}$$  \hfill (18.11)

and

$$F_3 = \frac{\psi_T}{\Delta Y} - \frac{\Delta Y}{2} C'_{NJ-1}$$

b. An asymmetric channel flow. The finite-difference representation of Equation (3.10) is

$$U_{NJ}^{i+1} = 0$$  \hfill (18.12)

and

$$\psi_{NJ}^{i+1} = \psi_T$$  \hfill (18.13)
Solving Equations (18.12) and (18.13) with Equations (4.5), (4.6) and (4.9) gives

\[
\chi = \frac{\Delta Y}{\Delta Y} \left( \frac{\psi_T}{2} \right)_{nj} - E'_{nj-1} - E'_{nj-1} \\
= \frac{\Delta Y}{\Delta Y} \left( \frac{\psi_T}{2} \right)_{nj} + D'_{nj-1}
\]

(18.14)
When a different numerical scheme is used for the boundary layer solution, the edge velocity, the edge stream function, and the pressure gradient can still be obtained in a manner similar to that discussed in Appendix I. In this section, a few numerical schemes discussed in Appendix E are considered.

1. Crank-Nicolson scheme with Newton linearization. When the Crank-Nicolson scheme is used, the pressure gradient $\chi$ is written in terms of $U_{N_J}$ as

$$\chi^{i+1} = \left( \frac{\Delta X}{\Delta X^+} \right) U_{N_J}^{i+1} - \frac{(U_{N_J}^{i+1})^2 + (U_{N_J}^i)^2}{2 \Delta X^+}$$

(19.1)

For the inverse method, Equation (19.1) is to be solved with Equations (4.5)-(4.7) and (4.9) for the pressure gradient. The pressure gradient $\chi$ then, becomes

$$\chi^{i+1} = \left( \frac{F_3}{F_1} \right) (U_{N_J}^{i+1})^2 - \frac{1}{2} \left( (U_{N_J}^{i+1})^2 + (U_{N_J}^i)^2 \right)$$

$$\Delta X^+ - \left( \frac{F_2}{F_1} \right) U_{N_J}^{i+1}$$

(19.2)

where

$$F_1 = Y_{N_J} - \frac{\Delta Y}{2} + \frac{\Delta Y}{2} B_{N_J-1} + \left( \frac{\Delta Y}{2} \right) A_{N_J-1}$$

$$F_2 = D_{N_J-1} + \left( \frac{\Delta Y}{2} \right) H_{N_J-1}$$

$$F_3 = E_{N_J-1} + \left( \frac{\Delta Y}{2} \right) C_{N_J-1}$$
with
\[ \Delta Y = Y_{N+1} - Y_{N-1} \]

\[ \frac{Y_{N+1} - Y_{N-1}}{\Delta X} \]

**Un** can be obtained by using
\[ u_{nj}^{i+1} = \left( \frac{F_2}{F_1} \right) \Delta Y + \left( \frac{F_3}{F_1} \right) \]

(19.3)

For the direct method, the streamfunction \( \psi_{nj} \) can be obtained by following the procedure discussed in Appendix J.

2. **Fully implicit scheme with lagged coefficients.** In this scheme, the pressure gradient is written in terms of \( U_{nj} \) as
\[ \frac{U_{nj}^{i+1} - U_{nj}^i}{\Delta X} \]

(19.4)

Similarly, for the inverse method, the above equation, Equation (19.4), is solved with Equations (4.5), (4.7) and (4.9) for the edge velocity. The result is
\[ u_{nj}^{i+1} = \frac{F_2}{F_1} \]

(19.5)

where
\[ F_1 = \left( \frac{\Delta Y}{2} \right) \left\{ 1 + A_{nj}^i + \frac{U_{nj}^i}{\Delta X} \right\} - \left( Y_{nj} - \Delta Y + 1 \right) \]
\[ + B_{nj}^i + D_{nj}^i \left( \frac{U_{nj}^i}{\Delta X} \right) \]
\[ F_2 = E_{nj}^i + \frac{(U_{nj}^i)^2}{2} - \frac{\Delta Y}{2} \left( \frac{(U_{nj}^i)^2}{\Delta X} - C_{nj}^i \right) \]

with
\[ \Delta Y = Y_{nj}^{i+1} - Y_{nj}^{i+1} \]
The pressure gradient \( \chi \) is, then, obtained by using Equation (19.4).

The solution procedure for the direct method is the same as case 1 discussed above.

It is interesting to note that no iterations are required when using the fully implicit scheme with lagged coefficients, even when reversed flow is present.
XX. APPENDIX L: TABULATION OF SOME TYPICAL TEST CASES
Table 20.1. Predicted velocity and streamfunction profiles in the separated flow region for laminar channel flow in a symmetric sudden expansion \((Re_H = 50, h/H_0 = 0.5)\) at \(x/h = 1.0\): fully developed inlet velocity profile

<table>
<thead>
<tr>
<th>(\frac{2v}{H_0})</th>
<th>(\frac{u}{u_{1,AVG}})</th>
<th>(\frac{2\psi}{\psi_T})</th>
<th>(\frac{2v}{H_0})</th>
<th>(\frac{u}{u_{1,AVG}})</th>
<th>(\frac{2\psi}{\psi_T})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.525</td>
<td>0.37011</td>
<td>0.00810</td>
</tr>
<tr>
<td>0.025</td>
<td>-0.02823</td>
<td>-0.00071</td>
<td>0.550</td>
<td>0.44427</td>
<td>0.02847</td>
</tr>
<tr>
<td>0.050</td>
<td>-0.05211</td>
<td>-0.00272</td>
<td>0.575</td>
<td>0.52289</td>
<td>0.05267</td>
</tr>
<tr>
<td>0.075</td>
<td>-0.07169</td>
<td>-0.00581</td>
<td>0.600</td>
<td>0.60497</td>
<td>0.08088</td>
</tr>
<tr>
<td>0.100</td>
<td>-0.08701</td>
<td>-0.00978</td>
<td>0.625</td>
<td>0.68918</td>
<td>0.11326</td>
</tr>
<tr>
<td>0.125</td>
<td>-0.09808</td>
<td>-0.01441</td>
<td>0.650</td>
<td>0.77399</td>
<td>0.14986</td>
</tr>
<tr>
<td>0.150</td>
<td>-0.10492</td>
<td>-0.01949</td>
<td>0.675</td>
<td>0.85778</td>
<td>0.19069</td>
</tr>
<tr>
<td>0.175</td>
<td>-0.10748</td>
<td>-0.02481</td>
<td>0.700</td>
<td>0.93899</td>
<td>0.23564</td>
</tr>
<tr>
<td>0.200</td>
<td>-0.10568</td>
<td>-0.03014</td>
<td>0.725</td>
<td>1.01628</td>
<td>0.28455</td>
</tr>
<tr>
<td>0.225</td>
<td>-0.09941</td>
<td>-0.03527</td>
<td>0.750</td>
<td>1.08858</td>
<td>0.33721</td>
</tr>
<tr>
<td>0.250</td>
<td>-0.08852</td>
<td>-0.03997</td>
<td>0.775</td>
<td>1.15509</td>
<td>0.39334</td>
</tr>
<tr>
<td>0.275</td>
<td>-0.07286</td>
<td>-0.04401</td>
<td>0.800</td>
<td>1.21525</td>
<td>0.45263</td>
</tr>
<tr>
<td>0.300</td>
<td>-0.05226</td>
<td>-0.04714</td>
<td>0.825</td>
<td>1.26872</td>
<td>0.51478</td>
</tr>
<tr>
<td>0.325</td>
<td>-0.02662</td>
<td>-0.04911</td>
<td>0.850</td>
<td>1.31527</td>
<td>0.57942</td>
</tr>
<tr>
<td>0.350</td>
<td>-0.00415</td>
<td>-0.04967</td>
<td>0.875</td>
<td>1.35478</td>
<td>0.64622</td>
</tr>
<tr>
<td>0.375</td>
<td>-0.04004</td>
<td>-0.04857</td>
<td>0.900</td>
<td>1.38716</td>
<td>0.71481</td>
</tr>
<tr>
<td>0.400</td>
<td>0.08119</td>
<td>-0.04554</td>
<td>0.925</td>
<td>1.41237</td>
<td>0.78485</td>
</tr>
<tr>
<td>0.425</td>
<td>0.12775</td>
<td>-0.04031</td>
<td>0.950</td>
<td>1.43040</td>
<td>0.85596</td>
</tr>
<tr>
<td>0.450</td>
<td>0.17986</td>
<td>-0.03261</td>
<td>0.975</td>
<td>1.44122</td>
<td>0.92780</td>
</tr>
<tr>
<td>0.475</td>
<td>0.23764</td>
<td>-0.02217</td>
<td>1.000</td>
<td>1.44483</td>
<td>1.00000</td>
</tr>
<tr>
<td>0.500</td>
<td>0.30111</td>
<td>-0.00869</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 20.2. Predicted velocity and streamfunction profiles in the redeveloping flow region for laminar channel flow in a symmetric sudden expansion ($Re_h = 50, h/H_1 = 0.5$) at $x/h = 7.5$: fully developed inlet velocity profile

<table>
<thead>
<tr>
<th>$2\nu H_0$</th>
<th>$u_{1,AVG}$</th>
<th>$2\psi H_0$</th>
<th>$u_{1,AVG}$</th>
<th>$2\psi H_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.525</td>
<td>0.51676</td>
<td>0.19927</td>
</tr>
<tr>
<td>0.025</td>
<td>0.00438</td>
<td>0.00111</td>
<td>0.550</td>
<td>0.55735</td>
</tr>
<tr>
<td>0.050</td>
<td>0.01092</td>
<td>0.00496</td>
<td>0.575</td>
<td>0.59804</td>
</tr>
<tr>
<td>0.075</td>
<td>0.01961</td>
<td>0.0126</td>
<td>0.600</td>
<td>0.63856</td>
</tr>
<tr>
<td>0.100</td>
<td>0.03046</td>
<td>0.00251</td>
<td>0.625</td>
<td>0.67862</td>
</tr>
<tr>
<td>0.125</td>
<td>0.04348</td>
<td>0.00436</td>
<td>0.650</td>
<td>0.71791</td>
</tr>
<tr>
<td>0.150</td>
<td>0.05867</td>
<td>0.00691</td>
<td>0.675</td>
<td>0.75615</td>
</tr>
<tr>
<td>0.175</td>
<td>0.07602</td>
<td>0.01028</td>
<td>0.700</td>
<td>0.79303</td>
</tr>
<tr>
<td>0.200</td>
<td>0.09552</td>
<td>0.01457</td>
<td>0.725</td>
<td>0.82825</td>
</tr>
<tr>
<td>0.225</td>
<td>0.11716</td>
<td>0.01990</td>
<td>0.750</td>
<td>0.86153</td>
</tr>
<tr>
<td>0.250</td>
<td>0.14091</td>
<td>0.02635</td>
<td>0.775</td>
<td>0.89259</td>
</tr>
<tr>
<td>0.275</td>
<td>0.16672</td>
<td>0.03405</td>
<td>0.800</td>
<td>0.92115</td>
</tr>
<tr>
<td>0.300</td>
<td>0.19454</td>
<td>0.04308</td>
<td>0.825</td>
<td>0.94699</td>
</tr>
<tr>
<td>0.325</td>
<td>0.22430</td>
<td>0.05356</td>
<td>0.850</td>
<td>0.96986</td>
</tr>
<tr>
<td>0.350</td>
<td>0.25591</td>
<td>0.06558</td>
<td>0.875</td>
<td>0.98957</td>
</tr>
<tr>
<td>0.375</td>
<td>0.28925</td>
<td>0.07921</td>
<td>0.900</td>
<td>1.00593</td>
</tr>
<tr>
<td>0.400</td>
<td>0.32418</td>
<td>0.09456</td>
<td>0.925</td>
<td>1.01881</td>
</tr>
<tr>
<td>0.425</td>
<td>0.36057</td>
<td>0.11169</td>
<td>0.950</td>
<td>1.02809</td>
</tr>
<tr>
<td>0.450</td>
<td>0.39823</td>
<td>0.13067</td>
<td>0.975</td>
<td>1.03368</td>
</tr>
<tr>
<td>0.475</td>
<td>0.43696</td>
<td>0.15157</td>
<td>1.000</td>
<td>1.03553</td>
</tr>
<tr>
<td>0.500</td>
<td>0.47655</td>
<td>0.17442</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 20.3. Prediction of flow development for laminar channel flow downstream of a symmetric sudden expansion ($hu_{i,\text{max}}/v = 56$, $h/H_{i} = 1.0$)

<table>
<thead>
<tr>
<th>$x/h$</th>
<th>$\frac{u_{CL}}{u_{i,\text{max}}}$</th>
<th>$c_{f} \times 10^{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0236</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0625</td>
<td>1.0156</td>
<td>-0.2765</td>
</tr>
<tr>
<td>0.1250</td>
<td>1.0073</td>
<td>-0.5264</td>
</tr>
<tr>
<td>0.1875</td>
<td>0.9989</td>
<td>-0.7555</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.9902</td>
<td>-0.9653</td>
</tr>
<tr>
<td>0.3750</td>
<td>0.9722</td>
<td>-1.3313</td>
</tr>
<tr>
<td>0.5000</td>
<td>0.9536</td>
<td>-1.6333</td>
</tr>
<tr>
<td>0.7500</td>
<td>0.9151</td>
<td>-2.0778</td>
</tr>
<tr>
<td>1.0000</td>
<td>0.8759</td>
<td>-2.3501</td>
</tr>
<tr>
<td>1.2500</td>
<td>0.8370</td>
<td>-2.5041</td>
</tr>
<tr>
<td>1.5000</td>
<td>0.7987</td>
<td>-2.7241</td>
</tr>
<tr>
<td>2.0000</td>
<td>0.7249</td>
<td>-1.7869</td>
</tr>
<tr>
<td>2.5000</td>
<td>0.6564</td>
<td>-1.1627</td>
</tr>
<tr>
<td>3.00</td>
<td>0.5949</td>
<td>-0.5247</td>
</tr>
<tr>
<td>3.50</td>
<td>0.5421</td>
<td>0.0207</td>
</tr>
<tr>
<td>4.00</td>
<td>0.4988</td>
<td>0.4467</td>
</tr>
<tr>
<td>5.0</td>
<td>0.4369</td>
<td>0.7709</td>
</tr>
<tr>
<td>6.0</td>
<td>0.3996</td>
<td>1.1889</td>
</tr>
<tr>
<td>7.0</td>
<td>0.3781</td>
<td>1.4110</td>
</tr>
<tr>
<td>8.0</td>
<td>0.3659</td>
<td>1.5277</td>
</tr>
<tr>
<td>10.0</td>
<td>0.3556</td>
<td>1.6215</td>
</tr>
<tr>
<td>12.0</td>
<td>0.3527</td>
<td>1.6473</td>
</tr>
<tr>
<td>14.0</td>
<td>0.3518</td>
<td>1.6551</td>
</tr>
<tr>
<td>16.0</td>
<td>0.3515</td>
<td>1.6573</td>
</tr>
<tr>
<td>20.0</td>
<td>0.3514</td>
<td>1.6583</td>
</tr>
<tr>
<td>25.0</td>
<td>0.3514</td>
<td>1.6583</td>
</tr>
<tr>
<td>30.0</td>
<td>0.3514</td>
<td>1.6583</td>
</tr>
<tr>
<td>35.0</td>
<td>0.3514</td>
<td>1.6583</td>
</tr>
<tr>
<td>40.0</td>
<td>0.3514</td>
<td>1.6583</td>
</tr>
</tbody>
</table>
Table 20.4. Prediction of laminar flow over a rearward-facing step with viscous-inviscid interaction (\(hu_{\max}/\nu = 412, h = 0.01016 m\))

<table>
<thead>
<tr>
<th>(x/h)</th>
<th>(u_{e,L}/u_{e,0})</th>
<th>(u_{e,U}/u_{e,0})</th>
<th>(\delta^*_L/h)</th>
<th>(\delta^*_U/h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.5</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.3335</td>
<td>0.3335</td>
</tr>
<tr>
<td>12.6</td>
<td>1.0015</td>
<td>1.0010</td>
<td>0.3405</td>
<td>0.3425</td>
</tr>
<tr>
<td>11.7</td>
<td>1.0031</td>
<td>1.0018</td>
<td>0.3476</td>
<td>0.3512</td>
</tr>
<tr>
<td>10.8</td>
<td>1.0046</td>
<td>1.0026</td>
<td>0.3545</td>
<td>0.3598</td>
</tr>
<tr>
<td>9.9</td>
<td>1.0062</td>
<td>1.0033</td>
<td>0.3612</td>
<td>0.3682</td>
</tr>
<tr>
<td>9.0</td>
<td>1.0078</td>
<td>1.0040</td>
<td>0.3676</td>
<td>0.3765</td>
</tr>
<tr>
<td>8.1</td>
<td>1.0094</td>
<td>1.0045</td>
<td>0.3737</td>
<td>0.3847</td>
</tr>
<tr>
<td>7.2</td>
<td>1.0111</td>
<td>1.0050</td>
<td>0.3795</td>
<td>0.3930</td>
</tr>
<tr>
<td>6.3</td>
<td>1.0129</td>
<td>1.0054</td>
<td>0.3847</td>
<td>0.4011</td>
</tr>
<tr>
<td>5.4</td>
<td>1.0148</td>
<td>1.0057</td>
<td>0.3897</td>
<td>0.4092</td>
</tr>
<tr>
<td>4.5</td>
<td>1.0169</td>
<td>1.0060</td>
<td>0.3936</td>
<td>0.4174</td>
</tr>
<tr>
<td>3.6</td>
<td>1.0191</td>
<td>1.0062</td>
<td>0.3977</td>
<td>0.4257</td>
</tr>
<tr>
<td>2.7</td>
<td>1.0219</td>
<td>1.0063</td>
<td>0.3994</td>
<td>0.4340</td>
</tr>
<tr>
<td>1.8</td>
<td>1.0246</td>
<td>1.0062</td>
<td>0.4022</td>
<td>0.4425</td>
</tr>
<tr>
<td>0.9</td>
<td>1.0285</td>
<td>1.0061</td>
<td>0.4001</td>
<td>0.4511</td>
</tr>
<tr>
<td>0.0</td>
<td>1.0324</td>
<td>1.0059</td>
<td>0.4004</td>
<td>0.4598</td>
</tr>
<tr>
<td>0.9</td>
<td>1.0321</td>
<td>1.0056</td>
<td>1.3894</td>
<td>0.4688</td>
</tr>
<tr>
<td>1.8</td>
<td>1.0309</td>
<td>1.0051</td>
<td>1.3725</td>
<td>0.4778</td>
</tr>
<tr>
<td>2.7</td>
<td>1.0292</td>
<td>1.0046</td>
<td>1.3533</td>
<td>0.4870</td>
</tr>
<tr>
<td>3.6</td>
<td>1.0270</td>
<td>1.0041</td>
<td>1.3285</td>
<td>0.4964</td>
</tr>
<tr>
<td>4.5</td>
<td>1.0242</td>
<td>1.0034</td>
<td>1.3016</td>
<td>0.5059</td>
</tr>
<tr>
<td>5.4</td>
<td>1.0210</td>
<td>1.0027</td>
<td>1.2696</td>
<td>0.5155</td>
</tr>
<tr>
<td>6.3</td>
<td>1.0172</td>
<td>1.0018</td>
<td>1.2360</td>
<td>0.5252</td>
</tr>
<tr>
<td>7.2</td>
<td>1.0129</td>
<td>1.0010</td>
<td>1.1981</td>
<td>0.5350</td>
</tr>
<tr>
<td>8.1</td>
<td>1.0082</td>
<td>1.0001</td>
<td>1.1591</td>
<td>0.5448</td>
</tr>
<tr>
<td>9.0</td>
<td>1.0030</td>
<td>0.9991</td>
<td>1.1170</td>
<td>0.5546</td>
</tr>
<tr>
<td>9.9</td>
<td>0.9976</td>
<td>0.9981</td>
<td>1.0748</td>
<td>0.5644</td>
</tr>
<tr>
<td>10.8</td>
<td>0.9920</td>
<td>0.9971</td>
<td>1.0312</td>
<td>0.5740</td>
</tr>
<tr>
<td>11.7</td>
<td>0.9867</td>
<td>0.9962</td>
<td>0.9891</td>
<td>0.5835</td>
</tr>
<tr>
<td>12.6</td>
<td>0.9818</td>
<td>0.9952</td>
<td>0.9481</td>
<td>0.5928</td>
</tr>
<tr>
<td>13.5</td>
<td>0.9777</td>
<td>0.9942</td>
<td>0.9101</td>
<td>0.6019</td>
</tr>
<tr>
<td>14.4</td>
<td>0.9743</td>
<td>0.9933</td>
<td>0.8753</td>
<td>0.6109</td>
</tr>
<tr>
<td>15.3</td>
<td>0.9719</td>
<td>0.9923</td>
<td>0.8443</td>
<td>0.6192</td>
</tr>
<tr>
<td>16.2</td>
<td>0.9702</td>
<td>0.9915</td>
<td>0.8171</td>
<td>0.6274</td>
</tr>
<tr>
<td>17.1</td>
<td>0.9693</td>
<td>0.9906</td>
<td>0.7935</td>
<td>0.6353</td>
</tr>
<tr>
<td>18.0</td>
<td>0.9689</td>
<td>0.9899</td>
<td>0.7735</td>
<td>0.6427</td>
</tr>
<tr>
<td>18.9</td>
<td>0.9689</td>
<td>0.9892</td>
<td>0.7566</td>
<td>0.6498</td>
</tr>
<tr>
<td>19.8</td>
<td>0.9693</td>
<td>0.9884</td>
<td>0.7426</td>
<td>0.6566</td>
</tr>
<tr>
<td>20.7</td>
<td>0.9699</td>
<td>0.9878</td>
<td>0.7311</td>
<td>0.6630</td>
</tr>
<tr>
<td>21.6</td>
<td>0.9707</td>
<td>0.9872</td>
<td>0.7218</td>
<td>0.6691</td>
</tr>
<tr>
<td>22.5</td>
<td>0.9715</td>
<td>0.9867</td>
<td>0.7143</td>
<td>0.6746</td>
</tr>
</tbody>
</table>
Table 20.4. Continued

