Dynamics and control of spin-stabilized spacecraft with sloshing fluid stores

Daniel Eugene Hill

Iowa State University

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DYNAMICS AND CONTROL OF SPIN-STABILIZED SPACECRAFT WITH
SLOSHING FLUID STORES

Iowa State University

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Dynamics and control of spin-stabilized spacecraft with sloshing fluid stores

by

Daniel Eugene Hill

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of the Requirements for the Degree of DOCTOR OF PHILOSOPHY

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SUMMARY

Sloshing fluid stores have been suspected as a source of dynamic instability in the launch of the STAR 48 Communications Satellites. An analysis of flight data indicated the satellite was not behaving as a single rigid body.

This study used an equivalent mechanical pendulum model of free surface fluid motion coupled with the dynamics of the main body and rocket motor as an approximation to the system. A comparison of the simulation using the model and the experimental flight data showed similar behavior.

This study also used the spacecraft model as a basis for the development of a linear feedback control law. The control law was formulated as a linear quadratic tracking problem (LQTP). Numerical solution of the LQTP provided a control law which could be used for counteracting the dynamic instability caused by the fluid slosh and also for earth pointing maneuvers. Reaction jet thrusters were used as the control mechanism. A possible implementation of the control system was also outlined.

In the course of this study, four computer programs were developed. The first program simulated the flight of the spacecraft during the launch phase. The second program linearized the equations of motion for use in the third program which computed the control law. The fourth program simulated
the response of the nonlinear system using the control law. The programs were written in FORTRAN IV and implemented on NAS AS/6 and NAS 9160 mainframe computers.
CHAPTER I.
INTRODUCTION

Launchings of several of the STAR 48 Communications Satellites from the Space Shuttle have consistently resulted in a nutating motion of the spacecraft. Flight data from roll, pitch, and yaw axis rate gyros indicated a constant frequency, equal amplitude, sinusoidal oscillation about the pitch and yaw axis. The vector combination of these two components of vibration resulted in a coning motion of the satellite about the roll axis. The vehicle was spin stabilized at launch, having a one revolution per second roll velocity imparted to it.

After launching from the shuttle in the perigee phase of its orbit, the satellite's power assist module (PAM) fired its thruster to establish a geosynchronous earth orbit. It is this axial thrust that gives rise to the coning which predominates after PAM motor burnout. Consistently, flight data from rate gyros indicated the steady state coning and a one-half cycle per second small amplitude disturbance superimposed on the one revolution per second roll velocity.

Spacecraft designers thought that combustion instabilities in the PAM rocket motor were thought to be the source of a side force which would induce the coning motion. In order to investigate the presence of any such combustion instabilities, a STAR 48 motor was fired at the Engine Test Facility, Arnold
Engineering Development Center, Arnold Air Force Station. A test fixture having lateral and axial load cells was utilized, and the rig allowed the PAM to be spun at one revolution per second during firing. A spectral analysis was completed of the resulting load cell records obtained during firing. The test results indicated no significant forces at the required frequency (one-half cycle per second) and it was concluded that combustion instabilities were not the source of moments causing coning motion.

A preliminary analysis of the payload (communication satellite) was completed indicating that a 55 ft-lb external moment would induce the coning motion. It was suspected that sloshing motion of liquid stores in the vehicle was the mechanism for creating the nutation of the spacecraft.

Sloshing of fluid stores has been a problem which received much attention in the early years of space flight. Large liquid fueled rocket boosters have failed because of fluid slosh excited by attitude control systems.\(^1\) The early launch failures motivated researchers to try to understand the complicated behavior of free surface fluid motion. Analytical models of the fluid motion were developed for various tank geometries.\(^2,3,4\) The analyses were similar but the boundary conditions imposed by different tank geometries made each problem unique. Stability of the fluid motion was studied and unstable modes were identified.
Experimental studies of free surface fluid motion of different tank geometries were also undertaken. Sumner\textsuperscript{5} developed an equivalent lumped parameter mechanical model of the complex fluid behavior. The model was applicable to spherical and oblate-spherical containers. Sumner and Stofan\textsuperscript{6} also investigated the effect of viscous damping in spherical tanks. Stability boundaries were identified for the equivalent lumped parameter mechanical model by Sayar and Baumgarten.\textsuperscript{7} The stability boundaries established the range of validity of the model.

The analytical and experimental analysis of fluid slosh provided a basis for the investigation of how to prevent unstable fluid motion. Anderson\textsuperscript{8} and Stephens, et al.\textsuperscript{9} studied the damping of sloshing fluid by use of baffle systems. Baffling is an effective way of damping oscillations but any added weight is costly in terms of payload reduction.

The modeling of fluid slosh is extensive and has been used by researchers to study its effect on space vehicle motion. Abramson\textsuperscript{10} studied the response of a planar model of a launch vehicle with sloshing fluid stores. Stability boundaries were identified so that the control frequency of the gimbaled rocket motor could be designed far away from the fundamental slosh frequency. Michelini et al.\textsuperscript{11} outlined a procedure for developing the equations of motion of a spinning satellite containing fluid stores. The equations of motion were not
presented but the study supplied the analytical background for the experimental identification of the dynamic model. Experimental results showed that small amplitude free surface wave motion does not cause instabilities in the vehicle. Instabilities were found to be generated by the first mode natural frequency which is not excited by small disturbances. The consequence of the first mode natural frequency causing instability in the vehicle justifies the use of an equivalent spherical pendulum model of the fluid slosh.

The natural frequency of the first mode can be near the coning or control frequency of the vehicle which would result in unstable motion. Baur\textsuperscript{12} and Eide, et al.\textsuperscript{13} have also analyzed the stability of launch vehicles with sloshing fluid stores and discussed the need for a control system which would generate control forces that are phase shifted with the liquid motion stabilizing the vehicle.

The research on the dynamics and control of space vehicles with sloshing fluid stores has been concerned with establishing design methods to either constrain the fluid motion or build the vehicle and control system so that fluid slosh is not excited. It is quite possible that because of design constraints a vehicle may naturally tend to excite the fluid slosh.

The first part of this study modeled the vehicle coupled with the sloshing fluid stores using an equivalent spherical
pendulum model for the fundamental slosh mode. The simulation of the launch phase of the satellite was then conducted.

The second part of this study was the development of a control law which may be applied to a spin-stabilized spacecraft with sloshing fluid stores without baffling or changing the design of the spacecraft. A closed loop feedback control law was developed which stabilizes the spacecraft and may be used for earth pointing maneuvers. A closed loop control system tries to maintain a prescribed relationship of one system variable to another by comparing functions of these variables and using the difference as a means of control.\textsuperscript{14}

Control systems for spacecraft fall into two general categories; internal and external torque devices. Internal torque devices consist of momentum wheels or other movable masses. A momentum wheel is a disk which is driven by a motor attached to the satellite body. The reactive torque of the motor on the spacecraft creates an internal moment which, when three wheels are used, can control the spacecraft attitude. Vadali and Junkins\textsuperscript{15} showed that momentum wheels can be used for flat spin recovery and attitude maneuvers of a spacecraft. The advantage of momentum wheels is their precise control while the disadvantages are slower response and higher cost than external torque devices. Kane and Sobala\textsuperscript{16} have shown that an internal mass moving with a prescribed motion can be used to
bring a spacecraft into simple spin from an arbitrary state of motion.

External torque devices consist of reaction jets which eject a fluid to apply the torque to the body. External torque devices which are normally used on spacecraft which are spin-stabilized, are an inexpensive method of control. The method of control for this study was chosen to be external torque reaction jets, which is compatible with the STAR 48 design.

The third part of this study discusses the implementation of the control system developed. The control system would consist of a digital or analog computer, A/D (analog to digital) and D/A (digital to analog) conversion, two servo-valves and two propellant tanks. Modulated pulsing of bi-directional servo-valves would create a timed impulse on the structure, ultimately controlling the device.
CHAPTER II.
DERIVATION OF THE EQUATIONS OF MOTION

The equations of motion for the dynamic system were derived using two different methods. The first formulation was derived using D'Alembert's form of Lagrange's equation, or Kane's equation, while the second formulation was made using the classical Lagrange equation. The two methods are quite different in form and a term for term match provided confidence that the equations of motion were correct.

A schematic diagram of the system is shown in Figure 1. The reference frames a, b, and n are inertial, body fixed to the main body and pendulum, respectively. The inertial frame corresponds to the center of the earth. A set of generalized coordinates to describe the position and orientation of the system were chosen by inspection. The coordinates used to describe the location of the center of mass (G) of the main body relative to the inertial frame were a set of cartesian coordinates defined as,

\[ x_1 \hat{a}_1 \] rectilinear distance along \( a_1 \) \hspace{1cm} (1)
\[ x_2 \hat{a}_2 \] rectilinear distance along \( a_2 \) \hspace{1cm} (2)
\[ x_3 \hat{a}_3 \] rectilinear distance along \( a_3 \) \hspace{1cm} (3)

and shown in Figure 1. The position of the satellite relative to the earth may also be defined by using a length and two
Figure 1. Model of spacecraft with spherical pendulum
angles where,

\[ R \triangleq \text{altitude + radius of earth} = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} \]  

\[ \phi \triangleq \text{azimuth angle} = \tan^{-1} \frac{x_3}{(x_1^2 + x_2^2)^{\frac{1}{2}}} \]  

\[ \theta \triangleq \text{orbital angle} = \tan^{-1} \frac{x_2}{x_1} \]  

The generalized coordinates describing the main body orientation and pendulum orientation are given by,

\[ \theta_1 \triangleq \text{angle of rotation about } a_3 \]  

\[ \theta_2 \triangleq \text{angle of rotation about } b_1 \]  

\[ \theta_3 \triangleq \text{angle of rotation about } b_2 \]  

\[ \alpha \triangleq \text{angle of rotation about } b_3 \]  

\[ \beta \triangleq \text{angle of rotation about } b_1 \]  

and are shown in Figures 2 and 3. The total number of degrees of freedom is equal to \(6 + 2N\) where \(N\) corresponds to the number of spherical pendulums. Coordinate transformations were derived using successive right hand 3-1-2 rotations, i.e., rotate the body fixed frame about its 3 axis, 1 axis, and 2 axis into the final position. The transformations are given by,

\[ c_{ij} \triangleq a_i \cdot b_j \quad (i, j = 1, 2, 3) \]
Figure 2. Reference frame a to b transformation
Figure 3. Reference frame b to n transformation
\[
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{bmatrix} =
\begin{bmatrix}
  c_{11} & c_{12} & c_{13} \\
  c_{21} & c_{22} & c_{23} \\
  c_{31} & c_{32} & c_{33}
\end{bmatrix}
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix}
\]

\( sn \triangleq \sin \quad cs \triangleq \cos \)

\( c_{11} = cs^{\theta_1}cs^{\theta_3} - sn^{\theta_1}sn^{\theta_2}sn^{\theta_3} \)  \hspace{1cm} (14)

\( c_{12} = -sn^{\theta_1}cs^{\theta_2} \)  \hspace{1cm} (15)

\( c_{13} = cs^{\theta_1}sn^{\theta_3} + sn^{\theta_1}sn^{\theta_2}cs^{\theta_3} \)  \hspace{1cm} (16)

\( c_{21} = sn^{\theta_1}cs^{\theta_3} + cs^{\theta_1}sn^{\theta_2}sn^{\theta_3} \)  \hspace{1cm} (17)

\( c_{22} = cs^{\theta_1}cs^{\theta_2} \)  \hspace{1cm} (18)

\( c_{23} = sn^{\theta_1}sn^{\theta_3} - cs^{\theta_1}sn^{\theta_2}cs^{\theta_3} \)  \hspace{1cm} (19)

\( c_{31} = -cs^{\theta_2}sn^{\theta_3} \)  \hspace{1cm} (20)

\( c_{32} = sn^{\theta_2} \)  \hspace{1cm} (21)

\( c_{33} = cs^{\theta_2}cs^{\theta_3} \)  \hspace{1cm} (22)

\( t_{ij} \triangleq \frac{n_i \cdot b_j}{(i, j = 1, 2, 3)} \)  \hspace{1cm} (23)

\[
\begin{bmatrix}
  n_1 \\
  n_2 \\
  n_3
\end{bmatrix} =
\begin{bmatrix}
  t_{11} & t_{12} & t_{13} \\
  t_{21} & t_{22} & t_{23} \\
  t_{31} & t_{32} & t_{33}
\end{bmatrix}
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix}
\]

\( t_{11} = sn^\alpha \)  \hspace{1cm} (24)

\( t_{12} = cs^\alpha \)  \hspace{1cm} (25)

\( t_{13} = 0 \)  \hspace{1cm} (26)

\( t_{21} = -sn^\alpha cs^\beta \)  \hspace{1cm} (27)

\( t_{22} = cs^\alpha cs^\beta \)  \hspace{1cm} (28)

\( t_{23} = sn^\beta \)  \hspace{1cm} (29)

\( t_{31} = sn^\alpha sn^\beta \)  \hspace{1cm} (30)

\( t_{32} = -cs^\alpha sn^\beta \)  \hspace{1cm} (31)
The equations of motion were first formulated using Kane's equations by developing expressions for angular velocities, velocities, angular accelerations, accelerations, partial velocities, and then assembling the generalized inertia forces and generalized active forces. In order to simplify the writing of the kinematics, a notation for writing angular velocities was chosen. A bracketed quantity with a superscript \( m \) denotes that all angular velocities within the bracket are with respect to the pendulum, otherwise they are with respect to the main body. The remaining notation is given by,

- \( a_x \): Acceleration vector of point \( x \) in the inertial reference frame
- \( a_{x/y} \): Acceleration vector of point \( x \) relative to point \( y \) in the inertial reference frame
- \( v_{a_x} \): Acceleration vector of point \( x \) relative to reference frame \( y \)
- \( A \): Earth or inertial reference frame
- \( a_1, a_2, a_3 \): Inertial reference frame dextral set of orthogonal unit vectors
- \( B \): Main rigid body reference frame on which the pendulums are attached
- \( b_1, b_2, b_3 \): Body B reference frame dextral set of orthogonal unit vectors
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$D_\alpha$</td>
<td>Damping coefficient associated with $\alpha$ degree of freedom</td>
</tr>
<tr>
<td>$D_\beta$</td>
<td>Damping coefficient associated with $\beta$ degree of freedom</td>
</tr>
<tr>
<td>$F_G$</td>
<td>Gravitational attraction force vector</td>
</tr>
<tr>
<td>$F_T^*$</td>
<td>Rocket motor thrust vector</td>
</tr>
<tr>
<td>$F_B^*$</td>
<td>Inertia force on body $B$</td>
</tr>
<tr>
<td>$F_i^*$</td>
<td>Inertia force on pendulum mass $i$</td>
</tr>
<tr>
<td>$F_r^*$</td>
<td>Generalized active force with respect to degree of freedom $r$</td>
</tr>
<tr>
<td>$F_r^*$</td>
<td>Generalized inertia force with respect to degree of freedom $r$</td>
</tr>
<tr>
<td>$I$</td>
<td>Inertia tensor</td>
</tr>
<tr>
<td>$K$</td>
<td>Kinetic energy</td>
</tr>
<tr>
<td>$K_G$</td>
<td>Gravitation attraction parameter</td>
</tr>
<tr>
<td>$K_\alpha$</td>
<td>Spring rate associated with $\alpha$ degree of freedom</td>
</tr>
<tr>
<td>$K_\beta$</td>
<td>Spring rate associated with $\beta$ degree of freedom</td>
</tr>
<tr>
<td>$L$</td>
<td>Pendulum length</td>
</tr>
<tr>
<td>$L_{x/y}$</td>
<td>Position vector of pendulum point $x$ relative to point $y$</td>
</tr>
<tr>
<td>$M$</td>
<td>Mass of main body</td>
</tr>
<tr>
<td>$m$</td>
<td>Mass of pendulum</td>
</tr>
<tr>
<td>$\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$</td>
<td>Body $m$ reference frame dextral set of orthogonal unit vectors</td>
</tr>
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\( Q_r \) Generalized force associated with virtual displacement \( r \)

\( q_r \) Generalized coordinate

\( r_{x/y} \) Position vector on main body of point \( x \) relative to point \( y \)

\( T_B^B \) Pendulum torque vector on body \( B \)

\( T_m^m \) Pendulum torque vector on body \( m \)

\( T_r^r \) Rocket motor torque vector

\( T_B^* \) Inertia torque vector on body \( B \)

\( u_j \) Partial velocity coefficient \( j \)

\( V_x \) Velocity vector of point \( x \) in the inertial reference frame

\( V_{x/y} \) Velocity vector of point \( x \) relative to point \( y \) in the inertial reference frame

\( V_{x/y}^x \) Velocity vector of point \( x \) relative to reference frame \( y \)

\( V_r \) Generalized speed \( r \)

\( V_{yX}^x \) Partial velocity with respect to generalized speed \( V_r \). \( (\partial V_{yX}^x / \partial V_r) \)

\( V_{\omega X}^x \) Angular velocity vector of body \( x \) relative to reference frame \( y \)

\( V_{\omega V_r}^x \) Partial angular velocity with respect to generalized speed \( V_r \). \( (\partial V_{\omega X}^x / \partial V_r) \)

- Single underscore denotes vector
Dot over a variable denotes differentiation with respect to time.

Primed quantities denote the intermediate rotated orientation of the body or reference frame.

The angular velocity of body B is given by,

\[ \omega_B = \omega_B'' + \omega B' + B' \omega \]  

\[ = \hat{\delta}_1 b_1 + \hat{\delta}_2 b_2 + \hat{\delta}_3 b_3 \]  

\[ = (\hat{\delta}_1 c_{31} + \hat{\delta}_2 c_{32} \hat{\delta}_3) b_1 + (\hat{\delta}_1 c_{32} + \hat{\delta}_3) b_2 \]  

\[ + (\hat{\delta}_1 c_{33} + \hat{\delta}_2 s_{3} \hat{\delta}_3) b_3 \]  

\[ = \omega_1 b_1 + \omega_2 b_2 + \omega_3 b_3 \]  

where,

\[ \omega_1 = \hat{\delta}_1 c_{31} + \hat{\delta}_2 c_{32} \hat{\delta}_3 \]  

\[ \omega_2 = \hat{\delta}_1 c_{32} + \hat{\delta}_3 \]  

\[ \omega_3 = \hat{\delta}_1 c_{33} + \hat{\delta}_2 s_{3} \hat{\delta}_3. \]  

The angular velocity of m is given by,

\[ \omega_m = \omega_B + \omega_m \]  

\[ = \omega_B + \dot{\delta} b_3 + \dot{\delta}_3 n_1 \]  

\[ = (\omega_1 + \dot{\delta}_{31}) b_1 + (\omega_2 + \dot{\delta}_{32}) b_2 \]  

\[ + (\omega_3 + \dot{\delta}_3) b_3. \]  

The velocity of the pendulum mass may be written as,

\[ V_m = V_G + \dot{V}_0/G + \dot{V}_{m/o} \]  

\[ V_G = \dot{x}_1 a_1 + \dot{x}_2 a_2 + \dot{x}_3 a_3 \]  

\[ V_{0/G} = A \omega_B \times \vec{r}_{0/G} + B \dot{V}_0 \]
\[ v_m = (\omega_2 r_3 - \omega_3 r_2 - \dot{r}_1) b_1 + (\omega_3 r_1 - \omega_1 r_3 - \dot{r}_2) b_2 \\
+ (\omega_1 r_1 - \omega_2 r_2 - \dot{r}_3) b_3 \\
= [(r_3 c_{32} - r_2 c_{33}) \delta_1 - (r_2 s_{33}) \delta_2 + r_3 \delta_3 - \dot{r}_1] b_1 \\
+ [(r_1 c_{33} - r_3 c_{31}) \delta_1 + (r_1 s_{33} - r_3 c_{33}) \delta_2 - \dot{r}_2] b_2 \\
+ [(r_2 c_{31} - r_1 c_{32}) \delta_1 + (r_2 c_{33}) \delta_2 - r_1 \delta_3 - \dot{r}_3] b_3 \\
\]

\[ \frac{v_{m/o}}{m/o} = \frac{v_m}{m} \times \frac{m}{m/o} + n v_m \]  

\[ = (\omega_2 L_3 - \omega_3 L_2) m b_1 + (\omega_3 L_1 - \omega_1 L_3) m b_2 \\
+ (\omega_1 L_1 - \omega_2 L_2) m b_3 + \dot{L} n_2 \\
= [(L_3 c_{32} - L_2 c_{33}) \delta_1 + (-L_2 s_{33}) \delta_2 + L_3 \delta_3 \\
+ (L_3 t_{12}) \delta - L_2 \dot{a} + \dot{L} t_{21}] b_1 \\
+ [(L_1 c_{33} - L_3 c_{31}) \delta_1 + (L_1 s_{33} - L_3 c_{33}) \delta_2 \\
+ (-L_3 t_{11}) \delta + L_1 \dot{a} + \dot{L} t_{22}] b_2 \\
+ [(L_2 c_{31} - L_1 c_{32}) \delta_1 + (L_2 c_{33}) \delta_2 - L_1 \delta_3 \\
+ (L_2 t_{11} - L_1 t_{12}) \delta + \dot{L} t_{23}] b_3 \\
\]  

\[ L_1 \triangleq L t_{21} \]  

\[ L_2 \triangleq L t_{22} \]  

\[ L_3 \triangleq L t_{23}. \]  

The velocity components may be assembled and written as,

\[ \begin{align*}
V_m & = [\dot{x}_{1c_{11}} + \dot{x}_{2c_{21}} + \dot{x}_{3c_{31}} \\
& + ((r_3 + L_3)c_{32} - (r_2 + L_2)c_{33}) \delta_1 \\
& + (- (r_2 + L_2)s_{33}) \delta_2 + (r_3 + L_3) \delta_3 \\
& + (L_3 t_{12}) \delta - L_2 \dot{a} - \dot{r}_1 + \dot{L} t_{21}] b_1 \\
& + [\dot{x}_{1c_{12}} + \dot{x}_{2c_{22}} + \dot{x}_{3c_{32}} \\
& + ((r_1 + L_1)c_{33} + (r_3 + L_3)c_{31}) \delta_1]
\end{align*} \]
\[ + ((r_1 + L_1) \text{sn} \theta_3 - (r_3 + L_3) \text{cs} \theta_3) \dot{\theta}_2 \]
\[ + (-L_3 t_{11}) \dot{\theta} + L_1 \ddot{\theta}_2 + L \dot{t}_{22} b_2 \]
\[ + [\dot{x}_1 c_{13} + \dot{x}_2 c_{23} + \dot{x}_3 c_{33} \]
\[ + ((r_2 + L_2) c_{31} - (r_1 + L_1) c_{32}) \dot{\theta}_1 \]
\[ + ((r_2 + L_2) \text{cs} \theta_3) \dot{\theta}_2 + (-r_1 + L_1) \dot{\theta}_3 \]
\[ + (L_2 t_{12} - L_1 t_{12}) \dot{\theta} - \dot{r}_3 + L \dot{t}_{23} b_3. \]

The partial velocities and partial angular velocities are given by,

\[ \frac{A_v}{x_1} = c_{11} b_1 + c_{12} b_2 + c_{13} b_3 \]  
\[ (47) \]

\[ \frac{A_v}{x_2} = c_{21} b_1 + c_{22} b_2 + c_{23} b_3 \]  
\[ (48) \]

\[ \frac{A_v}{x_3} = c_{31} b_1 + c_{32} b_2 + c_{33} b_3 \]  
\[ (49) \]

\[ \frac{A_v}{\dot{\theta}_1} = [(r_3 + L_3) c_{32} - (r_2 + L_2) c_{33}] b_1 \]
\[ + [(r_1 + L_1) c_{33} - (r_3 + L_3) c_{31}] b_2 \]
\[ + [(r_2 + L_2) c_{31} - (r_1 + L_1) c_{32}] b_3 \]  
\[ (50) \]

\[ \frac{A_v}{\dot{\theta}_2} = [-(r_2 + L_2) \text{sn} \theta_3] b_1 \]
\[ + [(r_1 + L_1) \text{sn} \theta_3 - (r_3 + L_3) \text{cs} \theta_3] b_2 \]
\[ + [(r_2 + L_2) \text{cs} \theta_3] b_3 \]  
\[ (51) \]

\[ \frac{A_v}{\dot{\theta}_3} = [r_3 + L_3] b_1 + [0] b_2 + [-(r_1 + L_1)] b_3 \]  
\[ (52) \]

\[ \frac{A_v}{\dot{\theta}_3} = [L_3 t_{12}] b_1 + [-L_3 t_{11}] b_2 + [L_2 t_{12} - L_1 t_{12}] b_3 \]  
\[ (53) \]

\[ \frac{A_v}{\dot{\theta}_3} = [-L_2] b_1 + [L_1] b_2 + [0] b_3 \]  
\[ (54) \]

\[ \frac{A_v}{x_1} = 0 \quad (i = 1, 2, 3) \]  
\[ (55) \]

\[ \frac{A_v}{x_1} = c_{31} b_1 + c_{32} b_2 + c_{33} b_3 \]  
\[ (56) \]
\[
\begin{align*}
A \cdot B \cdot \omega_{\theta_2} &= \cos \theta_3 b_1 + \sin \theta_3 b_3 \\
A \cdot B \cdot \omega_{\theta_3} &= b_2 \\
A \cdot m \cdot \omega_{\beta} &= t_{11} b_1 + t_{12} b_2 \\
A \cdot B \cdot \omega_a &= b_3.
\end{align*}
\]

These quantities may be defined as,

\[
\begin{align*}
u_1 &= c_{11} \\
u_2 &= c_{12} \\
u_3 &= c_{13} \\
u_4 &= c_{21} \\
u_5 &= c_{22} \\
u_6 &= c_{23} \\
u_7 &= c_{31} \\
u_8 &= c_{32} \\
u_9 &= c_{33} \\
u_{10} &= (r_3 + L_3)c_{32} - (r_2 + L_2)c_{33} \\
u_{11} &= (r_1 + L_1)c_{33} - (r_3 + L_3)c_{31} \\
u_{12} &= (r_2 + L_2)c_{31} - (r_1 + L_1)c_{32} \\
u_{13} &= -(r_2 + L_2)\sin \theta_3 \\
u_{14} &= (r_1 + L_1)\sin \theta_3 - (r_3 + L_3)\cos \theta_3
\end{align*}
\]
The angular acceleration of the pendulum mass is given by,

\[
\frac{d^2 \omega_m}{dt^2} = \frac{d}{dt} \left[ (\omega_1 + \dot{\theta}_t) \beta_1 + (\omega_2 + \dot{\theta}_t) \beta_2 \right] + (\omega_3 + \dot{\theta}_t) \beta_3
\]

\[
= \left[ \dot{\omega}_1 + \ddot{\theta}_t + \dot{\beta}(\dot{\theta}_t - t_1 \omega_3) + \beta \omega_2 \right] \beta_1
\]

\[
+ \left[ \dot{\omega}_2 + \ddot{\theta}_t + \dot{\beta}(\dot{\theta}_t + t_1 \omega_3) - \beta \omega_1 \right] \beta_2
\]

The acceleration of the pendulum mass is given by,

\[
\ddot{a}_m = \ddot{a}_G + \ddot{a}_{o/G} + \ddot{a}_{m/o}
\]

where,

\[
\ddot{a}_G = x_1 \ddot{a}_1 + x_2 \ddot{a}_2 + x_3 \ddot{a}_3
\]

\[
\ddot{a}_{o/G} = A_w B \times (A_w B \times \ddot{r}_o/G) + \frac{d}{dt} A_w B \times \ddot{r}_o/G
\]
\[ + 2A_ω^B x B_v^O + B_{a^O} \]
\[ = [ω_1^2 r_2 - r_1(ω_2^2 + ω_3^2) + ω_1 ω_2 r_3 \]
\[ + 2r_3 - ω_2 r_2 - 2(ω_2 r_3 - ω_3 r_2) - r_1 h_1] \]
\[ + [ω_2 ω_3 r_3 - r_2(ω_1^2 + ω_3^2) + ω_1 ω_2 r_1 \]
\[ + 2r_3 - ω_1 r_3 - 2(ω_3 r_1 - ω_1 r_3) - r_2 h_2] \]
\[ + [ω_1^2 r_1 - r_3(ω_1^2 + ω_2^2) + ω_2 ω_3 r_2 \]
\[ + 2r_3 - ω_2 r_1 - 2(ω_1 r_2 - ω_2 r_1) - r_3 h_3] \]
\[ a_{m/o} = \frac{d}{dt} A_ω^m x \frac{L^m}{O} + \frac{d}{dt} \frac{A_ω^m x L^m}{O} \]
\[ + 2(\frac{A_ω^m x n_y^m}{O}) + n_{a^m} \]
\[ = [(ω_1 ω_2 L_2 - L_1 ω_2^2 + ω_3^2) + ω_1 ω_3 L_3]^m \]
\[ + L_3(ω_2 + \ddot{L}_{12} + \ddot{L}(t_{11} + t_{21}ω_3) - a_ω_1) \]
\[ - L_2(ω_3 + \ddot{L}_{11} + \ddot{L}(t_{12}ω_1 - t_{11}) \]
\[ + 2L(ω_2 t_{23} - ω_3 t_{22})^m + \ddot{L}_{21} h_1 \]
\[ + [(ω_2 ω_3 L_3 - L_2(ω_1^2 + ω_3^2) + ω_1 ω_2 L_1)^m \]
\[ + L_1(ω_3 + \ddot{L}_{11} + \ddot{L}(t_{12}ω_1 - t_{11}) \]
\[ - L_3(ω_1 + \ddot{L}_{11} + \ddot{L}(-t_{12} + t_{12}ω_3) + a_ω_2) \]
\[ + 2L(ω_3 t_{21} - ω_1 t_{23})^m + \ddot{L}_{22} h_2 \]
\[ + [(ω_1 ω_3 L_1 - L_3(ω_1^2 + ω_2^2) + ω_2 ω_3 L_2)^m \]
\[ + L_2(ω_1 + \ddot{L}_{11} + \ddot{L}(t_{12}ω_1 - t_{12}) \]
\[ - L_1(ω_1 + \ddot{L}_{11} + \ddot{L}(t_{12}ω_1 - t_{11}) - a_ω_1) \]
\[ + 2L(ω_1 t_{22} - ω_2 t_{21})^m + \ddot{L}_{23} h_3 \]

The angular acceleration components of \( \frac{d^2 A_ω}{dt^2} \) may be written as,

\[ \dot{ω}_1 = \ddot{\theta}_1 c_3 + \ddot{\theta}_2 c_3 + e_1 \]
\[ \dot{ω}_2 = \ddot{\theta}_1 c_3 + \ddot{\theta}_3 + e_2 \]
\[ \ddot{\theta}_3 = \theta_1 c_{33} + \theta_2 s_{33} + e_3. \quad (92) \]

where,

\[ e_1 \triangleq \theta_1 (\dot{\theta}_2 c_{32} s_{3} - \dot{\theta}_3 c_{33}) - \dot{\theta}_2 \dot{\theta}_3 s_{33} \quad (93) \]
\[ e_2 \triangleq \dot{\theta}_1 \dot{\theta}_2 c_{\theta_2} \quad (94) \]
\[ e_3 \triangleq \theta_1 (-\dot{\theta}_2 c_{32} c_{\theta_3} + \dot{\theta}_3 c_{31}) + \dot{\theta}_2 \dot{\theta}_3 c_{\theta_3}. \quad (95) \]

Substituting equations 90-95 into 86-89 the acceleration of the pendulum mass is given by,

\[ a_m = \sum [u_1 x_1 + u_4 x_2 + u_7 x_3 + u_{10} \theta_1 + u_{13} \theta_2 + u_{16} \theta_3 + u_{19} \theta + u_{22} \theta + B_1] d_1 \]
\[ + [u_2 x_1 + u_5 x_2 + u_8 x_3 + u_{11} \theta_1 + u_{14} \theta_2 + u_{17} \theta_3 + u_{20} \theta + u_{23} \theta + B_2] d_2 \]
\[ + [u_3 x_1 + u_6 x_2 + u_9 x_3 + u_{12} \theta_1 + u_{15} \theta_2 + u_{18} \theta_3 + u_{21} \theta + u_{24} \theta + B_3] d_3 \]

where,

\[ B_1 \triangleq (r_3 + L_3)e_2 - (r_2 + L_2)e_3 \quad (97) \]
\[ + L_3 (\dot{\theta}_t_{12} + t_{12} \omega_3) - \ddot{\omega}_1 \]
\[ - L_2 (\dot{\theta}_t_{11} + t_{11} \omega_2) \]
\[ + \omega_1 \omega_2 r_2 - r_1 (\omega_2 \omega_3 + \omega_3 \omega_3) - \omega_1 \omega_3 r_3 \]
\[ + (\omega_1 \omega_2 L_2 - L_1 (\omega_2 \omega_3 + \omega_3 \omega_3) + \omega_1 \omega_3 L_3)^m \]
\[ - 2(\omega_2 r_3 - \omega_3 r_2) - r_1 \]
\[ + 2L (\omega_2 t_23 - \omega_3 t_{22})^m + \omega_2 \]

\[ B_2 \triangleq (r_1 + L_1)e_3 - (r_3 + L_3)e_1 \quad (98) \]
\[ + L_1 (\dot{\theta}_t_{12} + t_{12} \omega_2) \]
\[ - L_3 (\dot{\theta}_t_{12} + t_{12} \omega_3) + \ddot{\omega}_2 \]
The next step in formulating the equations of motion is the derivation of the generalized inertia forces. The quantities are given by,

\[
(F_r^*) = \sum_{r} A_{vB} \cdot F_r^* + A_{\omega B} \cdot T_B^* + \sum_{i=1}^{N} A_{vmi} \cdot F_i^* \tag{100}
\]

where,

\[
F_B^* = -M a_G \tag{101}
\]

\[
T_B^* = \mathbf{\dot{r}}_B \cdot A_{\omega B} + (\mathbf{\dot{r}}_B \cdot A_{\omega B}) \times A_{\omega B} - \mathbf{\dot{r}}_B \cdot \frac{d}{dt} A_{\omega B} \tag{102}
\]

\[
F_i^* = -m_i a_{mi}, \quad i = 1, \ldots, N \text{ pendulums}. \tag{103}
\]

The inertia forces and inertia torques for body B are given by,

\[
F_B = -M(x_1 a_1 + x_2 a_2 + x_3 a_3) \tag{104}
\]

\[
T_B = \begin{pmatrix}
- (I_{11} c_{31} + I_{12} c_{32} + I_{13} c_{33}) \dot{\theta}_1 \\
- (I_{11} s c_{31} + I_{13} s n c_{33}) \dot{\theta}_2 - I_{12} \dot{\theta}_3
\end{pmatrix} \tag{105}
\]
The generalized inertia forces for body B may be written as,

\[ \mathbf{F}^* \mathbf{x}_1 = \mathbf{A}_{V^*} \mathbf{x}_1 \mathbf{a}_G = -m \mathbf{x}_1 \]  
\[ \mathbf{F}^* \mathbf{x}_2 = -m \mathbf{x}_2 \]  
\[ \mathbf{F}^* \mathbf{x}_3 = -m \mathbf{x}_3 \]  
\[ \mathbf{F}_{\theta 1}^* = \mathbf{A}_{\theta 1^*} \mathbf{T}^* \mathbf{B} \]

\[ = g_{11} \ddot{\theta}_1 + g_{12} \ddot{\theta}_2 + g_{13} \ddot{\theta}_3 + d_{1c_{31}} + d_{2c_{32}} + d_{3c_{33}} \]
\[ F_{\theta 2} = g_{21} \theta_1 + g_{22} \theta_2 + g_{23} \theta_3 \]
\[ F_{\theta 3} = g_{31} \theta_1 + g_{32} \theta_2 + g_{33} \theta_3 + d_2 \]
\[ g_{11} \triangleq - I_{11} c_{31}^2 - 2 I_{12} c_{31} c_{32} - 2 I_{13} c_{31} c_{33} \]
\[ I_{22} c_{32}^2 - 2 I_{23} c_{33} c_{32} - I_{33} c_{33} \]
\[ g_{12} \triangleq - I_{11} c_{31} s_{\theta 3} - I_{12} c_{32} c_{\theta 3} \]
\[ - I_{13} (c_{33} c_{\theta 3} + c_{31} s_{\theta 3}) - I_{23} c_{32} s_{\theta 3} \]
\[ - I_{33} c_{33} s_{\theta 3} \]
\[ g_{13} \triangleq - I_{12} c_{31} - I_{22} c_{32} - I_{23} c_{33} \]
\[ g_{21} \triangleq - I_{11} c_{31} c_{\theta 3} - I_{12} c_{32} c_{\theta 3} \]
\[ - I_{13} (c_{33} c_{\theta 3} + c_{31} s_{\theta 3}) - I_{23} c_{32} s_{\theta 3} \]
\[ - I_{33} c_{33} s_{\theta 3} \]
\[ g_{22} \triangleq - I_{11} c_{2}^2 s_{\theta 3} - 2 I_{13} c_{\theta 3} s_{\theta 3} - I_{33} s_{\theta 3}^2 \]
\[ g_{23} \triangleq - I_{12} c_{\theta 3} - I_{23} s_{\theta 3} \]
\[ g_{31} \triangleq - I_{12} c_{31} - I_{22} c_{32} - I_{23} c_{33} \]
\[ g_{32} \triangleq - I_{12} c_{\theta 3} - I_{23} s_{\theta 3} \]
\[ g_{33} \triangleq - I_{22} \]
\[ d_1 \triangleq -(I_{11} e_1 + I_{12} e_2 + I_{13} e_3) \]
\[ -(I_{11}^2 + I_{12}^2 + I_{13}^2) \]
\[ + I_{12}^2 \omega_2 + I_{23} \omega_3 \]
\[ d_2 \triangleq -(I_{12} e_1 + I_{22} e_2 + I_{23} e_3) \]
\[ -(I_{12}^2 + I_{22}^2 + I_{23}^2) \]
The generalized inertia forces associated with the pendulums are written in a similar manner.

The remaining step in formulating the equations of motion is the derivation of the generalized active forces. The quantities were formulated as,

\[ (F_r) = \frac{A_r}{V_r} \cdot (F_G + F_T) + \frac{A_r}{V_r} \cdot (T_T + T_B) \]

\[ + \frac{A_r}{V_r} \cdot \tau^m \quad r = 1, \ldots, 6 + 2N. \]

The generalized active forces were formulated separately for the gravitational, thrust, spring, and damper forces and then assembled using equation 124.

Gravitational forces on the main body were modeled as a point mass with the force being directly proportional to the main body mass and inversely proportional to the square of the distance from the center of the earth. The gravitational force on the pendulum was neglected because the mass is much less than the main body mass.
Thrust forces were modeled as an equivalent single force and couple applied at the outlet of the rocket motor.

Spring and damper forces were included in the pendulum model of the sloshing fluid store. The spring provides a restoring force to the position of the pendulum while the damper dissipates energy.\(^6,7\)

The four force groups were derived as follows:

**Gravitational force**

\[
K_G = 4.3735 \times 10^{14} \text{ M lb}_f
\]  
(125)

\[
F_G = -K_G \cdot \frac{x_1 a_1 + x_2 a_2 + x_3 a_3}{[x_1^2 + x_2^2 + x_3^2]^{3/2}}
\]  
(126)

\[
F_{x_1} = \frac{A V G}{x_1} \cdot F_G = -K_G x_1
\]  
(127)

\[
F_{x_2} = \frac{-K_G x_2}{[x_1^2 + x_2^2 + x_3^2]^{3/2}}
\]  
(128)

\[
F_{x_3} = \frac{-K_G x_3}{[x_1^2 + x_2^2 + x_3^2]^{3/2}}
\]  
(129)

**Thrust force**

\[
F_T = F_1 b_1 + F_2 b_2 + F_3 b_3
\]  
(130)

\[
F_{x_1} = \frac{A V G}{x_1} \cdot F_T
\]  
(131)

\[
= (c_{11} b_1 + c_{12} b_2 + c_{13} b_3) \cdot (F_1 b_1 + F_2 b_2 + F_3 b_3)
\]

\[
= c_{11} F_1 + c_{12} F_2 + c_{13} F_3
\]

\[
F_{x_2} = c_{21} F_1 + c_{22} F_2 + c_{23} F_3
\]  
(132)
\[ F_{x_3} = c_{31}F_1 + c_{32}F_2 + c_{33}F_3 \]  

**Thrust torque**  
\[ T_T = \frac{T}{G} \cdot \mathbf{T} \]  
\[ = (r_2F_3 - r_3F_2)T_1 + (r_3F_1 - r_1F_3)T_2 + (r_1F_2 - r_2F_1)T_3 \]  
\[ T_1 = r_2F_3 - r_3F_2 \]  
\[ T_2 = r_3F_1 - r_1F_3 \]  
\[ T_3 = r_1F_2 - r_2F_1 \]  
\[ F_\theta^1 = A_{\omega^B_\theta} \cdot T_T \]  
\[ = c_{31}T_1 + c_{32}T_2 + c_{33}T_3 \]  
\[ F_\theta_2 = T_1\cos\theta_3 + T_3\sin\theta_3 \]  
\[ F_\theta_3 = T_2 \]  

**Pendulum spring and damper torques**  
\[ \tau^m = -(K_a(a - a_0)^3 + D_a\dot{a})b_3 \]  
\[ = -(K_\beta(b - b_0)^3 + D_\beta\dot{b})n \]  
\[ = -(K_\beta(b - b_0)^3 + D_\beta\dot{b})t_{11}b_1 \]  
\[ - (K_\beta(b - b_0)^3 + D_\beta\dot{b})t_{12}b_2 \]  
\[ - (K_a(a - a_0)^3 + D_a\dot{a})b_3 \]  
\[ \tau^m = \tau^B \]  
\[ F_\theta^1 = A_{\omega^B_\theta} \cdot \tau^B + A_{\omega^m_\theta} \cdot \tau^m = 0 \]  
\[ F_\theta_2 = 0 \]
\[ F_{\beta_3} = 0 \]  
(145)

\[ F_{\beta} = 0 + (t_{11}\beta_1 + t_{12}\beta_2) \cdot T^m \]  
(146)

\[ = -K_\beta (\beta - \beta_0)^3 + D_\beta \dot{\beta} \]

\[ F_{\alpha} = 0 + D_3 \cdot T^m \]  
(147)

\[ = -K_\alpha (\alpha - \alpha_0)^3 + D_\alpha \dot{\alpha} \].

The generalized active forces may be assembled and written as,

\[ F_{x_1} = -K_1 \frac{x_1}{x_1^2 + x_2^2 + x_3^2}^{3/2} + c_{11}F_1 + c_{12}F_2 + c_{13}F_3 \]  
(148)

\[ F_{x_2} = -K_2 \frac{x_2}{x_1^2 + x_2^2 + x_3^2}^{3/2} + c_{21}F_1 + c_{22}F_2 + c_{23}F_3 \]  
(149)

\[ F_{x_3} = -K_3 \frac{x_3}{x_1^2 + x_2^2 + x_3^2}^{3/2} + c_{31}F_1 + c_{32}F_2 + c_{33}F_3 \]  
(150)

\[ F_{\theta_1} = c_{31}T_1 + c_{32}T_2 + c_{33}T_3 \]  
(151)

\[ F_{\theta_2} = c_{53}T_1 + \sin \theta_3 T_3 \]  
(152)

\[ F_{\theta_3} = T_2 \]  
(153)

\[ F_{\beta} = -K_\beta (\beta - \beta_0)^3 + D_\beta \dot{\beta} \]  
(154)

\[ F_{\alpha} = -K_\alpha (\alpha - \alpha_0)^3 + D_\alpha \dot{\alpha} \].  
(155)

The equations of motion were obtained by assembling the generalized inertia forces and generalized active forces as given by,

\[ F_r + F_r^* = 0 \quad r = 1, \ldots, 6 + 2N. \]  
(156)
The equations of motion for the system are given by,

\[
[-M \Sigma m_i] \ddot{x}_1 + [0] \ddot{x}_2 + [0] \ddot{x}_3 + [-\Sigma m_i (u_{110} + u_{111} + u_{312})] \ddot{\theta}_1 + [-\Sigma m_i (u_{113} + u_{114} + u_{315})] \ddot{\theta}_2 + [-\Sigma m_i (u_{116} + u_{117} + u_{318})] \ddot{\theta}_3 + [-\Sigma m_i (u_{119} + u_{120} + u_{321})] \ddot{\theta}_i + [-\Sigma m_i (u_{122} + u_{123} + u_{324})] \ddot{\alpha}_i + [-\Sigma m_i (u_{31} + u_{22} + u_{33})] + c_{11} F_1 + c_{12} F_2 + c_{13} F_3 - K_G x_1 = 0
\]

\[
\frac{[x_1^2 + x_2^2 + x_3^2]^{3/2}}{[x_1^2 + x_2^2 + x_3^2]^{3/2}}
\]

\[
[0] x_1 + [-M \Sigma m_i] x_2 + [0] x_3 + [-\Sigma m_i (u_{410} + u_{511} + u_{612})] \ddot{\alpha}_1 + [-\Sigma m_i (u_{413} + u_{514} + u_{615})] \ddot{\alpha}_2 + [-\Sigma m_i (u_{416} + u_{517} + u_{618})] \ddot{\alpha}_3 + [-\Sigma m_i (u_{419} + u_{520} + u_{621})] \ddot{\alpha}_i + [-\Sigma m_i (u_{422} + u_{523} + u_{624})] \ddot{\alpha}_i + [-\Sigma m_i (u_{31} + u_{22} + u_{33})] + c_{21} F_1 + c_{22} F_2 + c_{23} F_3 - K_G x_2 = 0
\]

\[
\frac{[x_1^2 + x_2^2 + x_3^2]^{3/2}}{[x_1^2 + x_2^2 + x_3^2]^{3/2}}
\]
\[ [0] x_1 + [0] x_2 + [-M - \Sigma m_1] x_3 + [-\Sigma m_1 (u_7 u_10 + u_8 u_{11} + u_9 u_{12})] \hat{\theta}_1 + [-\Sigma m_1 (u_7 u_{13} + u_8 u_{14} + u_9 u_{15})] \hat{\theta}_2 + [-\Sigma m_1 (u_7 u_{16} + u_8 u_{17} + u_9 u_{18})] \hat{\theta}_3 + [-\Sigma m_1 (u_7 u_{19} + u_8 u_{20} + u_9 u_{21})] \hat{\theta}_4 + [-\Sigma m_1 (u_7 u_{22} + u_8 u_{23} + u_9 u_{24})] \hat{\theta}_5 + [-\Sigma m_1 (u_7 b_1 + u_8 b_2 + u_9 b_3)] + c_{31} f_1 + c_{32} f_2 + c_{33} f_3 - K_g x_3 = 0 \]

\[ \frac{[x_1^2 + x_2^2 + x_3^2]^{3/2}}{} \]

\[ [-\Sigma m_1 (u_{10} u_1 + u_{11} u_2 + u_{12} u_3)] x_1 + [-\Sigma m_1 (u_{10} u_4 + u_{11} u_5 + u_{12} u_6)] x_2 + [-\Sigma m_1 (u_{10} u_7 + u_{11} u_8 + u_{12} u_9)] x_3 + [g_{11} - \Sigma m_1 (u_{10}^2 + u_{11}^2 + u_{12}^2)] \hat{\theta}_1 + [g_{12} - \Sigma m_1 (u_{10} u_{13} + u_{11} u_{14} + u_{12} u_{15})] \hat{\theta}_2 + [g_{13} - \Sigma m_1 (u_{10} u_{16} + u_{11} u_{17} + u_{12} u_{18})] \hat{\theta}_3 + [-\Sigma m_1 (u_{10} u_{19} + u_{11} u_{20} + u_{12} u_{21})] \hat{\theta}_4 + [-\Sigma m_1 (u_{10} u_{22} + u_{11} u_{23} + u_{12} u_{24})] \hat{\theta}_5 + [-\Sigma m_1 (u_{10} b_1 + u_{11} b_2 + u_{12} b_3)] + d_1 c_{31} + d_2 c_{32} + d_3 c_{33} + c_{31} t_1 c_{32} t_2 + c_{33} t_3 = 0 \]

\[ [-\Sigma m_1 (u_{13} u_1 + u_{14} u_2 + u_{15} u_3)] x_1 + [-\Sigma m_1 (u_{13} u_4 + u_{14} u_5 + u_{15} u_6)] x_2 + [-\Sigma m_1 (u_{13} u_7 + u_{14} u_8 + u_{15} u_9)] x_3 \]
\[ + [g_{21} - \Sigma_1 (u_{13}u_{10} + u_{14}u_{11} + u_{15}u_{12})]\theta_1 \\
+ [g_{22} - \Sigma_1 (u_{11}^2 + u_{14}^2 + u_{15}^2)]\theta_2 \\
+ [g_{23} - \Sigma_1 (u_{13}u_{16} + u_{14}u_{17} + u_{15}u_{18})]\theta_3 \\
+ [-\Sigma_1 (u_{13}u_{19} + u_{14}u_{20} + u_{15}u_{21})]\beta_i \\
+ [-\Sigma_1 (u_{13}u_{22} + u_{14}u_{23} + u_{15}u_{24})]\alpha_i \\
+ [-\Sigma_1 (u_{13}B_1 + u_{14}B_2 + u_{15}B_3)] \\
+ d_1 \cos^3 + d_3 \sin^3 + \cos^3 T_1 + \sin^3 T_3 = 0 \\
\]

\[ [-\Sigma_1 (u_{16}u_{1} + u_{17}u_{2} + u_{18}u_{3})]x_1 \] (162)

\[ + [\Sigma_1 (u_{16}u_{4} + u_{17}u_{5} + u_{18}u_{6})]x_2 \]
\[ + [-\Sigma_1 (u_{16}u_{7} + u_{17}u_{8} + u_{18}u_{9})]x_3 \]
\[ + [g_{31} - \Sigma_1 (u_{16}u_{10} + u_{17}u_{11} + u_{18}u_{12})]\theta_1 \\
+ [g_{32} - \Sigma_1 (u_{16}u_{13} + u_{17}u_{14} + u_{18}u_{15})]\theta_2 \\
+ [g_{33} - \Sigma_1 (u_{16}^2 + u_{17}^2 + u_{18}^2)]\theta_3 \\
+ [-\Sigma_1 (u_{16}u_{19} + u_{17}u_{20} + u_{18}u_{21})]\beta_i \\
+ [-\Sigma_1 (u_{16}u_{22} + u_{17}u_{23} + u_{18}u_{24})]\alpha_i \\
+ [-\Sigma_1 (u_{16}B_1 + u_{17}B_2 + u_{18}B_3)] + d_2 + T_2 = 0 \\
\]

\[ [-\Sigma_1 (u_{19}u_{1} + u_{20}u_{2} + u_{21}u_{3})]x_1 \] (163)

\[ + [-\Sigma_1 (u_{19}u_{4} + u_{20}u_{5} + u_{21}u_{6})]x_2 \]
\[ + [-\Sigma_1 (u_{19}u_{7} + u_{20}u_{8} + u_{21}u_{9})]x_3 \]
\[ + [-\Sigma_1 (u_{19}u_{10} + u_{20}u_{11} + u_{21}u_{12})]\theta_1 \\
+ [-\Sigma_1 (u_{19}u_{13} + u_{20}u_{14} + u_{21}u_{15})]\theta_2 \\
+ [-\Sigma_1 (u_{19}u_{16} + u_{20}u_{17} + u_{21}u_{18})]\theta_3 \\
+ [-\Sigma_1 (u_{19}^2 + u_{20}^2 + u_{21}^2)]\beta_i \\
+ [-\Sigma_1 (u_{19}u_{22} + u_{20}u_{23} + u_{21}u_{24})]\alpha_i \]
+ [-m_1(u_{19}B_1 + u_{20}B_2 + u_{21}B_3)]
- (K_a(b_i - b_{i0})^3 + D_a b_i) = 0

\[ \begin{align*}
- m_1(u_{22}u_1 + u_{23}u_2 + u_{24}u_3) & x_1 \\
- m_1(u_{22}u_4 + u_{23}u_5 + u_{24}u_6) & x_2 \\
- m_1(u_{22}u_7 + u_{23}u_8 + u_{24}u_9) & x_3 \\
- m_1(u_{22}u_{10} + u_{23}u_{11} + u_{24}u_{12}) & \ddot{\theta}_1 \\
- m_1(u_{22}u_{13} + u_{23}u_{14} + u_{24}u_{15}) & \ddot{\theta}_2 \\
- m_1(u_{22}u_{16} + u_{23}u_{17} + u_{24}u_{18}) & \ddot{\theta}_3 \\
- m_1(u_{22}u_{19} + u_{23}u_{20} + u_{24}u_{21}) & \dddot{\theta}_1 \\
- m_1(u_{22}^2 + u_{23}^2 + u_{24}^2) & \dddot{\theta}_1 \\
+ [-m_i(u_{22}B_1 + u_{23}B_2 + u_{24}B_3)]
\end{align*} \]

\[ \begin{align*}
- m_1(u_{22}u_1 + u_{23}u_2 + u_{24}u_3) \\
- m_1(u_{22}u_4 + u_{23}u_5 + u_{24}u_6) \\
- m_1(u_{22}u_7 + u_{23}u_8 + u_{24}u_9) \\
- m_1(u_{22}u_{10} + u_{23}u_{11} + u_{24}u_{12}) & \ddot{\theta}_1 \\
- m_1(u_{22}u_{13} + u_{23}u_{14} + u_{24}u_{15}) & \ddot{\theta}_2 \\
- m_1(u_{22}u_{16} + u_{23}u_{17} + u_{24}u_{18}) & \ddot{\theta}_3 \\
- m_1(u_{22}u_{19} + u_{23}u_{20} + u_{24}u_{21}) & \dddot{\theta}_1 \\
- m_1(u_{22}^2 + u_{23}^2 + u_{24}^2) & \dddot{\theta}_1 \\
+ [-m_i(u_{22}B_1 + u_{23}B_2 + u_{24}B_3)]
\end{align*} \]

\[ \begin{align*}
- (K_{\gamma}(a_i - a_{i0})^3 + D_{a_1}) & = 0,
\end{align*} \]
i = 1, \ldots, N pendulums.