<table>
<thead>
<tr>
<th>x</th>
<th>$u_{e,L}$</th>
<th>$u_{e,U}$</th>
<th>$\frac{\xi_h}{h}$</th>
<th>$\frac{\xi_U}{h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.4</td>
<td>0.9725</td>
<td>0.9862</td>
<td>0.7085</td>
<td>0.6800</td>
</tr>
<tr>
<td>24.3</td>
<td>0.9734</td>
<td>0.9858</td>
<td>0.7041</td>
<td>0.6850</td>
</tr>
<tr>
<td>25.2</td>
<td>0.9744</td>
<td>0.9855</td>
<td>0.7008</td>
<td>0.6898</td>
</tr>
<tr>
<td>26.1</td>
<td>0.9753</td>
<td>0.9851</td>
<td>0.6986</td>
<td>0.6943</td>
</tr>
<tr>
<td>27.0</td>
<td>0.9763</td>
<td>0.9849</td>
<td>0.6971</td>
<td>0.6986</td>
</tr>
<tr>
<td>27.9</td>
<td>0.9772</td>
<td>0.9847</td>
<td>0.6964</td>
<td>0.7025</td>
</tr>
<tr>
<td>28.8</td>
<td>0.9781</td>
<td>0.9845</td>
<td>0.6962</td>
<td>0.7065</td>
</tr>
<tr>
<td>29.7</td>
<td>0.9789</td>
<td>0.9844</td>
<td>0.6965</td>
<td>0.7101</td>
</tr>
<tr>
<td>30.6</td>
<td>0.9798</td>
<td>0.9844</td>
<td>0.6972</td>
<td>0.7137</td>
</tr>
<tr>
<td>31.5</td>
<td>0.9806</td>
<td>0.9843</td>
<td>0.6983</td>
<td>0.7172</td>
</tr>
<tr>
<td>32.4</td>
<td>0.9814</td>
<td>0.9843</td>
<td>0.6995</td>
<td>0.7207</td>
</tr>
<tr>
<td>33.3</td>
<td>0.9821</td>
<td>0.9843</td>
<td>0.7010</td>
<td>0.7239</td>
</tr>
<tr>
<td>34.2</td>
<td>0.9829</td>
<td>0.9843</td>
<td>0.7026</td>
<td>0.7272</td>
</tr>
<tr>
<td>35.1</td>
<td>0.9836</td>
<td>0.9843</td>
<td>0.7043</td>
<td>0.7304</td>
</tr>
<tr>
<td>36.0</td>
<td>0.9844</td>
<td>0.9844</td>
<td>0.7061</td>
<td>0.7336</td>
</tr>
</tbody>
</table>
Table 20.5. Prediction of turbulent flow over a rearward-facing step with viscous-inviscid interaction (reference flow): the $k-\varepsilon$ turbulence model

<table>
<thead>
<tr>
<th>$x/h$</th>
<th>$u_{e,L}/u_{e,0}$</th>
<th>$u_{e,U}/u_{e,0}$</th>
<th>$e^*/L/h$</th>
<th>$e^*/U/h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4.0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.0265</td>
<td>0.0265</td>
</tr>
<tr>
<td>-3.6</td>
<td>1.0012</td>
<td>1.0012</td>
<td>0.0277</td>
<td>0.0276</td>
</tr>
<tr>
<td>-3.2</td>
<td>1.0023</td>
<td>1.0024</td>
<td>0.0288</td>
<td>0.0288</td>
</tr>
<tr>
<td>-2.8</td>
<td>1.0033</td>
<td>1.0035</td>
<td>0.0298</td>
<td>0.0298</td>
</tr>
<tr>
<td>-2.4</td>
<td>1.0045</td>
<td>1.0045</td>
<td>0.0309</td>
<td>0.0309</td>
</tr>
<tr>
<td>-2.0</td>
<td>1.0058</td>
<td>1.0056</td>
<td>0.0320</td>
<td>0.0320</td>
</tr>
<tr>
<td>-1.6</td>
<td>1.0070</td>
<td>1.0066</td>
<td>0.0330</td>
<td>0.0331</td>
</tr>
<tr>
<td>-1.2</td>
<td>1.0085</td>
<td>1.0073</td>
<td>0.0340</td>
<td>0.0341</td>
</tr>
<tr>
<td>-0.8</td>
<td>1.0096</td>
<td>1.0080</td>
<td>0.0350</td>
<td>0.0352</td>
</tr>
<tr>
<td>-0.4</td>
<td>1.0149</td>
<td>1.0084</td>
<td>0.0355</td>
<td>0.0363</td>
</tr>
<tr>
<td>0.0</td>
<td>1.0149</td>
<td>1.0086</td>
<td>0.0375</td>
<td>0.0375</td>
</tr>
<tr>
<td>0.4</td>
<td>1.0099</td>
<td>1.0084</td>
<td>1.0314</td>
<td>0.0387</td>
</tr>
<tr>
<td>0.8</td>
<td>1.0098</td>
<td>1.0079</td>
<td>1.0299</td>
<td>0.0399</td>
</tr>
<tr>
<td>1.2</td>
<td>1.0098</td>
<td>1.0071</td>
<td>1.0277</td>
<td>0.0411</td>
</tr>
<tr>
<td>1.6</td>
<td>1.0097</td>
<td>1.0061</td>
<td>1.0251</td>
<td>0.0424</td>
</tr>
<tr>
<td>2.0</td>
<td>1.0093</td>
<td>1.0047</td>
<td>1.0219</td>
<td>0.0438</td>
</tr>
<tr>
<td>2.4</td>
<td>1.0089</td>
<td>1.0030</td>
<td>1.0181</td>
<td>0.0452</td>
</tr>
<tr>
<td>2.8</td>
<td>1.0083</td>
<td>1.0006</td>
<td>1.0135</td>
<td>0.0468</td>
</tr>
<tr>
<td>3.2</td>
<td>1.0076</td>
<td>0.9976</td>
<td>1.0078</td>
<td>0.0484</td>
</tr>
<tr>
<td>3.6</td>
<td>1.0065</td>
<td>0.9938</td>
<td>1.0010</td>
<td>0.0502</td>
</tr>
<tr>
<td>4.0</td>
<td>1.0052</td>
<td>0.9890</td>
<td>0.9921</td>
<td>0.0522</td>
</tr>
<tr>
<td>4.4</td>
<td>1.0036</td>
<td>0.9830</td>
<td>0.9814</td>
<td>0.0546</td>
</tr>
<tr>
<td>4.8</td>
<td>1.0011</td>
<td>0.9757</td>
<td>0.9674</td>
<td>0.0574</td>
</tr>
<tr>
<td>5.2</td>
<td>0.9976</td>
<td>0.9669</td>
<td>0.9498</td>
<td>0.0607</td>
</tr>
<tr>
<td>5.6</td>
<td>0.9909</td>
<td>0.9564</td>
<td>0.9268</td>
<td>0.0645</td>
</tr>
<tr>
<td>6.0</td>
<td>0.9768</td>
<td>0.9444</td>
<td>0.8963</td>
<td>0.0690</td>
</tr>
<tr>
<td>6.4</td>
<td>0.9538</td>
<td>0.9312</td>
<td>0.8554</td>
<td>0.0742</td>
</tr>
<tr>
<td>6.8</td>
<td>0.9227</td>
<td>0.9175</td>
<td>0.8064</td>
<td>0.0801</td>
</tr>
<tr>
<td>7.2</td>
<td>0.8887</td>
<td>0.9036</td>
<td>0.7504</td>
<td>0.0866</td>
</tr>
<tr>
<td>7.6</td>
<td>0.8619</td>
<td>0.8904</td>
<td>0.6966</td>
<td>0.0934</td>
</tr>
<tr>
<td>8.0</td>
<td>0.8427</td>
<td>0.8780</td>
<td>0.6504</td>
<td>0.1004</td>
</tr>
<tr>
<td>8.4</td>
<td>0.8288</td>
<td>0.8667</td>
<td>0.6098</td>
<td>0.1071</td>
</tr>
<tr>
<td>8.8</td>
<td>0.8200</td>
<td>0.8567</td>
<td>0.5761</td>
<td>0.1133</td>
</tr>
<tr>
<td>9.2</td>
<td>0.8147</td>
<td>0.8481</td>
<td>0.5491</td>
<td>0.1191</td>
</tr>
<tr>
<td>9.6</td>
<td>0.8113</td>
<td>0.8407</td>
<td>0.5267</td>
<td>0.1242</td>
</tr>
<tr>
<td>10.0</td>
<td>0.8092</td>
<td>0.8345</td>
<td>0.5086</td>
<td>0.1287</td>
</tr>
<tr>
<td>10.4</td>
<td>0.8078</td>
<td>0.8292</td>
<td>0.4938</td>
<td>0.1327</td>
</tr>
<tr>
<td>10.8</td>
<td>0.8068</td>
<td>0.8249</td>
<td>0.4815</td>
<td>0.1361</td>
</tr>
<tr>
<td>11.2</td>
<td>0.8061</td>
<td>0.8212</td>
<td>0.4714</td>
<td>0.1390</td>
</tr>
<tr>
<td>11.6</td>
<td>0.8056</td>
<td>0.8182</td>
<td>0.4631</td>
<td>0.1414</td>
</tr>
<tr>
<td>12.0</td>
<td>0.8051</td>
<td>0.8157</td>
<td>0.4560</td>
<td>0.1435</td>
</tr>
</tbody>
</table>
Table 20.5. Continued

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{u_{e,L}}{u_{e,0}} )</th>
<th>( \frac{u_{e,U}}{u_{e,0}} )</th>
<th>( \frac{\delta^*_L}{h} )</th>
<th>( \frac{\delta^*_U}{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.4</td>
<td>0.8046</td>
<td>0.8134</td>
<td>0.4501</td>
<td>0.1451</td>
</tr>
<tr>
<td>12.8</td>
<td>0.8043</td>
<td>0.8116</td>
<td>0.4452</td>
<td>0.1465</td>
</tr>
<tr>
<td>13.2</td>
<td>0.8040</td>
<td>0.8100</td>
<td>0.4411</td>
<td>0.1476</td>
</tr>
<tr>
<td>13.6</td>
<td>0.8038</td>
<td>0.8086</td>
<td>0.4394</td>
<td>0.1486</td>
</tr>
<tr>
<td>14.0</td>
<td>0.8036</td>
<td>0.8075</td>
<td>0.4345</td>
<td>0.1495</td>
</tr>
<tr>
<td>14.4</td>
<td>0.8035</td>
<td>0.8066</td>
<td>0.4320</td>
<td>0.1503</td>
</tr>
<tr>
<td>14.8</td>
<td>0.8034</td>
<td>0.8056</td>
<td>0.4297</td>
<td>0.1510</td>
</tr>
<tr>
<td>15.2</td>
<td>0.8034</td>
<td>0.8049</td>
<td>0.4277</td>
<td>0.1517</td>
</tr>
<tr>
<td>15.6</td>
<td>0.8035</td>
<td>0.8043</td>
<td>0.4259</td>
<td>0.1524</td>
</tr>
<tr>
<td>16.0</td>
<td>0.8036</td>
<td>0.8036</td>
<td>0.4242</td>
<td>0.1532</td>
</tr>
</tbody>
</table>
XXXI. APPENDIX M: COMPUTER CODE "KSTEP"
THIS PROGRAM IS FOR THE VISCOUS-INVISCID INTERACTION METHOD. IN THE METHOD, VISCOUS SOLUTIONS ARE OBTAINED BY SOLVING THE BOUNDARY-LAYER EQUATIONS, AND THE SOLUTIONS ARE MATCHED WITH THE INVISCID SOLUTION WHICH IS OBTAINED BY SOLVING THE LAPLACE EQUATION FOR STREAMFUNCTION. THE BOUNDARY-LAYER EQUATIONS ARE SOLVED INVERSELY IN A COUPLED MANNER. THE LAPLACE EQUATION IS SOLVED USING AN ADI METHOD WITH SOR.

THE FOLLOWING LIST CONTAINS AN EXPLANATION OF ALL NECESSARY INPUT PARAMETERS, QUANTITIES WHICH MUST ALWAYS BE INPUT.

CONDTN CONTROL PARAMETER; IF LESS THAN 0.0, ONLY BOUNDARY-LAYER SOLUTION IS CALCULATED: IF DIRECT SOLUTION OF B.L. EQUATIONS IS REQUIRED, I.E. DISPLACEMENT THICKNESS IS NOT NEEDED TO BE SPECIFIED, THEN CONDTN SHOULD BE LESS THAN -10.0. IF INVERSE SOLN. OF B.L. EQUATIONS IS REQUIRED, I.E. DISPLACEMENT TH. IS REQUIRED TO BE SPECIFIED, THEN IT MUST BE ANY NUMBER BETWEEN -10.0 AND 0.0. ***** IF CONDTN.LT. -10.0, THE DATA FOR XU(I) AND YU(I) FOR I=1, LPOP ARE REQUIRED TO BE PUT. ********

WRITE CONTROL PARAMETER FOR PRINT OUT: IF LARGER THAN 1, ALMOST OF THE DATA EXCEPT FOR THE INTERMEDIATE CALC. RESULTS FOR THE INVISCID SOLN. WILL BE PRINTED OUT.

ITERTN NUMBER OF THE GLOBAL ITERATION. FOR THE START OF THE INTERACTION PROCEDURE IT MUST BE 0.

HIGHT CHANNEL INLET HEIGHT (FT)

RELROR CONVERGENCE CRITERIA

RCONST RELAXATION FACTORS FOR INVISCID SOLUTION

XSTPT,XENDPT START AND END POINTS FOR STORING B.L. SOLNS. THESE SOLNS. ARE USED FOR THE INPUT DATA OF
C INVISCID CALCULATION.  (FT)
C
C UEDST  VELOCITY AT THE INTERACTION STARTING POINT
         (FT/SEC)
C
C DSTOST  DISPLACEMENT THICKNESS AT THE INTERACTION
         START POINT*1000.0  (FT)
C
C RHOINF  DENSITY AT CHANNEL INLET(LBM/FT**3)
C
C XMUINF  VISCOSITY AT CHANNEL INLET(LBM/FT/SEC)
C
C RFLBM  RELAXATION FACTOR FOR UPDATING D*
C
C UEREFO  REFERENCE VELOCITY ( = U AVG.) (FT/SEC)
C
C NADD  NUMBER OF ADDITIONAL D*'S TO DINV OR DOLD
C        IN THE REGION WHERE THE METHOD FOR THE B.L. SOLNS. CHANGES
C        FROM DIRECT TO INVERSE.
C
C NSTOP  CONTROL PARAMETER FOR TERMINATING CALC.;
C        THE INTERACTION CALC. WILL STOP AT NSTOP ITERATIONS.
C
C IPRINT  CONTROL PARAMETER FOR PRINTING DATA OUT.;
C         MOST OF THE PARAMETERS WILL BE OUT AT IPRINT ITERATIONS
C         IF INITIALLY IWRITE WAS SET LESS THAN 0.
C
C NXIC  NUMBER OF DATA POINTS FOR INVISCID SOLN.
C
C NYIC  NO. OF GRID POINTS IN Y-DIRECTION FOR
C        THE INVISCID SOLN.
C
C ICON1  IF EQUAL TO 1, BOUNDARY LAYER SOLUTIONS
C        WILL NOT CALCULATED. THE INVISCID SOLN. WILL BE OBTAINED
C        USING THE INPUT.
C
C ICON2  IF EQUAL TO 1, THE DISPLACEMENT TH. IS
C        NOT COMPUTED; THE INTERACTION PROCEDURE WILL NOT EXECUTED.
C
C ICON3  SET TO 1
C
C ICON4  IF LESS THAN 0, XTW AND YTW ARE READ
C         FROM DISK.
C
C*** LPOP  = NUMBER OF FREE STREAM U*'S INPUT FOR
C PRESS. GRADIENT: IF .GT. 0, XU(J), YU(J) AND SIGMA ARE
C REQUIRED TO BE SPECIFIED. IF .LE.0 DESIGNATES FREE STREAM
C IS CONSTANT.
C
C LADD  ADDITIONAL XU AND YU (LESS THAN 20)
LDF MUST BE LESS THAN 0. ABS(LDF) DESIGNATES MCOUNT WHERE THE INVERSE PROCEDURE BEGINS.

IWALL, IWALLU IF .EQ.-1, THERE IS A STEP ON THE WALL
IF IWALL = 0, THE CHANNEL HAS A SYMMETRIC EXPANSION AND THE CENTERLINE IS USED AS THE UPPER BOUNDARY SO THAT DSTUIC'S = 0.

C*** LINOP LT.0 SETS EQUAL Y SPACING. GE.0 VARIABLE GRID. IF .GE.0, LINOP DESIGNATES WHERE THE VARIABLE GRID TERMINATES AND A UNIFORM GRID IS ADOPTED THEREABOVE.

LINOPU LINOP FOR UPPER WALL
LINOPL LINOP FOR LOWER WALL

INTNL IF .EQ.1, IMPLICIT SCHEME IS TO BE USED FOR INTERNAL FLOW. PRESS. GRAD. WILL BE OBTAINED BY USING THE MASS FLOW RATE CONSTRAINT BOUNDARY COND.

HSTEP HEIGHT OF REARWARD-FACING STEP (FT)
XSTEP LOCATION OF THE STEP IN X-DIRECTION (FT)
TOLVP TOLERANCE FOR INVISCID SOLUTION

XIC(I) INPUT DATA POINT FOR INVISCID CALC. (I = 1,NXIC) (FT)

DSTUIC(I), DSTLIC(I) INITIAL DISPLACEMENT THICKNESS OR D* OBTAINED AT THE PREVIOUS CALC. FOR BOTH THE UPPER AND LOWER WALLS. (FT) (I=1,NXIC)

XU(J) J=1,LP0P X LOCATIONS OF FREE STREAM VELOCITY INPUT (FT)
YU(J) J=1,LP0P FREE STREAM VELOCITIES CORRESPONDING TO XU(J) VALUES (FT/SEC)

XTW(J) J=1, NADD, X LOCATIONS OF DELSTAR INPUT(FT)
YTW(J) J=1, NADD, D*(FT) MULTIPLIED BY 1000.0 CORRESPONDING TO XTW(J). ** XTW AND YTW ARE NOT REQUIRED WHEN ICON4 IS LESS THAN 0 ***

JYSTEP ADDITIONAL GRIDS FOR A REARWARD-FACING STEP. ***** Y(JYSTEP + 1) = HSTEP *****
YSTEP(J) ADDITIONAL GRIDS FOR THE STEP(FT) (J=1, JYSTEP).
**********FOR SUBROUTINE BLMAIN **********

JAM   NO LONGER USED. SET NOT EQUAL TO 0.

MIMMY   SET EQUAL TO ANY POSITIVE INTEGER.

LOT VALUE OF MCOUNT AT WHICH INVERSE PROCEDURE TERMINATES.

LDST   DETERMINES IF DELSTAR IS TO BE COMPUTED AND PRINTED AT EVERY STEP EVEN IN DIRECT MODE. YES IF LDST.LT.0

KETST   DETERMINES IF TURBULENT KINETIC ENERGY EQ. IS TO BE SOLVED. YES IF KETST.LT.0

IFPR   IF IFPR( NE. 0, INPUT DATA IS PRESSURE GRADIENT AND NOT VELOCITY DISTRIBUTION.

ZAP   IF ZAP.NE.1.0 PRESSURE GRADIENT AND OTHER NECESSARY INFORMATION IS PRINTED AT EVERY X STEP.

PORN   RESEARCH TOOL TO SET PRESSURE TO ZERO IMMEDIATELY UPON EXECUTION. IF PORN.GE.0.0, ZERO PRESS. GRAD. WILL BE ASSIGNED IF THE PRESS.GRAD. AT THE PREVIOUS X LOCATION IS NEGATIVE.

GAMMTR   FOR NO TRANSITION, I.E., COMPLETE LAMINAR OR TURBULENT FLOW EQUAL. TO 1.0. WHEN TRANSITION IS EXPECTED SET EQUAL TO 0.0

XCHA   X DISTANCE IN FT. AFTER WHICH THE STEP SIZE IS TO BEGIN DECREASING.

XCHA2   DISTANCE IN FT. PAST WHICH THE STEP SIZE WILL BEGIN INCREASING.

TOLERC   IS THE ERROR ABOUT ZERO WHICH IS ALLOWED FOR CONVERGENCE IN THE INVERSE SOLUTION METHOD.

NLMT   CONTROL PARAMETER TO AVOID EXCESSIVE RUN TIMES IF ERROR OCCURS. IF NUMBER OF X STEPS (MCOUNT).GE. NLMT SOLUTION STOPS.

NNNEG   COUNTER WHICH DESIGNATES NUMBER OF X STEPS TO BE TAKEN PAST SEPARATION (U(1,2).LT.0.0) MAY BE DISREGARDED DEPENDING ON MRPO.
US = REF. VELOCITY FOR NONDIMENSIONALIZATION.
ANY VALUE OKAY FOR EQUAL Y SPACING OPTION (LINOP.LT.0).
FOR UNEQUAL Y SPACING OPTION (LINOP.GE.0) TYPICALLY USED FOR
TURBULENT FLOW, US IS INTERPRETED AS A REPRESENTATIVE
FRICTION VELOCITY TO ESTABLISH Y GRID. (FT/SEC)

XMUS = FREE STREAM INFINITY ABS. VISCOSITY USED
FOR NONDIMENSIONALIZATION (LBM/FT-SEC)

RHOS = FREE STREAM INFINITY DENSITY (LBM/FT**3)
USED FOR NONDIMENSIONALIZATION.

DELY = DELTA Y GRID SPACING IN FT, VALUE ONLY
USED FOR EQUAL SPACING (LINOP.LT.0)

VW = NORMAL VELOCITY AT THE WALL, FOR BLOWING
OR SUCTION, HAS NEVER BEEN USED FOR THIS THESIS. (FT/SEC)

UREF = FREE STREAM INFINITY U VELOCITY (FT/SEC)
WILL VARY AS SOLUTION PROCEEDS, US IS FIXED.

TEST VALUE USED TO CHECK FOR EDGE OF B.L. IF
(U(J)*US/UREF).GE.TEST, EDGE IS LOCATED AT THAT J VALUE.
TYPICALLY 0.9995.

RF,RFU,RFDS RESEARCH PARAMETERS NO LONGER USED,RFU
SHOULD BE .LT. 0 IF RF AND RFDS CAN HAVE ANY VALUE

PRK = TURB PRANDTL # FOR K. E., ONLY NEEDED IF
KETST.LT.0

CKE = CONSTANT IN TKE,ONLY NEEDED IF KETST.LT.0

FST = FREESTream TURB LEVEL FOR TKE EQN., ONLY
NEEDED IF KETST.LT.0

MROP LOGIC PARAMETER IF(MROP.LT.0) OUTPUT LOCAL
TION PAST SEPARATION IS BASED ON DISTANCE OR OTHER INPUT.
IF .GE.0, CALCULATION WILL STOP AT SEPARATION OR ACCORDING
TO NNEG.

LORI = LAMINAR OR TURBULENT. LT.0 FOR LAMINAR

LOUTD = IF(LOUTD.LE.0) PROFILE DATA IS TO BE OUTPUT
FOR START UP DOWNSTREAM OF THE LEADING EDGE.

LCOMP = ALWAYS .GE.0 . LT.0 WILL USE ENERGY EQN.

NPRINT = DETERMINES HOW OUTPUT IS DETERMINED.
If (NPRINT.LE.0) VALUES OF NP1,NP2,XP3(J),NPCR,XP4(J) ARE
C NEEDED. IF .GT. 0 EQUALS NUMBER OF X STEPS BETWEEN C PRINOUTS. I.E., IF NPRINT=25, EVERY 25 STEPS OUTPUT WILL C OCCUR.
C LOOP SAFETY PARAMETER. IF (NPRINT.LE.0) LOOP.
C LE. 0, IF(NPRINT.GT.0) LOOP SHOULD TAKE ON LARGEST MCOUNT C ALLOWED BEFORE TERMINATION.
C LVOP IF(LVOP.EQ.1) INITIAL PROFILE IS NONUNIFORM C AND IS DATA INPUT FROM DOWNSTREAM OF L.E. (SEE STATEMENT C 3000)
C NJ TAKES VALUE OF EDGE LOCATION. THIS IS USU C ALLY USED AS UPPER BOUND ON INTEGRATION IN Y DIRECTION.
C XE LARGEST ALLOWABLE X DISTANCE BEYOND WHICH C THE CALCULATION WILL STOP (FT)
C NP1 IF NPRINT WAS .LE.0, THEN NP1 EQUALS NUMBER C OF VALUES READ IN TO BE USED TO FIND OUTPUT LOCATIONS.
C NP2 IF.*LT.0, XDISTANCE DETERMINES OUTPUT LOCA C TION. IF *EQ.0,REYNOLDS NUMBER BASED X DETERMINES OUTPUT C LOCATION. IF.*GT.0, REYNOLDS # BASED ON THETA DETERMINES C OUTPUT LOCATION.
C XP3(J) J=1,NP1, NUMERICAL VALUES OF OUTPUT LOCA C TIONS ACCORDING TO NP2 (FT)
C NPCR NUMBER OF VALUES OF X DISTANCE READ IN TO C BE USED TO LOCATE OUTPUT OF PROFILES ON DATA CARDS.
C XP4(J) J=1,NPCR, NUMERICAL VALUES CORRESPONDING C TO X DISTANCE. I.E., THESE ARE DISTANCES WHERE DATA ON C CARDS IS TO BE OUTPUT (FT).
C NOPTION ANY INTEGER WILL BE OKAY FOR THE POINT C TRANSITION PROBLEMS STUDIED AT PRESENT.
C XTRFPT SET LARGE VALUE 5*10**6
C CNSTDX IF *LT. 0.0, CONSTANT DELX IS TAKEN FOR C IMPLICIT METHOD AND ABS(CNSTDX) IS THE MAGNITUDE OF DELX C (FT).
C XFCTR A FACTOR FOR CHANGING STEP SIZE. IF X C LOCATES BETWEEN XCHA AND XCHA2, DELX = ABS(CNSTDX)*XCONV* C XFCTR.
C RESETX THE LOCATION FROM WHICH Y-GRID CHANGES
C TO A NEW GRID SYSTEM (FT). (SEE SUBROUTINE XRESET)
C
SIGMA NOT DESIREABLE TO USE THIS VARIABLE AS IT
CAUSES FURTHER DELX DEPENDENCE. SET A LARGE VALUE E.G. 100.
*** SEE LINE # 66, BETWEEN STATEMENT # 232 AND 2804
C IN SUBROUTINE BLMAIN *********
C
************** FOR DISKS **************
C
SOME OF THE FOLLOWING DISKS ARE NEEDED IN ORDER TO EXECUTE
C THIS PROGRAM:
C
DISK PURPOSE
C
$$$$09,11 THIS IS REQUIRED TO READ THE STORED DATA
C ON THE DISK WHEN THE CALCULATION FOR THE B*L. SOLN. STARTS
C DOWNSTREAM OF THE L.E. (SEE THE STATEMENT # 3000 IN
C SUBROUTINE BLMAIN ).
C
$$$$10 THIS IS REQUIRED TO STORE ALL DATA OF THE
C B*L. SOLN. AT THE SPECIFIED POINT SO THAT CALCULATION FOR
C THE B*L. SOLN. CAN START FROM THAT POINT THEREAFTER. (REFER
C TO NPCR AND XP4 ).
C
$$$$12 THIS IS NEEDED, WHEN XTW AND YTW ARE READ
C DIRECTLY FROM THE DISK $$$$13 (SEE ICON4).
C
$$$$13 THIS IS USED TO STORE XTW AND YTW AND
C ALWAYS NEEDED.
C
C******************************************************************************
C
C******************** MAIN PROGRAM ************
C
COMMON/MSTEP/ICON1,CONDTN,ITERTN,RERROR,DSTOST,RHOINF,
1XMUIINF,RFBLM,UREF0,NGIVEN,NADD,NSTOP,PRINT,ICON2,
2ICON3,ICON4,LADD,IVALLL,IVALLU,LINOPU,LINOPL,ULAVG,
3DSTLIC(60),DSTUIC(60),XTWL(70),YTUW(70),UEBLLL(60),
4UEBLU(60),XUP(20),YUP(20)
COMMON/TVEL/DELX,ITS,UREFl,A(200),B(200),C(200),D(200),
1XMU(200),XKE(200),XKE1(200),Y(200),V(200),U(200),
2AP(200),RHO(200),XL(200),U1(200),CMAX,PCON,PRI,CKE,FST,
3KEJ,RHOS,XMUS,DELTI,PRS,US,TEST,DELT,NJ,MCOUNT,LDF,
4LJDEL,KJDEL,NU,NEG,U,ITER,NU2,MXITER,NEG2,UPDATE,MITER
COMMON/MIXLE/V1(200),XL1(200),AST,PTZ,UST,TAU,GAMMR,
COMMON/OKEY1/XABL(400),UABL(400),DABL(400),WRITE,
SUBROUTINE DINPUT
C THIS PROGRAM IS FOR READING INPUT PARAMETERS.
COMMON/MSPE/ICONI,CONDTN,ITERTN,ERROR,DSST,DSTF,RMOINF.
1XMUINF,RFBLM,UREFO,NAG,NAOT,STP,IPRINT,ICON2.
2ICON3,ICON4,LADD,IAWALL,IWALLU,LINOPL,LINOL,UAVG.
3DSLICIC60),STOPIC60),YTWLIC70),YTWFIC70),UELWIC60).
4UELWIC60),XUPIC20),YUPIC20)
COMMON/TVEL/DELX,TTS,UREF1,A(200),B(200),C(200),D(200).
1XMU(200),XKEI(200),XKEII(200),Y(200),V(200),U(200),
2AP(200),RHO(200),XL(200),UI(200),CMAX,PCON,PRK,CKE,FST.
3KEJ,RHOS,XMUS,DELT1,PRS,UR,TEST,DELT,J,AU3,DEL,J,MCOUNT,MDF.
4LJDEL,KJDEL,NOU,NEGU,ITER,NOU2,MIXTER,NEG2,UPDT,MITER
COMMON/MIXLE/VIC200),XLI(200),AST,PTZ,USBTT,AGAMM
COMMON/OKEY1/XABLIC400),UABL(400),DABL(400),DABL(400),WRITE.
1JBLNO,XTWIC70),YTIC70),XSTPT,XENDPT,NREAD,JAN,MINMY.
2NSTART,KSTART,NFORM,NAVE,KTRAKR,LOT,LDST,KETST,LFP.
3AVEAGE,ZAP,HERMAN,PORN,ERROR,XCHA,PERC,SMALL,BIG,XCHA2.
4OUTPUT,TOLE,TAC
1UREF,RF,RFDS,MROP,LORT,LINOP,LOUT,LCOMP,IPRINT.
2LOOP,LVOP,LPOP,MKITER,CSNDX,YSTEPIC150),JYSTEP,PGRAD1.
3PGRAD2,IPG,CONXL,CONU
COMMON/OKEY3/NE,CNP1,CP2,CP3(30),NPIC30),NDEL.
1NQNTRXTRFPT,SIGMA,XUIC70),YUIC70),NTOP,NTIN,NJT,TE,TWS.
2CP,PI,PRT,CON,STA,INTNL,XFCTR,RESETX.
COMMON/LPLACE/STUIC60),STULC60),XIC60),NXIC,NYIC.
1CPOLIC4,60),OINLIC60),OINVIC60),UEOVIC60),OEST,TOLE,VRCNS.
2PSIIC60,60),PSI0IC60,60)
COMMON/STEP/HSTEP,STEP,ITWALL,IDEY,HGRAPH
CALL DINPUT
CALL STPFLO
STOP
END
2PSI(60,60),PSIO(60,60)
COMMON/STEP/HSTEP,XSTEP,IWALL,IDERLY,HIGHT
COMMON/STEP1/QFLOW
IDERLY=0
ICON1=-1
NREAD=0
KSTART=8000
NFORM=-1
NAVE=0
AVEAGE=10.0
OUTPUT=2.0
MING=1
MKITER=10
UPDATE=-1.0
WRITE(6,400)
READ(5,201) CONDTN,WRITE,ITERTN
WRITE(6,401) CONDTN,WRITE,ITERTN
READ(5,201) HIGHT,RError,CONST,XSTEP,TXENDPT,UEOST
WRITE(6,402) HIGHT,RError,CONST,XSTEP,TXENDPT,UEOST
READ(5,201) DSTOST,RHOINF,XMUINF,RFBLM,UEREFO
WRITE(6,403) DSTOST,RHOINF,XMUINF,RFBLM,UEREFO
READ(5,201) NADD,NSTOP,IPRINT,NXIC,NYIC
WRITE(6,404) NADD,NSTOP,IPRINT,NXIC,NYIC
DSTOST=DSTOST/1000.
QFLOW=UEOST*(HIGHT-2.0*DSTOST)
READ(5,201) ICON1,ICON2,ICON3,ICON4,LADD,LDF,IWALLL
IWWALLU
WRITE(6,405) ICON1,ICON2,ICON3,ICON4,LADD,LDF,IWALLL
IWWALLU
READ(5,201) LINOPU,LINOP,INTNL
WRITE(6,406) LINOPU,LINOP,INTNL
IF(CONDTN.LT.-10.0) GO TO 999
READ(5,201) HSTEP,XSTEP,TOLEVP
WRITE(6,407) HSTEP,XSTEP,TOLEVP
READ(5,301) (XIC(I),I=1,NXIC)
READ(5,301) (DSTLC(I),I=1,NXIC)
DO 36 1=1,NXIC
DSTLC(I)=DSTLC(I)/1000.0
36
DSTLC(I)=DSTLC(I)/1000.0
WRITE(6,304)
WRITE(6,203) (XIC(I),I=1,NXIC)
WRITE(6,305)
WRITE(6,203) (DSTLC(I),I=1,NXIC)
WRITE(6,315)
WRITE(6,203) (DSTLC(I),I=1,NXIC)
NADDP1=NADD+1
NPADD=NXIC+NADD
IF(LADD.LT.0) GO TO 658
READ(5,301) (XU(I),I=1,LADD)
READ(5,301) (YU(I),I=1,LADD)
WRITE(6,316)
WRITE(6,203) (XU(I),I=1,LADD)
WRITE(6,317)
WRITE(6,203) (YU(I),I=1,LADD)

CONTINUE
IF(ICON4.LT.0) GO TO 508
DO 41 K=1,NXIC
J=NPADD-K+1
XTW(J)=XIC(J-NADD)
YTWL(J)=DSTLIC(J-NADD)
41 YTWU(J)=DSTUIC(J-NADD)
READ(5,301) (XTW(I),I=1,NADD)
READ(5,301) (YTW(I),I=1,NADD)
DO 42 J=1,NADD
42 YTW(J)=YTW(J)/1000.0
WRITE(6,318)
WRITE(6,203) (XTW(I),I=1,NADD)
WRITE(6,319)
WRITE(6,203) (YTW(I),I=1,NADD)
DO 660 I=1,NADD
YTWU(I)=YTW(I)
660 GO TO 999

CONTINUE
DO 509 J=1,NPADD
READ(12) XTW1, YTW1, YTW2
XTW(J)=XTW1
YTWL(J)=YTW1
509 YTWU(J)=YTW2

CONTINUE
IF(IWALLL.NE.-1) GO TO 60
READ(5,110) JYSTEP
WRITE(6,214) JYSTEP
IF(JYSTEP.EQ.0) GO TO 60
READ(5,301) (YSTEP(J),J=1,JYSTEP)
WRITE(6,203) (YSTEP(J),J=1,JYSTEP)
CONTINUE
IF(ICON1.EQ.1) GO TO 3
LPOP=LADD
IF(LPOP.LT.0) GO TO 657
IF(CONDTN.GE.-10.0) GO TO 657
READ(5,301) (XU(I),I=1,LPOP)
READ(5,301) (YU(I),I=1,LPOP)
WRITE(6,316)
WRITE(6,203) (XU(I),I=1,LPOP)
WRITE(6,317)
WRITE(6,203) (YU(I),I=1,LPOP)
CONTINUE
READ(5,100)
WRITE(6,100)
C........... INPUT DATA FOR BOUNDARY-LAYER SOLUTION ..........
READ(5,110)JAM,MIMMY,LOT,LDIST,KETST,IFPR
WRITE(6,408)JAM,MIMMY,LOT,LDIST,KETST,IFPR
READ(5,105)ZAP,PORN,GAMMTR
WRITE(6,409)ZAP,PORN,GAMMTR
READ(5,105)XCHA,XCHA2,TOLERC
WRITE(6,410)XCHA,XCHA2,TOLERC
READ(5,110)NLMT,NNEG
WRITE(6,411)NLMT,NNEG
READ(5,105)US,XMUS,RHOS,DELY,VW,UREF,TEST,RF,RFU,RFDS
WRITE(6,412)US,XMUS,RHOS,DELY,VW,UREF,TEST,RF,RFU,RFDS
PRK=1.0
CKE=0.0
FST=0.0
IF(KETST.LT.0) READ(5,105)PRK,CKE,FST
IF(KETST.LT.0) WRITE(6,413)PRK,CKE,FST
READ(5,110)MROP,LORT,LOUTD,LCOMP,NPRINT,LOOP,LVOP,NJ
WRITE(6,414)MROP,LORT,LOUTD,LCOMP,NPRINT,LOOP,LVOP,NJ
IF(LOOP)10,10,11
10 CONTINUE
READ(5,105)XE
WRITE(6,417)XE
11 CONTINUE
IF(NPRINT)221,221,222
221 CONTINUE
READ(5,110)NP1,NP2
WRITE(6,418)NP1,NP2
READ(5,105)(XP3(J),J=1,NP1)
WRITE(6,419)
WRITE(6,203)(XP3(J),J=1,NP1)
READ(5,110)NPCR
WRITE(6,420)NPCR
READ(5,105)(XP4(J),J=1,NPCR)
WRITE(6,203)(XP4(J),J=1,NPCR)
222 CONTINUE
READ(5,680)NOPTN,XTRFPT,
CNSTDX, XFCTR, RESEX
WRITE(6,421)NOPTN,XTRFPT,
CNSTDX, XFCTR, RESEX
3 CONTINUE
UAVG=UREFO
REYSTP=UAVG*HSTEP*RHOINF/XMUINF
WRITE(6,422)QFLOW,REYSTP
100 FORMAT(72H
1 )
105 FORMAT(7G10.4)
110 FORMAT(1216)
201 FORMAT(7G10.8)
203 FORMAT(10X,6G16.6)
214 FORMAT(/,10('.*,',) JYSTEP ='*,I5,10('.*,',)/,5X,5('**'),
1* YSTEP '=',5('**'),/)

SUBROUTINE STPFLO
C THIS PROGRAM PROVIDES THE VISCOUS, INVISID AND
C VISCOUS-INVISCO INTERACTION SOLUTIONS.
C THE PROGRAM CALLS BLMAIN FOR THE VISCOUS SOLUTIONS AND
C INVIS FOR THE INVISCO SOLUTION.
COMMON/MSTEP/IION1,ICON2,ITERN,RERROR,DSTOST,RHOINF,
1XMINF,RFBLM,UREF0,NGIVEN,NADD,NSTOP,IPRINT,ICON3,
2ICON4,LADD,LDF,IWALLL,IWALLU
COMMON/MSTEP/1CON1,ICON2,ITERN,RERROR,DSTOST,RHOINF,
XMINF,RFBLM,UREF0,NGIVEN,NADD,NSTOP,IPRINT,ICON3,
ICON4,LADD,LDF,IWALLL,IWALLU
RETURN
3DSTLIC(60), DSTUIC(60), YTWL(70), YTWU(70), UEBLL(60),
4UEBLU(60), XPUP(20), YUP(20)
COMMON/TVEL/DELTX, TTS, UREF1, A(200), B(200), C(200), D(200),
XUM(200), XKE(200), XKE1(200), Y(200), V(200), U(200),
2AP(200), RHO(200), XL(200), U1(200), CMAX, PCON, PRK, CKE, FST,
3KEJ, RHOX, XMUS, DELTI, PRS, US, TEST, DELT, NJ, MCOUNT, LDF,
4LJDEL, KJDEL, NOUG, TITER, NOU2, MXITER, NERO, UPDATE, METER
COMMON/MIXLE/V1(200), XL1(200), AST, PTZ, UST, TAU, GAMMR
COMMON/OKEY1/XABL(400), UABL(400), DABL(400), IWRITE,
1JBLNO, XTW(70), YTW(70), XSTPT, XENDPT, NREAD, JAM, MIMMY,
2NSTART, KSTART, NFORM, NA, KTRAKR, LQT, LDST, KETST, IFPR,
3AVEAGE, ZAP, HERMAN, PORN, ERROR, XCHA, PERCG, SMALL, BIG, XCHA2,
4OUTPUT, TOLER, AFCR
COMMON/OKEY2/LMT, NLMT, NNEG, NXTRAP, MING, DELY, VW, DXF, DX7,
1UREF, RF, RFU, RFOS, MRO, LORT, LINDP, LOUDT, LCOMP, NPRINT,
2LOOP, LVOP, LPOP, MKITER, CNSTDX, YSTEP(150), JSTEP, PGRAD, 3PGRAD2, IPG, CONXL, CONUI
COMMON/OKEY3/XE, NP1, NP2, XP3(30), NPCR, XP4(30), NDEL,
1NOPTN, XTRFP, SIGMA, XU(70), YU(70), NTOP, NTIN, NJT, TE, TWS,
2CPS, P, IP, PRT, RCON, STA, INTNL, XFCR, RETEX
COMMON/LPLACE/DSTUIC(60), DSTLI(60), XIC(60), NXIC, NYIC,
1CPOL(4, 60), UINV(60), UINVU(60), UEOST, TOLVEP, RCONST,
2PSI(60, 60), PSI(60, 60)
COMMON/STEP/HSTEP, XSTEP, IWALL, IDELAY, HIGHT
COMMON/STEP1/QFLOW
ICONVG=0
IUSTP=0
JUSTP=0
NADDP1=NADD+1
NPADD=NXIC+NADD
999 CONTINUE
NREAD=NREAD+1
IF(NREAD.EQ.1) GO TO 1000
REWIND 9
1000 CONTINUE
JBLNO=0
IF(ICON1.EQ.1) GO TO 3
C............................................ CALCULATION OF BOUNDARY LAYER SOLUTION ON THE
C LOWER WALL ............................................
NPADD=NXIC+NADD
IF(NPADD.EQ.0) GO TO 713
DO 661 J=1, NPADD
661 YTW(J)=YTWL(J)
IF(NREAD.GT.1) GO TO 713
IF(ICON4.LT.0) GO TO 713
IF(IWALLL.NE.-1) GO TO 713
DO 714 J=1, NPADD
IF(XTW(J).GT.XSTEP) GO TO 713
YTW(J)=YTW(J)+HSTEP
YTWL(J)=YTW(J)
CONTINUE
NDEL=NPADD
IWALL=IWALLL
LINOP=LINOPL
LPOP=LADD
ITERTN=ITERTN+1
WRITE(6,314) ITERTN
CALL BLMAIN(IUSTP)
IF(IUSTP.EQ.100) GO TO 501
XCONV=RHOS*US/XMUS
WRITE(6,316) JBLNO
IF(JBLNO.LE.1) GO TO 716
WRITE(6,310)
WRITE(6,310) (XABL(I),UABL(I),DABL(I),I=1,JBLNO)
IF(IWALLL.NE.-1) GO TO 711
710 DABL(J)=DABL(J)+HSTEP
DO 54 J=1,NXIC
XDSIGN=XIC<J)
CALL POLFIT(JBLNO,XABL,J,EABL,XDSIGN,0DSIGN)
CALL POLFIT(JBLNO,XABL,UABL,XDSIGN,UOSIGN)
DSTLIC(J)=DDSIGN
UEBLL(J)=UOSIGN
54 CONTINUE
IF(IWALLL.NE.-1) GO TO 716
DO 715 J=1,NXIC
IF(XIC(J).GT.XSTEP) GO TO 716
715 DSTLIC(J)=DSTLIC(J)-HSTEP
716 CONTINUE
IF(C0NDT.LT.0.0) GO TO 501
IF(IWALLU.EQ.0) GO TO 1
C********** CALCULATION OF BOUNDARY SOLUTION ON THE
C UPPER WALL ************
JBLNO=0
IWALL=IWALLU
LINOP=LINOPU
DO 651 J=1,NPADD
651 YTW(J)=YTWU(J)
REWIND 9
IF(NREAD.EQ.1) NREAD=0
CALL BLMAIN(IUSTP)
IF(IUSTP.EQ.100) GO TO 501
IF(NREAD.EQ.0) NREAD=1
WRITE(6,316) JBLNO
WRITE(6,310)
WRITE(6,310) (XABL(I),UABL(I),DABL(I),I=1,JBLNO)
DO 653 J=1,NXIC
XDIGN=XC(J)
CALL POLFIN(JBLNO,XABL,DABL,XDIGN,DDSIGN)
CALL POLFIN(JBLNO,XABL,UABL,XDIGN,UDSIGN)
DSTUIC(J)=DDSIGN
UEBLU(J)=UDSIGN
653 CONTINUE
3 CONTINUE
   DDIF1=DSTLIC(1)-DSTOST
   DDIF2=DSTUIC(1)-DSTOST
   UEDF1=UEBLU(1)-UEOST
   UEDF2=UEBLU(1)-UEOST
   DO 55 I=1,NXIC
   DSTLIC(I)=DSTLIC(I)-DDIF1
   DSTUIC(I)=DSTUIC(I)-DDIF2
   UEBLUI(I)=UEBLU(I)-UEDF2
55 UEBLL(I)=UEBLU(I)-UEDF1
GO TO 4
1 CONTINUE
   QFLOW0=QFLOW
   QFLOW=QFLOW/2.0
   HIGHT0=HIGHT
   HIGHT=HIGHT/2.0
   DDIF1=DSTLIC(1)-DSTOST
   UEDF1=UEBLU(1)-UEOST
   DO 2 I=1,NXIC
   DSTLIC(I)=DSTLIC(I)-DDIF1
   UEBLL(I)=UEBLU(I)-UEDF1
2 CONTINUE
   WRITE(6,210)
   WRITE(6,204)
   WRITE(6,212) (XIC(I),DSTLIC(I),DSTUIC(I),UEBLL(I),
   UEBLU(I),I=1,NXIC)
C.............. CAL CULATION OF INVISCID SOLUTION ..............
   CALL INVISO(JUSTP,DSTUIC,DSTLIC,ITERTN,NREAD)
   IF(IWALLU.EQ.0) QFLOW=QFLOW0
   IF(JUSTP.EQ.100) GO TO 501
   IF(ICON1.EQ.1) GO TO 501
   THIGHT=HIGHT+HSTEP
   DO 19 J=1,NXIC
19   DSTUJ(J)=THIGHT-DSTUJ(J)
   IF(IWALLU.EQ.0) HIGHT=HIGHT0
C.............. UPDATING DTL FOR N+1 ITERATION ..............
   IF(ICON2.EQ.1) GO TO 17
   DO 16 I=1,NXIC
16   DSTLIC(I)=DSTUJ(I)*UEBLU(I)/UINVU(I)
17   DSTLIC(I)=DSTLIC(I)*UEBLU(I)/UINVU(I)
C.............. CHECKING CONVERGENCE ......................
   IF(ICONVG.EQ.100) GO TO 501
   WRITE(6,210)
WRITE(6,206)
DO 18 J=1,NXIC
  RELUL=ABS(UEBL(U(J))-UINVL(J))
  RELUL=RELUL/UINVL(J)
  IF(IWALLU.EQ.0) UEBLU(J)=UINVU(J)
  RELUU=ABS(UEBLU(J)-UINVU(J))
  RELUU=RELUU/UINVU(J)
WRITE(6,207) XIC(J),UEBL(J),UINVL(J),RELUL,UEBLU(J),
*UINVU(J),RELUU,OSTLIC(J),OSTUIC(J)
IF(J.GT.2) .AND. (ICONVG.EQ.0)) GO TO 18
IF(RELUU.LE.RERROR.AND.RELUL.LE.RERROR) GO TO 654
  ICONVG=0
  GO TO 18
654 ICONVG=100
18 CONTINUE
WRITE(6,210)
WRITE(6,213)
DO 997 I=1,NXIC
  D STLIP =RFLBM*DSTLIC(I)+(1.0-RFLBM)*DSTLI(I)
  DSTUIP=DSTUIC(I)
WRITE(6,205) XIC(I),DSTLI(I),DSTLIC(I),DSTLIP,
1DSTUI(I),DSTUIC(I),DSTUIP
DSTLI(I)=DSTLIP
DSTUI(I)=DSTUIP
997 CONTINUE
IF(ICONVG.EQ.0) GO TO 21
WRITE(6,309)
DO 57 I=1,NXIC
  CPLINV=UINVL(I)*UINVL(I)/(UEREFO*UEREFO)
  CPUINV=UINVU(I)*UINVU(I)/(UEREFO*UEREFO)
  XPHSTP=(XIC(I)-XSTEP)/HSTEP
WRITE(6,203) XIC(I),XPHSTP,CPLINV,CPUINV
57 CONTINUE
WRITE(6,209) ITERTN
IWRITE=1
GO TO 17
21 WRITE(6,208)
17 CONTINUE
C*************** UPDATING DSTLIC AND DSTUIC **************
DO 43 K=1,NXIC
  J=NPADD-K+1
  YTWL(J)=DSTLI(J-NADD)
  YTWU(J)=DSTUI(J-NADD)
43 CONTINUE
IF(IWRITE.LT.0) GO TO 504
WRITE(6,210)
WRITE(6,307) (XTW(I),YTWL(I),YTWU(I),I=1,NPADD)
504 CONTINUE
IF(ICONVG.NE.100) GO TO 659
IF(IWRITE.GT.1) GO TO 501
GO TO 999