A classical Lagrangian formulation was written as a check on the equations of motion (see Appendix A). An assumption was made by not allowing the pendulum attachment points to move relative to the center of mass (G) of the main body. This assumption can be removed using Kane's equations quite easily but, as an equation match is desired, this simplifies the algebra required. The actual construction of the rocket motor is such that it burns radially and its center of mass coincides with the center of mass of the structure and payload so the assumption is valid. A second assumption is that the pendulum mass does not move relative to the arm it is on. This assumption is valid unless mass is being expelled from the fuel
tanks which would change the equivalent mechanical pendulum length. Using Kane's equations, this again is not a problem, but unnecessarily complicates the Lagrangian formulation.

It can be seen from equations 148-155 and 39-46 of Appendix A that the generalized active forces in Kane's equations match the generalized forces using the principle of virtual work. The generalized inertia forces of Kane's equations can be shown to match the negative of the time derivative part of Lagrange's equation after considerable algebraic manipulation. Trigonometric functions must be identified and simplified to obtain the match. Both methods result in symmetry about the diagonal of the coefficients of the highest derivatives for the equations of motion. Kane's method is clearly a more efficient way to formulate the equations of motion than the Lagrangian approach.

The utility of Kane's method becomes more apparent when the concept of generalized speeds is used. In the first derivation using Kane's equations, the generalized speeds were chosen as,

\[ \begin{align*}
V_1 & \triangleq \dot{x}_1 \\
V_2 & \triangleq \dot{x}_2 \\
V_3 & \triangleq \dot{x}_3 \\
V_4 & \triangleq \dot{\theta}_1 \\
V_5 & \triangleq \dot{\theta}_2 \\
V_6 & \triangleq \dot{\theta}_3 \\
V_7 & \triangleq \dot{\delta}_1 \\
V_8 & \triangleq \dot{\delta}_2 \\
V_9 & \triangleq \dot{\delta}_3 \\
V_{5+2N} & \triangleq \dot{\delta}_N
\end{align*} \]  

(165)

A better choice of generalized speeds yields a much simpler form for the equations of motion. A formulation using a new set of generalized speeds was derived using,
The angular velocities, velocities, angular accelerations, accelerations, and partial velocities were rewritten in terms of this new set of generalized speeds.

The velocity of the pendulum mass becomes,

\[
\mathbf{v}_m = [\dot{x}_1 c_{11} + \dot{x}_2 c_{21} + \dot{x}_3 c_{31} \\
+ \dot{\omega}_2 r_3 - \omega_3 r_2 + (L_3(\omega_2 + \dot{\theta}_{t2}) - L_2(\omega_3 + \dot{\theta}) - \dot{\hat{r}}) + \dot{L}_{t21}b_1 + (\dot{x}_1 c_{12} + \dot{x}_2 c_{22} + \dot{x}_3 c_{32}) + \dot{\omega}_3 r_1 - \omega_1 r_3 + (L_1(\omega_3 + \dot{\theta}) - L_3(\omega_1 + \dot{\theta}_{t1})) - \dot{\hat{r}}_2 + \dot{L}_{t22}b_2 + (\dot{x}_1 c_{13} + \dot{x}_2 c_{23} + \dot{x}_3 c_{33}) + \dot{\omega}_1 r_2 - \omega_2 r_1 + (L_2(\omega_1 + \dot{\theta}_{t1}) - L_1(\omega_2 + \dot{\theta}_{t1})) - \dot{\hat{r}}_3 + \dot{L}_{t23}b_3 = [c_{11} \dot{x}_1 + c_{21} \dot{x}_2 + c_{31} \dot{x}_3 + (r_3 + L_3)\omega_1 - (r_2 + L_2)\omega_3 + L_3 t_{12} \dot{\theta} - L_2 \dot{\hat{\theta}} - \dot{\hat{\theta}}_1 + \dot{L}_{t22}b_1 + (c_{12} \dot{x}_1 + c_{22} \dot{x}_2 + c_{32} \dot{x}_3 + (r_1 + L_1)\omega_3 - (r_3 + L_3)\omega_1 - L_3 t_{11} \dot{\theta} + L_1 \dot{\hat{\theta}} - \dot{\hat{\theta}}_2 + \dot{L}_{t23}b_2 + (c_{13} \dot{x}_1 + c_{23} \dot{x}_2 + c_{33} \dot{x}_3 + (r_2 + L_2)\omega_1 - (r_1 + L_1)\omega_2 + (L_2 t_{11} - L_1 t_{12}) \dot{\theta}]
\]
The partial velocities and partial angular velocities become,

\[
\begin{align*}
\frac{A_{y_1}^m}{x_1} &= c_{11}b_1 + c_{12}b_2 + c_{13}b_3 \\
\frac{A_{y_2}^m}{x_2} &= c_{21}b_1 + c_{22}b_2 + c_{23}b_3 \\
\frac{A_{y_3}^m}{x_3} &= c_{31}b_1 + c_{32}b_2 + c_{33}b_3 \\
\frac{A_{\omega_1}}{x_1} &= -(r_3 + L_3)b_2 + (r_2 + L_2)b_3 \\
\frac{A_{\omega_2}}{x_2} &= (r_3 + L_3)b_1 - (r_2 + L_2)b_3 \\
\frac{A_{\omega_3}}{x_3} &= -(r_2 + L_2)b_1 + (r_1 + L_1)b_2 \\
\frac{A_{\omega_1}}{\beta} &= L_3t_{12}b_1 - L_3t_{11}b_2 + (L_2t_{11} - L_1t_{12})b_3 \\
\frac{A_{\omega_2}}{\alpha} &= -L_2b_1 + L_1b_2 \\
\frac{\omega_{Bi}}{x_i} &= 0 \quad (i = 1, 2, 3) \\
\frac{\omega_{B1}}{\omega_1} &= b_1 \\
\frac{\omega_{B2}}{\omega_2} &= b_2 \\
\frac{\omega_{B3}}{\omega_3} &= b_3 \\
\frac{A_{\omega_1}}{\beta} &= t_{11}b_1 + t_{12}b_2 \\
\frac{A_{\omega_3}}{\alpha} &= b_3.
\end{align*}
\]
These quantities may be defined as,

\[ u_1 \triangleq c_{11} \]  
\[ u_2 \triangleq c_{12} \]  
\[ u_3 \triangleq c_{13} \]  
\[ u_4 \triangleq c_{21} \]  
\[ u_5 \triangleq c_{22} \]  
\[ u_6 \triangleq c_{23} \]  
\[ u_7 \triangleq c_{31} \]  
\[ u_8 \triangleq c_{32} \]  
\[ u_9 \triangleq c_{33} \]  
\[ u_{10} \triangleq 0 \]  
\[ u_{11} \triangleq -(r_3 + L_3) \]  
\[ u_{12} \triangleq (r_2 + L_2) \]  
\[ u_{13} \triangleq (r_3 + L_3) \]  
\[ u_{14} \triangleq 0 \]  
\[ u_{15} \triangleq -(r_1 + L_1) \]  
\[ u_{16} \triangleq -(r_2 + L_2) \]  
\[ u_{17} \triangleq (r_1 + L_1) \]  
\[ u_{18} \triangleq 0 \]  
\[ u_{19} \triangleq L_3 t_{12} \]
The acceleration of the pendulum mass becomes,

$$\ddot{a}_m = \left[ u_{1x_1} + u_{4x_2} + u_{7x_3} + u_{1001} \right. + u_{13x_2} + u_{16x_3} + u_{19x_3} + u_{22x_3} + B_1b_1 \\
+ \left. \left[ u_{2x_1} + u_{5x_2} + u_{8x_3} + u_{11x_1} + u_{14x_2} \right. \right. + u_{17x_3} + u_{20x_3} + u_{23x_3} + B_2b_2 \\
+ \left. \left[ u_{3x_1} + u_{6x_2} + u_{9x_3} + u_{12x_1} + u_{15x_2} \right. \right. + u_{18x_3} + u_{21x_3} + u_{24x_3} + B_3b_3 \right. \]$$

where,

$$B_1 \equiv L_3(\ddot{\theta}(\dot{t}_{11} + t_{11}\omega_3) - \ddot{\omega}_1)$$

$$B_2 \equiv L_1(\ddot{\theta}(\dot{t}_{12}\omega_1 - t_{11}\omega_2))$$

$$= L_2(\ddot{\theta}(t_{12}\omega_1 - t_{11}\omega_2)) + \omega_1\omega_2\phi_2 - \phi_1(\omega_1^2 + \omega_2^2) + \omega_1\omega_3\phi_3$$

$$+ (\omega_1\omega_2L_2 - L_1(\omega_1^2 + \omega_2^2) + \omega_1\omega_3L_3)b_3$$

$$= (\omega_1\omega_2L_2 - L_1(\omega_1^2 + \omega_2^2) + \omega_1\omega_3L_3)^m$$

$$- 2(\omega_3\phi_3 - \omega_3\phi_2) - \phi_1$$

$$+ 2L_2(\omega_3\phi_2 - \omega_3\phi_2)^m + L_1t_{21}$$

$$B_2 \equiv L_1(\ddot{\theta}(\dot{t}_{12}\omega_1 - t_{11}\omega_2))$$

$$= L_3(\ddot{\theta}(\dot{t}_{12}\omega_1 - t_{11}\omega_3) + \ddot{\omega}_2)$$

$$+ \omega_1\omega_2\phi_2 - \phi_1(\omega_1^2 + \omega_2^2) + \omega_1\omega_3\phi_3$$

$$+ (\omega_1\omega_2L_2 - L_1(\omega_1^2 + \omega_2^2) + \omega_1\omega_3L_3)b_3$$

$$= (\omega_1\omega_2L_2 - L_1(\omega_1^2 + \omega_2^2) + \omega_1\omega_3L_3)^m$$

$$- 2(\omega_3\phi_3 - \omega_3\phi_2) - \phi_1$$

$$+ 2L_2(\omega_3\phi_2 - \omega_3\phi_2)^m + L_1t_{21}$$

$$u_{20} \equiv -L_3t_{11}$$

$$u_{21} \equiv L_2t_{11} - L_1t_{12}$$

$$u_{22} \equiv -L_2$$

$$u_{23} \equiv L_1$$

$$u_{24} \equiv 0.$$
The equations of motion were assembled in the same manner as before by deriving generalized inertia forces and generalized active forces.

The equations of motion of the system are given by,

\[ [-M - \Sigma m_i] \ddot{x}_1 + [0] \ddot{x}_2 + [0] \ddot{x}_3 \]

\[ + [-\Sigma m_i (u_{11}^2 + u_{12}^2)] \dddot{x}_1 \]

\[ + [-\Sigma m_i (u_{13}^2 + u_{15}^2)] \dddot{x}_2 \]

\[ + [-\Sigma m_i (u_{16}^2 + u_{17}^2)] \dddot{x}_3 \]

\[ + [-\Sigma m_i (u_{19}^2 + u_{20}^2 + u_{21}^2)] \dddot{\theta}_1 \]

\[ + [-\Sigma m_i (u_{22}^2 + u_{23}^2)] \dddot{\alpha}_i \]

\[ + [-\Sigma m_i (u_{1} B_1 + u_{2} B_2 + u_{3} B_3)] \]

\[ + u_1 F_1 + u_2 F_2 + u_3 F_3 - K_G x_1 \]

\[ = 0 \]

\[ \frac{x_1^2 + x_2^2 + x_3^2}{[x_1^2 + x_2^2 + x_3^2]^{3/2}} \]
\[\begin{align*}
\dot{\omega}_1 &= \epsilon_1 (u_5 u_1 + u_6 u_2) \\
\dot{\omega}_2 &= \epsilon_1 (u_4 u_3 + u_6 u_1) \\
\dot{\omega}_3 &= \epsilon_1 (u_4 u_6 + u_5 u_7) \\
\dot{\beta}_i &= \epsilon_1 (u_4 u_1 + u_5 u_{20} + u_6 u_{21}) \\
\dot{\alpha}_i &= \epsilon_1 (u_4 u_{22} + u_5 u_{23}) \\
\dot{\gamma}_i &= \epsilon_1 (u_4 B_1 + u_5 B_2 + u_6 B_3) \\
u_4 F_1 + u_5 F_2 + u_6 F_3 - K_G x_2 &= 0 \\
\frac{[x_1^2 + x_2^2 + x_3^2]^{3/2}}{[x_1^2 + x_2^2 + x_3^2]^{3/2}}
\end{align*}\]
+ [−I_{13} - \Sigma m_i (u_{11}u_{17})] \ddot{\omega}_3 \\
+ [−\Sigma m_i (u_{11}u_{20} + u_{12}u_{21})] \ddot{\beta}_i \\
+ [−\Sigma m_i (u_{11}u_{23})] \alpha_i \\
+ [−\Sigma m_i (u_{11}B_2 + u_{12}B_3)] + d_1 + T_1 = 0

[−\Sigma m_i (u_{13}u_{1} + u_{15}u_3)] x_1
+ [−\Sigma m_i (u_{13}u_{4} + u_{15}u_6)] x_2
+ [−\Sigma m_i (u_{13}u_{7} + u_{15}u_9)] x_3 \\
+ [−I_{12} - \Sigma m_i (u_{15}u_{12})] \dot{\omega}_1 \\
+ [−I_{22} - \Sigma m_i (u_{13}^2 + u_{15}^2)] \dot{\omega}_2 \\
+ [−I_{23} - \Sigma m_i (u_{13}u_{16})] \dot{\omega}_3 \\
+ [−\Sigma m_i (u_{13}u_{19} + u_{15}u_{21})] \ddot{\beta}_i \\
+ [−\Sigma m_i (u_{13}u_{22})] \alpha_i \\
+ [−\Sigma m_i (u_{13}B_1 + u_{15}B_3)] + d_2 + T_2 = 0

[−\Sigma m_i (u_{16}u_{1} + u_{17}u_2)] x_1
+ [−\Sigma m_i (u_{16}u_{4} + u_{17}u_5)] x_2
+ [−\Sigma m_i (u_{16}u_{7} + u_{17}u_8)] x_3 \\
+ [−I_{13} - \Sigma m_i (u_{17}u_{11})] \dot{\omega}_1 \\
+ [−I_{23} - \Sigma m_i (u_{16}u_{13})] \dot{\omega}_2 \\
+ [−I_{33} - \Sigma m_i (u_{16}^2 + u_{17}^2)] \dot{\omega}_3 \\
+ [−\Sigma m_i (u_{16}u_{19} + u_{17}u_{20})] \ddot{\beta}_i \\
+ [−\Sigma m_i (u_{16}u_{22} + u_{17}u_{23})] \alpha_i \\
+ [−\Sigma m_i (u_{16}B_1 + u_{17}B_2)] + d_3 + T_3 = 0

[−m_i (u_{19}u_1 + u_{20}u_2 + u_{21}u_3)] x_1

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(214)
\[ + [-m_i(u_{19}u_4 + u_{20}u_5 + u_{21}u_6)]x_2 \\
+ [-m_i(u_{19}u_7 + u_{20}u_8 + u_{21}u_9)]x_3 \\
+ [-m_i(u_{20}u_{11} + u_{21}u_{12})] \omega_1 \\
+ [-m_i(u_{19}u_{13} + u_{21}u_{15})] \omega_2 \\
+ [-m_i(u_{19}u_{16} + u_{20}u_{17})] \omega_3 \\
+ [-m_i(u_{19}^2 + u_{20}^2 + u_{21}^2)] \delta_i \\
+ [-m_i(u_{19}u_{22} + u_{20}u_{23})] a_i \\
+ [-m_i(u_{19}B_1 + u_{20}B_2 + u_{21}B_3)] \\
+ [-K_8(\beta_i - \beta_{10})^3 - D_8 \ddot{\beta}_i] = 0 \]

\[ [\ddot{u}_{22} + \ddot{u}_{23}u_2] x_1 \]

\[ + [-m_i(u_{22}u_4 + u_{23}u_5)] x_2 \]

\[ + [-m_i(u_{22}u_7 + u_{23}u_8)] x_3 \]

\[ + [-m_i(u_{23}u_{11})] \omega_1 \]

\[ + [-m_i(u_{22}u_{13})] \omega_2 \]

\[ + [-m_i(u_{22}u_{16} + u_{23}u_{17})] \omega_3 \]

\[ + [-m_i(u_{22}u_{19} + u_{23}u_{20})] \delta_i \]

\[ + [-m_i(u_{22}^2 + u_{23}^2)] a_i \]

\[ + [-m_i(u_{22}B_1 + u_{23}B_2)] \\
+ [-K_8(\alpha_i - \alpha_{10})^3 - D_8 \ddot{\alpha}_i] = 0, \]

\( i = 1, \ldots, N \) pendulums where,

\[ d_1 \Delta = -(\ddot{\dot{I}}_{11}\omega_1 + \ddot{\dot{I}}_{12}\omega_2 + \ddot{\dot{I}}_{13}\omega_3) \]

\[ + I_{12}\omega_1\omega_3 + I_{22}\omega_2\omega_3 + I_{23}\omega_3^2 \]

\[ - (I_{13}\omega_1\omega_2 + I_{23}\omega_2^2 + I_{33}\omega_2\omega_3) \]

\[ d_2 \Delta = -(\ddot{\dot{I}}_{12}\omega_1 + \ddot{\dot{I}}_{22}\omega_2 + \ddot{\dot{I}}_{23}\omega_3) \]
The generalized speeds were chosen in the first formulation to coincide with their respective generalized coordinate. This choice makes the algebra simpler when obtaining the equation match with the Lagrangian approach. Lagrange's equation has no counterpart to the concept of generalized speeds and this is where the advantage of Kane's method comes about.

A second set of generalized coordinates which simplifies the equations of motion are Euler parameters. Euler's theorem states that every change in the relative orientation of two rigid bodies or reference frames A and B can be produced by means of a simple rotation of B in A. Morton et al.\textsuperscript{19} derived relations between Euler parameters and Euler angles which are useful in that Euler parameters are difficult to interpret graphically or physically.

Using Euler parameters, which are generalized coordinates, the transformation matrix may be written as,

\[
\begin{bmatrix}
\mathbf{a}_1 \\
\mathbf{a}_2 \\
\mathbf{a}_3
\end{bmatrix} =
\begin{bmatrix}
c_{11} & c_{12} & c_{13} \\
c_{21} & c_{22} & c_{23} \\
c_{31} & c_{32} & c_{33}
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]
where,

\[ \varepsilon = \lambda \sin \theta / 2 \]  
(220)

\[ \varepsilon_4 = \cos \theta / 2 \]  
(221)

\[ \lambda \triangleq \text{unit vector along the axis of rotation} \]  
(222)

\[ \theta \triangleq \text{simple rotation angle of B in A} \]  
(223)

\[ \varepsilon_i \triangleq \varepsilon \cdot \mathbf{a}_i = \varepsilon \cdot \mathbf{b}_i \quad (i = 1, 2, 3) \]  
(224)

\[ c_{ij} \triangleq \mathbf{a}_i \cdot \mathbf{b}_j \quad (i, j = 1, 2, 3) \]  
(225)

\[ c_{11} \triangleq 1 - 2(\varepsilon_2^2 + \varepsilon_3^2) \]  
(226)

\[ c_{12} \triangleq 2(\varepsilon_1 \varepsilon_2 - \varepsilon_3 \varepsilon_4) \]  
(227)

\[ c_{13} \triangleq 2(\varepsilon_1 \varepsilon_3 + \varepsilon_2 \varepsilon_4) \]  
(228)

\[ c_{21} \triangleq 2(\varepsilon_1 \varepsilon_2 + \varepsilon_3 \varepsilon_4) \]  
(229)

\[ c_{22} \triangleq 1 - 2(\varepsilon_3^2 + \varepsilon_1^2) \]  
(230)

\[ c_{23} \triangleq 2(\varepsilon_2 \varepsilon_3 - \varepsilon_1 \varepsilon_4) \]  
(231)

\[ c_{31} \triangleq 2(\varepsilon_3 \varepsilon_1 - \varepsilon_2 \varepsilon_4) \]  
(232)

\[ c_{32} \triangleq 2(\varepsilon_2 \varepsilon_3 + \varepsilon_1 \varepsilon_4) \]  
(233)

\[ c_{33} \triangleq 1 - 2(\varepsilon_1^2 + \varepsilon_2^2) \]  
(234)

The Euler angle to Euler parameter transformation for the 3-1-2 body rotation is given by,

\[ \varepsilon_1 = \cos^\theta_1 \sin^\theta_2 \cos^\theta_3 - \sin^\theta_1 \cos^\theta_2 \sin^\theta_3 \]  
(235)

\[ \varepsilon_2 = \cos^\theta_1 \cos^\theta_2 \sin^\theta_3 + \sin^\theta_1 \sin^\theta_2 \cos^\theta_3 \]  
(236)

\[ \varepsilon_3 = \sin^\theta_1 \cos^\theta_2 \cos^\theta_3 + \cos^\theta_1 \sin^\theta_2 \sin^\theta_3 \]  
(237)
\[ \varepsilon_4 = \cos^2 \theta_1 \cos^2 \theta_2 \cos^2 \theta_3 - \sin^2 \varepsilon_1 \sin^2 \varepsilon_2 \sin^2 \varepsilon_3. \]  
\hspace{1cm} (238)

The direction cosine to Euler angle transformation is given by,
\[ \theta_1 = \tan^{-1} \left( -\frac{c_{12}}{c_{22}} \right) \]  
\hspace{1cm} (239)
\[ \theta_2 = \sin^{-1} (c_{32}) \]  
\hspace{1cm} (240)
\[ \theta_3 = \tan^{-1} \left( -\frac{c_{31}}{c_{33}} \right). \]  
\hspace{1cm} (241)

A very useful property of Euler parameters is the relationship between Euler parameter rates and the body fixed angular rates. This relation is given by,
\[ \dot{\varepsilon} = \mathbf{E} \begin{bmatrix} \mathbf{A} \mathbf{w} \\ \mathbf{B} \mathbf{e} \end{bmatrix} \]  
\hspace{1cm} (242)

where,
\[ \mathbf{E} = \begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ -\varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4 \end{bmatrix} \]  
\hspace{1cm} (243)

The Euler parameter rates have no singular points as do Euler angle rates. This eliminates any need to switch to another set of Euler angles when a singularity is encountered.

The remaining equations to describe the system are given by,
\[ \dot{\varepsilon}_1 = [\varepsilon_4 \omega_1 - \varepsilon_3 \omega_2 + \varepsilon_2 \omega_3] \]  
\hspace{1cm} (244)
\[ \dot{\varepsilon}_2 = [\varepsilon_3 \omega_1 + \varepsilon_4 \omega_2 - \varepsilon_1 \omega_3] \]  
\hspace{1cm} (245)
\[ \dot{\varepsilon}_3 = [-\varepsilon_2 \omega_1 + \varepsilon_1 \omega_2 + \varepsilon_4 \omega_3] \]  
\hspace{1cm} (246)
\[ \dot{\varepsilon}_4 = [-\varepsilon_1 \omega_1 - \varepsilon_2 \omega_2 - \varepsilon_3 \omega_3]. \]  
\hspace{1cm} (247)
Equations 210-215 and 244-247 describe the motion of the system and are a system of nonlinear coupled ordinary differential equations. These equations must be solved using numerical techniques and pose a formidable numerical problem.
CHAPTER III. 
NUMERICAL SIMULATION OF THE EQUATIONS OF MOTION

The equations of motion must be solved by numerical methods because an analytical solution is not available. Numerical methods calculate the solution of the equations of motion at discrete points in time. The numerical solution is an approximation to the exact solution and will converge to the exact solution as the number of discrete points and arithmetic precision approaches infinity. A finite number of points will cause error in the numerical solution.

Numerical errors can be lumped into two categories; truncation error and roundoff error. Truncation error is the error induced by the algorithm used and may be defined as $e_n = y_n - y(t_n)$, where $y_n$ denotes the number produced by the algorithm using infinite precision arithmetic and $y(t_n)$ is the true solution. Roundoff error is the error induced by finite precision arithmetic and may be defined as $r_n = \tilde{y}_n - y_n$ where $\tilde{y}_n$ is the number calculated. The sum of both of these errors results in the total error between the numerical and exact solutions.

The equations of motion were formulated for numerical solution by defining new variables so that the differential equations are all first order. State variables chosen are given by,

$$y_1 = x_1$$
\[ y_2 \triangleq x_2 \]
\[ y_3 \triangleq x_3 \]
\[ y_4 \triangleq \omega_1 \]
\[ y_5 \triangleq \omega_2 \]
\[ y_6 \triangleq \omega_3 \]
\[ y_7 \triangleq \beta_1 \]
\[ y_8 \triangleq \alpha_1 \]
\[ \vdots \]
\[ y_5 + 2N \triangleq \beta_N \]
\[ y_6 + 2N \triangleq \alpha_N \]
\[ y_7 + 2N \triangleq x_1 \]
\[ y_8 + 2N \triangleq x_2 \]
\[ y_9 + 2N \triangleq x_3 \]
\[ y_{10} + 2N \triangleq \epsilon_1 \]
\[ y_{11} + 2N \triangleq \epsilon_2 \]
\[ y_{12} + 2N \triangleq \epsilon_3 \]
\[ y_{13} + 2N \triangleq \epsilon_4 \]
\[ y_{14} + 2N \triangleq \beta_1 \]
\[ y_{15} + 2N \triangleq \alpha_1 \]
\[ \vdots \]
\[ y_2(6 + 2N) \stackrel{\Delta}{=} \beta_N \]

\[ y_2(6 + 2N) + 1 \stackrel{\Delta}{=} \alpha_N \]

where \( N \) is the number of pendulums.

There are many algorithms available which solve ordinary differential equations. The methods employed in this study are referred to as GEAR's methods. They are also known as backward difference methods with the software developed by Hindmarsh. The programs are easy to use and well-documented. Local truncation error is an input parameter to the program which reduces the time step to control the error. Global or accumulated error is not estimated or controlled. Roundoff error is generally much smaller than truncation error and can be made smaller by using double precision arithmetic on a large word length computer.

The equations of motion were coded for numerical solution using GEAR's method (see Appendix B). Rocket motor thrust and inertia data were curve fitted using cubic splines. A comparison of the curve fit with thrust and inertia data is shown in Figures 4 and 5. The curve fit enables the program to calculate thrust, inertia and inertia derivatives as a function of time. The vehicle mass and moments of inertia are both time varying as rocket motor propellant is expelled from the vehicle during the 85.3 second burn.

The pendulum configuration used is shown in Figures 6 and 7 with system parameters given by,
Figure 4. Rocket motor thrust data and curve fit
Figure 5. Main body moments of inertia and curve fit
Figure 6. Pendulum configuration (top view)

Figure 7. Pendulum configuration (side view)
The pendulum parameters \( m \) and \( L \) were chosen to represent the first mode sloshing frequency. The values \( r_1, r_2, \) and \( r_3 \) denote the location of the center of the fuel tanks relative to the center of mass of the main body and were estimated. The values \( K_\alpha, K_\beta, D_\alpha \) and \( D_\beta \) were chosen to enhance the model of the first mode sloshing frequency. Spring constant values, \( K_\alpha \) and \( K_\beta \), were set equal to zero because the fluid is in a gravity free environment and the gravitational restoring force is absent. Damping coefficient values, \( D_\alpha \) and \( D_\beta \), were chosen by analyzing a single degree of freedom pendulum and using logarithmic decrement data from experiments. The initial and final vehicle masses were taken from rocket motor test data.

Initial conditions on the system were chosen such that the spacecraft would be orbiting at an altitude of 200 miles with a
period of 1 1/2 hours. The spacecraft orientation was aligned with the inertial frame and given a 1 rev/sec roll velocity. Pendulum positions were chosen as shown in Figure 7 because the fluid would align in a sense opposite to the longitudinal acceleration of the vehicle after spin up and ejection from the shuttle. Pendulum relative angular velocities were set equal to zero.

The results of the first simulation, without damping, are shown in Figures 8-10. Figure 8 shows the body fixed angular rates vs. time which are seen to increase about the roll, pitch, and yaw axes. The increase in roll angular velocity can be attributed to the decrease in the moment of inertia about the roll axis. Momentum is conserved about the roll axis, so the roll angular velocity increases as the moment of inertia about the roll axis decreases. Perturbations on the roll angular velocity are caused by the pendulums which also accounts for the increase in the velocities in pitch and yaw. The body fixed angular rates show a small amount of increase after the PAM burnout at 85.3 seconds and then the system vibrates in a steady state. Figure 9 shows the global position vs. time and the nonzero azimuth angle indicates the satellite would not achieve the proper orbit. Figure 10 shows the energy of the main body and pendulums relative to the main body fixed reference frame during the simulation. The energy increases during the PAM burn because mass is being expelled from the
Figure 8. Body fixed angular rates vs. time without pendulum damping and with spin-stabilization about the minor axis.
Figure 9. Global position vs. time without pendulum damping and with spin-stabilization about the minor axis
Figure 10. Energy of system vs. time without pendulum damping and with spin-stabilization about the minor axis
system. After PAM burnout, the system is conservative, thus the total energy of the system remains constant. The energy after PAM burnout can be seen to have a mean value with a small fluctuation about the mean. This small fluctuation can be attributed to roundoff error which is statistical in nature.

The results of the second simulation are shown in Figures 11-15. This simulation was different from the first in that damping of the pendulums was included. Figure 11 shows the body fixed angular rates vs. time which behave in a similar fashion when compared to Figure 8 except that after PAM burnout the pitch and yaw rates continue to increase and the roll rate decreases, i.e., note the scales. This behavior illustrates energy transfer from the roll axis into the pitch and yaw axes. The pendulum damping action causes energy to be dissipated which in turn decreases the roll rate. A decrease in the roll rate then causes the system to be less gyroscopically stiff and the pendulum action in turn causes a greater reaction torque to be imparted to the main body. The increased reaction torque causes the pitch and yaw rates to continue to increase which also drives the pendulums more causing the system to be unstable. Figure 12 shows the main body location relative to the earth and is seen to be similar to the undamped case. Figure 13 shows the energy dissipation which is seen to be large after PAM burnout. Figures 14 and 15 show the pendulum relative position angles. The pendulums are excited by the
Figure 11. Body fixed angular rates vs. time with pendulum damping and spin-stabilization about the minor axis
Figure 12. Global position vs. time with pendulum damping and spin-stabilization about the minor axis
Figure 13. Energy of the system vs. time with pendulum damping and spin-stabilization about the minor axis.
Figure 14. Pendulum relative position angles ($\alpha_1$, $\beta_1$, $\alpha_2$, $\beta_2$) vs. time with pendulum damping and spin-stabilization about the minor axis
Figure 15. Pendulum relative position angles (α₁, β₁, α₄, β₄) vs. time with pendulum damping and spin-stabilization about the minor axis
initial thrust of the rocket motor and then decrease until PAM burnout when large amplitude oscillations occur. Kaplan describes unstable behavior of satellite systems with energy dissipation by a heuristic approach referred to as the energy sink theory. The energy sink theory states that a semirigid system, i.e., system with dissipation, is stable only when spin-stabilized about the major axis. Various satellite systems in the past have shown rapid spin decay and reorientation caused by dissipation of various kinds. Agrawal has also analyzed the stability of spinning spacecraft with liquid filled tanks using Liapunov's method and concluded that the motion will be stable if the ratio of the spin to transverse moments of inertia is greater than \((1 + c)\), where \(c\) is a positive definite function of spacecraft parameters.

A verification of the energy dissipation theory was made by interchanging the major and minor axis inertias. This allowed the vehicle to be spin-stabilized about the major axis and the flight simulation conducted. Figures 16-20 show the results of the flight simulation with the vehicle spin-stabilized about the major axis. Figure 16 shows the body fixed angular rates which were perturbed after PAM ignition and burnout but the system always returns to the initial state. Figure 17 shows the global position and it can be seen that the azimuth angle remains zero. Comparison of Figures 9, 12 and 17 shows that
Figure 16. Body fixed angular rates vs. time with pendulum damping and spin-stabilization about the major axis.
Figure 17. Global position vs. time with pendulum damping and spin-stabilization about the major axis
Figure 18. Energy of the system vs. time with pendulum damping and spin-stabilization about the major axis.
Figure 19. Pendulum relative position angles ($\alpha_1$, $\beta_1$, $\alpha_2$, $\beta_2$) vs. time with pendulum damping and spin-stabilization about the major axis.
Figure 20. Pendulum relative position angles ($\alpha_3, \beta_3, \alpha_4, \beta_4$) vs. time with pendulum damping and spin-stabilization about the major axis.
the unstable system alters the flight trajectory of the vehicle because of the nonzero azimuth angle growth while the stable system remains on the desired course. Figure 18 shows the energy of the stable system during the flight and it can be seen to remain constant after PAM burnout. Figures 19 and 20 show the pendulum relative position angles and they can be seen to have oscillations after PAM ignition and burnout but the response is stable.

The flight simulation of the spacecraft with sloshing fluid stores has been shown to have a stable and unstable mode of operation. Stable operation may be produced by using the major axis as the spin axis.

Figure 21 shows flight test data telemetered from onboard rate gyros. The pitch and yaw body fixed angular rates have approximately equal amplitude and are 90° out of phase, indicating a coning behavior. Perturbations are seen on the roll axis rate gyro with a slight decrease in magnitude over time. If the system were behaving as a single rigid body, then conservation of energy would require that there be no perturbation at the frequency observed or decay in the roll rate. It can be concluded from the telemetered flight data that the system is not behaving as a rigid body.
Figure 21. Flight data—RCA-C¹
CHAPTER IV.
CONTROL SYSTEM ANALYSIS

After obtaining a suitable model for the system, the next step is to control the system. Controlling a satellite system means maintaining prescribed angular rate and attitude of the main body. The system model is highly nonlinear, which complicates the control analysis. In many cases, a nonlinear system may be linearized and a feedback control law developed using linear control theory to control the nonlinear system. A feedback control system uses an error signal to drive the control, and as long as the system is "somewhere" near its operating point (nominal flight condition), the nonlinear system will be controlled. Linear control theory was used in this study to obtain a control law which, when applied to the nonlinear system, brings the system to the desired angular rate and attitude.

The linearized equations are autonomous only if the body fixed angular rates are to be controlled, and nonautonomous if the attitude is to be controlled while the satellite is spin stabilized. Assumptions made to simplify the analysis were,

1. Gravitational attraction relative to earth is neglected, i.e., control interest is with respect to the satellite attitude of the main body.
2. The rocket motor has burned out and the inertias of the
main body remain constant.

3. The mass of the pendulums, therefore the equivalent
length, does not change as fuel is expelled for control.

4. Two pendulums were used instead of four and only two
control thrusters to make the implementation of the
controller less expensive to construct.

Linear optimal control theory is a well-developed tool for
approaching modern control problems. The problem to be solved
in this study is referred to as the Linear Quadratic Tracking
Problem (LQTP). This consists of minimizing the functional,

\[
J = \frac{1}{2} \left[ \mathbf{y}(t_f) - \mathbf{r}(t_f) \right]^T H(t_f) \left[ \mathbf{y}(t_f) - \mathbf{r}(t_f) \right] \\
+ \frac{1}{2} \int_{t_0}^{t_f} \left[ \mathbf{y}(t) - \mathbf{r}(t) \right]^T Q(t) \left[ \mathbf{y}(t) - \mathbf{r}(t) \right] \\
+ \mathbf{u}^T(t) R(t) \mathbf{u}(t) \right] dt 
\]  

subject to,

\[
\dot{\mathbf{x}}(t) = A(t) \mathbf{x}(t) + B(t) \mathbf{u}(t),
\]

which is the linearized system of the equations of motion. The
variables in the functional are defined as,

\[
\mathbf{r}(t) \triangleq \text{Reference or desired state} \quad (250) \\
\mathbf{y}(t) \triangleq \text{State variable vector} \quad (251) \\
H(t) , Q(t) \triangleq \text{Positive semidefinite weighting arrays} \quad (252) \\
R(t) \triangleq \text{Positive definite weighting array} \quad (253) \\
\mathbf{u}(t) \triangleq \text{Control variable vector} \quad (254) 
\]
t ≡ Time \hspace{2cm} \text{(255)}

t_o ≡ Initial time \hspace{2cm} \text{(256)}

t_f ≡ Final time. \hspace{2cm} \text{(257)}

Kirk\textsuperscript{28} derived the necessary conditions for optimality which results in the following equations:

\[ \dot{P}(t) = -P(t)A(t) - A^T(t)P(t) - Q(t) \]
\[ \quad + P(t)B(t)R^{-1}(t)B^T(t)P(t) \] \hspace{2cm} \text{(258)}

\[ \dot{S}(t) = -[A^T(t) - P(t)B(t)R^{-1}(t)B^T(t)]S(t) \]
\[ \quad + Q(t)\xi(t) \] \hspace{2cm} \text{(259)}

with boundary conditions,

\[ P(t_f) = H(t_f) \]
\[ \text{(260)} \]

\[ S(t_f) = -H(t_f)\xi(t_f). \]
\[ \text{(261)} \]

Equation 258 is known as the matrix Riccati equation. \( P(t) \) is symmetric and consists of \( n(n + 1)/2 \) values where \( n \) is the dimension of the system. \( S(t) \) is an \( n \) vector. If the solutions to equations 258-261 are found, the optimal control law is given by,

\[ u^*(t) = -R^{-1}(t)B^T(t)P(t)x(t) - R^{-1}(t)B^T(t)S(t). \] \hspace{2cm} \text{(262)}

\[ \xi \triangleq F(t)x(t) + \xi(t) \]

The optimal control law may be found, by integrating the \( [n(n + 1)/2] + n \) system of equations backward in time and storing the values of \( F(t) \) and \( \xi(t) \). The control law is linear in the state \( x \) and the closed-loop system can be shown to be asymptotically stable.\textsuperscript{28}
The first step in controlling the system is to linearize the equations of motion. This was accomplished by choosing a nominal operating point and defining perturbation variables. The nominal operating point and perturbation variables are given by

\[ \begin{align*}
\omega_1 &= \omega_{10} + \omega_1 = \omega_1 \\
\omega_2 &= \omega_{20} + \omega_2 = \Omega + \omega_2 \\
\omega_3 &= \omega_{30} + \omega_3 = \omega_3 \\
\beta_1 &= \beta_{10} + \beta_1 = \beta_1 \\
\beta_2 &= \beta_{20} + \beta_2 \\
\vdots & \\
\beta_N &= \beta_{N0} + \beta_N = \beta_N \\
\dot{\alpha}_1 &= \dot{\alpha}_{10} + \dot{\alpha}_1 = \dot{\alpha}_1 \\
\dot{\alpha}_2 &= \dot{\alpha}_{20} + \dot{\alpha}_2 \\
\vdots & \\
\dot{\alpha}_N &= \dot{\alpha}_{N0} + \dot{\alpha}_N = \dot{\alpha}_N \\
\ddot{\beta}_1 &= \ddot{\beta}_{10} + \ddot{\beta}_1 \\
\ddot{\beta}_2 &= \ddot{\beta}_{20} + \ddot{\beta}_2 \\
\vdots & \\
\ddot{\beta}_N &= \ddot{\beta}_{N0} + \ddot{\beta}_N \\
\ddot{a}_1 &= \ddot{a}_{10} + \ddot{a}_1 = \ddot{a}_1 \\
\ddot{a}_2 &= \ddot{a}_{20} + \ddot{a}_2 \\
\vdots & \\
\ddot{a}_N &= \ddot{a}_{N0} + \ddot{a}_N = \ddot{a}_N.
\end{align*} \]

where,

\( \omega \) denotes nominal roll velocity

' denotes unperturbed variable

\( \circ \) denotes nominal variable
The linearized pendulum length components are given by,

\[ L_1 = -Ls\alpha_0cs_\beta_0 + (Ls\alpha_0sn_\beta_0)\beta \]
\[ + (-Lcs_\alpha_0cs_\beta_0)a \]  (264)

\[ L_2 = Lcs_\alpha_0cs_\beta_0 + (-Lcs_\alpha_0sn_\beta_0)\beta \]
\[ + (-Lsn_\alpha_0cs_\beta_0)a \]  (265)

\[ L_3 = Lsn_\beta_0 + (Lcs_\beta_0)\beta. \]  (266)

The linearized acceleration terms, \( B_1, B_2, B_3 \), are given by,

\[ B_1 = [(r_2 + L_2)\Omega] \omega_1 + [-2(r_1 + L_1)\Omega] \omega_2 \]
\[ + [2Lcs_\beta_0]\beta + [(Lsn_\alpha_0sn_\beta_0)\Omega^2]a \]
\[ + [(-Lcs_\alpha_0cs_\beta_0)\Omega^2]a \]
\[ + [-Lcs_\alpha_0cs_\beta_0]a \]  (267)

\[ B_2 = [(r_1 + L_1)\Omega] \omega_1 + [(r_3 + L_3)\Omega] \omega_3 \]  (268)

\[ B_3 = [-2(r_3 + L_3)\Omega] \omega_2 + [(r_2 + L_2)\Omega] \omega_3 \]
\[ + [-2L_3t_{12}a]\beta + [2L_2a]a \]
\[ + [-Lcs_\beta_0\Omega^2]a + [-(r_3 + Lsn_\beta_0)\Omega^2]. \]  (269)

The linearized partial velocity coefficients are given by,

\[ u_{10} = 0 \]  (270)

\[ u_{11} = -(r_3 + Lcs_\beta_0) - (Lcs_\beta_0)\beta \]  (271)

\[ u_{12} = (r_2 + Lcs_\alpha_0cs_\beta_0) + (-Lcs_\alpha_0sn_\beta_0)\beta \]
\[ + (-Lsn_\alpha_0cs_\beta_0)a \]  (272)

\[ u_{13} = (r_3 + Lcs_\beta_0) + (Lcs_\beta_0)\beta \]  (273)

\[ u_{14} = 0 \]  (274)

\[ u_{15} = -(r_1 - Lsn_\alpha_0cs_\beta_0) + (-Lsn_\alpha_0sn_\beta_0)\beta \]
\[ + (Lcs_\alpha_0cs_\beta_0)a \]  (275)
The linearized equations of motion were assembled by substituting equations 263-284 into equations 210-215 and ignoring higher order terms. The linearized system may be written as:

\[
\begin{bmatrix}
\dot{\omega}_1 \\
\dot{\omega}_2 \\
\dot{\omega}_3 \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
h_{11} & h_{12} & \cdots & h_{1,3+4N} \\
h_{21} & h_{22} & \cdots & h_{2,3+4N} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\vdots
\end{bmatrix}
\]
where,

\( Z \triangleq \) Coefficient matrix of highest derivative terms \ (286) \\
\( h \triangleq \) Assembled coefficient matrix of linearized terms \ (287) \\
\( f \triangleq \) Assembled coefficient matrix of control force terms\ (288) \\
\( u \triangleq \) Control variable vector \ (289) \\
\( g \triangleq \) Assembled steady state linearized forcing term vector. \ (290) \\

The linearized equations of motion were put in the desired form by solving for the highest derivatives and are given by,

\[
\dot{y} = Ay + Bu + C
\]  

where,
The \( f \) matrix was formed by deriving the generalized active forces resulting from the reaction jet thrusters. The reaction jet thruster configuration used is shown in Figures 22 and 23. Generalized active forces for this force system are given by,

\[
F_{\omega_1} = -RF_1^2 \\
F_{\omega_2} = 0 \\
F_{\omega_3} = -RF_2^2.
\]

The moment arm, \( R \), was given a value of 6 feet. Equations 295-296 are linear in the control variables, \( F_1^2-F_2^2 \), and were added to the equations of motion. The components of the \( f \) matrix were identified by inspection.

The remaining equations of motion, equations 246-247, were linearized by formulating the Jacobian matrix. The Jacobian matrix is defined as,

\[
JC_{ij} \triangleq \frac{\delta^2}{\delta y_i \delta y_j} \quad (i = 1, \ldots, 4), (j = 1, \ldots, 7+4N)
\]

where the components are given by,

\[
JC_{11} = \frac{\epsilon_4}{2} \quad JC_{12} = \frac{-\epsilon_3}{2} \quad JC_{13} = \frac{\epsilon_2}{2} \\
JC_{21} = \frac{\epsilon_3}{2} \quad JC_{22} = \frac{\epsilon_4}{2} \quad JC_{23} = \frac{-\epsilon_1}{2}
\]
Figure 22. Reaction jet configuration (top view)

Figure 23. Reaction jet configuration (side view)
\[ \begin{align*}
JC_{31} &= \frac{-\varepsilon_2}{2} \\
JC_{32} &= \frac{\varepsilon_1}{2} \\
JC_{33} &= \frac{\varepsilon_4}{2} \\
JC_{41} &= \frac{-\varepsilon_1}{2} \\
JC_{42} &= \frac{-\varepsilon_2}{2} \\
JC_{43} &= \frac{-\varepsilon_3}{2} \\
JC_{1,4+4N} &= 0 \\
JC_{1,5+4N} &= \frac{\omega_3}{2} \\
JC_{1,6+4N} &= \frac{-\omega_2}{2} \\
JC_{1,7+4N} &= \frac{\omega_1}{2} \\
JC_{2,4+4N} &= \frac{-\omega_3}{2} \\
JC_{2,5+4N} &= 0 \\
JC_{2,6+4N} &= \frac{-\omega_1}{2} \\
JC_{2,7+4N} &= \frac{\omega_2}{2} \\
JC_{3,4+4N} &= \frac{\omega_1}{2} \\
JC_{3,5+4N} &= \frac{-\omega_1}{2} \\
JC_{3,6+4N} &= 0 \\
JC_{3,7+4N} &= \frac{\omega_3}{2} \\
JC_{4,4+4N} &= \frac{-\omega_1}{2} \\
JC_{4,5+4N} &= \frac{-\omega_2}{2} \\
JC_{4,6+4N} &= \frac{-\omega_3}{2} \\
JC_{4,7+4N} &= 0,
\end{align*} \]

with all other components equal to zero. The augmented set of linearized equations are given by,

\[
\dot{\mathbf{y}} = \begin{bmatrix} A \\ JC \end{bmatrix} \mathbf{y} + \begin{bmatrix} B \\ 0 \end{bmatrix} \mathbf{u} + \begin{bmatrix} C \\ 0 \end{bmatrix}.
\]

The augmented state vector, \( \mathbf{y} \), is defined as,

\[
\begin{align*}
\mathbf{y}_1 &\triangleq \omega_1 \\
\mathbf{y}_2 &\triangleq \omega_2 \\
\mathbf{y}_3 &\triangleq \omega_3 \\
\mathbf{y}_4 &\triangleq \beta_1 \\
\mathbf{y}_5 &\triangleq \alpha_1 \\
\vdots &\vdots \\
\mathbf{y}_{2+2N} &\triangleq \beta_N \\
\mathbf{y}_{3+2N} &\triangleq \alpha_N
\end{align*}
\]
\[ Y_{4+2N} \triangleq \beta_1 \]
\[ Y_{5+2N} \triangleq \alpha_1 \]
\[ \ldots \]
\[ Y_{2+4N} \triangleq \beta_N \]
\[ Y_{3+4N} \triangleq \alpha_N \]
\[ Y_{4+4N} \triangleq \epsilon_1 \]
\[ Y_{5+4N} \triangleq \epsilon_2 \]
\[ Y_{6+4N} \triangleq \epsilon_3 \]
\[ Y_{7+4N} \triangleq \epsilon_4 \]
CHAPTER V.

NUMERICAL SIMULATION OF THE CONTROL SYSTEM

The simulation of the controlled system requires the computation of the control law and the nonlinear system response. Control law computation was accomplished by solving equations 258-262 which required integrating the Riccati equation backward in time. The time varying feedback gain matrix, \( F(t) \), and command vector, \( V(t) \), were computed after calculating the solution to the Riccati equation. Program LINRIZ (see Appendix C) was written to compute the linearized system matrices while CONTRL (see Appendix D) calculated the feedback gain matrix and command vector. The main program (see Appendix E) simulates the nonlinear system response with the linear feedback control law applied to it.

The first simulation controls the body fixed angular rates and attitude of the system using the same moments of inertia as the flight simulation. A coning state was initiated by setting the roll, pitch, and yaw rates equal to 360, 0, and 35 degrees per second, respectively. The pitch angle \( (\theta_1) \) was set to zero while the yaw angle \( (\theta_2) \) was given a 15 degree offset. Figures 24 and 25 indicate that the coning motion is suppressed and the attitude approaches the desired final orientation of simple spin with the pitch and yaw angles equal to zero. Figures 26 and 27 show the thrust and command forces required to suppress the coning motion. The command forces are zero.
Figure 24. Body fixed angular rates vs. time with all state variables observable
Figure 25. Main body angles vs. time with all state variables observable
Figure 26. Thrust forces vs. time with all state variables observable.
Figure 27. Command forces vs. time
Figures 28-35 show the feedback gain components of the \( F \) matrix of equation 262 associated with their respective states. Comparison of Figures 28 and 32 reflect the geometry of the thruster configuration in that the pitch and yaw gains are similar for thrusters 1 and 2. The roll gains are seen to vary approximately linearly and approach zero as the end of the control interval is approached. Figures 29 and 30 and 33-34 show the gains associated with the pendulum angular rate and position, respectively, and are seen to be identical when the states associated with the alpha and beta degrees of freedom are compared. The similarity of the gains is a reflection of the symmetry in the system geometry. Figures 31 and 35 show the feedback gains associated with the Euler parameter states. The gains are damped sinusiods with a frequency of one half cycle per second. An interesting result of the feedback gain computation is that all the gains except those associated with the roll and Euler parameter states approach constant values. The time varying nature of the gains associated with the roll and Euler parameter states is a result of the main body rotation about the roll axis.

The second control simulation consists of an earth pointing maneuver in which the rotating body is commanded to reach a 10 degree pitch angle and a 0 degree yaw angle. Pitch and yaw angles are initially zero degrees. Figures 36 and 37 show the system response and indicate the desired final orientation is
Figure 28. Body fixed angular rate gains vs. time for thruster 1
Figure 29. Pendulum angular rate gains vs. time for thruster 1
Figure 30. Pendulum angular position gains vs. time for thruster 1
Figure 31. Euler parameter gains vs. time for thruster 1
Figure 32. Body fixed angular rate gains vs. time for thruster 2
Figure 33. Pendulum angular rate gains vs. time for thruster 2
Figure 34. Pendulum angular position gains vs. time for thruster 2
Figure 35. Euler parameter gains vs. time for thruster 2
approached. Figures 38 and 39 show the thrust and command forces required to point the system. The command force is seen to be sinusoidal with a one cycle per second frequency. All feedback gains are the same as the first simulation.