659 IF(IPRINT.EQ.1) GO TO 512
ICON3=ICON3+1
IF(ICON3.NE.IPRINT) GO TO 511
ICON3=0
IWRITE=10
GO TO 512

511 IWRITE=-10
512 CONTINUE
IF(NREAD.GE.NSTOP) GO TO 507
GO TO 999

507 WRITE(6,210)
DO 100 I=1,NPADD
XTW1P=XTW(I)
YTW1P=YTWL(I)
YTW2P=YTWU(I)
WRITE(13)XTW1P,YTW1P,YTW2P
WRITE(6,307)XTW1P,YTW1P,YTW2P
100 CONTINUE
IF(NREAD.GE.NSTOP) GO TO 507
GO TO 999

203 FORMAT(10X,6G16.6)
204 FORMAT(14X,"X(INPUT)",8X,"D*(L.B.)",9X,"D*(U.B.)",9X,
**UE(L.B.)",9X,"UE(U.B.)",/) 
205 FORMAT(7G15.6)
206 FORMAT(10X,"REL DU(L)",7X,"UEB(L)",7X,"UEINV(L)",7X,
**REL DU(U)",7X,"UEB(U)",7X,"UEINV(U)",5X,"REL DU(U)",
$7X,"D*(L)",7X,"D*(U)",/) 
207 FORMAT(9G14.6)
208 FORMAT(///,"***) DOES NOT CONVERGE ***) 
209 FORMAT(///,10("**") , " CONVERGES AFTER ",I4, 
** ITERATIONS ",10("**") )
210 FORMAT(///)
212 FORMAT(10X,6G16.6)
213 FORMAT(10X,"X(INPUT)",8X,"DSTLN",9X,"DSTLN+1",9X,
**DSTLN+1P",9X,"DSTUN",9X,"DSTUN+1",9X,"DSTUN+1P",/) 
302 FORMAT(///,"DATA HAVE BEEN OUT ON THE DISK",///)
307 FORMAT(5X,3G16.6,5X,3G16.6)
309 FORMAT(14X,"X(INPUT)",8X,"X/HSTEP",8X,"CP(L) ",
$9X,"CP(U) ",/)
"(B,U)",12X,"D",/) 
314 FORMAT(///,15("**") , "TH ITERATION ",15("**") ,///) 
316 FORMAT(///,10X ,JBLNO='',I4) 
501 RETURN
END
SUBROUTINE UVNDST(INTNL,UREF2,MSTEP,ASTSTP,DLSTEP)

C THIS PROGRAM IS FOR THE CALCULATION OF THE VELOCITIES
C AND DISPLACEMENT THICKNESS. THE PROGRAM HAS SUBPROGRAMS
C SUCH AS UVNPSI AND DSTNM.

COMMON/MIXLE/V1(200),XL1(200),AST,PTZ,UST,TAU,GAMMTR
COMMON/TVEL/DELX,DDLT,TTS,UREF1,A(200),B(200),C(200),D(200),
1 XMU(200),XKE(200),Y(200),V(200),U(200),
2 AP(200),RHOS(200),XL(200),UL(200),C(200),PR,KE,FST,
3 KEJ,RHOS,XMUS,DELT,I,PR,US,TEST,DELT,NJ,MCOUNT,LDL,
4 JLDEL,KJDEL,NOU,NEG,ITER,NOU2,MXITER,NEGO,UPDATE,MITER
COMMON/OKEY/XABL(400),UBLB(400),DABL(400),IWRITE,
1 JBNO,ITW(70),YTW(70),XSTPT,XENDPT,NREAD,JAM,MINMY,
2 NSTART,KSTART,IFORM,KTRAKR,LQT,LDST,KETST,IFPR,
3 AveAge,ZAP,HERMAN,PORN,ERROR,XCHA,PERCG,SMALL,BIG,XCHA2,
4 OUTPUT,TOLER,AFCTR

COMMON/OKEY2/LMT,NLMT,NEG,NXTRAP,MING,DELT,WWWQDX,DX7,
1 UREF,RF,RFDS,MRP,LORT,LINOP,LOUTD,LCOMP,NPRINT,
2 LOOP,LVOP,LPOP,MITER,CNSTDX,YSTEP(150),JYSTEP,PGRAD1,
3 PGRAD2,IP,CONXL,CONY1

COMMON/UVNST/PCON3,UP1,UP2,DP4,IP,LVREF,STMI,DM2,
1 RH01(200),PCONB,DFIX,UT1,PX,PX,XCONV,PRH01,
2 AFLM,W*XW,XF(200),E(200),JUSTP

COMMON/STEP/ISTP,XSTEP,IWALL,DELV,HIGHT

COMMON/VEL/PSIN(200),PSIO(200)

IF(MSTEP.NE.0) GO TO 80

ASSTP=AST
DLSTEP=Y(KJDEL)

80 CONTINUE

MSTP1=MSTEP

KJDELP=KJDEL

JUSTP=0

ASSTP=AST

IF(MSTEP.NE.10) GO TO 90

KJDELP=KJDEL+JYSTEP

90 CONTINUE

KJDEL=KJDELP

IF(IWALL.NE.1) GO TO 10

IF(XX*GT.XSTEP) GO TO 9

DFIX=DFIX-HSTEP*XCONV

GO TO 10

9

DXXF=(XXF-XSTEP)*XCONV

IF(DXXF.LT.DE莉) DFIX=DFIX-HSTEP*XCONV

10 CONTINUE

U1(1) = 0.0

V1(1) = VW*UREF1/US

DELT1 = -1.0

DELT = -1.0

CALL UVNPSI(JUSTP,LORT,DFIX,TOLER,UREF,RH01,INTNL, *AFLM,IP,LQT,IWRITE)

C********** CALCULATION OF U, V, AND PSI IN UVNPSI **********
IF(JUSTP.EQ.100) GO TO 724
IF(LCOMP)510,45,45
510 CALL TEMP
45 CONTINUE
INTN=-1

C. CALCULATING DISPLACEMENT THICKNESS
CALL DSTNM(NJ,Y,U1,RHO1,US,RHRF1,UREF1,SUM1,INTN)
DST1=SUM1
IF(INTN.NE.1) GO TO 726

C. CALCULATING MASS FLOW RATE
INTN=1
CALL DSTNM(NJ,Y,U1,RHO1,US,RHRF1,UREF1,SUM1,INTN)
SUM1=SUM1/XCONV+RHRF1*UREF1*(HIGHT/2.0-Y(NJ)/XCONV)
AFLOW1=SUM1
726 RHRF1=RHO1(NJ)
IF(LDST.GE.0) GO TO 8374
X=DST1/XCONV
IF(IWRITE.LT.0) GO TO 2990
WRITE(6,8373)MCOUNT,XXF,X,AFLOW,AFLOW1
2990 CONTINUE
IF(XXF.LT.XSTPT) GO TO 2991
IF(XXF.GT.XENDPT) GO TO 2991
JBLNO=JBLNO+1
XABL(JBLNO)=XXF
YABL(JBLNO)=X
UABL(JBLNO)=UREF1
2991 CONTINUE
8373 FORMAT(‘MCOUNT,X,DST,AFLOW,AFLOW1 =‘,5G12.5)
8374 CONTINUE
8104 ITER=0
8284 MITER=0
F3=XC
PCON3=PCON
IF(ZAP.GE.1.0) GO TO 724
IF(MCOUNT.EQ.1) GO TO 3
DP4=DST1/XCONV
UREF=V1(NJ)
PX=UREF*US
IF(IWRITE.LT.0) GO TO 2997
WRITE(6,8321)XXF,DP4,PX
2997 CONTINUE
3 CONTINUE
8321 FORMAT(‘IX,XDIST,G14.5,DST,G14.5,UREF,G14.5)
724 CONTINUE
RETURN
END
SUBROUTINE ENGIN
C
C THIS PROGRAM INITIALIZES THE NECESSARY PARAMETERS FOR
C THE SOLUTION OF THE ENERGY EQUATION.
C
COMMON/TVEL/DELX,TTS,UREFI,A(200),B(200),C(200),D(200),
1 XMU(200),XKE(200),Y(200),V(200),U(200),
2 AP(200),RHO(200),X(200),U1(200),CMAX,PCON,PRK,CKE,FST,
3 KEJ,RHOS,XMUS,DEL1,PRS,US,TEST,DEL,T,JMOUNT,LDF,
4 LDE,14,15,16,NOU,NEG,ITER,NOU2,MP,ITER,NEGO,UPDATE,ITER
COMMON/UDST/PCON3,UP1,UP2,DP4,IP4,IVREF,DT1,DT2,
1 RHO(200),PCON6,PCONPB,DFIX,UT1,PU,PD,XCONV,RHFR1,
2 AFLOW,X,XXF,D2(200),E(200),JUSTP
COMMON/OKEY1/XABL(400),UABL(400),DABL(400),WRITE,
1 NBLNO,XTW(70),YT(70),XSTPT,XENDPT,NREAD,JAM,NIMMY,
2 NSTART,KSTART,NFORM,NAVE,KTRK,KQT,LDT,KDT,KETST,IPPR,
3 SAVEAGE,2AP,HERMAN,PORN,ERRDR,XCHA,PERC,SMALL,BIG,XCHA2,
4 0UTPUT,TOLERC,AFCR
COMMON/INITL/BLT,F1,F2,F3,GCON,JCOUNT,JH,KCOUNT,KLJN,
1 KFJDEL,KTRK,LTD,LCONTP,LCOUNT,MNM,MSTEP,NCOD,
2 NCOUE,NPCC,NSTEP,NEG,NCARDS,NCODN9,NRESET,NNP,NPD,
3 PCON1,PCON2,PP,RADMT,UT,UT2,VRFF,UREF2,XC,TRIP
COMMON/NDUM/IEK,INPR,LQT,DINC,XDINC,D6,DX9,D10,
1 PERCB,MM,MDIS,UREF2,DIS,T,RHFR,TH1,EP1,DST,
2 IP,DIS3,TH2,NC,NT1,NT2,DFX,XMU(200),R,RE,RET,BETA,CF,
3 CFR,USTP,TAUP,H1,GUF2,UF1
COMMON/OKEY2/LMT,NLMT,NEG,NXTRAP,MIN,DELY,VW,DXF,DX7,
1 UREF,RF,RFU,RDFS,NRDP,LRT,LINOP,LOUT,LOCMP,NPRJ,
2 LQP,LQP,LQP,MKITER,STEP(150),YSTEP,PERG,1,
3 PGRAD2,IP,CXNL,CONF
COMMON/OKEY3/XE,NP1,NP2,XP3(30),NPCR,XP4(30),NDL,
1 NQPN,T,SIGMA,XU(70),YU(70),NTOP,NTIN,NTJ,TE,TWS,
2 CPS,PI,PRT,RCN,STA,INTNL,XCTR,RESETX
COMMON/ENG1/HS,H(100),E1,T(100),XVAR,H(100),HH(100)
IF(NREAD.NE.1) GO TO 2805
READ(5,110) NTOP,NTIN,NTJ
READ(5,105) TE,TWS,CPS,PRC,PI,PRT,RCN,STA
110 FORMAT(1216)
105 FORMAT(1216)
WRITE(6,111) NTOP,NTIN,NTJ
WRITE(6,106) TE,TWS,CPS,PRC,PI,PRT,RCN,STA
111 FORMAT(6('**'),**INPUT DATA FOR ENERGY EQUATION**,5('**'),
106 FORMAT(10X,**TE,TWS,CPS,PRC=*,4G13.5,/10X,**PI,PRT,RCN,
*STA =*,4G13.5))
2805 CONTINUE
HS = CPS*TE + UREF*UREF/(778.165*2.0*GCON)
H(1) = CPS*TWS/HS
ES = US*US/(HS*778.165*32.174)
T(1) = TWS
XVAR = 10.0**(-9.0/T(1))
XMU(1)=2.32E-8*GCON* SQRT(T(1))/(1.0+(220.0/T(1))*XVAR)
RHRF = PI/(RCON*TE)
RHRF1 = RHRF
DO 104 J= 2,200
H(J) = 1.0
H2(J)=1.0
HH(J)=1.0
T(J) = H(J)*HS/CPS - U(J)*U(J)*US*US/(2.0*GCON*778.165*
1CPS)
RHO(J) = PI/(RCON*T(J))
XVAR=10.0**(-9.0/T(J))
XMU(J)=2.32E-8*GCON* SQRT(T(J))/(1.0+(220.0/T(J)))*XVAR)
RHO1(J)=RHO(J)
104 XMU1(J)=XMU(J)
IF(NTOP)103,103,234
234 IF(NREAD.NE.1) GO TO 2826
READ(5,105) (XTW(J),J=1,NTOP)
READ(5,105) (YTW(J),J=1,NTOP)
WRITE(6,107)
WRITE(6,108) (XTW(J),J=1,NTOP)
WRITE(6,109)
WRITE(6,108) (YTW(J),J=1,NTOP)
107 FORMAT(5X,´***** XTW *****´)
108 FORMAT(5X,´10G12.4´)
109 FORMAT(5X,´***** YTW *****´)
2826 CONTINUE
103 RETURN
END

SUBROUTINE POLFIT(N,X,Y,AX,AY)
C THIS IS THE PROGRAM FOR INTERPOLATING VARIABLES USING
C POLYNOMIAL CURVE FITTING.
DIMENSION X(N),Y(N)
IF(AX.LT.X(1)) GO TO 10
DO 14 I=2,N
JJ=I
IF(AX-X(I)) 12,13,14
14 CONTINUE
XONE=X(N-2)
XTWO=X(N-1)
XTHREE=X(N)
M=N
WRITE(6,15)N
15 FORMAT(´/`, *** WARNING*** Y IS EXTRAPOLATED WHEN `,`*
,**N IS`,,15)
GO TO 16
13 AY=Y(JJ)
RETURN
12 IF(JJ.EQ.0.2) I=3
XONE=X(I-2)
XTWO=X(I-1)
XTHREE=X(I)
M=1
GO TO 16
10 XONE=X(1)
XTWO=X(2)
XTHREE=X(3)
M=3
WRITE(6,15)
16 AL1=(AX-XTWO)*(AX-XTHREE)*((XONE-XTWO)*(XONE-XTHREE))
AL2=(AX-XTHREE)*(AX-XONE)*((XTWO-XTHREE)*(XTWO-XONE))
AL3=(AX-XONE)*(AX-XTWO)*((XTHREE-XONE)*(XTHREE-XTWO))
AY=AL1*Y(M-2)+AL2*Y(M-1)+AL3*Y(M)
RETURN
END

SUBROUTINE TEMP
C THIS PROGRAM IS FOR THE SOLUTION OF THE ENERGY EQUATION.
C IN THE PRESENT STUDY THIS PROGRAM IS NOT USED.
RETURN
END

SUBROUTINE STARTN(CDO,CPO)
C THIS PROGRAM INITIALIZES ALL THE NECESSARY PARAMETERS
C FOR THE BOUNDARY-LAYER SOLUTIONS.
DIMENSION CDO(4,70),CPO(4,70)
COMMON/MIXLE/V1(200),XL1(200),AST,PTZ,UST,TAU,GAMMTR
COMMON/TVEL/DELX,TTS,UREF1,A(200),B(200),C(200),D(200),
1 XMU(200),XKE1(200),XKE(200),V(200),U(200),
2 AP(200),RHO(200),XL(200),UI(200),CMAK,PCON.PRK,CHE,FST,
3 KEJ,RHOS*XMUS*DEL1,PRSD,US,TEST,DELT,NJ,MCOUNT,LDF,
4 LQDEL,KQDEL,NOU,NEG,ITER,NOU2,MXITER,NEG,UPDATE,MITER
COMMON/UVST/DPS3,PCCN3,UP1,UP2,DP4,IIIP,VIREF,DIR1,DIR2,
1 RHO1(200),PCON6,PCONPB,DFIX,UT1,XP,PD,XCONV,RHRF1,
2 AFLOW,X,XXF,D2(200),E(200),JUSTP
COMMON/INITL/LBT,FI,FJ,GCON,JCOUNT,JH,KCOUNT,KLJN,
1 KFDEL,KTRK,KK,LDCONT,LDCOUNT,MIN,MSTP,NCOU,
2 NCUE,NPCC,NSTEP,NEG,NCARD,NCOND,NRESET,NPP,NPD,
3 PCON1,PCON2,PP,PRAMDR,UT,UT2,VRFF,VREF2,XC,XTRIP
COMMON/NONDIM/KEK,NIPR,LOI,DXINC,DXINC,DX8,DX9,DX10,
1 PERCGB,MM,DXDIS,UREF2,XDIST,RHRF,TH,EP(200),TH1,DST,
2 IPC,DST3,TH2,NPC,NPTI,DFT,XMU1(200),REX,RET,BETA,CF,
3 CFP,USTP,TAUP,HI,G,UF2,UF1
F2=0.0
F1=0.0
PCON1=0.0
PCON2=0.0
PCON3=0.0
V1REF=0.0
VRFF=0.0
VREF2=0.0
BLT1 = 1.0
JH = 1
GCON = 32.17
PP = 0.0
NCOND9=0
XTRIPT=0.0
RAMDTR=0.0
MSTEP=0
NRESET=0
C COMPUTE NONDIMENSIONALIZATION FACTORS
IKE=IABS(KETST)
XCONV = RHOS*US/XMUS
N1PR = NPRINT-2
LOT=1
NJ=NJ
IIP=IABS(LDF)-5
DXINC = 0.0
XDINC = DXINC
DX8 = DX7*XCONV
C COMPUTE LIMITING VALUES OF MULTIPLIERS OF STEP SIZE.
DX9=SMALL*DX7
DX10=BIG*DX7
PERCG8=(1.0-PERCG)+1.0
MM = 0
DELY = DELY*XCONV
C INITIALIZE X DISTANCE TO 0.0
DXDIS = 0.0
UREF1 = UREF
UREF2 = UREF
XDIST = 0.0001
RHRF = RHOS
RHRF1 = RHOS
TH = 0.01575*XCONV
EPS1(1) = 0.0
TH1 = TH
DST = TH
IPC=0
DST1 = DST
DST2=DST1
DST3=DST1
IF(NPRINT)221,221,222
221 CONTINUE
IF(NDEL.LE.0) GO TO 8380
WRITE(6,1004)(XTW(J),J=1,NDEL)
WRITE(6,1004)(YTW(J),J=1,NDEL)
CALL SPLICO(XTW,YTW,NDEL,CDQ)

8380 CONTINUE
NPC = 1
NPT1 = -1
222 CONTINUE
PRS=1.0
DFT = 1.0
C SET WALL CONDITIONS
XL(1) = 0.0
XKE(1)=0.0
XKE1(1)=0.0
V(1) = 0.0
U(1) = 0.0
V1(1) = 0.0
U1(1) = 0.0
RHO(1) = RHOS
XMU(1) = XMUS
D2(1)=0.0
XMUI(1) = XMUS
RHO1(1) = RHOS
XL1(1)=0.0
E(1)=RHOS
D(1)=0.0
IF(LVOP.EQ.1)GO TO 3011
C NONDIMENSIONALIZE
DO 15 J=2,200
U(J) = UREF/US
V(J) = 0.0
XKE(J)=FST*UREF*UREF/US/US
XKE1(J)=XKE(J)
D2(J)=0.0
U1(J)=U(J)
D(J) = U(J)
EPS(J) = 1.0
RHO(J) = RHOS
XMU(J) = XMUS
XL(J) = 0.0
XMUI(J) = XMUS
RHO1(J) = RHOS
E(J) = RHOS
XL1(J) = 0.0
15 CONTINUE
C SET Y GRID SPACING
3011 Y(1)=0.0
IF(LINOP) 417,418,418
417 DO 419 J=2,200
419 Y(J) = (J-1)*DELY
GO TO 3012

418  Y(2)=1.0
LINOPM=LINOP-1
DO 390 J=2,LINOPM
390  Y(J+1)=Y(J)+1.15*(Y(J)-Y(J-1))
DOVARI=Y(LINOP)-Y(LINOPM)
DO 391 J=LINOP,199
391  Y(J+1)=Y(J)+D0VARI
3012  CONTINUE
1004  FORMAT(3X,9G12.4)
RETURN
END

SUBROUTINE DSTNM(NJ,Y,U1,RHO1,US,RHRF1,UREF1,SUM1.
*INTNA)
C THIS SUBROUTINE PROVIDES THE DISPLACEMENT THICKNESS AND
C THE MASS FLOW RATE USING THE TRAPEZOIDAL RULE.
DIMENSION Y(200),U1(200),RHO1(200)
WRITE(6,1) (U1(J),J=1,NJ)
1  FORMAT(9G14.4)
LLJ=NJ
SUM1=0.0
DO 8313 J=1,LLJ
IF(J.EQ.LLJ) GO TO 92
YDE1=Y(J+1)-Y(J)
IF(J.EQ.1) YDE2=YDE1
IF(J.EQ.LLJ) GO TO 91
92  YDE2=Y(J)-Y(J-1)
91  CONTINUE
IF(J.EQ.LLJ) YDE1=YDE2
YDE3=(YDE1+YDE2)/2.
Y1=1.0-RHO1(J)*US*U1(J)/(RHRF1*UREF1)
SUM1=SUM1+Y1*YDE3
IF(J.EQ.1 OR J.EQ.LLJ) SUM1=SUM1-Y1*YDE3/2.
8313  CONTINUE
WRITE(6,1) SUM1
RETURN
END