The first two control simulations assumed that the entire state vector could be measured for calculation of the control law. Body fixed angular rates and attitude measurement may be accomplished using rate gyros and horizon sensors but the pendulum positions and angular rates would be difficult or impossible to measure. Techniques for estimating the entire state vector have been developed with the Kalman-Bucy filter being a very popular algorithm, often implemented for estimation tasks. A controller which would neither have to measure or estimate the pendulum states would be much easier and inexpensive to implement. An investigation of the sensitivity of the system response to a reduced order controller was then made.

A third simulation was performed with coning initial conditions identical to the first coning maneuver, but, in this case only the main body fixed angular rates and Euler parameters were used as feedback to the control law. The system response was virtually identical to the first simulation indicating that the control system is not sensitive to the pendulum angular rates and positions.
Figure 36. Body fixed angular rates vs. time with all state variables observable
Figure 37. Main body angles vs. time with all state variables observable
Figure 38. Thrust forces vs. time with all state variables observable
Figure 39. Command forces vs. time
A fourth simulation was made repeating the pointing maneuver but again only measurable states were included in the control law. The system response showed virtually identical behavior to the second simulation again indicating that the control system is not sensitive to the pendulum states.

An assumption which was made in the development of the control law was that the pendulum lengths, i.e., fluid levels, remain constant. Fluid is expended for control so a check must be made as to the sensitivity of pendulum length on the system response. A calculation was made of the fuel expended for control by computing the total impulse of the thrusters and assuming a back pressure on the fluid of 5000 psia. The amount of fuel expended for the first simulation was estimated at 20% for each of the initially 90% full tanks. Simulations for the coning suppression and earth pointing maneuver were made using the control law developed for the reduced level tanks and applied to the near full tanks. Figures 40-42 and 43-45 show the responses of the system to the coning suppression and earth pointing maneuvers, respectively. Comparison with Figures 24-26 and 36-38 show small differences but the final desired system state is still approached.

The desired system response was achieved by adjusting the weighting arrays $H$, $Q$, and $R$ in equations 252-254 and the final time of the integration interval. There are many combinations of the weighting parameters and final times which can shape the
Figure 40. Body fixed angular rates vs. Time with partial state observation.

<table>
<thead>
<tr>
<th>TIME (SECONDS)</th>
<th>ROLL (DEG/SEC)</th>
<th>PITCH (DEG/SEC)</th>
<th>YAW (DEG/SEC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>355.20</td>
<td>-40.00</td>
<td>40.00</td>
</tr>
<tr>
<td>2.00</td>
<td>355.30</td>
<td>-40.00</td>
<td>40.00</td>
</tr>
<tr>
<td>4.00</td>
<td>355.40</td>
<td>-40.00</td>
<td>40.00</td>
</tr>
<tr>
<td>6.00</td>
<td>355.50</td>
<td>-40.00</td>
<td>40.00</td>
</tr>
</tbody>
</table>
Figure 41. Main body angles vs. time with partial state observation and sensitivity to fuel tank level.
Figure 42. Thrust forces vs. time with partial state observation and sensitivity to fuel tank level
Figure 43. Body fixed angular rates vs. time with partial state observation and sensitivity to fuel tank level.
Figure 44. Main body angles vs. time with partial state observation and sensitivity to fuel tank level.
Figure 45. Thrust forces vs. time with partial state observation and sensitivity to fuel tank level
desired response. It is a choice for the designer to decide if the response is satisfactory. The weighting arrays used in this study were chosen as,

\[ Q = H \]

\[ Q_{11} = Q_{33} = Q_{j+4N, j+4N} = 50000 \quad (j = 4, \ldots, 7) \] (301)

\[ R_{ij} = 0.05 \quad (j = 1, 2) \] (302)

and the off diagonal terms were set to zero. Control simulations were performed using DVERK for solution of the Riccati equation and the nonstiff version of DGEAR for computation of the nonlinear system response.²⁹

The results generated from the simulation of the controlled system may be applied to the implementation of an on-board control system using either digital or analog electronic circuitry. The controller would consist of sensors, a computational device and reaction jet thrusting hardware.

A set of rate gyros would be used to measure the body fixed angular rates and horizon sensors used to measure the main body orientation. The computational device could either be a microcomputer with D/A (digital to analog) and A/D (analog to digital) conversion capability or alternatively an analog controller could be implemented consisting of simple analog computing elements such as operational amplifiers, adders, and integrators. The reaction jet thrusting hardware would be based around two bi-directional servovalves with the capability of imparting the thrust forces required by the simulation.
The rate gyros and horizon sensors would measure the states necessary for the reduced order controller to compute the required thrust forces. Pendulum angular rate and position feedback was shown not to effect the control of the spacecraft response appreciably and may be ignored. The signals measured would then be processed by the digital or analog controller to actuate the servovalves. The servovalves would in turn modulate the fluid flow rate from the pressurized tanks. The controller is relatively simple to implement and in feedback form.
CHAPTER VI.  
CONCLUSIONS AND RECOMMENDATIONS

This study has shown that sloshing fluid stores carried in the main rigid body of a spacecraft can be a source of dynamic instability. The fluid has been modeled as an equivalent spherical pendulum and only the first mode of fluid oscillation has been analyzed.

The major conclusions and recommendations drawn from this study are listed as follows:

1. The dynamic response equations for this complex dynamic system were derived using two different methods and are presented here. The equations prove to be highly nonlinear and are heavily coupled dynamically.

2. Digital computer programs were developed to solve the dynamic response equations and are presented here. The resulting numerical flight simulations show similar behavior to the telemetered flight data. It is concluded that the thrust from the PAM can cause fluid sloshing and that sloshing fluid stores can be a source of dynamic instability.

3. The spacecraft analyzed was spin-stabilized about its minor axis. If the spacecraft were spun about its major axis, numerical simulation shows that the dynamic effect of the sloshing fluid stores is dissipated and the coning problem
is eliminated. It is recommended that spacecraft carrying fluids on board be spun about the major axis.

4. Numerical control system simulations show that a linear optimal feedback control system can effectively control this highly nonlinear system. The control system presented here uses easily measured state variables (only roll, pitch and yaw rates and attitude angles). It is not necessary to measure the dynamic state of the fluids. The control system developed here can eliminate the coning even though the spacecraft is spun about a minor axis. Pointing maneuvers were also successfully accomplished by this control system.

5. This study outlines a design of a control system which uses few control variables and eliminates feedback loops of the less effective state variables. The design presented here could be implemented with readily available hardware.

6. A study was completed of the effect of depleting liquid fuel on the control system response. The control system was shown to be effective over a wide range of liquid volume.
APPENDIX A.

OUTLINE OF THE LAGRANGIAN FORMULATION OF THE EQUATIONS OF MOTION
The Lagrangian formulation was derived by using,
\[
\frac{d}{dt} \hat{a}K - \hat{a}K = Q_r \quad r = 1, \ldots, 6 + 2N \tag{1}
\]
which is the classical Lagrange equation.

The kinetic energy of the system may be written as,
\[
K = \frac{1}{2} M \mathbf{v}_G \cdot \mathbf{v}_G + \frac{1}{2} A \mathbf{w} \cdot \mathbf{w} \tag{2}
\]

\[+ \sum_{i=1}^{N} m_i \mathbf{v}_{mi} \cdot \mathbf{v}_{mi}
\]

The velocities \(\mathbf{v}_G\) and \(\mathbf{v}_m\) are given by equations 40 and 46. The inertia dyadic is given by,
\[
\mathbf{I} = \sum_{j=1}^{3} \sum_{k=1}^{3} b^*_{jk} l_{jk} b_{jk} \tag{3}
\]

The rotational kinetic energy of the main body may be written as,
\[
\frac{1}{2} A \mathbf{w} \cdot \mathbf{w} = \frac{1}{2} \left[ \omega_1 I_{11} + 2 \omega_1 \omega_2 I_{12} + 2 \omega_1 \omega_3 I_{13} + \omega_2^2 I_{22} + 2 \omega_2 \omega_3 I_{23} + \omega_3^2 I_{33} \right] \tag{4}
\]

The total kinetic energy, after simplification, may be written as,
\[
K = \frac{1}{2} [(M + E m_i) \mathbf{x}_1 \cdot \mathbf{x}_1 + (M + E m_i) \mathbf{x}_2 \cdot \mathbf{x}_2 + (M + E m_i) \mathbf{x}_3 \cdot \mathbf{x}_3 \tag{5}
\]

\[+ \frac{1}{2} (E m_i c_1 c_{31}^2 + 2 I_{12} c_{31} c_{32} + 2 I_{13} c_{31} c_{33} + I_{22} c_{32}^2
\]

\[+ 2 I_{23} c_{32} c_{33} + I_{33} c_{33}^2
\]

\[+ E m_i (c_1 c_{31}^2 + c_2 c_{32}^2 + c_3 c_{33}^2 - 2 (r_1 + L_1)(r_2 + L_2)c_{31} c_{32}
\]

\[+ 2 (r_1 + L_1)(r_3 + L_3)c_{31} c_{33} - 2 (r_2 + L_2)(r_3 + L_3)c_{32} c_{33})
\]

\[+ \frac{1}{2} (I_{11} c_{2}^2 \theta_3 + 2 I_{13} c_{2} \theta_3 s \theta_3 + I_{33} s \theta_3 \theta_3)
\]

\[+ E m_i (c_1 c_{2}^2 \theta_3 + c_3 s \theta_3 \theta_3 - 2 (r_1 + L_1)(r_3 + L_3) c_{2} \theta_3 s \theta_3) \]
\[ + \delta_3^2(I_{22} + \Sigma_m(c_2)) \\
+ \delta_1^2(2\Sigma_m(-c_1 c_3 L_3 + c_1 c_3 L_2) \\
+ c_{11} c_{32}(r_3 + L_3) - c_{13} c_{32}(r_1 + L_1) - c_{33} c_{11}(r_2 + L_2) \\
+ c_{22} c_{12}(r_1 + L_1)) \\
+ \delta_2^2(2\Sigma_m(-c_{12} cs_3(r_3 + L_3) + c_{13} cs_3(r_2 + L_2) \\
- c_{11} sn_3(r_2 + L_2) + c_{12} sn_3(r_1 + L_1)) \\
+ \delta_3^2(2\Sigma_m(c_{11}(r_3 + L_3) - c_{13}(r_1 + L_1)) \\
+ \delta_1^2(2\Sigma_m(c_{11} t_{12} L_3 - c_{12} t_{11} L_3 + c_{13}(L_2 t_{11} - L_1 t_{12})) \\
+ \delta_2^2(2\Sigma_m(-c_{12} cs_3(r_3 + L_3) + c_{23} cs_3(r_2 + L_2) \\
+ c_{21} c_{32}(r_3 + L_3) - c_{23} c_{32}(r_1 + L_1) - c_{33} c_{21}(r_2 + L_2) \\
+ c_{33} c_{22}(r_1 + L_1)) \\
+ \delta_2^2(2\Sigma_m(-c_{22} cs_3(r_3 + L_3) + c_{23} cs_3(r_2 + L_2) \\
- c_{21} sn_3(r_2 + L_2) + c_{22} sn_3(r_1 + L_1)) \\
+ \delta_3^2(2\Sigma_m(c_{21}(r_3 + L_3) - c_{23}(r_1 + L_1)) \\
+ \delta_1^2(2\Sigma_m(c_{21} t_{12} L_3 - c_{22} t_{11} L_3 + c_{23}(t_{11} L_2 - t_{12} L_1)) \\
+ \delta_2^2(2\Sigma_m(-c_{21} L_2 + c_{22} L_1)) \\
+ \delta_3^2(2\Sigma_m(-c_{32} cs_3(r_3 + L_3) + c_{33} cs_3(r_2 + L_2) \\
- c_{31} sn_3(r_2 + L_2) + c_{32} sn_3(r_1 + L_1)) \\
+ \delta_1^2(2\Sigma_m(c_{31}(r_3 + L_3) - c_{33}(r_1 + L_1)) \\
+ \delta_2^2(2\Sigma_m(c_{31} t_{12} L_3 - c_{32} t_{11} L_3 + c_{33}(L_2 t_{11} - L_1 t_{12})) \\
+ \delta_3^2(2\Sigma_m(-c_{31} L_2 + c_{32} L_1)) \\
+ \delta_1^2(2(c_{31} cs_3 I_{11} + c_{32} cs_3 I_{12} + (c_{33} cs_3 + c_{31} sn_3) I_{13} \\
+ c_{32} sn_3 I_{23} + c_{33} sn_3 I_{33} + \Sigma_m(-r_1 + L_1)(r_2 + L_2) c_{32} cs_3 \\
-(r_1+L_1)(r_3+L_3)(c_{33} cs_3 + c_{31} sn_3) - (r_2 + L_2)(r_3 + L_3) c_{32} sn_3)
\[ + c_1 c_3 \cosh^3 + c_3 c_3 \sinh^3 \) \\
+ \delta_1 \delta_3 (2(c_3 I_{12} + c_3 I_{22} + c_3 I_{23} \\
+ \Sigma \beta_3 (-(r_1 + L_1)(r_2 + L_2)c_3_1 - (r_2 + L_2)(r_3 + L_3)c_3_3 + c_3 c_3 \Sigma \beta_2 (2\Sigma \beta_1 (c_3 I_{11} I_3 (r_3 + L_3) + c_3 I_{11} I_2 (r_2 + L_2) \\
-c_3 t_{12} I_1 (r_2 + L_2) + c_3 t_{12} I_2 (r_3 + L_3) - c_3 t_{11} I_2 (r_1 + L_1) \\
+c_3 t_{12} I_1 (r_1 + L_1) - c_3 t_{12} I_3 (r_2 + L_2) - c_3 t_{11} I_3 (r_1 + L_1) \\
+ \delta_1 \delta_3 (2\Sigma \beta_1 (c_3 I_{11} I_1 (r_3 + L_3) - c_3 I_{12} I_2 (r_1 + L_1)) \\
+ c_3 I_{12} (r_2 + L_2) + c_3 I_{11} I_1 (r_1 + L_1)) \\
+ \delta_2 \delta_3 (2\Sigma \beta_1 (c_3 I_{11} I_1 (r_3 + L_3) + \Sigma \beta_1 (-(r_1 + L_1)(r_2 + L_2) c_3 \cosh^3 \\
- (r_2 + L_2)(r_3 + L_3)c_3 \cosh^3) \\
+ \delta_2 \delta_3 (2\Sigma \beta_1 (t_{11} c_3 \cosh^3 I_3 (r_3 + L_3) + t_{11} c_3 \cosh^3 I_2 (r_2 + L_2) \\
-t_{12} c_3 \cosh^3 I_1 (r_2 + L_2) - t_{12} c_3 \cosh^3 I_3 (r_2 + L_2) - t_{11} c_3 \cosh^3 I_3 (r_1 + L_1) \\
+ \delta_2 \delta_3 (2\Sigma \beta_1 (-c_3 I_{11} I_1 (r_3 + L_3) + c_3 \cosh^3 I_2 (r_2 + L_2) \\
+ c_3 \cosh^3 I_1 (r_1 + L_1)) \\
+ \delta_3 \delta_3 (2\Sigma \beta_1 (t_{12} I_3 (r_3 + L_3) - t_{11} I_2 (r_1 + L_1) + t_{12} I_1 (r_1 + L_1)) \\
+ \delta_3 \delta_3 (2\Sigma \beta_1 (-L_2 (r_3 + L_3)) \\
+ \delta_3 \delta_3 (2\Sigma \beta_1 ((L_3 t_{11})^2 + (L_3 t_{11})^2 + (L_2 t_{11} - L_1 t_{12})^2) \\
+ \delta_3 \delta_3 (2\Sigma \beta_1 (L_1^2 + L_2^2) \\
+ \delta_3 \delta_3 (2\Sigma \beta_1 (-L_2 L_3 t_{12} - L_1 L_3 t_{11})) \\
\text{where,} \\
\begin{align*} \\
c_1 &\triangleq (r_2 + L_2)^2 + (r_3 + L_3)^2 \\
c_2 &\triangleq (r_1 + L_1)^2 + (r_3 + L_3)^2 \\
c_3 &\triangleq (r_1 + L_1)^2 + (r_2 + L_2)^2. \end{align*} \]
The next step in formulating the equations of motion using the Lagrangian approach is the formulation of the generalized force, \( Q_r \), associated with the respective virtual displacement. This is done by using the principle of virtual work.

The principle of virtual work may be written as,

\[
\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{p}_i 
\]

where,

\( \mathbf{F}_i \) Active force or torque

\( \mathbf{p} \) Position or orientation vector of \( \mathbf{F}_i \)

\( \delta \mathbf{p} \) First variation of \( \mathbf{p} \).

The four force groups are given by,

**Gravitational force**

\[
\mathbf{F}_G = -K_G \frac{x_1a_1 + x_2a_2 + x_3a_3}{[x_1^2 + x_2^2 + x_3^2]^{3/2}}
\]

\[
= -K_G \frac{x_1a_1 + x_2a_2 + x_3a_3}{[x_1^2 + x_2^2 + x_3^2]^{3/2}}
\]

\[
\mathbf{p} = x_1a_1 + x_2a_2 + x_3a_3
\]

\[
\delta \mathbf{p} = \delta x_1a_1 + \delta x_2a_2 + \delta x_3a_3
\]

\[
\delta W = \mathbf{F}_G \cdot \delta \mathbf{p} = -K_G \frac{[x_1\delta x_1 + x_2\delta x_2 + x_3\delta x_3]}{[x_1^2 + x_2^2 + x_3^2]^{3/2}}
\]

\[
= Q_x \delta x_1 + Q_x \delta x_2 + Q_x \delta x_3
\]

\[
Q_x = -\frac{K_G x_1}{[x_1^2 + x_2^2 + x_3^2]^{3/2}}
\]
\[ Q_{x_2} \triangleq \frac{-K_G x_2}{\left[ x_1^2 + x_2^2 + x_3^2 \right]^{3/2}} \]  
\[ Q_{x_3} \triangleq \frac{-K_G x_3}{\left[ x_1^2 + x_2^2 + x_3^2 \right]^{3/2}} \]  

**Thrust force**

\[ F_T = F_{1b_1} + F_{2b_2} + F_{3b_3} \]  
\[ p = x_1 a_1 + x_2 a_2 + x_3 a_3 \]  

\[ \delta p = (c_{11} b_1 + c_{12} b_2 + c_{13} b_3) \delta x_1 \]  
\[ + (c_{21} b_1 + c_{22} b_2 + c_{23} b_3) \delta x_2 \]  
\[ + (c_{31} b_1 + c_{32} b_2 + c_{33} b_3) \delta x_3 \]  

\[ \delta W = F_T \cdot \delta p \]  
\[ = (F_{1c_{11}} + F_{2c_{12}} + F_{3c_{13}}) \delta x_1 \]  
\[ + (F_{1c_{21}} + F_{2c_{22}} + F_{3c_{23}}) \delta x_2 \]  
\[ + (F_{1c_{31}} + F_{2c_{32}} + F_{3c_{33}}) \delta x_3 \]  

\[ Q_{x_1} \triangleq F_{1c_{11}} + F_{2c_{12}} + F_{3c_{13}} \]  
\[ Q_{x_2} \triangleq F_{1c_{21}} + F_{2c_{22}} + F_{3c_{23}} \]  
\[ Q_{x_3} \triangleq F_{1c_{31}} + F_{2c_{32}} + F_{3c_{33}} \]  

**Thrust torque**

\[ P = \theta_1 a_3 + \theta_2 b_1 + \theta_3 b_2 \]  
\[ = \theta_1 (c_{31} b_1 + c_{32} b_2 + c_{33} b_3) \]  
\[ + \theta_2 (c \theta_3 b_1 + s \theta_3 b_3) \]
\[ \delta \tau = (c_{31} b_1 + c_{32} b_2 + c_{33} b_3) \delta \theta_1 + (c \theta_3 b_1 + \sin \theta_3 b_3) \delta \theta_2 + \delta \theta_3 b_2 \]

\[ \delta \omega = T_T \cdot \delta \tau = (c_{31} T_1 + c_{32} T_2 + c_{33} T_3) \delta \theta_1 + (c \theta_3 T_1 + \sin \theta_3 T_3) \delta \theta_2 + T_2 \delta \theta_3 \]

\[ = Q_{\theta_1} \delta \theta_1 + Q_{\theta_2} \delta \theta_2 + Q_{\theta_3} \delta \theta_3 \]

\[ Q_{\theta_1} \triangleq c_{31} T_1 + c_{32} T_2 + c_{33} T_3 \]

\[ Q_{\theta_2} \triangleq c \theta_3 T_1 + \sin \theta_3 T_3 \]

\[ Q_{\theta_3} \triangleq T_2 \]

**Pendulum spring and damper torques**

\[ T^m_i = -(K_\beta (\beta - \beta_o)^3 + D_\beta \dot{\beta}) t_{11} b_1 \]

\[ - (K_\beta (\beta - \beta_o)^3 + D_\beta \dot{\beta}) t_{12} b_2 \]

\[ - (K_\alpha (a - a_o)^3 + D_\alpha \dot{a}) b_3 \]

\[ \delta \tau = a b_3 + \dot{\beta} n_1 \]

\[ = \beta t_{11} b_1 + \beta t_{12} b_2 + a b_3 \]

\[ \delta \omega = T^m_i \cdot \delta \tau \]

\[ = -(K_\beta (\beta - \beta_o)^3 + D_\beta \dot{\beta}) \delta \beta \]

\[ - (K_\alpha (a - a_o)^3 + D_\alpha \dot{a}) \delta a \]
The coordinates \( x_1, x_2, x_3, \theta_1, \theta_2, \theta_3, \alpha, \) and \( \beta \) are independent of each other, and is the reason the differentials may be written as they were.

The generalized forces may be assembled and written as,

\[
F_{x_1} = -K_G x_1 + c_{11} F_1 + c_{12} F_2 + c_{13} F_3
\]

\[
F_{x_2} = -K_G x_2 + c_{21} F_1 + c_{22} F_2 + c_{23} F_3
\]

\[
F_{x_3} = -K_G x_3 + c_{31} F_1 + c_{32} F_2 + c_{33} F_3
\]

\[
F_{\theta_1} = c_{31} T_1 + c_{32} T_2 + c_{33} T_3
\]

\[
F_{\theta_2} = c s \theta_3 T_1 + s n \theta_3 T_3
\]

\[
F_{\theta_3} = T_2
\]

\[
F_\beta = -(K_\beta (\beta - \beta_o)^3 + D_\beta \ddot{\beta})
\]

\[
F_\alpha = -(K_\alpha (\alpha - \alpha_o)^3 + D_\alpha \ddot{\alpha})
\]

The equations of motion were formulated by applying equation 1 to equations 5 and 39-46. After algebraic manipulation the Lagrangian and Kaneian formulations were consistent, giving a term for term match.
APPENDIX B.

PROGRAM OF THE NUMERICAL SIMULATION
OF THE EQUATIONS OF MOTION
MAIN PROGRAM

THIS PROGRAM SIMULATES THE MOTION OF A RIGID BODY WITH AN EQUIVALENT SPRING, MASS, DAMPER SYSTEMS ATTACHED TO THE BODY AS SPHERICAL PENDULUMS.