SUBROUTINE BLMAIN(IUSTP)
C THIS PROGRAM IS FOR THE BOUNDARY-LAYER SOLUTIONS. THE
C PROGRAM INCLUDES 13 DIFFERENT SUBPROGRAMS. THEY ARE
C UVNDEST, ENGVIN, TEMP, STARTN, DSTNM, TKE, MIXLEN, PRINUT,
C UPDATN, XRESET, SPLICO, SFINT, AND TRAN9.
DIMENSION VV(200),CPO(4,70),CDQ(4,70),DU(200),DV(200),
*DY(200)  
COMMON/TVEL/DELX,TTS,UREF1,A(200),B(200),C(200),D(200),  
1XMU(200),XXE(200),XXE1(200),Y(200),V(200),U(200),  
2AP(200),RHDS(200),XL(200),UL(200),CMAX,PCON,PRK,CXE,FST,  
3KEJ,RHOS,XMUS,DELT1,PRG,US,TEST,DELT,NJ,MCOUNT,LDF,  
4LJDEL,KJDEL,NOU,NEG,ITER,NOU2,NXITER,NEG,UPDATE,ATER  
COMMON/MIXLE/V1(200),XL1(200),AST,PTZ,UST,TAU,GAMMTR  
COMMON/DKEY1/XABL(400),UABL(400),DABL(400),IWRITE,  
1JBLNO,XTW(70),YTW(70),XSTEMP,NREAD,JAM,NUMY,  
2START,KSTART,NFORM,NAVE,KTRAKR,LOT,LOST,KETST,IFPR,  
3AVEAGE,ZAP,HERMAN,PORR,ERROR,XCHA,PERCG,SMALL,BIG,XCHA2,  
4OUTPUT,TOLER,AFCTR  
COMMON/DKEY2/LMT,NLMT,NEG,NXTRAP,MING,DELY,WV,DXF,DX7,  
1UREF,RF,RFD,RDFS,RHOD,RQME,LQRT,LOUT,LQRTM,PRINT,  
2LOOP,LOME,LMPE,MKITER,CMSTDX,YSTEP(150),JYSTEP,PGRAD1,  
3PGRAD2,IMG,CONXL,CONUI  
COMMON/DKEY3/XE,NP1,NP2,XP3(30),NPCR,XP4(30),NDEL,  
1NPOTN,XTREFT,SIGMA,XU(70),YU(70),NTOPL,NTIN,NTJ,TE,TWS,  
2CPS,PI,PRT,RCON,STA,INTNL,XFCTR,RESETX  
COMMON/STEP1/QFLOW  
COMMON/VEL/PSIN(200),PSIQ(200)  
COMMON/STEP/HSTEP,XSTEP,IWALL,IDELY,HIGHT  
COMMON/UVDST/PCON3,UP1,UP2,DP4,IP1,VIREF,OST1,OST2,  
1RH01(200),PCON6,PCONPB,DFIX,UT1,PD,XCONV,RHFL1,  
2AFLOW,X*X,F1(200),E(200),USTP  
COMMON/INITL/BLT,F1,F2,F3,GCN,JCQUT,JKOUNT,JKJN,  
1KJDEL,KTRAK,KKK,LTD,Licont,LCOUNT,MMN,MSTEP,MCOUNT,  
2NCQE,NCPC,NSTEP,NEG,NCARDN,NCOUNT2,NRESET,NNP,NPO,  
3PCON1,PCON2,PP,RANDTR,UT1,UT2,VRFF,RVF2,XC,XTREPT  
COMMON/NONDIM/IKE,NPR,LQRT,DXINC,DXINC,DXB,DX9,DX10,  
1PERCG,MM,DXDIS,UREF2,XDIST,RHRL,THEPS(200),TH1,DST,  
2IPC,DST3,TH2,NPC,NPT1,DFT,XMU1(200),RXX,RET,BETA,CF,  
3CFP,USTP,TAP,HS,UF2,UF1  
COMMON/ENG1/HS,H(100),ES,T(100),XVAR,H2(100),HH(100)  
C INITIALIZING COUNTERS AND LOGIC PARAMETERS  
AFLOW=QFLOW*RHOS/2.0  
DELY=LUT  
CALL STARTN(CDO,CPO)  
UREFOR=UREF  
LQRTOR=LQRT  
NJOE=NJ  
105 FORMAT(7G10.4)  
IF(LVOP.EQ.1)GO TO 3000  
PSIN(1)=0.0  
PSIO(1)=0.0  
DO 7710 J=2,200  
PSIO(J)=PSIO(J-1)+(U(J)+U(J-1))*(Y(J)-Y(J-1))/2.0  
7710 PSIN(J)=PSIO(J)  
GO TO 420  
C READ IN DOWNSTREAM PROFILE IF CALCULATION STARTED
C DOWNSTREAM OF LEADING EDGE

3000 CONTINUE

READ(9) BLT1, DFT, DELX, DX8, XDIST, XDINC, DX7, AST, PIZ, UREF,  
1UREF2, RRHF, RRHF2, TH2, TH, DST, DST2, DST3, VRFF, UT, UT2,  
2MCOUNT, NSTEP, JCOUNT, KCOUNT, KLJN, NEG, LJDDEL, NJ, N2J, MM,  
3Y, D, D, D, XL, LCOUNT, VREF2, NGO, NOU, PCON3, PCONPB, XKE,  
4KEJ, NCOND9, LORT, XTRIPT, RAMDTR, GAMMTR, NSTEP, ASTSTP,  
5DLSTEP, PS10, PNS

C PCON3 IS PRESSURE GRADIENT AT PREVIOUS X
C PCONPB IS PRESSURE AT TWO STEPS BACK

8411 DX11=DELX
   IF(KETST.GE.1) KEJ=0
   JPIST=MCOUNT

7006 FORMAT(3X, 'INITIAL PROFILE NONUNIFORM, HAS BE STARTED',  
*DOWNSTREAM')
   WRITE(6,7007) BLT1, DFT, DELX, DX8, XDIST, XDINC, DX7, AST, PIZ
   WRITE(6,7008) UREF, UREF2, RRHF, RRHF2, TH2, TH, DST
   WRITE(6,7009) MCOUNT
   WRITE(6,7770) PCON3, PCONPB

690 FORMAT('0 NCOND9= ',I3,' LORT= ',I3,' XTRIPT= ',G15.5,  
*RAMBDA= ',G15.5,' GAMMA= ',G15.5)  
7770 FORMAT('0 PCON AT INPUT= ',G14.5,' PCONPB= ',G14.5)  
*G14.5,2X,G14.5)  
*2X,G14.5,2X,G14.5)  
7009 FORMAT('0',MCOUNT= ',I5)  

IF(LCOMP) 7003, 7002, 7002

7002 DO 7004 J=2,200
   EPS(J)=1.0
   E(J)=RHOS
   RHOS(J)=RHOS
   XMU(J)=XMUS
   XMUS(J)=XMUS
   RHOS(J)=RHOS
   IF(KETST.GE.0) XKE(J)=0.0
   IF(KETST.GE.0) XKE1(J)=0.0

7004 XL1(J)=0.0
   DO 8341 J=1,N2J
   UI(J)=U(J)
8341 V1(J)=V(J)
   GO TO 420

7003 CONTINUE

420 WRITE(6,1004) (Y(J),J=1,NJ)
   WRITE(6,1004) (U(J),J=1,NJ)
   IF(LPOP) 231, 231, 232

232 IF(NREAD.NE.1) GO TO 2804

READ(5,105) SIGMA
2804 CONTINUE
WRITE(6,1000) SIGMA
1000 FORMAT(*0SIGMA = 'G14.5,/)  
IF(LPOP.LT.0) GO TO 231
WRITE(6,1004)(XU(J),J=1,LPOP)
WRITE(6,1004)(YU(J),J=1,LPOP)
CALL SPLICO(XU,YU,LPOP,CPO)
231 LDELT = NJ
IF(LCOMP) 102,103,102
C THESE STATEMENTS FOR ENERGY EQUATION, NOT PRESENTLY
C USED IN INVERSE MODE ............................
102 CONTINUE
CALL ENGYIN
103 CONTINUE
C BEGIN COMPUTATION LOOP
191 CMAX = 0.0
NJ = NJ
C MOUNT = NUMBER OF STEPS IN X TAKEN.  
MCOUNT = MOUNT + 1
LCOUNT = LCOUNT + 1
TTS = DELX
1300 CONTINUE
XXF = XDIST/XCONV
FCTR = FCTR
IF(XXF.LT.XCHA OR XXF.GT.XCHA) FCTR = 1.0
DELX = ABS(CNSTDX)*XCONV*FCTR
381 XDIST = DELX+XDIST
IF(MOUNT.EQ.1) XDIST = DELX
PCON = 0.0
XXF = XDIST/XCONV
XDEL = DELX+TTS
IF(INVL.EQ.1) GO TO 8283
IF(LPOP) 8283,8283,233
233 IF(MOUNT.GT.LQT) GO TO 8381
IF(MOUNT.LT.IABS(LDF)) GO TO 8381
CALL SFINT(XTW,XXF,NPD,NDEL,CDO,DFIX)
DFIX = DFIX*XCONV
GO TO 8460
8381 CALL SFINT(XU,XXF,NPP,LPOP,CPO,UREF1)
IF(MCOUNT.EQ.1) GO TO 8283
8460 CONTINUE
IF(MCOUNT.EQ.IABS(LDF)) WRITE(6,1111) MOUNT, XDIST
1111 FORMAT(*OMOUNT Хотя *16,'*XDIST(NONDIM)=*G14.5,  
*INVERSE SCHEME')
8283 CONTINUE
CALL UVNDST(INVL,UREF2,MSTEP,F3,ASTTP,DLSTEP)
JUSTP = JUSTP
IF(JUSTP.EQ.100) GO TO 180
719 CONTINUE
C COME TE Theta
LLJ=NJ
SUM2=0.0
DO 322 J=1,LLJ
IF(J.EQ.LLJ) GO TO 93
YDE1=Y(J+1)-Y(J)
IF(J.EQ.1) YDE2=YDE1
IF(J.EQ.1) GO TO 94
93 YDE2=Y(J)-Y(J-1)
94 CONTINUE
IF(J.EQ.LLJ) YDE1=YDE2
YDE3=(YDE1+YDE2)/2.
YY1=RHO1(J)*US*U1(J)*(1.0-US*U1(J)/UREF1)/(RHRF1*UREF1)
SUM2=SUM2+YY1*YDE3
IF(J.EQ.1 OR J.EQ.LLJ) SUM2=SUM2-YY1*YDE3/2.0
322 CONTINUE
TH1 = SUM2
REX=RHRF1*UREF1*XDIST/(XMU1(NJ)*XCOND)
RET=REX*TH1/XDIST
IF(GAMMTR.LT.0.99 ) GO TO 612
GAMMTR=1.0
GO TO 610
612 YTRAN9=Y(KJDEI)
CALL TRAN9(REX,RET,GAMMTR,NCOND,LORT,XTRIPT,AST,
*RAMTR,U1,XDIST,XTRFPT,NOPTN,YTRAN9,DST1)
610 CONTINUE
C COMPUTE SHEAR AND SHAPE FACTORS ETC.
IF(MCOUNT=1) 382,382,383
382 UST = US
TAU = UST*UST*RHO1(1)/GCON
T2 = SQRT(TAU)
GO TO 384
383 XDEL = DELX+ITS
ZX = XMU1(1)*U1(2)*US*XCOND/(RHO1(1)*Y(2))
ZXP=(18.0*U1(2)-9.0*U1(3)+2.0*U1(4))/6.0
ZXP=ZX*ZXP/U1(2)
ZX = ABS(ZX)
ZXP=ABS(ZXP)
UST = SQRT(ZX)
USTP=SQRT(ZXP)
TAU = UST*UST*RHO1(1)/GCON
TAUP=USTP*USTP*RHO1(1)/GCON
H1 = DST1/TH1
CF=TAU*2.0*GCON/(RHRF1*UREF1*UREF1)
CFP=CF*TAUP/TAU
BETA = RHO1(1)*US*US*PCON*DIST/(TAU*GCON)
XG = CF/2.0
XG = ABS(XG)
XG = SQRT(XG)
G=(H1-1.0)/(H1*XG)
DXDISD=XDIST/XCOND
9080 IF(ZAP.GE.1.0)GO TO 888
   IF(IWRITE.LT.0) GO TO 2998
   WRITE(6,7777)PCON,MXITER,DXDISD,MCOUNT,VI(NJ-7),DX7*
   *DELFX,JDFX,KJDEL
2998 CONTINUE
   UW=UI(2)*US
   DDIX=DFIX/XCONV
   IF(IWRITE.LT.0) GO TO 2999
   WRITE(6,7778)UREF1,UW,TAU,TAUP,CF,GAMMTR
   WRITE(6,8405)DDIX,F3
2999 CONTINUE
7778 FORMAT(4X,' UE ',G14.5,' U(2) ',G14.5,' TAU ',G14.5,
      ** TAUP ',G14.5,' CF ',G14.5,' GAMMA ',G14.5)
7777 FORMAT(2X,'PCON ','G14.5, ' MXIT ',I2,' DXDIS ',G14.5,
      1' MCOUNT ',I4,' V1 ',G12.4,' DX7 ',G12.4,' DELX ',G14.5,
      2' LJ ',I3,' KJ ',I3)
8405 FORMAT(4X,'DFIX ',G14.5,' F3 ',G14.5)
C DETERMINE PRINTOUT LOCATION
888 IF(UI(2))815,815,384
815 NEG=NEG+1
   IF(NEG.EQ.2.AND.LORT.LT.0) GO TO 61
   IF(MROP.LT.0)GO TO 384
   GO TO 61
384 IF(LORT)333,331,331
331 CALL MIXLEN(ASTSTP,DLSTEP,UREF)
   IF(MSTEP.EQ.10) GO TO 65
   IF(KETST.GE.0) GO TO 718
   IF(MCOUNT.LT.IKE) GO TO 718
   IF(MCOUNT.GT.IKE) GO TO 8501
   P=FST*UREF1*UREF1/US/US
   DO 8502 J=2,NJ
      XKE(J)=XL(J)*(U(J+1)-U(J-1))*(U(J+1)-U(J-1))/Y(J+1)-
      Y(J-1))/(Y(J+1)-Y(J-1))/XKE/CKE
      IF(XKE(J).LT.P)XKE(J)=P
8502 CONTINUE
8501 CALL TKE
718 CONTINUE
65 CONTINUE
   IF(MSTEP.EQ.10) GO TO 61
333 IF(NPRINT)223,223,224
223 IF(NPT1)229,61,61
229 IF(NPC-NP1)230,230,180
230 IF(NP2)225,226,227
225 DXDIS = XDIST/XCONV
   IF(DXDIS-XP3(NPC))60,228,228
226 REX = RHRF1*UREF1*XDIST/(XMU1(NJ)*XCONV)
   IF(REX-XP3(NPC))60,228,228
227 RET = RHRF1*UREF1*TH1/(XMU1(NJ)*XCONV)
   IF(RET-XP3(NPC))60,228,228
228 NPT1 = 1
NPC=NPC+1
GO TO 61

224 IF(LCOUNT - NPRINT) 60, 61, 61
61 CONTINUE
CALL PR1NOT(DU, DY, DV, NSTART)
IF(LORT.LT.0) GO TO 60
GO TO 333

60 CONTINUE
IF(IWALL.GE.0) GO TO 25
IF(NRESET.EQ.-1) GO TO 25
IF(DXDIS.LT.RESETX) GO TO 25
NRESET=-1
CALL XRESET(U1, V1, XKE1, RH01, XMU1, XL1, Y, U, V, LJDEL, NJ, 1N2J, LINOP, NSTART, PSIN)

NJT=NJ
DO 26 J=1, NJ
DY(J)=Y(J)/XCONV
DU(J)=U1(J)*US
26 DV(J)=V1(J)*US
WRITE(6, 27) DXDIS
27 FORMAT(*,5(*,*)**, Y GRID SPACING WAS CHANGED SO THAT**, ** ALL VARIABLES ARE INTERPOLATED AT X =*, G16.5, 5(*,*)/)
WRITE(6, 1002) (DY(J), J=1, NJ)
WRITE(6, 1003)
WRITE(6, 1004) (DU(J), J=1, NJ)
WRITE(6, 1005)
WRITE(6, 1004) (DV(J), J=1, NJ)
WRITE(6, 28)
WRITE(6, 1004) (XL1(J), J=1, NJ)
28 FORMAT(*,2X,5HXL(J))

C ADD 7 POINTS TO NJ TO BE SURE EOG& IS WITHIN
C CALCULATION RANGE ..............................
N1J = NJ+1
N2J=MAX0(NJ, NJT)
N2J=N2J+7
DO 770 J=N1J, N2J
D(J) = D(NJ)
E(J)=E(NJT)
U(J)=U(NJ)
V(J)=V(NJ)
U1(J)=U1(NJ)
V1(J)=V1(NJ)
XL1(J)=XL1(NJ)
XKE(J)=XKE(NJ)
XKE1(J)=XKE1(NJ)
D2(J)=D2(NJ)
770 XL(J)=XL(NJ)
C UPDATING ALL VARIABLES
804 CONTINUE
CALL UPDATN(OHDEL,BUREF2,BVREF2,N2J,DY,DU,DV)

CONTINUE

DXDIS= XDIST/XCONV
IF(DXDIS.LT.XCHA)GO TO 807
NSTEP=NSTEP+1
IF(NSTEP.GT.LMT)GO TO 800
GO TO 807

800 NSTEP=0
IF(DXDIS.LT.XCHA2)GO TO 813
IF(DX7.LT.DX10)GO TO 814
DX7=DX10
GO TO 807

814 DX7=PERCG*DX7
GO TO 807

813 IF(DX7.LE.DX9)GO TO 812
DX7=PERCG*DX7
GO TO 807

812 DX7=DX9

807 DX82=DX8
PTZ=Y(KJDEL)
DX8=DX7*Y(KJDEL)
IF(LPOP.LE.0)GO TO 790
IF(PCON.LE.0.0)GO TO 790
IF(ABS((UREF1-UREF2)/UREF1).LT.0.0001)GO TO 790
ZZ= ABS(UREF1*UREF*SIGMA/(PCON*US*US))
IF(ZZ.GT.0X810)GO TO 791
DX7=DX7/10.0
GO TO 790

791 DX8=ZZ
KCOUNT=KCOUNT+1
IF(KCOUNT.GE.1)GO TO 790
JCOUNT=MCOUNT
WRITE(6,1)DX8,DELX,MCOUNT
1 FORMAT(' SIGMA HAS TAKEN EFFECT DX8= ',G14.5,' DELX=',G14.5,' MCOUNT=',I5)

790 NJ=LJDEL+7
ICOUNT=MCOUNT-JCOUNT+1
IF(ICOUNT-KCOUNT).LT.3)GO TO 5
GO TO 3

5 KCOUNT=0
JCOUNT=0
WRITE(6,2)DX8,DELX,MCOUNT
2 FORMAT(' SIGMA HAS DISCOUNTED DX8= ',G14.5,' DELX=',G14.5,' MCOUNT= ',I5)

3 LJDEL=NJ
C WRITE DATA ON DISK IF DESIRED FOR STARTING
C PROFILE OF SUBSEQUENT RUN ..................
IF(LOUTD).LT.9)GO TO 9


SUBROUTINE INVISCJUSTP, OSTUIC, DSTLIC, ITERTN, NREAD
C THIS PROGRAM PROVIDES THE INVISCID FLOW SOLUTION. THE
C LAPLACE EQUATION FOR STREAMFUNCTION IS SOLVED USING
C AN ADI METHOD WITH SOR.
C DIMENSION ZETA(60), ETA(60), AA(60), BB(60), CC(60), DD(60),
1 DSTUIC(60), DSTLIC(60)
2 COMMON/LPLACE/DSTUIC(60), DSTLIC(60), XIC(60), NXIC, NYIC,
3 CPOL(4,60), UINV(60), UINVU(60), UEDST, TOLEV, RCONST,
4 PSI(60,60), PSI0(60,60)
COMMON/STEP1/QFLOW
COMMON/STEP/HSTEP,XSTEP,IWALL,IDELY,HIGHT
ITER=0
NLIMIT=80
NXM=NXIC-1
NXM2=NXIC-2
NYM=NYIC-1
NYM2=NYIC-2
THIGHT=HIGHT+HSTEP
DO 2000 I=1,NXIC
   DSTUI(I)=THIGHT-DSTUIC(I)
   DSTLI(I)=DSTLIC(I)
   IF(IXC(I).LE.XSTEP) DSTLI(I)=DSTLIC(I)+HSTEP
2000 CONTINUE
C.............. SPECIFYING BOUNDARY CONDITIONS AND INITIALIZING
C PSI'S

   DZETA=1.0/NXM
   DETA=1.0/NYM
   ZETA(1)=0.0
   ZETA(NXIC)=1.0
   ETA(1)=0.0
   ETA(NYIC)=1.0
   DO 2 J=2,NYM
      ETA(J)=ETA(J-1)+DETA
   DO 3 J=2,NXM
      ZETA(J)=ZETA(J-1)+DZETA
   IF(ITERN.LE.1) GO TO 6
   IF(NREAD.GT.1) GO TO 7
   READ(12) PSI
   GO TO 7
6 CONTINUE
   DO 132 I=1,NXIC
      DO 132 J=1,NYIC
      PSI(I,J)=ETA(J)
132 CONTINUE
C............. PRINOUT INITIALIZED VALUES

   WRITE(6,5000)(ZETA(J),J=1,NXIC)
5000 FORMAT(3X,'ZETA : ' ,10(F10.6))
   WRITE(6,5001)(ETA(J),J=1,NYIC)
5001 FORMAT(3X,'ETA : ' ,10(F10.6))
   WRITE(6,5002)(PSI(I,1),I=1,NXIC)
5002 FORMAT(3X,'PSI(I,1) : ' ,10(F10.6))
   WRITE(6,5003)(PSI(1,NYIC),I=1,NXIC)
5003 FORMAT(3X,'PSI(1,NY) : ' ,10(F10.6))
   WRITE(6,5004)(PSI(1,J),J=1,NYIC)
5004 FORMAT(3X,'PSI(1,J) : ' ,10(F10.6))
   WRITE(6,5005)(DSTUI(J),J=1,NXIC)
5005 FORMAT(3X,'DSTUI : ' ,10(F10.6))
   WRITE(6,5006)(DSTLI(J),J=1,NXIC)
5006 FORMAT(3X,'DSTLI : ' ,10(F10.6))
C        ADI WITH SOR  

AL=XIC(NXIC)-XIC(1)  
DO 4 I=2,NXM  
CPOL(1,1)=DSTLI(I+1)-DSTLI(I-1)  
CPOL(2,1)=DSTUI(I+1)-DSTLI(I+1)-DSTUI(I-1)+DSTLI(I-1)  
CPOL(3,1)=DSTLI(I+1)-2.0*DSTLI(I)+DSTLI(I-1)  
CPOL(4,1)=DSTUI(I+1)-DSTLI(I+1)-2.0*DSTUI(I)+2.0*  
1DSTLI(I)+  
1DSTUI(I-1)-DSTLI(I-1)  
4 CONTINUE  
19 CONTINUE  
DMAX=0.0  
DO 8 J=2,NYM  
DO 5 I=2,NXM  
DSTUL=DSTUI(I)-DSTLI(I)  
C1=CPOL(1,1)  
C2=CPOL(2,1)  
C3=CPOL(3,1)  
C4=CPOL(4,1)  
SB=(4.0*DZETA*AL*DZETA*AL+(C1+ETA(J)*C2)*(C1+  
1ETA(J)*C2)/(4.0*DETA*DSTUL*DSTUL)  
SC=(C2*(C1+ETA(J)*C2)/(2.0*DSTUL*DSTUL)-(C3+ETA(J)*C4)/  
1DSTUL)/(2.0*DETA)  
SA=(C1+ETA(J)*C2)/(4.0*DSTUL*DETA)  
AA(I)=1.0  
BB(I)=1.0  
DD(I)=-2.0*(1.0+SB)  
5 CC(I)=SA*(PSI(I+1,J+1)-PSI(I+1,J-1)-PSI(I-1,J+1)+  
1PSI(I-1,J-1))-SB*(PSI(I,J+1)+PSI(I,J-1))+SC  
1*(-PSI(I,J)+1)+  
2PSI(I,J-1))  
AA(I)=0.0  
BB(I)=0.0  
CC(I)=PSI(I,J)  
DD(I)=1.0  
AA(NXIC)=0.0  
BB(NXIC)=0.0  
CC(NXIC)=PSI(NXIC,J)  
DD(NXIC)=1.0  
DO 11 I=2,NXIC  
BPERD=-BB(I)/DD(I-1)  
DD(I)=DD(I)+AA(I-1)*BPERD  
11 CC(I)=CC(I)+CC(I-1)*BPERD  
DO 12 I=2,NXM  
K=NXIC-I+1  
12 CC(K)=(CC(K)-AA(K)*CC(K+1))/DD(K)  
DO 13 I=2,NXM  
PSI(I,J)=PSI(I,J)  
13 PSI(I,J)=RCONST*CC(I)+(1.0-CONST)*PSI(I,J)
CONTINUE
DMAX1=0.0
DO 14 I=2,NXM
DO 15 J=2, NYM
DSTUL=DSTUL(I)-DSTLI(I)
C1=CPOL(1, I)
C2=CPOL(2, I)
C3=CPOL(3, I)
C4=CPOL(4, I)
SB=(4.0*OZETA*AL*DZETA*AL+CL+ETA(J)*C2)/(1.0+ETA(J)*C2)
SC=(C2*(C1+ETA(J)*C2)/(2.0*DSTUL*DSTUL)-(C3+ETA(J)*C4)/
(DSTUL)/(2.0*DETA)
SA=(C1+ETA(J)*C2)/(4.0*DSTUL*DETA)
AA(J)=SB+SC
BB(J)=SB-SC
DD(J)=-2.0*(1.0+SB)
CC(J)=SA*(PSI(I+1, J+1)-PSI(I+1, J-1)-PSI(I-1, J+1)+
PSI(I-1, J-1))-
1PSI(I+1, J)-PSI(I-1, J)
AA(1)=0.0
BB(1)=0.0
CC(1)=PSI(1, 1)
DD(1)=1.0
AA(NYIC)=0.0
BB(NYIC)=0.0
CC(NYIC)=PSI(I, NYIC)
DD(NYIC)=1.0
DO 16 J=2, NYIC
BPERD=-BB(J)/DD(J-1)
DD(J)=DD(J)+AA(J-1)*BPERD
CC(J)=CC(J)+CC(J-1)*BPERD
DO 17 J=2, NYM
K=NYIC-J+1
CC(K)=(CC(K)-AA(K)*CC(K+1))/DD(K)
DO 18 J=2, NYM
DPSI=(PSI(I, J)-CC(J))/CC(J)
DPSI1=(PSI(I, J)-CC(J))/CC(J)
DPSI=ABS(DPSI)
DPSI1=ABS(DPSI1)
IF(DPSI.GT.0.0*DMAX)DMAX=DMAX
IF(DPSI1.GT.0.0*DMAX1)DMAX1=DMAX1
PSI(I, J)=RCONST*CC(J)+(1.0-CONST)*PSI(I, J)
18 CONTINUE
14 CONTINUE
WRITE(6,111) ITER, DMAX, DMAX1
111 FORMAT(10X,' ITER, DMAX, DMAX1 = ', 3G15.6)
IF(DMAX.LE.TOLEVP) GO TO 20
IF(ITER.GE.NLIMIT) GO TO 30
ITER = ITER + 1
GO TO 19
30 JUSTP = 100
WRITE(6, 105)
105 FORMAT (/,'**', 'SOLUTION FAILED TO CONVERGE', '**')
GO TO 40
20 WRITE(6, 103) ITER

C ************ CALCULATION VELOCITIES ALONG BOUNDARIES ************
UINVU(1) = UEOSL
UINVL(1) = UEOSL
DO 21 I = 2, NXIC
DSTUL = DSTUI(I) - DSTLI(I)
UINVL(I) = QFLOW * (PSI(I, 2) - PSI(I, 1)) / (DETA * DSTUL)
21 UINVU(I) = QFLOW * (PSI(I, NYIC) - PSI(I, NYM)) / (DETA * DSTUL)

C ********** SMOOTHING USING THREE POINTS **********
DO 22 I = 2, NXM
UINVL(I) = (UINVL(I - 1) + UINVL(I) + UINVL(I + 1)) / 3.0
22 UINVU(I) = (UINVU(I - 1) + UINVU(I) + UINVU(I + 1)) / 3.0

C ********** PRINOUT INVISCID SOLUTIONS **********
WRITE(6, 106) (UINVL(I), I = 1, NXIC)
106 FORMAT (/,' ', 3X, ' UINVL ', 10(G12.5))
WRITE(6, 107) (UINVU(I), I = 1, NXIC)
107 FORMAT (/,' ', 3X, ' UINVU ', 10(G12.5))

DO 108 J = 1, NYIC
WRITE(6, 108) (PSI(I, J), I = 1, NXIC)
108 FORMAT (/,' ', 3X, 10(F10.6))

C SUBROUTINE UPDATE Boundary Layer SOLUTIONS
C
C THIS PROGRAM IS FOR UPDATING ALL THE PARAMETERS FOR
C THE BOUNDARY-LAYER SOLUTIONS BEFORE PROCEEDING TO THE
C NEXT STREAMWISE STATION.
C
DIMENSION DY(200), DU(200), DV(200)
COMMON/TVEL/DELX, BTS, UREF1, A(200), B(200), C(200), D(200),
1 MU(200), KE(200), KE1(200), Y(200), V(200), U(200),
2 AP(200), RHO(200), XU(200), U1(200), CMAP, PCON, PRK, CKE, FST,
3 KEJ, RHOJS, XMUS, DELTA, PRS, US, TEST, DELT, NJ, MCOUNT, DLF,
4 LDEL1, KJDEL, NOU, NNEG, ITER, NOU2, MXITER, NEGO, UPDATE, MITER
COMMON/MIXLE/V1(200), XL1(200), ASL, PTZ, UST, V1, GAMMTR
COMMON/INITL/BLT, F1, F2, F3, GCON, JCOUNT, JH, KCOUNT, KLJN,
1KFJDEL,KTRAK,KKK,LTD,LCOUNT,LCOUNT,MNM,MSTEP,NCOU,
2NCOUE,NPCC,NSTEP,NEG,NCARDS,NCNDA9,NRESET,NPP,NPD,
3PCON1,PCON2,PP,RAMDIR,UT,UT2,VRFF,VRFF,XC,XTIERT
COMMON/NONDIM/IKE,N1PR,LOT,DXINC,DXINC,DX8,DX9,DX10,
1PERCGB,MM,DXDIS,UREF2,XDIST,RHRF,TH,TH1,TH2,DST,
2IPC,DST3,TH2,NPC,NPT1,DFT,UXM1(200),RER,RET,BETA,CF,
3CFP,USTP,TAUP,HI,G,UF2,UF1
COMMON/UVST/PCON3,UP1,UP2,DP4,IP1,UREF,DST1,DST2,
1RH01(200),PCON6,PCONPB,DFx,UT1,PD,XCONV,RHRF1,
2AFLW,X,XDF,D2(200),E(200),JUSTP
COMMON/STEP/HSTEP,XSTEP,IWALL,IDE,YGON,HIGHT
COMMON/KEY2/LMT,NLMT,NNEG,NXTRP,MIN,DELAY,WW,DXF,DX7,
1UREF,RE,RFU,RFDS,MRP,LORT,INOP,LOUTD,LCOMP,NPRINT,
2LOOP,LVOP,LPDP,MKITER,CNSTDX,YSTEP(150),JYSTEP,PGRAD1,
3PGRAD2,IPG,CONXL,CONU1
COMMON/VEL/PSIN(200),PSIQ(200)
DMDEL=TT1
RHRF2 = RHRF
BUREF2=UREF2
PCONPB=PCON3
PCON3=PCON
UREF2 = UREF
RHRF = RHRF1
UREF=UREF1
UT2=UT
UT=UT1
TH2=TH
TH = TH1
DST = DST1
DST3=DST2
DST2=DST1
BVREF2=D2(NJ-7)
VRF2 =VRFF
VRFF=V1REF
DO 300 J = 1,N2J
D2(J)=V(J)
D(J) = U(J)
E(J) = RH0(J)
U(J) = U1(J)
V(J) = V1(J)
XKE(J)=XKE(J)
RHO(J) = RH0(J)
XMU(J) = XMU(J)
PSIQ(J)=PSIQ(J)
300 XL(J) = XL1(J)
WIDTH MODIFICATION FOR A REARWARD-FACING STEP
DUDX=DXDIST/XCONV
IF(MSTEP.EQ.10) MSTEP=100
IF(IWALL.GE.0) GO TO 806
IF(MSTEP.GT.0) GO TO 806
IF(DXDLS.LT.XSTEP) GO TO 806
MSTEP=10
DDELY=DELY/XCONV
IF(LINOP) 440,20,20

440 CONTINUE
ANYADD=HSTEP/DDELY+1.1
NYADD=ANYADD
NYADD=NYADD-1
GO TO 21
20 NYADD=JSTEP
21 LJDEL=LJDEL+NYADD
LJDELP=LJDEL-NYADD
DO 10 J=1,LJDELP
10 XL(J)=XL1(J)
DO 430 I=1,N2J
J=N2J-I+1
K=J+NYADD
DY(K)=Y(J)/XCONV+HSTEP
DU(K)=U1(J)*US
DV(K)=V1(J)*US
U1(K)=U1(J)
V1(K)=V1(J)
D(K)=D(J)
E(K)=E(J)
D2(K)=D2(J)
XKE1(K)=XKE1(J)
RH01(K)=RH01(J)
XMU1(K)=XMU1(J)
PSIN(K)=PSIN(J)

430 XL1(K)=XL1(J)
DY(1)=0.0
IF(LINOP)22,23,23
22 DO 432 J=2,NYADD
432 DY(J)=DY(J-1)+DDELY
GO TO 442
23 DO 24 J=2,NYADD
24 DY(J)=YSTEP(J)
442 CONTINUE
DO 433 J=1,NYADD
DU(J)=0.0
DV(J)=0.0
U1(J)=0.0
V1(J)=0.0
D(J)=0.0
E(J)=E(NYADD+1)
D2(J)=0.0
XKE1(J)=XKE1(NYADD+1)
RH01(J)=RH01(NYADD+1)
XMU1(J)=XMU1(NYADD+1)
PSIN(J)=0.0
SUBROUTINE UVNPSI(JUSTP,LORT,DFIX,TOLER,UREF,RHO1, 
INTNL,AFLOW,IIP,LQT,IWRITE) 
C THIS PROGRAM PROVIDES THE SOLUTION OF THE BOUNDARY-
C LAYER EQUATIONS FOR EXTERNAL AND INTERNAL FLOWS.
C THE CONTINUITY AND MOMENTUM EQUATIONS ARE SOLVED IN
C A COUPLED MANNER. THE PRESSURE GRADIENT IN THE INVERSE
C METHOD IS OBTAINED AS A SOLUTION. A FULLY IMPLICIT
C FINITE-DIFFERENCE ALGORITHM WITH NEWTON LINEARIZATION
C IS USED.
DIMENSION AA(200),BB(200),CC(200),DD(200),EE(200), 
SHH(200),RHO1(200) 
COMMON/VEL/PSIN(200),RHO(200) 
COMMON/TVEL/DELY,TTST,UREF1,T(200),B(200),C(200),D(200), 
1XMU(200),XKE(200),XKE1(200),Y(200),V(200),U(200), 
2AP(200),RHO(200),XL(200),U1(200),CMAIC,PCON,PRK,CKE,FST, 
3KEJ,RHOS,XMUS,DELT1,PRS,US,TEST,DELT,NJ,MCOUNT,LDF,
4LJ0EL,KJDEL,NOU,NEGU,ITER,NOU2,MXITER,NEGO,UPDATE,NITER
COMMON/V1(200),XL1(200),AST,PTZ,UST,TAU,GAMMT
COMMON/STEP/HSTEP,XSTEP,IWALL,IDELY,HIGHT
IIP5=IIP+5
NIT=0
NEGO=0
XCONV=US*RHOS/XMUS
XHITE=HIGHT*XCONV
LJ0M7=LJ0EL-7
C IF INTNL = 100, FLOW BECOMES FULLY DEVELOPED.
IF(INTNL.EQ.100) GO TO 800
IF((INTNL.EQ.100) .AND. (Y(LJ0M7).GE.XHITE/2.0))GO TO 801
LJ0ELP=LJ0EL+1
NJPl=NJ+1
NJP2=NJ+2
U(NJP1)=U(NJ)
U(NJP2)=U(NJ)
A(1)=XMU(1)/XMUS
XKE(NJP1)=XKE(NJ)
GO TO 802
801 INTNL=100
WRITE(6,803) MCOUNT
803 FORMAT("******** FLOW BECOMES FULLY DEVELOPED.*
1* AT MCOUNT =",15)
800 NJ=LJ0M7
NJPl=NJ+1
U(NJP1)=U(NJ-1)
XL(NJP1)=XL(NJ-1)
XKE(NJP1)=XKE(NJ-1)
Y(NJ)=XHITE/2.0
Y(NJ+1)=Y(NJ)+(Y(NJ)-Y(NJ-1))
PSIO(NJ)=AFLOW*XCONV/(RHOS*US)
PSIO(NJP1)=PSIO(NJ)+0.5*(U(NJ)+U(NJP1))*(Y(NJP1)-Y(NJ))
PSIN(NJ)=PSIO(NJ)
PSIN(NJP1)=PSIO(NJP1)
U(NJP1)=U(NJP1)
802 CONTINUE
DO 411 J=2,NJP1
IF(KEJ.EQ.0) GO TO 415
IF(J.LT.KEJ)GO TO 415
A(J)=XMU(J)/XMUS+RHO(J)*XL(J)*A7*GAMMT/(Y(J+1)-Y(J-1))
1*RHOS
GO TO 411
415 A7=ABS(U1(J+1)-U1(J-1))
A(J)=XMU(J)/XMUS+RHO(J)*XL(J)*A7*GAMMT/((Y(J+1)-Y(J-1))
1*RHOS)
411 CONTINUE
B(1)=A(1)
DO 413 J=2,NJ
413 B(J)=(A(J+1)+A(J-1))/3.0
B(NJP1)=A(NJP1)

IF(INTNL.EQ.100) GO TO 16

DO 12 K=LJDELP,NJP1

PSIO(K)=0.5*(U(K)+U(K-1))*Y(K)-Y(K-1)+PSIO(K-1)

12 CONTINUE

16 CONTINUE

DO 100 J=2,NJ

UHALF=U1(J)

DYP=Y(J+1)-Y(J)

DYM=Y(J)-Y(J-1)

DYT=DYP+DYM

CXP=0.5*(B(J+1)+B(J))/DYP

CXM=0.5*(B(J)+B(J-1))/DYM

AA(J)=-(PSIN(J)-PSIO(J))/DELX+2.0*CXP)/DYT

BB(J)=((PSIN(J)-PSIO(J))/DELX-2.0*CXM)/DYT

CNC=1.0

IF(UHALF.LT.0.0) CNC=0.0

CC(J)=CNC*U1(J)*U1(J)/DELX-PSIN(J)*U1(J+1)-$U1(J-1))/DELX*DYT)

100 EE(J)=-(U1(J+1)-U1(J-1))/(DELX*DYT)

AA(1)=0.0

BB(1)=0.0

CC(1)=0.0

DD(1)=0.0

EE(1)=0.0

HH(1)=0.0

DO 101 J=2,NJ

TA=AA(J)

TB=BB(J)

TC=CC(J)

TD=DD(J)

TE=EE(J)

TH=1.0

DYM=Y(J)-Y(J-1)

TP=DYM/2.0

TSC=0.0

SQ2=BB(J-1)+TP*(1.0+AA(J-1))

SQI=TD+(TB+TE*TP)*AA(J-1)+TE*(BB(J-1)+TP)

AA(J)=-TA/SQ1

BB(J)=AA(J)*SQ2

CC(J)=(TC-TB*CC(J-1)+TE*(TSC-TP*CC(J-1)-EE(J-1)))/SQI

HH(J)=(TH-TB*HH(J-1)-TE*(DD(J-1)+TP*HH(J-1)))/SQI

EE(J)=-TSC+EE(J-1)+TP*CC(J-1)+CC(J)*SQ2

101 DO( J)=+TP*HH(J-1)+DD(J-1)+HH(J)*SQ2

NJM=NJ-1

DMAX=0.0

IF(INTNL.NE.1) GO TO 20

C THE EVALUATION OF THE PRESSURE GRADIENT
AND CENTERLINE VELOCITY FOR A SYMMETRICALLY DEVELOPING CHANNEL FLOW

\[
FA = XHfTE/2.0 - Y(NJ) + BB(NJM) + TP*(1.0 + AA(NJM))
\]

\[
FB = -(DD(NJM) + TP*H(NJM))
\]

\[
FC = AFL0W*XCONV/(RHOS*US) - EE(NJM) - TP*CC(NJM)
\]

\[
FB = FB/FA
\]

\[
FC = FC/FA
\]

\[
PBETA = (FC*(2.0*U1(NJ) - U(NJ)) - U1(NJ)*U1(NJ))/DELX - 1FB*(2.0*U1(NJ) - U(NJ))
\]

\[
U1NJ = FB*PBETA + FC
\]

\[
PSINJ = AFL0W*XCONV/(RHOS*US) - U1NJ*(XHfTE/2.0 - Y(NJ))
\]

GO TO 71

20 CONTINUE

IF(INTNL.EQ.100) GO TO 804

IF(MCOUNT.GT.1IP5) GO TO 70

IF(MCOUNT.GT.LQT) GO TO 70

THE EVALUATION OF THE PRESSURE GRADIENT, EDGE VELOCITY AND EDGE STREAMFUNCTION FOR INVERSE BOUNDARY-LAYER SOLUTIONS.

\[
FA = Y(NJ) - DFIx - BB(NJM) - TP*(1.0 + AA(NJM))
\]

\[
FB = DD(NJM) + TP*H(NJM)
\]

\[
FC = EE(NJM) + TP*CC(NJM)
\]

\[
FB = FB/FA
\]

\[
FC = FC/FA
\]

\[
PBETA = (FC*(2.0*U1(NJ) - U(NJ)) - U1(NJ)*U1(NJ))/DELX - 1FB*(2.0*U1(NJ) - U(NJ))
\]

\[
U1NJ = FB*PBETA + FC
\]

\[
PSINJ = U1NJ*(Y(NJ) - OFIX)
\]

GO TO 71

70 CONTINUE

THE EVALUATION OF THE EDGE STREAMFUNCTION FOR A DIRECT BOUNDARY-LAYER SOLUTION.

\[
U1NJ = UREF1/US
\]

\[
PBETA = (1(2.0*U1(NJ) - U(NJ)))*U1NJ - U1(NJ)*U1(NJ))/DELX
\]

\[
U1NJM1 = AA(NJM)*U1NJ + HH(NJM)*PBETA + CC(NJM)
\]

\[
PSINJM = BB(NJM)*U1NJ + DD(NJM)*PBETA + EE(NJM)
\]

\[
PSINJ = PSINJM + (U1NJ + U1NJM1)*(Y(NJ) - Y(NJM))/2.0
\]

GO TO 71

804 CONTINUE

THE EVALUATION OF THE PRESSURE GRADIENT AND CENTERLINE VELOCITY FOR SYMMETRICALLY DEVELOPED CHANNEL FLOW.

\[
NJM2 = NJ-2
\]

\[
ALX = 4.0 - AA(NJM2)
\]

\[
AL1 = 1.0 - AA(NJM)*ALX/3.0
\]

\[
AL2 = (ALX*HH(NJM) - HH(NJM2))/3.0
\]

\[
AL3 = (ALX*CC(NJM) - CC(NJM2))/3.0
\]

\[
AM1 = 1.0 + BB(NJM) + TP + AA(NJM)
\]

\[
AM2 = -(HH(NJM) + DD(NJM))/TP
\]

\[
AM3 = AFL0W*XCONV/(RHOS*US) + TP - EE(NJM) + TP - CC(NJM)
\]
\[ PBETA = \frac{(AL1*AM3-AL3*AM1)}{(AL2*AM1-AL1*AM2)} \]

\[ UINJ = AL2*PBETA/AL1+AL3/AL1 \]

\[ PSINJ = AFLOW*XCONV/(RHOS*US) \]

71 CONTINUE

\[ DUJ = U1(NJ) - UINJ \]

\[ DPJ = PSIN(NJ) - PSINJ \]

\[ DUJ = DUJ/UINJ \]

\[ DPJ = DPJ/PSINJ \]

\[ U1(NJ) = UINJ \]

\[ PSIN(NJ) = PSINJ \]

\[ IF(ABS(DUJ)*GT.DMAX) DMAX = ABS(DUJ) \]

\[ IF(ABS(DPJ)*GT.DMAX) DMAX = ABS(DPJ) \]

C THE CALCULATION OF U AND PSI ACROSS THE BOUNDARY LAYER

C OR THE CHANNEL. THE MAXIMUM VARIATION OF THE U AND

C PSI IN TWO SUCCESSIVE ITERATIONS IS ALSO EVALUATED.