IMPLICIT REAL*8 (A-Z)
REAL*4 TVEC, WVEC, ZVEC, HVEC, TVEC, PVEC, ENGVEC, 
1 PVECA1, PVECA2, PVECA3, PVECA4, PVECA5, PVECA6, PVECA7, PVECA8, TVEC, WVEC, ZVEC
INTEGER I, J, K, CMAT, JMAT, NPN, RSTATE, NPNST2, NPNZ, NEQS,
1 NPTS, NPTS, T, IER, INDEX, ISTATE, H, HNTER, KRT, KPS
EXTERNAL FCT, PCTI
DIMENSION Y(45), DER(45), W(LY(45), WK(2728),
1 KSPNBG(8), KSPNBG(8), DAMP(8), DAMP(8), ALPH(8), BETA(8),
1 ALPH(8), BETA(8), ALPHDT(8), BETA(8), R(8), R(8), R(8), R(8), R(8)
DIMENSION TMDATA(68),
1 FDATA(68), BF(68), CF(68), DF(68), TDATA(14), TDATA(14),
1 TDATA(14), TDATA(14), TDATA(14), TDATA(14), BI11(9),
1 CI11(10), DI11(9), BI12(9), CI12(10), DI12(9), BI13(9), CI13(10),
1 DI13(9), BI22(9), CI22(10), DI22(9), BI23(9), CI23(10), DI23(9),
1 BI33(9), CI33(10), DI33(9)
DIMENSION TVEC(9500), WVEC(9500), ZVEC(9500), HVEC(9500),
1 TVEC(9500), TVEC(9500), TVEC(9500), TVEC(9500),
1 TVEC(9500), TVEC(9500), TVEC(9500), TVEC(9500)
DIMENSION XRECT(3), YRECT(3), ZRECT(3), YRECT(3)
COMMON /BLK1/ TMDATA, TDATA, BF, CF, DF, TDATA, TDATA, TDATA,
1 TDATA, TDATA, TDATA, TDATA, TDATA, TDATA, TDATA, TDATA,
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1 TDATA, TDATA, TDATA, TDATA, TDATA, TDATA, TDATA, TDATA,
1 TDATA, TDATA, TDATA, TDATA, TDATA, TDATA, TDATA, TDATA,
DO 10 J=1,8
   K=7* (J-1)
   READ (5,1) (FDATA(I+K), I=1,7)
   WRITE (6,1) (FDATA(I+K), I=1,7)
10 CONTINUE
DO 15 J=1,2
   K=7* (J-1)
   READ (5,1) (TIDATA(I+K), I=1,7)
   WRITE (6,1) (TIDATA(I+K), I=1,7)
15 CONTINUE
DO 20 J=1,2
   K=7* (J-1)
   READ (5,1) (IIZ2DATA(I+K), I=1,7)
   WRITE (6,1) (IIZ2DATA(I+K), I=1,7)
20 CONTINUE
READ (5,13) LPEND,RPEND,PHNRT,PHNED,KSMPGR,DAMP,PEN
WRITE (6,13) LPEND,RPEND,PHNRT,PHNED,KSMPGR,DAMP,PEN
13 FORM mat (8F9.2)
READ (5,14) HBNITL,HBNOUT,I13IN,I13SCL,I22SCL
WRITE (6,14) HBNITL,HBNOUT,I13IN,I13SCL,I22SCL
14 FORM mat (8F9.2)
READ (5,16) ALT8D,ORSPER,W1,W2,W3
WRITE (6,16) ALT8D,ORSPER,W1,W2,W3
16 FORM mat (8F9.2)
READ (5,17) THRSTE,ALPHT
WRITE (6,17) THRSTE,ALPHT
17 FORM mat (2F9.2)
READ (5,18) TOL,TINSTEP,TINITL,TFINAL,METH,HITER
WRITE (6,18) TOL,TINSTEP,TINITL,TFINAL,METH,HITER
18 FORM mat (4F9.5,2F9)
READ (5,19) TH1,TH2,TH3
WRITE (6,19) TH1,TH2,TH3
19 FORM mat (3F9.2)
NPEM2=NPEN+2
NPEM2=NPEM2
NSM2=2* (6+2*NPEN)+1
NEG0S=6+NPEM2
ICN=0
KC=4.3735D14
PI=3.141592654D0
RDPDG=PI/180.D0
DGPRD=1.D0/RDPDG
FTPIN=1.D0/12.D0
FTPHEL=5280.D0
GC=32.2D0
TBOUT=95.3D0
HBNGAT= (HBINLT-HBOUT) /TBOUT
REARTH=3963.D0*FTPHEL
WORBID=2.00*PI* (1.00/(ORBPER=3600.D0))
XSPRCL (1) =ALTD*FTPHEL*REARTH
VSPRCL (1) =0.D0
VSPRCL (2) =0.D0
VSPRCL (3) =0.D0
CALL RTPIXZ (XSPRCL, IRECT, VSPRCL, VRECT)
I1=IRECT (1)
I2=IRECT (2)
I3=IRECT (3)
I1DT=VRECT (1)
I2DT=VRECT (2)
I3DT=VRECT (3)

THRUST MISALIGNMENT

ALPHT=ALPHT/RDPDG
BETAT=0.D0
PCOSB1=-DSIN (ALPHT) *DCOS (BETAT)
PCOSB2=DCOS (ALPHT) *DCOS (BETAT)
PCOSB3=0.00
t1=0.00
t2=THRSTE
t3=0.00

INITIALIZING ARRAYS

Y (1)=I1DT
Y (2)=I2DT
Y (3)=I3DT
Y (4)=W1
Y (5)=W2
Y (6)=W3
Y (7+NPENT2)=X1
Y (8+NPENT2)=X2
Y (9+NPENT2)=X3
CALL EULER (TH1, TH2, TH3, ET, E2, E3, E4)
Y (10+NPENT2)=E1
Y (11+NPENT2)=E2
Y (12+NPENT2)=E3
Y(13+NPENT2)=E4
ANG=2.00=PI/DFLOAT(NPEN)
NPEND=NPEND+FTPIN
PENHT=PENHT+FTPIN
DO 40 I=1,NPEN
J=I-1
KRT=5+2*I
KPS=12+NPENT2+2*I
Y(KRT)=0.00
Y(KRT+1)=0.00
BETAO(I)=BETAO(I)*RDPDG
ALPH0(I)=ALPH0(I)*RDPDG
Y(KPS)=BETAO(I)
Y(KPS+1)=ALPH0(I)
KSPNGA(I)=KSPNGA
KSPNGB(I)=KSPNGA(I)
DAMPA(I)=DAMPCF
DAMPB(I)=DAMPA(I)
RT(I)=-RPEND*OCOS(DFLOAT(J)*ANG)
R2(I)=PENHT
R3(I)=RPEND*DSIN(DFLOAT(J)*ANG)
L(I)=LPEND
M(I)=NPEND
40 CONTINUE

GENERATING CUBIC SPLINE FIT OF STAR 48 MOTOR THRUST

CALL DCURVE(NPTST,0.00,0.00,TDATA,PDATA,PF,CF,DF)

GENERATING CUBIC SPLINE FIT OF INERTIAL PROPERTIES

DO 35 I=1,NPTST
111DAT(I)=111DAT(I)*GC*11SCL
35 112DAT(I)=112DAT(I)*GC*11SCL
111DTO=-14.400*GC
111DTN=-18.100*GC
CALL DCURVE(NPTST,111DTO,111DTN,111DATA,111DAT,BI11,CI11,DI11)
112DTO=1.000*GC
112DTN=-5.510*GC
CALL DCURVE(NPTST,112DTO,112DTN,111DATA,112DAT,BI22,CI22,DI22)

SOLVING EQUATIONS OF MOTION USING DGEAR

CALL OUTP(TINITL,T)
HSTEP=0.001DO
TIME=TINITL
TEND=TINITL+HSTEP
INDEX=1
21 CALL DGEAR(NSTATE,FCT,FCTJ,TIME,HSTEP,T,TEND,TOL,NETH,NITER,INDEX,IK,WK,ZER)
IF (IER.GT.128 .OR. INDEX .EQ. 1) GOTO 22
CALL OUTP (TIME, Y)
TH END = TIME * STEP
IF (TH END.GT.TFINAL) GOTO 23
GOTO 21

22 IF (IER.EQ.132) WRITE (6,3)
3 FORMAT ("INTEGRATION HALT (ERROR TEST FAILED) ")
IF (IER.EQ.133) WRITE (6,4)
4 FORMAT ("INTEGRATION HALT (CORRECTOR CONVERGENCE FAILED) ")
IF (IER.EQ.134) WRITE (6,6)
6 FORMAT ("INTEGRATION HALT (ERROR OR TOL. TEST FAILED) ")
IF (IER.EQ.135) WRITE (6,7)
7 FORMAT ("INCORRECT INPUT PARAMETER")
IF (IER.EQ.136) WRITE (6,8)
8 FORMAT ("INDEX IS NOT BEYOND 1")
IF (INDEX .EQ. 1) WRITE (6,9)
9 FORMAT ("INDEX EQUALS 1")
23 CONTINUE

PLOTTING SIMULATION RESULTS
CALL GRAY (ICNT)
STOP
END

SUBROUTINE GRAY (ICNT)
DIMENSION TVEC (9500), W1VEC (9500), W2VEC (9500), W3VEC (9500),
W1VEC (9500), TH1VEC (9500), PH1VEC (9500), ENGVEC (9500),
PVEC1 (9500), PVEC2 (9500), PVEC3 (9500), PVEC4 (9500),
PVEC5 (9500), PVEC6 (9500), PVEC7 (9500), PVEC8 (9500),
TH1VEC (9500), TH2VEC (9500), TH3VEC (9500),
COMMON/BLK3/TVEC,W1VEC,W2VEC,W3VEC,RVEC,THVEC,PHVEC,ENGVEC,
PVEC1,PVEC2,PVEC3,PVEC4,PVEC5,PVEC6,PVEC7,PVEC8,
TH1VEC,TH2VEC,TH3VEC
GRFHT=2.0
GRFLNM=18.0
BORDER=1.0
SPACE=0.75
CALL PLOTS (0,0,31)

PLOTTING BODY FIXED RATES VS TIME
CALL ORIGIN (BORDER, BORDER, 0)
CALL SCALE (TVEC, GRFLNM, ICNT, 1)
CALL AXIS (0.0, 0.0, 13, TIME (SECONDS), -13, GRFLNM, 0.0, TVEC (ICNT+1),
TVEC (ICNT+2))
TORGIN=BORDER+SPACE
CALL ORIGIN (BORDER, TORGIN, 0)
CALL SCALE (W2VEC, GRFHT, ICNT, 1)
CALL AXIS (0.0, 0.0, 13, ROLL (DEG/SEC), 13, GRFHT, 90.0, W2VEC (ICNT+1)
1, W2VEC (ICNT+2))
CALL LINE (TVEC, #2VEC, ICNT, 1, 0, 2)
TORGIN = BORDER + GRFHGT + 2.0*SPACE
CALL ORIGIN (BORDER, TORGIN, 0)
CALL SCALE (#2VEC, GRFHGT, ICNT, 1)
CALL AXIS (0.0, 0.0, 14, PITCH (DEG/SEC), 14, GRFHGT, 90.0, #3VEC (ICNT + 1)
#3VEC (ICNT + 2))
CALL LINE (#4VEC, #3VEC, ICNT, 1, 0, 2
TORGIN = BORDER + 2.0*GRFHGT + 3.0*SPACE
CALL ORIGIN (BORDER, TORGIN, 0)
CALL SCALE (#4VEC, GRFHGT, ICNT, 1)
CALL AXIS (0.0, 0.0, 12, HORIZ (DEG/SEC), 12, GRFHGT, 90.0, #5VEC (ICNT + 1),
#5VEC (ICNT + 2))
CALL LINE (#6VEC, #5VEC, ICNT, 1, 0, 2
TORGIN = BORDER + SPACE
CALL ORIGIN (XBOUND, TORGIN, 0)
CALL SCALE (#6VEC, GRFHGT, ICNT, 1)
CALL AXIS (0.0, 0.0, 10, HORIZ (MILES), 10, GRFHGT, 90.0, #7VEC (ICNT + 1)
#7VEC (ICNT + 2))
CALL LINE (#8VEC, #7VEC, ICNT, 1, 0, 2
TORGIN = BORDER + 2.0*GRFHGT + 2.0*SPACE
CALL ORIGIN (XBOUND, TORGIN, 0)
CALL SCALE (#8VEC, GRFHGT, ICNT, 1)
CALL AXIS (0.0, 0.0, 16, HORIZONTAL ANG (DEG), 16, GRFHGT, 90.0, #9VEC (ICNT + 1)
#9VEC (ICNT + 2))
CALL LINE (#10VEC, #9VEC, ICNT, 1, 0, 2
TORGIN = BORDER + 2.0*GRFHGT + 3.0*SPACE
CALL ORIGIN (XBOUND, TORGIN, 0)
PHIVEC (ICNT + 1) = -2.0
PHIVEC (ICNT + 2) = 2.0
CALL AXIS (0.0, 0.0, 15, HAZMUTH ANG (DEG), 15, GRFHGT, 90.0, #11VEC (ICNT + 1),
#11VEC (ICNT + 2))
CALL LINE (#12VEC, PHIVEC, ICNT, 1, 0, 2
XBOUND = XBORDR + 20.0
PLOTTING ORBITAL POSITION VS TIME

GRFHGT = 8.0
CALL ORIGIN (XBOUND, BORDER, 0)
CALL SCALE (#12VEC, GRFHGT, ICNT, 1)
CALL AXIS (0.0, 0.0, 13, HORIZ (SECONDS), 13, GRFHGT, 0.0, #13VEC (ICNT + 1),
#13VEC (ICNT + 2))
ENGVEC (ICNT + 1) = 0.0

PLOTTING ENERGY VS TIME
ENVEC(1CNT+2) = 100000.0
CALL AXIS(0.0,0.0,18,ENERGY(LBM-FT2/52),18,GRFHT,90.0,
1ENVEC(1CNT+1),ENVEC(1CNT+2))
CALL LINE(TVEC,ENVEC,1CNT,1,0,2)

PLOTTING PENDULUM POSITIONS VS TIME

XBOBDR=XBOBDR+20.0
GRFHT=1.5
CALL ORIGIN(XBOBDR,BORDER,0)
CALL SCALE(TVEC,GRFHT,1CNT,1)
CALL AXIS(0.0,0.0,13RTIME(SECONDS),-13,GRFHT,0.0,TVEC(1CNT+1),
1TVEC(1CNT+2))
FORIGIN=BORDER
CALL ORIGIN(XBOBDR,FORIGIN,0)
CALL SCALE(PVEC1,GRFHT,1CNT,1)
CALL AXIS(0.0,0.0,8HAL1(DEG),8,GRFHT,90.0,PVECB1(1CNT+1)
1,PVECB1(1CNT+2))
CALL LINE(TVEC,PVECB1,1CNT,1,0,2)
FORIGIN=BORDER+GRFHT*SPACE
CALL ORIGIN(XBOBDR,FORIGIN,0)
CALL Scale(PVECA1,GRFHT,1CNT,1)
CALL AXIS(0.0,0.0,8HAL1(DEG),8,GRFHT,90.0,PVECA1(1CNT+1)
1,PVECA1(1CNT+2))
CALL LINE(TVEC,PVECA1,1CNT,1,0,2)
FORIGIN=BORDER+3.0*GRFHT+SPACE
CALL ORIGIN(XBOBDR,FORIGIN,0)
CALL Scale(PVECA2,GRFHT,1CNT,1)
CALL AXIS(0.0,0.0,8HAL2(DEG),8,GRFHT,90.0,PVECA2(1CNT+1)
1,PVECA2(1CNT+2))
CALL LINE(TVEC,PVECA2,1CNT,1,0,2)
XBOBDR=XBOBDR+20.0
CALL ORIGIN(XBOBDR,BORDER,0)
CALL Scale(TVEC,GRFHT,1CNT,1)
CALL AXIS(0.0,0.0,13RTIME(SECONDS),-13,GRFHT,0.0,TVEC(1CNT+1),
1TVEC(1CNT+2))
FORIGIN=BORDER
CALL ORIGIN(XBOBDR,FORIGIN,0)
CALL Scale(PVECB3,GRFHT,1CNT,1)
CALL AXIS(0.0,0.0,8HAL3(DEG),8,GRFHT,90.0,PVECB3(1CNT+1)
1,PVECB3(1CNT+2))
CALL LINE(TVEC,PVECB3,1CNT,1,0,2)
FORIGIN=BORDER+GRFHT*SPACE
CALL ORIGIN(XBOBDR,FORIGIN,0)
END
SUBROUTINE FCT(NSTATE,TIME,DER)
IMPLICIT REAL*8 (A-Z)
INTEGER I,J,K,ICNT,JCNT,HPEN,NSTATE,NPENT2,NPTS,KPLUS1,
1NEQS,IER,ILJB,ICNG,ISWITCH,KPS,KRT
DIMENSION Y(45),DERY(45),ZVEC(990),Z(22,22),RHS(22),U(29,8),
1B(3,9),DET(2112),ICNG(68),WAREA(2068),INW(68),
1KSPNGA(8),KSPNGB(8),DAMPA(8),DAMPB(8),ALPHO(8),BETA0(8),
1ALPHTH(8),BETATH(8),ALPHDT(8),BETADT(8),RT(8),R2(8),R3(8),L(8),M(8)
DIMENSION TDATA(64),
1FDATA(64),BF(53),CF(64),DP(53),TDATA(14),TT1DAT(14),
1TT2DAT(14),TT3DAT(14),TT22DAT(14),TT33DAT(14),BI11(9),
1CI11(10),DI11(9),BI12(9),CI12(10),DI12(9),BI13(9),CI13(10),
1DI13(9),BI22(9),CI22(10),DI22(9),BI23(9),CI23(10),DI23(9),
1BI33(9),CI33(10),DI33(9)
COMMON/BX1/TDATA,FDATA,BF,CF,DP,TDATA,TT1DAT,TT2DAT,
1TT3DAT,TT22DAT,TT33DAT,BX11,CI11,DI11,BX12,CI12,DI12,BI13,
1CI13,DI13,BI22,CI22,DI22,BI23,CI23,DI23,BX33,CI33,DI33
COMMON/BX2/R1P,R2P,R3P,PI,PGRT,RPEND,TBNOUT,TBOUT,MRINTL,
1MNBRAT,GC,KC,KC0SB1,KC0SB2,FCS0B3,TC0SB3,TT11,TI11,T112,TI22,TI33,ISWITCH
COMMON/BX4/TT1,TI11,TI12,TI13,TI122,TI133,ISWITCH
CALCULATION OF STAR 48 MOTOR THRUST AND INERTIA VALUES
IF (TIME.LT.TBNOOUT) GOTO 21
FBTHRST=0.DO
HBODY=HBOUT
I1T=IT1DAT(NPTS)
I11DT=0.DO
I12=0.DO
I12DT=0.DO
I13=I13IN=GC
I13DT=0.DO
I22=IT22DAT(NPTS)
I22DT=0.DO
I23=I2.DO
I23DT=0.DO
I33=I3
I33DT=I31DT
GOTO 22
21 CALL DCUBIC(TIME,FBTHRST,FBTHD,TDDATA,FDATA,BF,CF,DP)
CALCULATING INERTIA VALUES
CALL DCUBIC(TIME,I11,I11DT,TDDATA,I11DAT,BI11,CI11,DI11)
I11=0.DO
I11DT=0.DO
I13=I13IN=GC
I13DT=0.D0
CALL DCUMBIC(TIME,I22,I22DT,TIDATA,I22DAT,BI22,CI22,DI22)
I23=0.D0
I23DT=0.D0
I33=I11
I33DT=IT1DT
HBDT=HBINTL-MBSTATE
22 IF (ISWITCH.EQ.1) GOTO 999

INITIALIZING Z ARRAY AND RHS VECTOR

DO 5 I=1,NEQS
RHS(I)=0.D0
DO 5 J=1,NEQS
5 Z(I,J)=0.D0
KG=KC+HBODT
XI0T=Z(1)
X20T=Z(2)
X30T=Z(3)
W1=Z(4)
W2=Z(5)
W3=Z(6)
XI=Z(7+MPENT2)
X2=Z(8+MPENT2)
X3=Z(9+MPENT2)
DO 10 I=1,NPEN
K=2*(I-1)
BETA(I)=Y(1+(I+MPENT2*K))
BETA0T(I)=Y(7+K)
ALPH(I)=Y(15+MPENT2*K)
10 ALPH0T(I)=Y(8+K)

TRANSFORMATION MATRICES

E1=Z(10+MPENT2)
E2=Z(11+MPENT2)
E3=Z(12+MPENT2)
E4=Z(13+MPENT2)
E1SQR=E1**2
E2SQR=E2**2
E3SQR=E3**2
E4SQR=E4**2
E1E2=E1*E2
E1E3=E1*E3
E1E4=E1*E4
E2E3=E2*E3
E2E4=E2*E4
E3E4=E3*E4
C11=1.D0-2.D0*(E2SQR+E3SQR)
C12=2.D0*(E1E2-E3E4)
C13=2.0 D1* (E1E3+E2E4)
C21=2.0 D1* (E2E2+E3E4)
C22=1.0 D0-2.0 D1* (E3SQR+E3SQR)
C23=2.0 D0* (E2E3-E1E2)
C31=2.0 D0* (E1E3-E2E4)
C32=2.0 D0* (E2E3+E1E2)
C33=1.0 D0-2.0 D1* (E1SQR+E2SQR)

CALCULATING PARTIAL RATES OF CHANGE OF POSITION

U1=C11
U2=C12
U3=C13
U4=C21
U5=C22
U6=C23
U7=C31
U8=C32
U9=C33

QUANTITIES RELATING MAIN BODY KINEMATICS

W1W2=W1=W2
W1W3=W1=W3
W2W3=W2=W3
W1SQR=W1=W2
W2SQR=W2=W2
W3SQR=W3=W3
IWWB2=I1*W1SQR+I2*W1W2*+I3*W1W3-(I1*I2W3+I2*W2W3+I2*W3SQR)
IWWB3=I1*W1W2*+I2*W2SQR+I3*W2W3-(I1*W1SQR+I2*W1W2+I3*W1W3)
IDTWB1=I1D1TW1*+I2D1TW2*+I3D1TW3
IDTWB2=I1D1TW1*+I2D1TW2*+I3D1TW3
IDTWB3=I1D1TW1*+I2D1TW2*+I3D1TW3
D1=IWWB1-DIDTW1
D2=IWWB2-DIDTW2
D3=IWWB3-DIDTW3

SUMMATION OF PENDULUM CONTRIBUTIONS TO EQUATIONS OF MOTION

DO 15 I=1,NPEM
CSALPH=DCOS(ALPH(I))
SNALPH=DSIN(ALPH(I))
CSBETA=DCOS(BETA(I))
SNBETA=DSIN(BETA(I))
T71=CSALPH
T72=SNALPH
T73=SNBETA
T21=-SNALPH-CSBETA
T22=CSALPH-CSBETA
T23=SNBETA
\[ T1DT = -\text{ALPHDT}(I) = \text{SNALPH} \]
\[ T2DT = \text{ALPHDT}(I) = \text{CSALPH} \]

**Body fixed angular rates (\( \text{RWH}(I) \))**

\[ W1 = W1 + \text{BETADT}(I) * T11 \]
\[ W2 = W2 + \text{BETADT}(I) * T12 \]
\[ W3 = W3 + \text{ALPHDT}(I) \]

\[ W1W2 = W1 * W2 \]
\[ W1W3 = W1 * W3 \]
\[ W2W3 = W2 * W3 \]

\[ W1SQR = \sqrt{W1^2} \]
\[ W2SQR = \sqrt{W2^2} \]
\[ W3SQR = \sqrt{W3^2} \]

**Calculating pendulum position and length in B basis**

\[ L1 = L(I) * T21 \]
\[ L2 = L(I) * T22 \]
\[ L3 = L(I) * T23 \]

\[ B1 = B1(I) = L1 \]
\[ B2 = B2(I) = L2 \]
\[ B3 = B3(I) = L3 \]

**Calculating partial rates of change of orientation.**

\[ 0(10, I) = 0.00 \]
\[ 0(11, I) = -H3L3 \]
\[ 0(12, I) = R212 \]
\[ 0(13, I) = R313 \]
\[ 0(14, I) = 0.00 \]
\[ 0(15, I) = -R1L1 \]
\[ 0(16, I) = -R2L2 \]
\[ 0(17, I) = -R3L3 \]
\[ 0(18, I) = 0.00 \]
\[ 0(19, I) = L3*T13 \]
\[ 0(20, I) = -L3*T11 \]
\[ 0(21, I) = L(I) * CSBETA \]
\[ 0(22, I) = L2 \]
\[ 0(23, I) = L1 \]
\[ 0(24, I) = 0.00 \]

**Assembling pendulum contributions to equations of motion**

\[ B(1, I) = L3 = (-\text{ALPHDT}(I) = W1 * \text{BETADT}(I) = (T12DT + T11 = W3)) - \]
\[ 1L2 * \text{BETADT}(I) = (-T11 * W2 + T12 * W1) + W1W2SQR(I) = \]
\[ 1R3(I) + W1W2 + L = SQR(W2SQR + W3SQR) + W1W3 = L3 \]
\[ B(2, I) = L3 * \text{BETADT}(I) = (-T11 * W2 + T12 * W1) - L3 = (\text{BETADT}(I) \]
\[ 1 = (T12DT - T12 = W3) + \text{ALPHDT}(I) = W2 + W2SQR(I) = R3(I) - B2(I) = (W3SQR + W1SQR) + \]
\[ 1W1W2 = R1(I) + W2W3 = L3 - L2 = (W3SQR + W1SQR) + W1W2 = L1 \]
\[ B(3, 1) = L_2^\alpha (\text{ALPHDT}(2) + 2^\beta \text{BETADT}(1) + (T_1 \text{DT} - T_1 T_2 = W_3) \]

\[ I - L_{11}^\alpha (1 - \text{ALPHDT}(1) + \text{BETADT}(1) (T_1 \text{DT} + T_1 T_2 = W_3) + W_1 W_3 \text{OR} (I) = R_3 (I) \]

\[ T + 2^\beta W_3 = L_3 (W_1 \text{SQR} + W_2 \text{SQR}) + 2^\beta W_3 = L_2 \]

\[ K = 5 + 2^\beta T \]

**KPLUSTL = K + 1**

\[ Z(1, 1) = Z(1, 1) - H(I) \]

\[ Z(1, 2) = Z(1, 2) - H(I) = (0^2 = (11, 1) + 0^2 = (12, 1)) \]

\[ Z(2, 2) = Z(2, 2) - H(I) = (0^2 = (13, 1) + 0^2 = (15, 1)) \]

\[ Z(3, 3) = Z(3, 3) - H(I) = (0^2 = (17, 1) + 0^2 = (19, 1)) \]

**RIGHT HAND SIDE ADDITION OF GENERALIZED ACTIVE FORCES FOR ALPHA AND BETA**
**RIGHT HAND SIDE ADDITION OF INERTIA FORCES FROM MAIN BODY**

\[
\text{RHS (4)} = \text{RHS (4)} \times D1 \\
\text{RHS (5)} = \text{RHS (5)} \times D2 \\
\text{RHS (6)} = \text{RHS (6)} \times D3
\]

**RIGHT HAND SIDE ADDITION OF GENERALIZED ACTIVE FORCES FROM MAIN BODY PLUS PREVIOUS INERTIA FORCES**

\[
\begin{align*}
F1 &= RHS (1) \\
F2 &= RHS (2) \\
F3 &= RHS (3) \\
T1 &= R1P \times F3 \times R2P \\
T2 &= R1P \times F3 \times R2P \\
T3 &= R1P \times F3 \times R2P
\end{align*}
\]

\[
\text{RSQR32} = (I1^2 + I2^2 + I3^2)^0.5D0
\]

\[
\begin{align*}
\text{RHS (1)} &= (\text{RHS (1)} + (\text{KG} \times \text{RSQR32} \times F1^2 + F2^2 + F3^2)^0.5D0) \times GC \\
\text{RHS (2)} &= (\text{RHS (2)} + (\text{KG} \times \text{RSQR32} \times F1^2 + F2^2 + F3^2)^0.5D0) \times GC \\
\text{RHS (3)} &= (\text{RHS (3)} + (\text{KG} \times \text{RSQR32} \times F1^2 + F2^2 + F3^2)^0.5D0) \times GC \\
\text{RHS (4)} &= (\text{RHS (4)} + T1 \times GC) \\
\text{RHS (5)} &= (\text{RHS (5)} + T2 \times GC) \\
\text{RHS (6)} &= (\text{RHS (6)} + T3 \times GC)
\end{align*}
\]