DO 102 I = 2, NJM

\[ J = NJ - I + 1 \]

\[ U1(J) = AA(J) * U1(J+1) + HH(J) * PBETA + CC(J) \]

\[ PSIJ = BB(J) * U1(J+1) + DD(J) * PBETA + EE(J) \]

\[ DUJ = ABS(U1(J) - U1(J)) \]

\[ DPJ = ABS(PSIN(J) - PSIJ) \]

\[ DUJ = DUJ/UINJ \]

\[ DPJ = DPJ/PSINJ \]

\[ U1(J) = UINJ \]

\[ PSIN(J) = PSIJ \]

\[ IF(DUJ.GT.DMAX) DMAX = DUJ \]

102 IF(DPJ.GT.DMAX) DMAX = DPJ

IF(IWRITE.LT.0) GO TO 9999

WRITE(6,120) NIT, DMAX, PBETA, U1(NJ)

9999 CONTINUE

\[ UREF = U1(NJ) * US \]

\[ IF(INTNL.EQ.100) GO TO 805 \]

C FINDING THE LOCATION OF THE BOUNDARY-LAYER EDGE.

\[ DELT1 = -1.0 \]

\[ DELT = -1.0 \]

DO 107 J = 2, NJ

107 IF(DELT1) 360, 360, 361

360 IF(U1(J)*US/UREF-0.99) 573, 47, 41

361 KJDEL = J

362 IF(U1(J) * US/UREF - TEST) 47, 41, 41

47 LJDNL = J

41 IF(U1(J) * US/UREF - TEST) 772, 772, 573

573 IF(DDEL) 400, 400, 713

400 IF(U1(J) * US/UREF - TEST) 47, 41, 41

41 LJDNL = J

713 U1(J) = UREF/US

47 IF(U1(J) * LE.0.0001) NEG = J

107 CONTINUE

\[ U1(NJP1) = U1(NJ) \]
GO TO 806
805 U1(NJP1)=U1(NJ-1)
806 CONTINUE
   IF(DMAX.LE.TOLER) GO TO 50
   NIT=NIT+1
   IF(NIT.GT.20) GO TO 55
   IF(DMAX.GT.1.0)WRITE(6,200)(U1(J),J=1,NJ)
200 FORMAT(10G12.4)
   GO TO 11
55 JUSTP=100
   WRITE(6,121)
120 FORMAT(' NIT,DMAX,PBETA(-PCON),U1(NJ)=',4G15.5)
121 FORMAT(' SOLUTION FAILED TO CONVERGE*')
50 CONTINUE
   PCON=-PBETA
C CALCULATION OF V'S FROM PSI'S OBTAINED ABOVE.
   DO 60 J=2,NJ
60 V1(J)=-(PSIN(J)-PSIO(J))/DELX
   IF(INTNL.EQ.100) PSIN(NJP1)=PSIN(NJ)+
   10.5*(U1(NJ)+U1(NJP1))*
   C(Y(NJP1)-Y(NJ))
   IF(INTNL.EQ.100) GO TO 807
   LJ=LJDEL+10
   DO 61 J=NJP1,LJ
   U1(J)=U1(NJ)
61 PSIN(J)=U1(J)*(Y(J)-DFIX)
807 IF(INTNL.EQ.100) LJDEL=NJ
RETURN
END

SUBROUTINE PRINOUT(DU,DY,DV,NSTART)
C THIS SUBROUTINE IS WRITTEN FOR PRINTING ALL THE
C IMPORTANT PARAMETERS AT A GIVEN LOCATIONS.
DIMENSION DU(200),DY(200),DV(200)
COMMON/N0DIM/IKE,NIPR,LOT,DXINC,XDINC,DX8,DX9,DX10,
1PERCGB,MN,DXDIS,UREF2,XDISTRHRF,TH,EPS(200),TH1,DST,
2IPC,INTNL,TH2,NPC,NPT1,DFT,XMU1(200),REX,RET,BETA,CF,
3CFP,USTP,TAUH1,G,UF2,UF1
COMMON/TVEL/DELX,TTS,UREF1,A(200),B(200),C(200),D(200),
1XMU(200),XKE(200),XKE1(200),Y(200),V(200),U(200),
2AP(200),RHO(200),XL(200),UI(200),CMAX,PCON,PRK,CFE,FST,
3KEJ,RHOS,XMUS,DELTI,PRS,US,TEST,DELN,MCOUNT,LDF,
4LJDEL,KJDEL,NU,NEGU,ITER,NU2,MXITER,NEGO,UPDATE,MITER
COMMON/OKEY2/LMT,NLMT,NNEG,NXTRAP,MING,DELV,DXF,DX7,
1UREF,RF,RFU,RFDS,MROP,LORT,LINOP,LOUTD,LCOMP,NPRINT,
2LOOP,LVOP,LPPOP,MKITER,CNSTDX,YSTEP(150),JYSTEP,PGRAD1,
3PGRAD2,IP,CONXL,CONU
COMMON/UVDSL/PCON3,UP1,UP2,DP4,IIP,V1REF,DST1,DST2.
RH01(200),PCON6,PCONPB,DFIX,UT1,PD,XCONV,RHRF1, 
AFLOW,X,XXF,D2(200),E(200),JUSTP
COMMON/MIXLE/V1(200),XL1(200),AST,PTZ,UST,TAU,GAMMTR
COMMON/INITL/BLT,F1,F2,F3,GCON,JCOUNT,JH,KCOUNT,JKJN,
KFDJEL,KTRAK,KLK,LTD,LCONTP,LCOUNT,MNN,MSTEP,NCQU,
NCOUNT,PCONP,PCON2,PP,RAMDTR,UT,UT2,VRFF,VREF2,VC,XTRIP,
COMMON/VEL/PSIN(200),PSIO(200)
NPT1=-1
WRITE(6,100)
100 FORMAT(125(*))
WRITE(6,1000)US,XMUS,RHOS,UREF,MCOUNT,DXF,XCONV,LJDEL,
1KJDEL
C        DIMENSIONALIZE.
DTH = TH1/XCONV
DST = DST1/XCONV
REX = RHRF1*UREF1*XDIST/(XMUI(NJ)*XCONV)
RET = REX*TH1/XDIST
201 DDELY = DELY/XCONV
DXDIS = XDISI/XCONV
DDELX = DELX/XCONV
DDXY=(DDELX/Y(2))*XCONV
DVW = VW
DUREF = UREF1
PK=-BETA*CFP*RH01(NJ)/(RH01(1)*RET*H1*2.0)
PPLUS=8ETA*UREF1/(RET*H1*USTP)
C PRINTOUT ALL NECESSARY VARIABLES.
63 CONTINUE
WRITE(6,1001)DDELY,DDELX,DVW,DUREF,DXDIS,TEST,DDXY
WRITE(6,1006)TAU,DTH,DDST,UST,CF,REX,H1,RET,BETA,G,AST
WRITE(6,1011)TAUP,USTP,CFP,PPLUS,PK,DX7,PCON3,F3
WRITE(6,1047)UF2,PCON2,UF1,PCON1,MXITER
1000 FORMAT(/5X,5HUS = ,F8.2,2X,6HMUS = ,G14.5,2X,
17HRHOS = ,G14.5,2X,7HUREF = ,F10.2,2X,10HMCOUNT = ,
2 15.2X,6HDXF = ,F7.2,5X,6HXCONV = ,G14.5,2X,8HLJDEL = ,
313.2X,8HKJDEL = ,13)
1001 FORMAT(/5X,7HDELY = ,G14.5,2X,7HDELX = ,G14.5,2X,
15HVW = ,G14.5,2X,7HUREF = ,G14.5,2X,8HXDIST = ,G14.5,
22X,7HTEST = ,F10.7,2X,7HDXY = ,G14.5/)
1006 FORMAT(5X,6HTAU = ,G14.5,2X,5HUTH = ,G14.5,2X,
16HDST = ,G14.5,2X,6HUST = ,G14.5,2X,5HCF = ,G14.5/2X,
26HREX = ,G14.5,2X,4HH = ,G14.5,2X,6HRET = ,G14.5,2X,
37HETA = ,F9.4,2X,4HG = ,F9.3,2X,6HAST = ,F9.3)
1011 FORMAT(/5X,7HTAU = ,G14.5,2X,7HSTP = ,G14.5,2X,
16HCFP = ,G14.5,2X,8HPPLUS = ,G14.5,2X,5HPK = ,G14.5,
22X,6HDX7 = ,G14.5,*PCON3 = ,G14.5,*F3 = ,G14.5/)
1047 FORMAT(5X,*F2 = ,G15.7,*PCON2 = ,G15.7,*F1 = ,G15.7,
1*PCON1 = ,G15.7,*NXITER = ,I6/)
EPS(NJ) = 0.0
211 DO 214 J=1,NJ
DY(J) = Y(J)/XCONV
XEP1=(U(J+2)-U(J))/(Y(J+2)-Y(J))
IF(KEJ.EQ.0) GO TO 15
IF(J.LT.KEJ) GO TO 16
X=CKE*SQRT(XL(J+1))/SQRT(XKE(J+1))*US*US*XEP1
GO TO 16
15 CONTINUE
X=XEP1*XEP1*US*US*XL(J+1)
16 CONTINUE
EPS(J+1)=X
DU(J) = U1(J)*US
214 DV(J) = V1(J)*US
EPS(1)=0.0
WRITE(6,1002)(OY(J),J=1,NJ)
WRITE(6,1003)
WRITE(6,1004)(DU(J),J=1,NJ)
WRITE(6,1005)
WRITE(6,1006)(PSIN(J),J=1,NJ)
66 FORMAT(5X,3HPSI,2X,9(G12.4))
1002 FORMAT(5X,1HY,2X,9(G12.4))
1003 FORMAT(/2X,4HU(J))
1004 FORMAT(3X,9G12.4)
1005 FORMAT(/2X,4HV(J))
LCOUNT = 0
IF(LORT) 60,365,365
365 DO 366 J=1,NJ
DY(J) = Y(J)*XMUS*RHO1(J)*UST/(RHOS*US*XMU1(J))
366 DU(J) = U1(J)*US/UST
WRITE(6,1007)
WRITE(6,1009)
WRITE(6,1010)
WRITE(6,1004) (EPS(J),J=1,NJ)
WRITE(6,8503)
1007 FORMAT(/2X,2HY)
1009 FORMAT(/2X,2HU+)
1010 FORMAT(/2X,32HREYNOLD SHEAR STRESS (FT/SEC)**2)
8503 FORMAT(/2X,'KINETIC ENERGY')
DO 8304 J=1,NJ
8304 DU(J) = XKE(J)*US*US/UREF1/UREF1
WRITE(6,1004) (DU(J),J=1,NJ)
60 RETURN
END

SUBROUTINE MIXLEN(ASTSTP,DLSTEP,UREF)
THIS SUBROUTINE PROVIDES THE TURBULENCE MIXING LENGTH
SCALE. PLETCHER'S LENGTH SCALE TRANSPORT EQUATION
IS SOLVED. AN EMPIRICAL SIMPLE ALGEBRAIC MIXING LENGTH
MODEL FOR THE SEPARATED AND REDEVELOPING FLOW REGIONS
DOWNSTREAM OF A REARWARD-FACING STEP IS ALSO INCLUDED
IN THIS PROGRAM.

COMMON/TVEL/DELX,TTS,UREF1,A(200),B(200),C(200),D(200),
1XME(200),XKE(200),XKE1(200),Y(200),U(200),
2AP(200),RHO(200),XL(200),U1(200),CMAJ,PCON,PK,CKE,FST,
3KEJ,RHOS,XMUS,DELT1,PR,US,TEST,DELT,NJ,MOUNT,LDF,
4LJDEL,KJDEL,NU,NEG,ITER,NOU2,MXITER,NEG,UPDATE,MIXER
COMMON/MIXLE/V1(200),XL1(200),AST,PTZ,UST,TAU,GAMMTR
COMMON/STEP/HSTEP,XSTEP,HWALL,IDENT,HIGHT
COMMON/UVDT/PCON3,UP1,UP2,DP4,IP1,V1REF,DST1,DST2,
1RH01(200),PCON6,PCONPB,DIX,UT1,PX,PD,XCONV,RHRF1,
2AFL0W,X,XXF,02(200),E(200),JUSTP
COMMON/VEL/PSIN(200),PSIO(200)
COMMON/NONDIM/IKE,NIPR,LT1,DXINC,DXINC,DX8,DX9,DX10,
1PERCGB,MM,DXIS,UREF2,XDIST,RHR,TH,EP1(200),TH1,DST,
2IPC1,DST3,TH2,NPCT1,DF1,XMU1(200),REX,RET,BETA,CF,
3CF1,USTP,TAMP,H1,G,UF2,UF1

C XL1=0.12
C XL2=0.8
C APP=1.0
NJM5= NJ-5
VST=UST
XL1(1)=0.0
PTZ=Y(KJDEL-1)+(Y(KJDEL)-Y(KJDEL-1))*(0.99*U1(NJ)-
1(U1(KJDEL)-U1(KJDEL-1)))/
1(U1(KJDEL)-U1(KJDEL-1))
PTZP=PTZ
XSTPP=XSTEP+DELX/XCCNV

C

ZX=XMU(1)*U1(2)*US*XCONV/Y(2)
TAUMAX=ABS(ZX)
TAU1=TAUMAX
JTAUMX=1
X1MX=0.0
DO 53 J=1,NJM5
XEP1=(U1(J+2)-U1(J))/(Y(J+2)-Y(J))
IF(KEJ.EQ.0) GO TO 15
IF(J-L,T,KEJ) GO TO 15
X1=CKE*SQR(X1MX)*XEP1*US*US
X=X1*RHO(J+1)+XNU(J+1)*XEP1*US*XCONV
GO TO 16
15 CONTINUE
X=X1*REX(J+1)+XEP1*US*XCONV*XMU(J+1)
16 CONTINUE
X=ABS(X)
IF(X1.GT.X1MX) X1MX=X1
IF(X.LE.TAUMAX) GO TO 53
TAUMAX=X
JTAUMX=J+1
53 CONTINUE
54 FORMAT(3X,'TAUMAX, JMAX, U*MAX, PTZ, AST(I-1) =
15G12.4)
CAPP=TAUMAX/TAU1
TAUMAX=TAUMAX/RHO(JTAUMX)
VST=SQR(TAUMAX)
TAV=VST*VST*RHO(JTAUMX)/32.174
PTZT=PTZ-Y(JTAUMX)
C
C
IF(IWALL.NE.-1) GO TO 49
IF(XXF.GT.XSTPP) GO TO 52
GO TO 49
52 CONTINUE
C
C
JY=1
JYU=1
DO 68 J=2,NJMS
IF(U1(J).LT.0.0) JYU=J
IF(PSIN(J).LT.0.0) JY=J
IF(JY.GT.0. AND. PSIN(J).GE.0.0) GO TO 65
68 CONTINUE
65 JYP=JY+1
YUZERO=Y(JY)+(Y(JYP)-Y(JY))*ABS(PSIN(JY))/
1(ABS(PSIN(JY)))
1PSIN(JYP))
PTZ1=PTZ
PTZ=PTZ-YUZERO
CXL1=0.1 *PTZ1/SQR(PTZT**2+Y(JTAUMX)**2)
PY1=PTZ
CXL1=CXL1*PTZ1/PTZ
CXA=0.08*( (XXF-XSTEP)*XCONV/PTZ1*HSTEP/
1(HSTEP+HIGHT)+1.0)
IF(CXA.GT.CXL1) GO TO 10
CXL1=CXA
GO TO 10
49 CONTINUE
IF(MCOUNT.EQ.25)AST=.089*PTZ/.12
PY1=AST+VST*(PTZ-AST)/(CXL2*UREF1*PTZP)*DELX*AST/PTZP
10 CONTINUE
AST=PY1
WRITE(6,54) TAV, JTAUMX, VST, PTZ, AST
54 IF(MCOUNT.LE.25)PY1=Y(KJDEL)*0.089/0.12
XLK=PY1*PY1*CXL1*CXL1
US1T=UST
PPLUS=PCON*US*US*US*RHOS/(UST*TAU*32.174)
TSIX=26.
YFIX=UST*RHOS/(XMUS*XCONV)
PFIX=1.
TATST=0.0
JM=NJ-5
DO 101 J=2,JM
TPL=(U1(J+1)-U1(J-1))/(Y(J+1)-Y(J-1))*US*XCONV
TPL=ABS(TPL)
101 TATST=AMAX1(TPL,TATST)
TEF=TATST*XMUS/32.174
JJ=NJ+1
US1T=SQRT(TEF*32.174/RHOS)
US3T=TATST*RHOS/XMUS
US3T=SQRT(US3T)

CAPPA=CAPP

JSW=0
DO 335 J=2,JJ
YTTT=Y(J)
TTST=YTTT*UST/US
PFIX=1.+PPLUS*YTTT*US1T/US
IF(PFIXLT.0.0)PFIX=1.
IF(PPLUS.GE.0.0)PFIX=1.
PFIX=SQRT(PFIX)
XPF=YTTT*US3T/(XCONV*TSIX)
IF(U1(2).GT.0.0)XPF=PFIX*XPF*PFIX
IF(XPF.GT.50.)XPF=50.
XPF=1.-EXP(-XPF)
UK=.41*XPF
XL1(J)=UK*UK*YTTT*YTTT
XL1(J)=XL1(J)*CAPPA
KEJP=J
IF(YTTT.LT.60.)KEJPP=J
IF(XL1(J)-XLK)335,335,339
339 CONTINUE
WRITE(6,5)J,KEJPP,XL1,X1MX
5 FORMAT(* J SWITCH, JSW, XL1, MAX REYSTRESS = ' ,
14G12.5)
IF(U1(2).LE.0.0) GO TO 401
XLK=AMAX1(XL1(J),XLK)
401 JP=J+1
XL1(J)=XLK
GO TO 340
335 CONTINUE
US2T=US1T
SUBROUTINE TKE
C THIS PROGRAM IS WRITTEN FOR THE SOLUTION OF A STANDARD
C TURBULENCE KINETIC ENERGY EQUATION. A FULLY
C IMPLICIT FINITE DIFFERENCE SCHEME IS EMPLOYED.
DIMENSION AA(200), BB(200), CC(200), DD(200)
COMMON/TVEL/DELX, TTS, UREF1, A(200), B(200), C(200), D(200),
1 XMU(200), XKE(200), XKE1(200), Y(200), V(200), U(200),
2 AP(200), RHO(200), XL(200), UI(200), CMAX, PCON, PK, CKE, FST,
3 KEJ, RHO*, XMUS, DELT1, PRS, US, TEST, DELT, NJ, MCOUNT, LDF,
4 LJDEL, KJDEL, NOU, NEGU, ITER, NOU2, MXITER, NEGO, UPDATE, MITER
COMMON/MIXLE/VI(200), XL1(200), AST, PTZ, UST, TAU
CDE=CKE*CKE*CKE
CNC=2
XCONV=US*RHO*/XMUS
NJP=NJ+1
DO 2 J=KEJ+1
GRAD=(UI(J+1)-UI(J-1))/((Y(J+1)-Y(J-1))*CKE)
XKE1(J)=XL1(J)*GRAD*GRAD
2 CONTINUE
AA(KEJ)=0.0
LO=KEJ+1
A(1)=1.0
DO 4 J=2, NJP
IF(KEJ.EQ.0) GO TO 11
IF(J.LT.KEJ) GO TO 11
A(J)=XMU(J)/XMUS+RHO(J)*XL1(J)*A7*GAMMTR/
1 ((Y(J+1)-Y(J-1))*RHO)
4 CONTINUE
B(1)=A(1)
DO 12 J=2, NJP
B(J)=(A(J+1)+A(J)+A(J-1))/3.0
B(NJP)=A(NJP)
DD(KEJ)=1.
C(KEJ)=XKE1(KEJ)
DO 3 J=LO,NJ
YDE1=Y(J+1)-Y(J)
YDE2=Y(J)-Y(J-1)
YDE3=YDE1+YDE2
PKE=SQRT(XKE(J))
PEL=SQRT(XL1(J))
GRAD=(U1(J,J)-U1(J-1))/YDE3
SORC=CKE*PEL*PKE*GRAD*GRAD
CXP=(.5*(B(J+1)+B(J))-1.*PRK)/(YDE1*PRK)
CXM=(.5*(B(J)+B(J-1))-1.*PRK)/(YDE2*PRK)
AFIX=XKE(J)
IF(U1(J),LT,0.0)GO TO 103
AA(J)=V1(J)/YDE3-2.*CXP/YDE3
IF(IA(J),GT,0.0) GO TO 108
BB(J)=-V1(J)/YDE3-2.*CXM/YDE3
DD(J)=2.0*(CXP+CXM)/YDE3+(U1(J))/DELX+V1(J)/YDE2
1+CDE*PKE/PEL
GO TO 104
103 CONTINUE
CC(J)=CNC*ABS(U1(J))*AFIX/DELX+SORC
IF(V1(J),LE,0.0) GO TO 105
AA(J)=-2.*CXP/YDE3
BB(J)=-V1(J)/YDE3-2.*CXM/YDE3
DD(J)=2.0*(CXP+CXM)/YDE3+(U1(J))/DELX+V1(J)/YDE1
1+CDE*PKE/PEL
GO TO 104
105 AA(J)=V1(J)/YDE3-2.*CXP/YDE3
BB(J)=2.*CXM/YDE3
DD(J)=2.0*(CXP+CXM)/YDE3+CNC*ABS(U1(J))/DELX-V1(J)/YDE1
1+CDE*PKE/PEL
GO TO 104
106 AA(J)=V1(J)/YDE3-2.*CXP/YDE3
BB(J)=2.*CXM/YDE3
DD(J)=2.0*(CXP+CXM)/YDE3+(U1(J))/DELX+V1(J)/YDE2
1+CDE*PKE/PEL
GO TO 104
104 CONTINUE
B8(NJ)=0.0
DD(NJ)=1.
CC(NJ) = FST*UREF1*UREF1/US/US
DO 10 I = LO, NJ
R = BB(I)/DD(I-1)
DD(I) = DD(I) - R*AA(I-1)
10 CC(I) = CC(I) - R*CC(I-1)
CC(NJ) = CC(NJ)/DD(NJ)
DO 20 I = LO, NJ
J = NJ - I + KEJ
CC(J) = (CC(J) - AA(J)*CC(J+1))/DD(J)
20 XKE1(J) = CC(J)
XKE1(NJ) = CC(NJ)
RETURN
END

SUBROUTINE XRESET(U1, V1, XKE1, RHO1, XMU1, XL1, Y, U, V,
ILJDEL, NJ, NJ2, LIN, NSTART, PSIN)
C THIS SUBPROGRAM IS FOR INTERPOLATING ALL THE NECESSARY
C VARIABLES FOR A NEW Y-GRID SYSTEM. THE CHANGE IN
C THE Y-GRID SIZE MAY BE REQUIRED DURING COMPUTATION
C ESPECIALLY WHEN THE BOUNDARY LAYER GROWS SO RAPIDLY
C THAT IT FINALLY EXCEEDS THE MAXIMUM SPECIFIED HEIGHT
C OF Y(200). HOWEVER, IN THE PRESENT CALCULATION THIS
C SUBROUTINE IS NOT USED.
DIMENSION U1(200), V1(200), XKE1(200), RHO1(200),
XMU1(200), XL1(200), Y(200), U(200), V(200), YP(200),
2D1(200), DS(200), D3(200), D4(200), PSIN(200)
RETURN
END

SUBROUTINE SFINT(XX, XF, NN, NS, CC, Y)
C THIS SUBPROGRAM PROVIDES A FUNCTIONAL VALUE AT A GIVEN
C LOCATION BASED ON THE CUBIC SPLINE CURVE FITTING.
DIMENSION XX(70), CC(4, 70)
NB = NN
IF(XX(NB) .GT. XF) NB = 1
DO 10 J = NB, NS
IF(XX(J) .LE. XF) GO TO 10
NN = J - 1
GO TO 12
10 CONTINUE
12 IF(NN .EQ. NS) NN = NN - 1
IF(NN .LT. 1) NN = 1
A = XX(NN + 1) - XF
B = XF - XX(NN)
Y = CC(1, NN) * A * A + CC(2, NN) * B * B + CC(3, NN) * A + CC(4, NN) * B
RETURN
SUBROUTINE SPLICO(X,Y,M,C)

C THIS SUBROUTINE AND SUBPROGRAM SFINT ARE WRITTEN FOR
C CUBIC SPLINE CURVE FITTING. THIS PROGRAM PROVIDES
C THE COEFFICIENTS OF THE CUBIC POLYNOMIAL.
C
DIMENSION X(70),Y(70),D(70),P(70),E(70),C(4,70),
* A(70,3),B(70),
1Z(70),W(70)

MM=M-1

2 DO K=1,MM
  D(K)=X(K+1)-X(K)
  P(K)=D(K)/6.
  E(K)=(Y(K+1)-Y(K))/D(K)
  DO 3 K=2,MM
  B(K)=E(K)-E(K-1)
  A(1,2)=-1.-D(1)/D(2)
  A(1,3)=D(1)/D(2)
  A(2,3)=P(2)-P(1)*A(1,3)
  A(2,2)=2.+(P(1)+P(2))-(P(1)*A(1,2)
  A(2,3)=A(2,3)/A(2,2)
  B(2)=B(2)/A(2,2)
  DO 4 K=3,MM
  A(K,2)=2.+(P(K-1)+P(K))-P(K-1)*A(K-1,3)
  B(K)=B(K)-P(K-1)*B(K-1)
  A(K,3)=P(K)/A(K,2)
  4 B(K)=B(K)/A(K,2)
  Q=D(M-2)/D(M-1)
  A(M,1)=1.+Q*A(M-2,3)
  A(M,2)=-Q*A(M,1)*A(M-1,3)
  B(M)=B(M-2)-A(M,1)*B(M-1)
  Z(M)=B(M)/A(M,2)
  MN=M-2

2 DO 6 K=1,MN
  Z(K)=B(K)-A(K,3)*Z(K+1)

6 Z(K)=B(K)-A(K,3)*Z(K+1)

K=M-1
  Z(1)=-A(1,2)*Z(2)-A(1,3)*Z(3)
  DO 7 K=1,MM

7 W(K)=E(K)-P(K)*((Z(K+1)+2.*Z(K))

RETURN
END
SUBROUTINE TRAN9(REX,RET,GAMMTR,NCOND9,LORT,
1XTRFPT,AST,AMDTR,U1,XDIST,XTRFPT,NQPTN,YTRAN9,DST1)
C THIS PROGRAM PREDICTS TRANSITION INITIATION POINT AND
C PROVIDES THE INTERMITTENCY FACTOR. IN THE PRESENT STUDY
C A POINT TRANSITION MODEL IS USED. THE TRANSITION
C INITIATION POINT IS SET EQUAL TO XTRFPT SO THAT WHENEVER
C XDIST EXCEEDS THE SPECIFIED XTRFPT FLOW BECOMES FULLY
C TURBULENT.
C *** FOR XTRFPT SEE THE INPUT DATA SET *****
DIMENSION U1(200)
IF(XDIST.LT.XTRFPT) GO TO 20
LORT=1
GAMMTR=1.0
AST=0.089*YTRAN9/0.12
20 RETURN
END
XXII. APPENDIX N: COMPUTER CODE "KSTEP-2"
THIS PROGRAM IS WRITTEN FOR THE SOLUTION OF A TWO-DIMENSIONAL CHANNEL FLOW IN A SUDDEN EXPANSION. FOR THE SOLUTION, THE BOUNDARY-LAYER EQUATIONS ARE SOLVED. ONLY LAMINAR FLOW IS CONSIDERED.

HI  HEIGHT OF CHANNEL AT UPSTREAM OF STEP
H2  HEIGHT OF STEP
DX  DELTA X
XSTEP POINT WHERE STEP LOCATES
DY  DELTA Y
*** DX SHOULD BE SPECIFIED SUCH THAT NJ IS AN ODD NO.
XEND CALCULATION END POINT
TOLERC CONVERGENCE CRITERIA FOR THE BOUNDARY-LAYER SOLUTIONS.
XDISK POINT WHERE DATA ON DISK IS TO BE OUTPUT
ISYM IF NE 0, SYMMETRIC CHANNEL
UAVG AVERAGE VELOCITY USED FOR REFERENCE VALUE
RHOINF DENSITY USED FOR REFERENCE VALUE
XMUINF VISCOISITY USED FOR REFERENCE VALUE
VWL V-VELOCITY AT LOWER WALL
VWU V-VELOCITY AT UPPER WALL
RHO  DENSITY
XMUS VISCOISITY
PRESS PRESSURE AT CALCULATION STARTING POINT
IVEL  CONTROL PARAMETER FOR VELOCITY INPUT
IF .EQ. 0, UNIFORM VELOCITY PROFILE
IF .EQ. 1, FULLY DEVELOPED VELOCITY PROFILE
OTHERWISE, VELOCITY PROFILE SHOULD BE SPECIFIED
AND UFACTR MUST BE PUT IN (SEE UFACTR).

LIMIT  MAX. NO. OF CALCULATION STATION IN X-DIRECTION
ISTART  CONTROL PARAMETER; IF .LT. 0, INITIAL PROFILE
IS UNIFORM AND DATA INPUT FROM DOWNSTREAM OF THE CALCULATION
STARTING POINT.

IWRITE  CONTROL PARAMETER FOR OUTPUT LOCATION; IF .GT. 0, Y, U1 AND V1 ARE PRINTED OUT AT EVERY IWRITE STEP
IF .EQ. 0, THE VALUES ARE PRINTED OUT AT EVERY STEP.
IF .LT. 0, THESE ARE PRINTED AT XWRITE

NWRITE  NO. OF XWRITE
XCHA  THE PLACE WHERE DX CHANGES.
DCHA  THE FACTOR FOR THE CHANGE OF DX
XWRITE  DESIGNATES THE OUTPUT LOCATION, IF .LT. 0, THESE MUST BE SPECIFIED.

UFACTR  IF IVEL IS NEITHER 0 NOR 1, NEED TO SPECIFY UFACTR. THIS IS A NONDIMENSIONALIZATION FACTOR FOR U'S.

DIMENSION YD(100),UD(100),VD(100),RH0(100),XMU1(100),
1XWRITE(20),VP(100)
COMMON/VEL/A(100),B(100),XMU(100),RH0(100),V(100),
1U(100),V(100),U1(100),V1(100),PCCN,ITER,XMUINF,NJ,
2RH0INF,MOUNT,MAXITER,OXND,UREF,TOLRP,VPP,PSIO(100),
3PSIN(100),AMASSN
WRITE(6,103)
100 FORMAT(8G10.6)
101 FORMAT(8I10)
102 FORMAT(' HI=*, F10.5,' H2=*, F10.5,' DX=*, F10.5,
1' DY=*, F10.5,' XSTEP=*, F10.5,' XEND=*, F10.5,
2F10.5)
103 FORMAT('I',6(' '*), ' INPUT DATA *.5(*')
104 FORMAT(' TOLERC=', F10.5,' ISYM=', I5, ' UAVG=', F10.5,
1' RH0INF=',
2G15.6,' MUINF=', G15.6,' TOLRP=', G15.6)
105 FORMAT(' VML=', F10.5,' VWU=', F10.5,' RHO=', G15.6,
1' XMU=', G15.6,' PRESS=', F10.5)
106 FORMAT(' IVEL=', I5,' LIMIT=', I5,' ISTART=', I5,)
1' IWRITE=",15", NWRITE=",15")
107 FORMAT(6(*''),' (STEP) REYNOLD NO=","16.5,
1' MASS FLOW RATE=",16.5)
108 FORMAT(6(*''),' DIMENSIONLESS VALUES ",6(*''),/
1' ...
109 FORMAT(8G16.6)
110 FORMAT(* U ".")
111 FORMAT(* V ".")
112 FORMAT(10F13.5)
113 FORMAT(* U1 ".")
114 FORMAT(* V1 ".")
115 FORMAT(6(*''),' DIMENSIONAL VALUES ",6(*''),/
1' ...
116 FORMAT( PCON =",15.5, * XC="",15.5,
$ * P="",15.5)
117 FORMAT( MCOUNT=",15, * PCON="",15.5,
$ * PCOND="",15.5,* P="",15.5,* PND="",15.5)
118 FORMAT( x="",F10.4," U(2)=",F10.5," U(1)=",F10.5,
1' TAU L="",F10.5.
2 * TAU U="",F10.5," CF L="",15.5," CF U="",15.5)
119 FORMAT(/)
120 FORMAT(* XWRITE ".")
121 FORMAT(* PSI ".")
122 FORMAT( XCHA =",16.6," DCHA =",16.6)
123 FORMAT(F15.5)
200 FORMAT( INITIAL PROFILE NONUNIFORM HAS",,
1' BEEN STARTED DOWNSTREAM")
201 FORMAT( XND="",16.6," PBND="",16.6," DXND="",16.6,
1' XSTPND="",16.6," DYN="",16.6)
203 FORMAT(6(*''),' DATA HAS BEEN OUT*:5(""*)")
204 FORMAT(6(*''),' NONDIMENSIONAL VALUES ",6(*''),)
READ(5,100) H1,H2,DX,DY,XSTEP,XEND,XDISK
READ(5,100) TOLERC,ISYM, UAVG,RHINF,XMUINF,TOLRP
READ(5,100) VWL,VWU,RHOS,XMUS,PRESS
READ(5,101) IVEL,LIMIT,ISTART,IWRITE,NWRITE
READ(5,100) XCHA,DCHA
WRITE(6,102) H1,H2,DX,DY,XSTEP,XEND,XDISK
WRITE(6,104) TOLERC,ISYM, UAVG,RHINF,XMUINF,TOLRP
WRITE(6,105) VWL,VWU,RHOS,XMUS,PRESS
WRITE(6,106) IVEL,LIMIT,ISTART,IWRITE,NWRITE
WRITE(6,122) XCHA,DCHA
IF(IWRITE.GE.0) GO TO 8
READ(5,100) (XWRITE(I),I=1,NWRITE)
WRITE(6,120)
WRITE(6,112) (XWRITE(J),J=1,NWRITE)
8 CONTINUE
GC=9.81
UREF=UAVG
XCONV=RHINF*UREF/XMUINF
\begin{verbatim}
AN1 = H1 / DY + 1.1
N1 = AN1
AN2 = H2 / DY + 1.1
N2 = AN2
N = N1 + N2 - 1
NJ = N1
REXSTP = H2 * UAVG * RHOINF / XmUINF
AMASS = UAVG * H1 * RHOINF
IF(ISYM.EQ.0) GO TO 60
N = N + N2 - 1
NJ = N / 2 + 1
AMASS = AMASS / 2.0
CONTINUE
WRITE(6, 107) REXSTP, AMASS
JUSTP = 0
NCMA = 0
IF(ISTART.LE.0) GO TO 41
C
C INITIALIZING
C
KC = 0
LC = 1
ITER = 0
XND = 0.
MCOUNT = 0
Y(1) = 0.
U(1) = 0.
U1(1) = 0.
V(1) = VWL
V1(1) = VWL
PB = PRESS
ISTEP = -10
IDISK = -10
C
C UPPER BOUNDARY CONDITIONS
C
IF(ISYM.NE.0) GO TO 61
U(NJ) = 0.
U1(NJ) = 0.
CONTINUE
V(NJ) = VWU
V1(NJ) = VWU
C
C NON-DIMENSIONALIZATION
C
PBND = PB * GCON / (RHOINF * UREF * UREF)
DXND = DX * XCONV
XSTPND = XSTEP * XCONV
DYND = DY * XCONV
\end{verbatim}
XEND = XEND * XCONV
XDISK = XDISK * XCONV

C INITIAL VELOCITY PROFILE

IF (IVEL.EQ.0) GO TO 10
IF (IVEL.EQ.1) GO TO 11
READ (5, 100) (U(J), J=1, NJ)
READ (5, 123) UFACTR
DO 52 J = 1, NJ
   U(J) = U(J) / UFACTR
52 V(J) = 0.
GO TO 12

11 NP = (NJ - 1) / 2 + 1
   NPM = NP - 1
   IF (ISYM .NE. 0) NP = NJ
   DO 13 J = 2, NP
      YPH = Y(J) / (XCONV * H1)
      U(J) = 6.0 * (YPH - YPH * YPH)
   13 V(J) = 0.
   IF (ISYM .NE. 0) GO TO 12
   DO 18 J = 1, NPM
      K = NP + J
      I = NP - J
      U(K) = U(I)
   18 V(K) = V(J)
   GO TO 12

10 DO 14 J = 1, NJ
   U(J) = 1.0
14 V(J) = 0.
12 CONTINUE

DO 17 J = 2, NJ
   U1(J) = U(J)
   V1(J) = V(J)
17 PSI0(J) = PSI0(J-1) + 0.5 * (U(J) + U(J-1)) * (Y(J) - Y(J-1))

18 PSI(NJ) = PSI0(NJ)
   IF (ISYM .EQ. 0) U1(NJ) = 0.0
   AMASSN = PSI0(NJ)
   GO TO 16

41 WRITE (6, 200)
   READ (9) XND, ISTEP, MCOUNT, PBND, NJ, N, U, U1, V, V1,
   1Y, DXND, XSTPND, DYND,
   $AMASSN, PSI0, PSI
WRITE (6, 201) XND, PBND, DXND, XSTPND, DYND
WRITE (6, 202) ISTEP, MCOUNT, NJ, N
READ (5, 100) DX
DXND = DX * XCONV
KC = 0
LC = 1
ITER = 0
IDISK = -10
XEND = XEND * XCONV
XDISK = XDISK * XCONV

DO 15 J = 1, NJ
    Y(J) = Y(J) / XCONV
    XMUS(J) = XMUS
    XMUS1(J) = XMUS
    RHO(J) = RHOS
    RHO1(J) = RHOS
    UD(J) = U1(J) * UREF
15   VD(J) = V1(J) * UREF
WRITE(6, 108)
WRITE(6, 109) (Y(J), J = 1, NJ)
WRITE(6, 110)
WRITE(6, 109) (U(J), J = 1, NJ)
WRITE(6, 111)
WRITE(6, 109) (V(J), J = 1, NJ)
WRITE(6, 112)
WRITE(6, 109) (U1(J), J = 1, NJ)
WRITE(6, 113)
WRITE(6, 109) (V1(J), J = 1, NJ)
WRITE(6, 114)
WRITE(6, 119)
WRITE(6, 109) (PSIN(J), J = 1, NJ)

C**************
C COMPUTATION LOOP
C**************

999 MCOUNT = MCOUNT + 1
WRITE(6, 119)
IF(NCHA .EQ. -1) GO TO 50
IF(XND .LT. XCHA * XCONV) GO TO 50
NCHA = -1
DXND = DXND * DCHA
50 CONTINUE
XND = XND + DXND
IF(XND .GT. XEND) GO TO 1000
ISEC = -10
ILIMIT = 10
CALL KWOINC(JUSTP, RHO1, ILIMIT, ISEC, ISYM)
IF(JUSTP .NE. 100) GO TO 51
CALL KWOINCL(JUSTP, RHO1, ILIMIT, ISEC)
IF(JUSTP .EQ. 100) GO TO 1000
51 CONTINUE
PND = PCON * DXND + PBND
P = PNO * RHOINF * UREF * UREF / GCON
PCOND = PCON * RHOINF * UREF * UREF / GCON * XCONV
WRITE(6, 117) MCOUNT, PCON, PCOND, P, PND
ITER = 0
MITER = 0
NJM=NJ-1
ZXL=XMU1(1)*U1(2)*UREF*XCONV/(RHO1(1)*Y(2))
ZXU=XMU1(NJM)*U1(NJM)*UREF*XCONV/(RHO1(NJM)*Y(NJM)-Y(NJM))
ZXL=ABS(ZXL)
ZXU=ABS(ZXU)
USTL=SQRT(ZXL)
USTU=SQRT(ZXU)
TAUL=USTL*USTL*RHO1(1)/GCON
TAUU=USTU*USTU*RHO1(NJM)/GCON
CFL=TAUL*2.0*GCON/(RHOINF*UREF*UREF)
CFU=TAUU*2.0*GCON/(RHOINF*UREF*UREF)
X=XND/XCONV
UM=U1(2)*UREF
UNJM1=U1(NJM)*UREF
WRITE(6,118)X,UM,UNJM1,TAUL,TAUU,CFL,CFU
IF(ISTEP.GT.0) GO TO 30
IF(XND.LT.XSTPNO) GO TO 30
ISTEP=10
DO 32 I=1,NJ
J=NJ-I+1
K=J+N2-1
U1(K)=U1(J)
V1(K)=V1(J)
PSIN(K)=PSINCJ
RHO1(K)=RHO1(J)
32 XMU1(K)=XMU(J)
N2M=N2-1
NJ=N
DO 33 J=1,N2M
U1(J)=0.
V1(J)=0.
PSIN(J)=0.0
XMU1(J)=XMUS
33 RHO1(J)=RHO5
Y(J)=0.
DO 36 J=2,NJ
Y(J)=Y(J-1)+OYND
36 YD(J)=Y(J)/XCONV
CONTINUE
DO 34 J=1,NJ
UD(J)=U1(J)*UREF
34 VD(J)=V1(J)*UREF
IF(IWRITE.EQ.0) GO TO 47
IF(IWRITE.LT.0) GO TO 45
KC=KC+1
IF(KC.LT.IWRITE) GO TO 46
KC=0
GO TO 47
45 IF(X.LT.XWRITE(LC)) GO TO 46
SUBROUTINE KWONCL(JUSTP,RHO1,LIMIT,ISEC)
C THIS PROGRAM IS FOR THE SOLUTION OF THE BOUNDARY-LAYER
C EQUATIONS. A FULLY IMPLICIT FINITE-DIFFERENCE METHOD
C WITH LAGGED COEFFICIENTS IS USED.
C THE PRESSURE GRADIENT IS EVALUATED IN A COUPLED MANNER.
DIMENSION AA(200),BB(200),CC(200),DD(200),EE(200),
1HH(200),RHO1(200)
COMMON/VEL/A(100),B(100),XMU(100),RHO(100),Y(100),
1U(100),V(100),U1(100),V1(100),PCON,ITER,XMUINF,NJ,
2RHOINF,MCOUNT,MXITER,DXND,UREF,TOLRP,VPP,PSIQ(100),
3PSIN(100),AMASSN
JUSTP=0
DELF=DXND
US=UREF
PBETA = FA/FB
DMAX = 0.0
UI(NJ) = 0.0
PSINJ = PSIN(NJ)
DO 102 I = 2, NJM
J = NJ - I + 1
UIJ = AA(J) * UI(J+1) + HH(J) * PBETA + CC(J)
PSIJ = BB(J) * UI(J+1) + DD(J) * PBETA + EE(J)
DUJ = ABS(UI(J) - UIJ)
DPJ = ABS(PSIN(J) - PSIJ)
DPJ = DPJ / PSINJ
UI(J) = UIJ
PSIN(J) = PSIJ
IF(DIJ.GT.DMAX) DMAX = DIJ
102 IF(DPJ.GT.DMAX) DMAX = DPJ
WRITE(6,120) NIT, DMAX, PBETA, UI(NJ)
IF(DMAX.LE.0.0001) GO TO 50
NIT = NIT + 1
IF(NIT.GT.LIMIT) GO TO 56
200 FORMAT(10G12.4)
GO TO 11
56 JUSTP = 100
WRITE(6,121)
120 FORMAT(' NIT, DMAX, PBETA(-PCON), UI(NJ) =', 4G15.5)
121 FORMAT(' SOLUTION FAILED TO CONVERGE')
GO TO 65
50 CONTINUE
PCON = -PBETA
DO 60 J = 2, NJ
60 Vl(J) = -(PSIN(J) - PSIO(J))/DELX
65 CONTINUE
RETURN
END

SUBROUTINE KWONC(JUSTP,RHO1,LIMIT,ISEC,ISYM)
C THIS PROGRAM IS FOR THE SOLUTION OF THE BOUNDARY-LAYER
C EQUATIONS. A FULLY IMPLICIT FINITE-DIFFERENCE SCHEME
C WITH NEWTON LINEARIZATION IS USED. THE CONTINUITY
C AND MOMENTUM EQUATIONS ARE SOLVED IN A COUPLED MANNER.
DIMENSION AA(200), BB(200), CC(200), DD(200), EE(200),
1 HH(200), RHO(200)
COMMON/VEL/A(100), B(100), XMU(100), RHO(100), Y(100),
1 UI(100), V(100), UI(100), V(100), PCON, ITER, XMUINF, NJ,
2 RHOINF, MCOUNT, MXITER, DXND, UREF, TOLRP, VPP, PSIO(100),
3 PSIN(100), AMASSN
PBETA = -PCON
DELX = DXND
US = UREF
OMAX=0.0
IF(ISYM*NE.0) GO TO 2
FA=PSIN(NJ)-TP*CC(NJM)-EE(NJM)
FB=TP*HH(NJM)+DD(NJM)
P\beta =FA/FB
54 CONTINUE
U1(NJ)=0.0
PSINJ=PSIN(NJ)
GO TO 3
2 CONTINUE
PSINJ=PSIN(NJ)
NJM2=NJ-2
ALX=4.0-AA(NJM2)
AL1=1.0-AA(NJM)*ALX/3.0
AL2=(ALX*HH(NJM)-HH(NJM2))/3.0
AL3=(ALX*CC(NJM)-CC(NJM2))/3.0
AM1=1.0+BB(NJM)/TP+AA(NJM)
AM2=-(HH(NJM)+DD(NJM)/TP)
AM3=PSINJ/TP-EE(NJM)/TP-CC(NJM)
P\beta =((AL1*AM3-AL3*AM1)/(AL2*AM1-AL1*AM2)
U1(NJ)=AL2*P\beta /AL1+AL3/AL1
U1(NJ)=U1NJ
PSIN(NJ)=PSINJ
3 CONTINUE
DO 102 I=2,NJM
J=NJ-I+1
U1J=AA(J)*U1(J+1)+HH(J)*P\beta +CC(J)
PSIJ=BB(J)*U1(J+1)+DD(J)*P\beta +EE(J)
DUJ=ABS(U1(J)-U1J)
DPJ=ABS(PSINJ-PSIJ)
U1(J)=U1J
PSIN(J)=PSINJ
IF(DUJ*GT*DMAX)DMAX=DUJ
102 IF(DPJ*GT*DMAX)DMAX=DPJ
WRITE(6,120) NIT,OMAX,P\beta ,U1(NJ)
IF( DMAX*LE.0.0001) GO TO 50
NIT=NIT+1
IF(NIT*GT*LIMIT) GO TO 56
200 FORMAT(10G12.4)
GO TO 11
56 IF(ISEC*LT.0) GO TO 55
GO TO 50
55 JUSFP=100
DO 57 J=1,NJ
PSIN(J)=PSIN(J)
57 U1(J)=U(J)
WRITE(6,121)
120 FORMAT(* NIT,DMAX,P\beta (-PCON),U1(NJ)=*4G15.5)
121 FORMAT(* SOLUTION FAILED TO CONVERGE*)
GO TO 65
CONTINUE
PCON=-PBETA
DO 60 J=2,NJ
   VIC JJ=-(PSIN(J)-PSIQ(J))/DELX
60  CONTINUE
RETURN
END