**CALCULATING REMAINING RIGHT HAND SIDE TERMS**

\[
\begin{align*}
\text{DERY (7+NPENT2)} &= X1DT \\
\text{DERY (8+NPENT2)} &= X2DT \\
\text{DERY (9+NPENT2)} &= X3DT \\
\text{DERY (10+NPENT2)} &= 0.5D0 \times (E1 \times W1 - E2 \times W2 + E3 \times W3) \\
\text{DERY (11+NPENT2)} &= 0.5D0 \times (E1 \times W1 + E2 \times W2 - E3 \times W3) \\
\text{DERY (12+NPENT2)} &= 0.5D0 \times (-E2 \times W1 + E1 \times W2 + E3 \times W3) \\
\text{DERY (13+NPENT2)} &= 0.5D0 \times (-E1 \times W1 - E2 \times W2 - E3 \times W3)
\end{align*}
\]

DO 20 I=1,NPEN

K=I+NPENT2+2DT

DERY (K) = BETADT (I)

20 DER (K+1) = ALPHDT (I)
CALCULATION OF LEFT HAND SIDE TERMS

\[ Z(1,1) = Z(1,1) - HBODY \]
\[ Z(2,2) = Z(2,2) - HBODY \]
\[ Z(3,3) = Z(3,3) - HBODY \]
\[ Z(4,4) = Z(4,4) - I11 \]
\[ Z(4,5) = Z(4,5) - I12 \]
\[ Z(4,6) = Z(4,6) - I13 \]
\[ Z(5,5) = Z(5,5) - I22 \]
\[ Z(5,6) = Z(5,6) - I23 \]
\[ Z(6,6) = Z(6,6) - I33 \]

SOLVING SIMULTANEOUS LINEAR EQUATIONS (ZDERY=RHS) FOR DERY

JCNT = 0
DO 25 J = 1, NEQS
DO 25 I = 1, J
JCNT = JCNT + 1
25 ZVEC(JCNT) = Z(I, J)
IJOB = 0
CALL LEQ2S(ZVEC, NEQS, RHS, 1, 22, IJOB, WK, WKAREA, IER)
IF (IER.EQ.130) WRITE (6, 1)
1 FORMAT (1X, 'MATRIX Z IS ILL-CONDITIONED')
IF (IER.EQ.129) WRITE (6, 2)
2 FORMAT (1X, 'Z IS SINGULAR')
DO 30 I = 1, NEQS
30 DERY(I) = RHS(I)
999 RETURN

SUBROUTINE RTPYX(ISPRCL, XRECT, VSPRCL, VRECT)
IMPLICIT REAL*8 (A-H, O-Z)
DIMENSION ISPRCL(1), XRECT(1), VSPRCL(1), VRECT(1)
CSTH = COS (ISPRCL(2))
SNTH = SIN (ISPRCL(2))
CSPHI = COS (ISPRCL(3))
SNPHI = SIN (ISPRCL(3))
V11 = CSPHI * CSTH
V12 = CSPHI * SNTH
V13 = SNPHI
V21 = -SNTH
V22 = -CSTH
V23 = 0.0
V31 = -SNPHI * CSTH
V32 = -SNPHI * SNTH
V33 = CSPHI
XRECT(1) = ISPRCL(1) * V11
XRECT(2) = ISPRCL(1) * V12
XRECT(3) = ISPRCL(1) * V13
VTH = VSPRCL(1)
VTH = VSPRCL(1) * CSPHI

RETURN
END
TPHI = XSPHC1 (1) « VSPRCL (3)  
VRECT (1) = V11 = V21 = V31 = VPHI  
VRECT (2) = V12 = V22 = V32 = VPHI  
VRECT (3) = V13 = V23 = V33 = VPHI  
RETURN  
END  
SUBROUTINE XYZRTF (XSPRCL, XRECT, VSPRCL, VRECT)  
IMPLICIT REAL = 8 (A-H, O-Z)  
DIMENSION XSPRCL (1), XRECT (1), VSPRCL (1), VRECT (1)  
XSPRCL (1) = XRECT (1) « 2 + XRECT (2) 2 + XRECT (3) 2 = 0.50  
XSPRCL (2) = DATA2 (XRECT (2), XRECT (1))  
HARG = (XRECT (1) 2 + XRECT (2) 2) 0.50  
XSPRCL (3) = DATA2 (XRECT (3), HARG)  
CSTH = DCOS (XSPRCL (2))  
SNTH = DSIN (XSPRCL (2))  
CSPHI = DCOS (XSPRCL (3))  
SPHII = DSIN (XSPRCL (3))  
V11 = CSPHI 2 CSTH  
V13 = CSPHI 2 SNTH  
V21 = SNTH  
V22 = -CSTH  
V23 = 0.00  
V31 = -SNTH 2 CSTH  
V32 = -SNTH 2 SNTH  
V33 = CSPHI  
VSPRCL (1) = V11 2 XRECT (1) + V12 2 XRECT (2) + V13 2 XRECT (3)  
VSPRCL (2) = -(V12 2 XRECT (1) + V22 2 XRECT (2) + V23 2 XRECT (3)) / (XSPRCL (1) 2  
VSPRCL (3) = (V31 2 XRECT (1) + V32 2 XRECT (2) + V33 2 XRECT (3)) / XSPRCL (1)  
RETURN  
END  
SUBROUTINE DCRVE (N, FPX0, FPXN, X, P, B, C, D)  
IMPLICIT REAL = 8 (A-H, O-Z)  
DIMENSION P (1), X (1), B (1), C (1), D (1), ALPHA (60), EL (60), U (60), Z (60)  
NM = N - 1  
ALPHA (1) = 3.00 * (P (2) - P (1)) / (X (2) - X (1)) - 3.00  
ALPHA (N) = 3.00 * FPXN - 3.00 * (P (N) - P (N - 1)) / (X (N) - X (N - 1))  
DO 40 I = 2, NM  
ALPHA (I) = F (I) = (X (I) - X (I - 1)) - P (I) = (X (I + 1) - X (I - 1))  
D (I) = (X (I + 1) - X (I)) / (Z (I) - X (I - 1))  
U (I) = Z (I) = ALPHA (I) / D (I)  
DO 50 I = 2, NM  
U (I) = Z (I - 1) - X (I - 1) / Z (I - 1)  
DO 50 I = 2, NM  
Z (I) = (ALPHA (I) - (X (I) - X (I - 1)) 2 2 (I - 1)) / Z (I - 1) / Z (I)  
40 CONTINUE  
50 CONTINUE
\[
EL(N) = X(N) - X(N-1) = (2 \cdot D0 - 0) \cdot (N-1)
\]
\[
Z(N) = (\text{ALPHA}(N) - (X(N) - X(N-1)) \cdot Z(N-1)) / EL(N)
\]
\[
C(N) = Z(N)
\]
\[
J = N - 1
\]

60  C(J) = Z(J) - O(J) = C(J+1)

B(J) = \left( P(J+1) - P(J) \right) / \left( X(J+1) - X(J) \right) = (C(J+1) + 2 \cdot D0 \cdot C(J)) / 13.00

D(J) = (C(J+1) - C(J)) / (3.00 \cdot (X(J+1) - X(J)))

IF (J.GT.0) GOTO 60

RETURN

END

SUBROUTINE DCUBIC (TIME, VALUE, X, P, S, C, D)
IMPLICIT REAL (A-Z)
DIMENSION X(1), F(1), S(1), C(1), D(1)

J=1
10 IF (TIME.LT.X(J+1)) GOTO 20

J=J+1
GOTO 10
20 XIHXJ = TIME - X(J)

XIXJSQ = XIHXJ**2

XIXJCS = XIXJSQ / 3

VALUE = F(J) + B(J) \cdot XIHXJ + C(J) \cdot S(J) + D(J) \cdot XIHXJ2

RETURN

END

SUBROUTINE OUTP (T, I)
IMPLICIT REAL (A-Z)

INTEGER I, J, ICT, ISWCH, NPE, NPEK2, NPTS, ITIME, NSTATE
REAL TVEC, WVEC, W3VEC, BVEC, PRVEC, ENGVEC,

PVEC1, PVEC2, PVEC3, PVEC4, PVEC5, PVEC6,

T1VVEC, T2VVEC, T3VVEC

DIMENSION Y(45), DEBY (45),

XSPNGA(8), XSPNGB(8), DAMPA(8), DAMPB(8), ALPHO(8), BETA0(8),

1ALPH(8), BETA(8), BETADT(8), R1(8), R2(8), R3(8), N(8)

DIMENSION TVEC (9500), WVEC (9500), W3VEC (9500), BVEC (9500),

PRVEC (9500), ENGVEC (9500),

PVEC1 (9500), PVEC2 (9500), PVEC3 (9500), PVEC4 (9500),

PVEC5 (9500), PVEC6 (9500),

T1VVEC (9500), T2VVEC (9500), T3VVEC (9500)

DIMENSION ISPM2 (3), IRECT (3), VSPRC (3), VRECT (3)

COMMON/BLK3/TVEC, WVEC, W3VEC, R2, BVEC, PRVEC, ENGVEC,

PVEC1, PVEC2, PVEC3, PVEC4, PVEC5, PVEC6,

T1VVEC, T2VVEC, T3VVEC

COMMON/BLK4/RF1, RF2, RF3, RF4, R1, R2, R3, RCING, SOUT, NBOUT

COMMON/GC, KC, FCOSB1, FCOSB2, FCOSB3, IJ3IN, NPE, NPEK2, NPTS, ITIME

COMMON/BLK6/RF1, RF2, RF3, RF4, R1, R2, R3, RCING, SOUT, NBOUT, NBOUT, NBOUT, NBINTL

COMMON/GC, KC, FCOSB1, FCOSB2, FCOSB3, IJ3IN, NPE, NPEK2, NPTS, ITIME

COMMON/BLK7/T1, T2, T3, T4, T5, T6, T7, T8, T9, T10

COMMON/BLK8/T1, T2, T3, T4, T5, T6, T7, T8, T9, T10
ICNT=ICNT+1
IRECT (1) = Y (7*NPEN2)
IRECT (2) = Y (8*NPEN2)
IRECT (3) = Y (9*NPEN2)
VRECT (1) = Y (1)
VRECT (2) = Y (2)
VRECT (3) = Y (3)
CALL X2RTP (ISPRC1, IRECT, VSPRCL, VRECT)
HVEC (ICNT) = (ISPRCL (1) - REARTH) / PTHER
THVEC (ICNT) = ISPRCL (2) * DGPRD
PHVEC (ICNT) = ISPRCL (3) * DGPRD
W1 = Y (4)
W2 = Y (5)
W3 = Y (6)
W1VEC (ICNT) = W1 * DGPRD
W2VEC (ICNT) = W2 * DGPRD
W3VEC (ICNT) = W3 * DGPRD
TVEC (ICNT) = T
I1DT = Y (1)
I2DT = Y (2)
I3DT = Y (3)
W1 = Y (4)
W2 = Y (5)
W3 = Y (6)
I1 = Y (7*NPENT2)
I2 = Y (8*NPENT2)
I3 = Y (9*NPENT2)

RIGID BODY ROTATIONAL KINETIC ENERGY

QUANTITIES RELATING MAIN BODY KINEMATICS

ISWITCH = 1
NSTATE = 13 + 4*NPEN
CALL FCYT (NSTATE, T, DT, DERY)
ISWITCH = 0
W1W2 = W1 * W2
W1W3 = W1 * W3
W2W3 = W2 * W3
W1SQR = W1 ** 2
W2SQR = W2 ** 2
W3SQR = W3 ** 2
ENGRGB = 0.5 * (I11 = W1SQR + 2.0 * I12 = W1W2 + 2.0 * I13 = W1W3 + I22 = W2SQR + I33 = W3SQR)

PENDULUM KINETIC ENERGY

ENPEN = 0.5
DO 10 I = 1, NPEN
K = 2 * (I - 1)
BETA(I) = Y (14 * NPEMT2 + K)
BETADT(I) = Y (7 + K)
ALPH(I) = Y (15 * NPEMT2 + K)

10 ALPHDT(I) = Y (8 + K)

SUMMATION OF PENDULUM CONTRIBUTIONS TO KINETIC ENERGY

DO 15 I = 1, NPEMT
CSALPH = DCOS(ALPH(I))
SNALPH = DSIN(ALPH(I))
CSBET = DCOS(BETA(I))
SNBET = DSIN(BETA(I))

T11 = CSALPH
T12 = SNALPH
T21 = -SNALPH * CSBET
T22 = CSALPH * CSBET

CALCULATING PENDULUM POSITION AND LENGTH IN 3 BASIS

L1 = L(I) * T11
L2 = L(I) * T12
L3 = L(I) * T21

R1L1 = R1(I) * L1
R2L2 = R2(I) * L2
R3L3 = R3(I) * L3

VHSQRT = (R3L3 * R3L3 - R2L2 * R2L2 - R1L1 * R1L1)**2

VHSQR = VHSQRT + (R1L1 * L3 - R3L3 * L1)**2

BHGPE = BHGPE + 0.5 * D0 * H(I) * VHSQR

IS COHTIHOE

E5G7BC(ICT) = BHGRGB + BHGPE

CONTINUE

ENGVEC(ICT) = ENGREG + ENGPE

STORING PENDULUM POSITIONS AND EULER ANGLES

PYECS1(ICT) = Y (14 * NPEMT2) * DGPRD
PYECA1(ICT) = Y (15 * NPEMT2) * DGPRD
PYECS2(ICT) = Y (16 * NPEMT2) * DGPRD
PYECA2(ICT) = Y (17 * NPEMT2) * DGPRD
PYECS3(ICT) = Y (18 * NPEMT2) * DGPRD
PYECA3(ICT) = Y (19 * NPEMT2) * DGPRD
PYECS4(ICT) = Y (20 * NPEMT2) * DGPRD
PYECA4(ICT) = Y (21 * NPEMT2) * DGPRD

E1 = Y (10 * NPEMT2)
E2 = Y (11 * NPEMT2)
E3 = Y (12 * NPEMT2)
SUBROUTINE EULER(TH1, TH2, TH3, E1, E2, E3, P4)

IMPLICIT REAL*8 (A-H, O-Z)

CS1DV2 = DCOS (TH1/2.0D0)
CS2DV2 = DCOS (TH2/2.0D0)
CS3DV2 = DCOS (TH3/2.0D0)
SN1DV2 = DSIN (TH1/2.0D0)
SN2DV2 = DSIN (TH2/2.0D0)
SN3DV2 = DSIN (TH3/2.0D0)
E1 = CS1DV2 * SN2DV2 * CS3DV2 - SN1DV2 * CS2DV2 * CS3DV2
E2 = CS1DV2 * CS2DV2 * SN3DV2 - SN1DV2 * SN2DV2 * CS3DV2
E3 = SN1DV2 * CS2DV2 * CS3DV2 + CS1DV2 * SN2DV2 * SN3DV2
E4 = CS1DV2 * CS2DV2 * CS3DV2 - SN1DV2 * SN2DV2 * SN3DV2

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APPENDIX C.

PROGRAM OF THE LINEARIZATION
OF THE EQUATIONS OF MOTION
SUBROUTINE LINEARIZ (YROM, YREF, TIME)
IMPLICIT REAL=8 (A-H,O-Z)
REAL*8 M, L, I11, I12, I13, I22, I23, I33, MPEND, KSPNGB, KSPNGB, KSPNGA,
     I11, I12, I13, LCSTB, KAOCB, KBOCB
REAL*8 LCB, LSB, LCSB, LSCB, LACB, LSCB
DIMENSION P (3,15,15), R(15,15), P1U(15,15), T1J(15,15), UJ(15,15),
     T(15,15), P1U(15,11), BCOM(3,2), YROM(15), YREF(15,15),
     R1(2), R2(2), R3(2), L(2), M(2), KSPNGA(2),
     KSPNGB(2), DAPA(2), DAME(2), ALPH(2), BETA(2), ALPHDT(2), BETADT(2)
COMMON/BLK1/ R1, R2, R3, I11, I12, I13, I22, I23, I33, R2THST, R5THST, IFINAL
COMMON/BLK3/ MPED, MPENT2, MPENT4, MSTATE, MPENL, IFLAG, IINC
COMMON/BLK6/ TH1NOM, TH2NOM, TH3NOM, TH1REF, TH2REF, TH3REF
COMMON/BLK9/ IJ, OJ.
DO 50 1=1,KEGS
DO 50 J=1,KECS
CE 2 (I,J)=O.DO
50 CONTINUE
H1=TH0H (1)
B2=T1UOJ(2)
B3=TNOPR (3)
DO 55 I=1,KPEN
KRT=2*2*I
KPS=KRT+MPENT2
BETA (I)=YROM (KPS)
ALPH(I)=YROM(KPS+1)
BETADT(I)=YROM(KRT)
ALPHDT(I)=YROM(KRT+1)
55 CONTINUE
QUANTITIES RELATING MAIN BODY DYNAMICS
W2SQR=W2**2
SUMMATION OF PENDULUM CONTRIBUTIONS TO EQUATIONS OF MOTION
DO 60 I=1,MPEN
CSALPH=DCOS(ALPH(I))
SNALPH=DSIN(ALPH(I))
CSBETAT=DCOS(BETA(I))
SNBETAT=DSIN(BETA(I))
T11=CSALPH
194
Z(KPLUS1,XPLUS1)=—F(I)>({0(22,I)=2+0(23,I)=2})

LINEARIZATION OF RIGHT HAND SIDES OF THE EQUATIONS OF MOTION

ISTB=K+MPERTZ
ISTA=ISTF+1
ISTB0T=K
ISTADT=ISTBDET+1

COEFFICIENTS FOR THE RIGHT HAND SIDES OF THE LINEARIZED EQUATIONS OF MOTION

LCB=L(I) CSBETA
LSB=L(I) SNBETA
LSASB=L(I) CSALPH CSBETA
LCAB=L(I) CSALPH SSBETA

PO(10,ISTB)=0.00
PO(10,ISTA)=0.00
PO(11,ISTB)=—LCB
PO(11,ISTA)=0.00
PO(12,ISTB)=—LCASB
PO(12,ISTA)=0.00
PO(13,ISTB)=—LSAB
PO(13,ISTA)=0.00
PO(14,ISTB)=0.00
PO(14,ISTA)=0.00
PO(15,ISTB)=—LSASB
PO(15,ISTA)=0.00
PO(16,ISTB)=—LCACB
PO(16,ISTA)=0.00
PO(17,ISTB)=—LSACB
PO(17,ISTA)=0.00
PO(18,ISTB)=0.00
PO(18,ISTA)=0.00
PO(19,ISTB)=—LCACB
PO(19,ISTA)=0.00
PO(20,ISTB)=—LSASB
PO(20,ISTA)=0.00
PO(21,ISTB)=—LSB
PO(21,ISTA)=0.00
PO(22,ISTB)=—LCASB
PO(22,ISTA)=0.00
PO(23,ISTB)=—LSASB
PO(23,ISTA)=0.00
PO(24,ISTB)=0.00
PO(24,ISTA)=0.00
SCON(1,I)=—R1L^25QR
SCON(2,1)=0.00
BCON(3,1) = -E3L3*W2SQR
PB(1,1,1) = R2L2*W2
PB(1,2,1) = -2.00*R1L1*W2
PB(1,3,1) = 0.00
PB(1,ISTBDT,1) = 2.00*L(I)*CSBETA*W2
PB(1,ISTADT,1) = 0.00
PB(1,ISTP,1) = (L(I)*CSALPH*CSBETA)*W2SQR
PB(1,ISTA,1) = (L(I)*CSALPH*CSBETA)*W2SQR
PB(3,1,1) = R1L1*W2
PB(3,2,1) = 0.00
PB(3,3,1) = R2L2*W2
PB(3,ISTBDT,1) = 0.00
PB(3,ISTADT,1) = 0.00
PB(3,ISTA,1) = 0.00
PB(2,1,1) = R3L3*W2
PB(2,2,1) = 0.00
PB(2,3,1) = R3L3*W2
PB(2,ISTBDT,1) = 0.00
PB(2,ISTADT,1) = 0.00
PB(2,ISTP,1) = 0.00
PB(2,ISTA,1) = 0.00
PB(1,1,1) = R1L1*W2
PB(1,2,1) = 0.00
PB(1,3,1) = 0.00
PB(1,ISTBDT,1) = 0.00
PB(1,ISTADT,1) = 0.00
PB(1,ISTA,1) = 0.00
PB(2,1,1) = R1L1*W2
PB(2,2,1) = 0.00
PB(2,3,1) = R1L1*W2
PB(2,ISTBDT,1) = 0.00
PB(2,ISTADT,1) = 0.00
PB(2,ISTP,1) = 0.00
PB(2,ISTA,1) = 0.00
PB(3,1,1) = 0.00
PB(3,2,1) = R3L3*W2
PB(3,3,1) = R3L3*W2
PB(3,ISTBDT,1) = -2.00*L3*T12*W2
PB(3,ISTADT,1) = -2.00*L2*W2
PB(3,ISTA,1) = R3L3*W2SQR
PB(3,ISTA,1) = 0.00

60 CONTINUE

RIGID BODY INERTIA TERMS FOR LEFT SIDE OF THE EQUATIONS OF

INITIALIZING LINEARIZED EQUATION MATRICES

DO 65 I=1,NSTATE
DO 65 J=1,NSTATE
TJ(1,J) = 0.00
UJ(1,J) = 0.00
V(J) = 0.00

65 CONTINUE

ASSEMBLING PENDULUM CONTRIBUTIONS TO BODY FIXED RATE TERMS FOR

ASSEMBLING PENDULUM CONTRIBUTIONS TO BODY FIXED RATE TERMS FOR

LINEARIZATION OF THE RIGHT HAND SIDES OF THE EQUATIONS
OF MOTION

LINEARIZATION OF THE RIGHT HAND SIDES OF THE EQUATIONS
OF MOTION

DO 5 IRW=1,5
IEQ=IRW
MCNT=10*3*(IEQ-1)
DO 10 JCOL=1,3
TOTAL2=0.00
DO 15 I=1,NPEN
TOTAL1=0.0
KCNT=MCNT-1
DO 20 K=1,3
KCNT=KCNT+1
SUM=U(KCNT,I)*PB(K,JCOL,I)
TOTAL1=TOTAL1+SUM
20 CONTINUE
IF (IRON.GT.3) GOTO 16
TOTAL2=TOTAL2+B(I)*TOTAL1
GOTO 15
16 IEQU=IRON+2*(I-1)
TOTAL2=B(I)*TOTAL1
H(IEQU,JCOL)=TOTAL2
15 CONTINUE
H(IEQU,JCOL)=TOTAL2
10 CONTINUE
5 CONTINUE

ASSEMBLING PENDULUM CONTRIBUTIONS TO ALPH, BETA, ALPHDT, AND
BETADT TERMS FOR LINEARIZATION OF THE RIGHT HAND SIDES OF
THE EQUATIONS OF MOTION

DO 25 IROW=1,5
IEQU=IROW
MCNT=10*3*(IEQU-1)
DO 25 I=1,4
H(IEQU,JCOL)=H(IEQU,JCOL)*PB(I,JCOL)
30 CONTINUE
IF (IRON.GT.3) IEQU=IRON+2*(I-1)
H(IEQU,JCOL)=H(I)*TOTAL
H(IEQU,JCOLDT)=H(I)*TOTALDT
IF (MCNT.LT.19) GOTO 25
IF (MCNT.EQ.19) GOTO 21
H(IEQU,JCOL)=H(IEQU,JCOL)*KSPNGA(I)*G
H(IEQU,JCOLDT)=H(IEQU,JCOLDT)*TPA(I)*G
GOTO 25
21 H(IEQU,JCOL)=H(IEQU,JCOL)*KSPRGA(I)*G
H(IEQU,JCOLDT)=H(IEQU,JCOLDT)+DAMPB(I)*G
25 CONTINUE
ASSEMBLING RIGID BODY TERMS TO THE RIGHT SIDES OF THE LINEARIZED EQUATIONS OF MOTION

\[
H(1,3) = H(1,3) - (I22 - I33) = w2 \\
H(3,1) = H(3,1) - (I11 - I22) = w2
\]

COMPUTING LINEARIZED CONTROL MATRIX

\[
DO 45 I = 1, \text{NSTATE} \\
DO 45 J = 1, \text{NCTRL} \\
PFU(I, J) = 0.0 \\
45 \text{CONTINUE}
\]

\[
PFU(1,1) = -R2THST \\
PFU(1,2) = -R2THST \\
PFU(3,3) = R2THST \\
PFU(1,4) = R2THST \\
PFU(2,5) = -R5THST
\]

ASSEMBLING LINEARIZED EQUATIONS

\[
DO 70 I = 1, \text{NEQS} \\
DO 70 J = I, \text{NEQS} \\
Z(J, I) = Z(I, J) \\
70 \text{CONTINUE}
\]

\[
\text{CALL HATIST}(2, \text{NEQS}, 0, \text{ET}, I, \text{EERR}) \\
\text{IF}(\text{IERR} \neq 1) \text{B/tag}(6,3) \\
3 \text{FORMAT}(I, 'Z IS SINGULAR') \\
\text{NEUL} = 3 + \text{NPEMT} \\
\text{CALL MATHUL}(2, H, 3, \text{NEQ}, \text{NEQ}, \text{NEUL}) \\
\text{CALL MATHUL}(2, \text{PFU}, \text{UJ}, \text{NEQ}, \text{NEQ}, \text{NCTRL}) \\
\text{NEQSP1} = \text{NEQS} + 1 \\
DO 75 I = \text{NEQSP1}, \text{NEUL} \\
\text{YJ}(I, I - \text{NPEMTZ}) = 1.0 \\
75 \text{CONTINUE}
\]

\[
\text{WRITE}(6, 31) \\
\text{FORMAT}(I, 'TN0H IS') \\
\text{WRITE}(6, 32) (\text{TN0H}(I), I = 1, \text{NSTATE}) \\
\text{WRITE}(5, 33) \\
\text{FORMAT}(I, 'TREF IS') \\
\text{WRITE}(6, 32) (\text{TREF}(I, I), I = 1, \text{NSTATE}) \\
\text{FORMAT}(8F9.3)
\]

14 \text{CONTINUE}

COMPUTING EULER PARAMETER REFERENCE STATE

\[
\text{TH3NOM} = \text{TN0N}(2) = \text{TIME} \\
\text{TH3REF} = \text{TH3NOM} \\
\text{CALL EULER(TH1REF, TH2REF, TH3REF, E1REF, E2REF, E3REF, E4REF)} \\
\text{CALL EULER(TH1NOM, TH2NOM, TH3NOM, E1NOM, E2NOM, E3NOM, E4NOM)}
\]
ASSEMBLING JACOBIAN MATRIX FOR EULER RATE EQUATIONS

IEQU=IE3Q+1
TJ (IE30,1) = E3NOM/2.D0
TJ (IE30,2) = E2NOM/2.D0
TJ (IE30,3) = E1NOM/2.D0
TJ (IE30,5*NPENTQ) = W3/2.D0
TJ (IE30,6*NPENTQ) = W2/2.D0
TJ (IE30,7*NPENTQ) = W1/2.D0
IE3Q=IE3Q+1
TJ (IE3Q,1) = E3NOM/2.D0
TJ (IE3Q,2) = E2NOM/2.D0
TJ (IE3Q,3) = E1NOM/2.D0
TJ (IE3Q,5*NPENTQ) = W3/2.D0
TJ (IE3Q,6*NPENTQ) = W2/2.D0
TJ (IE3Q,7*NPENTQ) = W1/2.D0
IFLAG=1
RETURN
END
APPENDIX D.

PROGRAM OF THE FEEDBACK CONTROL LAW COMPUTATION
THIS SUBROUTINE COMPUTES THE FEEDBACK GAIN MATRIX AND COMMAND CONTROL VECTOR FOR USE AS A CONTROL LAW. THE MATRIX RICCATI EQUATION AND AUXILIARY EQUATION ARE SOLVED BY INTEGRATING BACKWARD IN TIME AND STORING THE FEEDBACK GAIN MATRIX AND COMMAND CONTROL VECTOR AT EACH TIME STEP.

SUBROUTINE CONTRLC(FOPT,VOPT,TSTEP,TOL,TSTEP,TINITL,FINISH,IMETH,MITER,ISTEP,NSTATE,NCNTRL,ROU,Q,YREF,INOM)

INITIALIZING RICCATI SOLUTION (P AND S)

ISTEP=0
ICNT=0
DO 5 I=1,NSTATE
DO 5 J=1,NSTATE
ICNT=ICNT+1
TP (ICNT)=Q(I,J)
CONTINUE
CALL MATKCL(Q,TREF,S,NSTATE,NSTATE,1)
CALL MATSCA(S,S,-1.0D0,NSTATE,1)
DO 10 I=1,NSTATE
ICNT=ICNT+1
TP (ICNT)=-S(I,1)
CONTINUE
RICCATI EQUATION SOLUTION

CALL OOTSIC (FINISH,TP,P,S,TSTEP,ISTEP,NSTATE,NCNTRL)

OPTIMAL CONTROL LAW CALCULATION

CALL LINRIZ (INOM,TREF,TIME)
CALL MATTM (UJ,UJTRN,NSTATE,NCNTRL)

DIMENSION P (15,15) ,S (15,15) ,Q(15,15),R2U (15,15),YREF(15,15),
T(15,15),OJ (15,15),UJTRN(15,15),R2UJTN (15,15),FGAIN(15,15),
2OFFSET (15,15),FOPT (5,15,500),VOPT (6,500),TSTEP (500)
DIMENSION TP (135),TOM (15),IMK (135),WK (2296),S (135,9),C(24)
COMMON/BLK9/TJ,OJ

IMPLICIT REAL38(A-H,0-2)
EXTERNAL HICF0N,2ICJ
DIMENSION P (15,15) ,S (15,15) ,Q(15,15),R2U (15,15),YREF(15,15),
T(15,15),OJ (15,15),UJTRN(15,15),R2UJTN (15,15),FGAIN(15,15),
2OFFSET (15,15),FOPT (5,15,500),VOPT (6,500),TSTEP (500)
DIMENSION TP (135),TOM (15),IMK (135),WK (2296),S (135,9),C(24)
COMMON/BLK9/TJ,OJ

INITIALIZING RICCATI SOLUTION (P AND S)

ISTEP=0
ICNT=0
DO 5 I=1,NSTATE
DO 5 J=1,NSTATE
ICNT=ICNT+1
TP (ICNT)=Q(I,J)
CONTINUE
CALL MATKCL(Q,TREF,S,NSTATE,NSTATE,1)
CALL MATSCA(S,S,-1.0D0,NSTATE,1)
DO 10 I=1,NSTATE
ICNT=ICNT+1
TP (ICNT)=-S(I,1)
CONTINUE
RICCATI EQUATION SOLUTION

CALL OOTSIC (FINISH,TP,P,S,TSTEP,ISTEP,NSTATE,NCNTRL)

OPTIMAL CONTROL LAW CALCULATION

CALL LINRIZ (INOM,TREF,TIME)
CALL MATTM (UJ,UJTRN,NSTATE,NCNTRL)
CALL MATMUL (R2U, UJTRN, R2UJTN, NCNTL, NCNTL, NSTATE)
CALL MATMUL (R2UJTN, P, FGAIN, NCNTL, NSTATE, NSTATE)
CALL MATMUL (R2UJTN, S, OFFSET, NCNTL, NSTATE, 1)
DO 15 I = 1, NCNTL
VOFT (I, ISTEP) = OFFSET (I, 1)
DO 15 J = 1, NSTATE
VOFT (I, J, ISTEP) = FGAIN (I, J)
15 CONTINUE
IF (TEND .LE. TINIT) GOTO 999
TEND = TIME - TSTEP
GOTO 21
999 RETURN
END

SUBROUTINE KICFUN
IMPLICIT REAL*A-Z
DIMENSION P (15, 15), O (15, 15), R(2(15, 15), A (15, 15), ATRN (15, 15),
1ATNP (15, 15), B (15, 15), BTRN (15, 15), TREF (15, 15), S (15, 15),
2R2 (15, 15), BR2BT (15, 15), PB2BT (15, 15), PB2BTP (15, 15),
3PA (15, 15), QR (15, 15), PDOT (15, 15), SDOT (15, 15), TJ (15, 15), GJ (15, 15)
DIMENSION YNOM (15), TP (135), DNP (135)
EQUIVALENCE (TJ (15, 15), A (15, 15)), (GJ (15, 15), S (15, 15))
COMMON/BLK3/YNOM, NCNTL
COMMON/BLK9/TJ, GJ

LINEARIZING PLANT EQUATIONS
NCNTL = NCNTL
NSTATE = NSTATE
CALL LINRIZ (YNOM, TREF, TIME)
ICMT = 0
DO 5 I = 1, NSTATE
DO 5 J = 1, NSTATE
ICMT = ICMT + 1
P (I, J) = TP (ICMT)
P (J, I) = P (I, J)
5 CONTINUE
DO 10 I = 1, NSTATE
ICMT = ICMT + 1
S (I, J) = TP (ICMT)
10 CONTINUE

COMPUTING PDOT
CALL MATMUL (B, BTRN, NSTATE, NCNTL)
CALL MATMUL (B, B2R, B2R, NSTATE, NCNTL, NCNTL)
CALL MATMUL (B2R, BTRN, B2BTM, NSTATE, NCNTL, NSTATE)
CALL MATMUL (P, B2BTP, B2BTP, NSTATE, NSTATE, NSTATE)
CALL MATMUL (B2BTM, P, B2BTP, NSTATE, NSTATE, NSTATE)
CALL MATMUL (P, PA, NSTATE, NSTATE, NSTATE)
CALL HATLIB (A,ATRN,NSTATE,NSTATE)
CALL HATTUL (ATRN,P,ATRP,NSTATE,NSTATE,NSTATE)
CALL MATSUB (PBZTP,PA,PDOT,NSTATE,NSTATE)
CALL MATSUB (PDOT,ATRN,PDOT,NSTATE,NSTATE)
CALL MATSUB (PDOT,Q,PDOT,NSTATE,NSTATE)

CALL HATRUL (P,THET,QR,NSTATE,NSTATE)
CALL HATRUL (Q,THET,QR,NSTATE,NSTATE)

ASSEMBLING DERP

ICNT=0
DO 15 I=1,NSTATE
   DO 15 J=1,NSTATE
      ICNT=ICNT+1
      DERP (ICNT)=PDOT (I,J)
   15 CONTINUE
DO 20 I=1,NSTATE
   DO 20 J=1,NSTATE
      ICNT=ICNT+1
      DERP (ICNT)=SDOT (I,J)
   20 CONTINUE

RETURN
END SUBROUTINE RICJ (NSTATE,TIME,Y,P)
IMPLICIT REAL*8 (A-H,O-Z)
END

SUBROUTINE OTRIC (TIME,YP,P,S,TSTEP,ISTEP,NSTATE,NCNTRL)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION P (135),S (15,15),TSTEP (500)
ISTEP=ISTEP+1
ICNT=0
DO 5 I=1,NSTATE
   DO 5 J=1,NSTATE
      ICNT=ICNT+1
      P (I,J)=YP (ICNT)
   5 CONTINUE
   S (I,1)=YP (ICNT)
   ICNT=ICNT+1
DO 10 I=1,NSTATE
   DO 10 J=1,NSTATE
      S (I,J)=YP (ICNT)
   10 CONTINUE
TSTEP (ISTEP)=TIME
RETURN
END
MATMUL multiplies matrices of compatible dimensions
ARRATA = left array to be multiplied
ARRATB = right array to be multiplied
ARRATC = product of left and right arrays
N = no. of rows of ARRATA
M = no. of columns of ARRATA
NC = no. of columns of ARRATB

SUBROUTINE MATMUL(ARRATA,ARRATB,ARRATC,N,M,NC)
 IMPLICIT REAL*8(A-H,O-Z)
 DIMENSION ARRATA(15,15),ARRATB(15,15),ARRATC(15,15)
 DO 50 I=1,N
 DO 50 J=1,NC
 ARRATA(I,J) = ARRATB(I,J)
 50 CONTINUE
 DO 55 I=1,N
 DO 55 J=1,M
 ARRATB(I,J) = ARRATA(I,J)
 55 CONTINUE
 DO 60 I=1,N
 DO 60 J=1,M
 SUM = 0.D0
 DO 70 K=1,NC
 SUM = SUM + ARRATA(I,K) * ARRATB(K,J)
 70 CONTINUE
 ARRATC(I,J) = SUM
 60 CONTINUE
 RETURN
 END

MATADD adds matrices of the same dimensions (N=M)

SUBROUTINE MATADD(ARRATA,ARRATB,ARRATC,N,M)
 IMPLICIT REAL*8(A-H,O-Z)
 DIMENSION ARRATA(15,15),ARRATB(15,15),ARRATC(15,15)
 DO 60 I=1,N
 DO 60 J=1,M
 ARRATC(I,J) = ARRATA(I,J) + ARRATB(I,J)
 60 CONTINUE
 RETURN
 END

MATSCL multiplies a matrix by a scalar

SUBROUTINE MATSCA(ARRATA,ARRATC,SCALN,N,M)
 IMPLICIT REAL*8(A-H,O-Z)
 DIMENSION ARRATA(15,15),ARRATC(15,15)
 DO 60 I=1,N

DO 60 J=1,N
ARRAYC(I,J) = SCALAR ARRAYA(I,J)
60 CONTINUE
RETURN
END

MATTHIHB TRANSPOSES A MATRIX

SUBROUTINE MATTHIHB(ARRAYA,ARRAYC,N,N)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION ARRAYA(15,15),ARRAYC(15,15)
DO 60 I=1,N
DO 60 J=1,N
ARRAYC(J,I) = ARRAYA(I,J)
60 CONTINUE
RETURN
END

MATSUB SUBTRACTS MATRICES C=A-B

SUBROUTINE MATSUB(ARRAYA,ARRAYB,ARRAYC,N,N)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION ARRAYA(15,15),ARRAYB(15,15),ARRAYC(15,15)
DO 60 I=1,N
DO 60 J=1,N
ARRAYC(I,J) = ARRAYA(I,J) - ARRAYB(I,J)
60 CONTINUE
RETURN
END

THE PURPOSE OF THIS SUBROUTINE IS TO INVERT AN N X N GENERAL MATRIX USING A GAUSS-JORDAN

SUBROUTINE MATINV(A,N,DET,IERR)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION A(15,15),L(15),R(15)
IERR=0
S=1.0D-7
IF (N.LE.0) IERR=1
IF (S.LE.0) GO TO 100

SCALAR INVERSION

IF (A.GT.1) GO TO 60
DET = A(1,1)
IF (DABS(DET),LT. 5) IERR=1
IF (IERR.EQ.1) GO TO 100
DET=1.0D0/DET
A(1,1) = DET
GO TO 100
60 DET=1.0D0
SEARCH FOR LARGEST PIVOT

DO 5 I=1,N
L(K)= K
M(K)= K
BIGA=A(K,K)
DO 10 I=K,N
DO 10 J=K,N
IF(DABS(BIGA).GE.DABS(A(I,J))) GO TO 10
BIGA= A(I,J)
L(K)= I
M(K)= J
10 CONTINUE

INTERCHANGING ROWS

DO 15 I=1,N
HOLD= -A(K,I)
A(K,I)= A(J,I)
A(J,I)= HOLD
15 CONTINUE

INTERCHANGING COLUMNS

DO 20 J=1,N
HOLD= -A(J,K)
A(J,K)= A(J,I)
A(J,I)= HOLD
20 CONTINUE

DIVIDING COLUMNS BY MINUS PIVOT

DO 25 I=1,N
IF(I.NE.K) A(I,K)=A(I,K)/(-A(K,K))
25 CONTINUE

DO 30 I=1,N
IF(I.EQ.K) GO TO 30
DO 35 J=1,N
IF(J.NE.K) A(I,J)=A(I,K)*A(K,J)
35 CONTINUE
75  DO 40 J=1,N
   IF(J.LE.K) A(K,J) = A(K,J)/A(K,K)
 40  CONTINUE
CC  COMPUTING DETERMINANT
   DET= DET* A(K,K)
   IF (DABS(DET) .LE. 5) IERR = 1
   IF (DABS(DET) .LE. 5) GO TO 100
CC  REPLACING PIVOT BY RECIPROCAL
   A(K,K) =1. DO/1(A(K,K)
   CONTINUE
CC  FINAL ROW AND COLUMN INTERCHANGE
   K=N
   K=K-1
   IF (K.LE.0) GO TO 100
   I=L(K)
   DO 45 J=1,N
      HOLD= A(J,K)
      A(J,K) = A(J,I)
      A(J,I) = HOLD
   45  CONTINUE
   J= K(K)
   IF (J.LE.K) GO TO 130
   DO 50 I=1,N
      HOLD= A(K,I)
      A(K,I) = -A(J,I)
      A(J,I) = HOLD
   50  CONTINUE
   GO TO 120
130  GO TO 120
100  RETURN
END
APPENDIX E.

PROGRAM OF THE NUMERICAL SIMULATION OF THE CONTROLLED SYSTEM
MAIN PROGRAM

THIS PROGRAM CALCULATES THE FEEDBACK GAINS AND OPTIMAL CONTROL
LAW FOR THE LINEAR TRACKING PROBLEM USING THUSTER CONTROL
OF THE PLANT AND THEN SOLVES THE NONLINEAR SYSTEM OF EQUATIONS
GOVERNING THE PLANT USING THE LINEARIZED CONTROL LAW.

IMPLICIT REAL*8(A-H,O-Z)
EXTERNAL FCT,FCTJ
REAL*8 R,L,I11,I12,I13,I22,I23,I33,HPEND,KSPNGR,KSPNGB,KSPNGA,
1LPEND
REAL*8 TVEC,W1VEC,W2VEC,W3VEC,T1VEC,T2VEC,T3VEC
DIMENSION Q(15,15),R2O(15,15),FOFT(5,15,500),70FT(5,500),
1TSTEP(500),FSTORE(5,500),T0TIRM(5),TREF(15,15),TMON(15)
DIMENSION R1(2),R2(2),R3(2),L(2),M(2),KSPNGA(2),
2KSPNGB(2),DAMP(2),DAMPB(2),ALPH(2),BETA(2),ALPH(2),BETA(2),
3ALPHT(2),BETAT(2),T(15),DBRT(15),DVB(15),W(256)
DIMENSION TVEC(500),W1VEC(500),W2VEC(500),W3VEC(500),
1TVEC(500),TVEC(500),TVEC(500)
COMMON/BLK1/R1,R2,R3,L,H,KSPNGA,KSPNGB,DAMP,DAMPB,ALPH,BETA,
1ALPH,BETA,ALPH,BETA
COMMON/BLK2/P1,GC,I11,I12,I13,I22,I23,I33,R2THST,R5THST,TFINAL
COMMON/BLK3/NPEN,NPENT2,NPENT3,NSTATE,NEQS,NCNTRL,IFLAG,JINC
COMMON/BLK4/R20,G,TREF,NST,NCNTRL
COMMON/BLK5/TH1NOM,TH2NOM,TH3N0M,TH1REF,TH2REF,TH3REF
COMMON/BLK6/GHPRD,TVEC,W1VEC,W2VEC,W3VEC,T1VEC,T2VEC,T3VEC,
1FSTORE,TSTEP
COMMON/BLK8/TSTEP,FOFT,FOFT

SETTING NOMINAL OPERATING POINT AND PARAMETERS

IFLAG=0
READ(5,1) (ALPH(I),I=1,2)
WRITE(6,1) (ALPH(I),I=1,2)
READ(5,1) (BETA(I),I=1,2)
WRITE(6,1) (BETA(I),I=1,2)
1 FORMAT(8F9.2)
READ(5,2) LPEND,RPEND,PENHT,RPEND,KSPNGR,DAMPCF,PENP,NCNTRL
WRITE(6,2) LPEND,RPEND,PENHT,RPEND,KSPNGR,DAMPCF,PENP,NCNTRL
2 FORMAT(8F9.2)
READ(5,3) W1,W2,W3,TH1NOM,TH2NOM,TH3NOM,WGETQ,WGETR
WRITE(6,3) W1,W2,W3,TH1NOM,TH2NOM,TH3NOM,WGETQ,WGETR
3 FORMAT(BF9.5)
READ(5,4) I11,I12,I13,I22,I23,I33,R2THST,R5THST
WRITE(6,4) I11,I12,I13,I22,I23,I33,R2THST,R5THST
4 FORMAT(BF9.2)
NPENT2=NPEN=2
NPENT3=NPEN=2
IISTATE=7 + !FPENTT
!iegs=3*2#*pe!f
NST=NST*TF
kchtlsircmthl
PI=3.141592653589793
SDPDG =pi
DGPRD=1 .DO/12.DO
FTP19=1.DO/12.DO
GC=32.2D0
R2THST=GC*R2THST*FTPIN
R3THST=GC*R3THST*FTPIN
READ(5,6) TOL,THSTEP,TFINA,T,NET,RITER
WRITE(6,6) TOL,THSTEP,TFINA,T,NET,RITER
6 FORMAT(4F9.5,2I9)
YNOM (1) =1
YNOM (2) =2
YNOM (3) =3

EULER ANGLE TO EULER PARAMETER TRANSFORMATION
CALL EULER (TH1OM, THHOM, THSOM, E1OM, E2OM, E3OM, E4OM)
YNOM (4-NPENT4) =E1OM
YNOM (5-NPENT4) =E2OM
YNOM (6-NPENT4) =E3OM
YNOM (7-NPENT4) =E4OM

PENDULUM INITIAL STATE, LENGTH, AND SPRING PRELOAD
ANG=2.DO*PI/DFLOAT (NPEN)
RTPE=NPEN*FTPIN
PEWHT=PEWHT*FTPIN
DO 5 I=1,NPEN
J=I-1
KRT=2*2*1
KPS=KRT*NPENT2
YNOM (KRT) =0.DO
YNOM (KRT+1) =0.DO
BETAO (I) =BETAO (I) #RDPDG
ALPHO (I) =ALPHO (I) #RDPDG
YNOM (KPS) =BETAO (I)
YNOM (KPS+1) =ALPHO (I)
KSPNGA (I) =KSPNGA (I)
DAMPA (I) =DAMPA (I)
DAMPB (I) =DAMPB (I)
R1 (I) =-RPEW#DCOS (DFLOAT (J) =ANG)
R2 (I) =PEWHT
R3 (I) =RPEW#DSIN (DFLOAT (J) =ANG)
L (I) =RPEW#FTPIN
M (I) =RPEW
5 CONTINUE

COST FUNCTION WEIGHTING MATRICES (Q AND R2U)

NEUL=3*XMPENT4
DO 10 I=1,NSTATE
DO 10 J=1,NSTATE
Q(I,J)=0.DO
IF (I.EQ.J .AND. I.GT.NEUL) Q(I,J)=WGHTQ
10 CONTINUE
Q(1,1)=WGHTQ
Q(3,3)=WGHTQ
DO 15 I=1,NCNTRL
DO 15 J=1,NCNTRL
R2U(I,J)=0.DO
IF (I.EQ.J) R2U(I,J)=WGHTR
15 CONTINUE

INVERTING R2U

CALL MATINV(R2U,NCNTRL,DET,IERR)
IF (IERR.EQ.1) WRITE (6,7)
7 FORMAT (1X,'R2U IS SINGULAR')

SETTING REFERENCE STATE

READ (5,3) W1REF,W2REF,W3REF,T1REF,T2REF,T3REF
WRITE (6,3) W1REF,W2REF,W3REF,T1REF,T2REF,T3REF
TREF(1,1)=W1REF
TREF(2,1)=W2REF
TREF(3,1)=W3REF
NEQSP2=NEQSP+2
READ (5,7) (TREF(I,1),I=NEQSP2,NEUL,2)
WRITE (6,7) (TREF(I,1),I=NEQSP2,NEUL,2)
NEQSP1=NEQSP+1
READ (5,1) (TREF(I,1),I=NEQSP1,NEUL,2)
WRITE (6,1) (TREF(I,1),I=NEQSP1,NEUL,2)
DO 20 I=NEQSP1,NEUL
TREF(I,1)=TREF(I,1)-RDPC
TREF(I,NPENT2,1)=0.DO
20 CONTINUE

CALL EULER(T1REF,T2REF,T3REF,E1REF,E2REF,E3REF,E4REF)
TREF(4+NPENT4,1)=E1REF-E1NON
TREF(5+NPENT4,1)=E2REF-E2NON
TREF(6+NPENT4,1)=E3REF-E3NON
TREF(7+NPENT4,1)=E4REF-E4NON

LINEAR CONTROL LAW CALCULATION

CALL CONTROL(P0FT,V0FT,TSTEP,TOL,TRSTEPS,TINITL,TFINAL,METH,
GOTO 21
23 CONTINUE

TOTAL THRUSTER IMPULSE CALCULATION

DO 50 K=1,NCTRL
  TOTIMP(K)=0.00
50 CONTINUE
DO 55 J=1,JSTEP
  DO 60 K=1,NCTRL
    SUM=TOTIMP+TSTORE(K,J)
    SUM=ABS(SUM)
    TOTIMP(K)=TOTIMP(K)+SUM
60 CONTINUE
55 CONTINUE

FORMAT (1X,'THE TOTAL IMPULSE (LBF-SEC) FOR THRUSTER I IS,')
WRITE (6, 36)
36 FORMAT (1X,' 1 2 3 5 9')
WRITE(6,37) (TOTIMP(I), I=1,NCTRL)
37 FORMAT (5F9.2)

CALL GRAP (JSTEP,NSTATE,NCTRL,TVEC,TVVEC,TVVEC,TVVEC
1TVVEC,TVVEC,TSTORE,FOPT,VOFT)
999 STOP

SUBROUTINE FCT (NST,TIME,N,DEBT)

IMPLICIT REAL*8(A-H,O-Z)
REAL*8 H, L, IT1, IT2, IT3, IT22, IT33, MPEND, KSPNGR, KSPNGB, KSPNGA,
1XAAACB. XBBABC, ITI, IT2, IT3, LCSBTA

DIMENSION R1(2), R2(2), R3(2), L(2), I(2), J(24, 3), B(3, 2), KSPNGA(2),
2KSPNGB(2), DAMPA(2), DAMPB(2), ALPHO(2), BETA0(2), ALPH(2), ETA(2),
3ALPHT(2), BETADT(2), T(15), DEBT(15), Z(15,15), ANS(7), ZVE(28)

DIMENSION INK(14), ILKBE(49)

DIMENSION FOPT(5,15,500), VOFT(5,500), TSTEP(500), YM(15), J(5)
COMMON/BLK2/IT1, IT2, IT3, L, IT33, MPEND, KSPNG, KSPNGB, KSPNGA,
1DAMPA, DAMPB, ALPHO, BETA0, ALPH, ETA, ALPHT, BETADT
COMMON/BLX2/PL, GC, IT1, IT2, IT3, IT22, IT33, RZSTST, RZST, RSZST, RJST
COMMON/BLK2/NPEN, NPENT2, NPENT4, NST, NSTAT, NEQS, NCTRL, IFLAG, JINC
COMMON/BLK5/TMON
COMMON/BLK2/TMON, TH2NOM, TH3NOM, THR3, THR3E, THR3E
COMMON/BLK2/TSTEP, POFT, VOFT
THM=O+TM0N (2)*TIME
CALL GRFR(THMOM, TH2NOM, TH3NOM, TH2N, E1, E2, E3, E4)
TM0N (1+NPENT4) = E1
TM0N (5+NPENT4) = E2
TM0N (6+NPENT4) = E3
TM0N (7+NPENT4) = E4
IF (TIME.LT.RFINAL) GOTO 16
JINC = 2
GOTO 17
16 IF (TIME.LT.TSTEP(JINC-1)) GOTO 17
  JINC=JINC-1
GOTO 16
17 DO 35 I=1,NCNTRL
  F(I)=0.DO
  DO 40 J=1,MAXSTATE
  DELTAY=Y(J)-THON(J)
  F(I)=F(I)+VOFT(I,J,JINC)=DELTAY
40 CONTINUE
  F(I)=F(I)*VOFT(I,JINC)
35 CONTINUE
W1=Y(1)
W2=Y(2)
W3=Y(3)
DO 5 I=1,NEOS
  RMS(I)=0.DO
  DO 5 J=1,NEOS
    Z(I,J)=0.DO
5 CONTINUE
DO 10 I=1,NPEN
  KRT=2*PI
  KPS=KRT*NPENT2
  BETA(I)=Y(KPS)
  ALPH(I)=Y(KPS+1)
  ALPHDT(I)=Y(KRT)
10 CONTINUE
QUANTITIES RELATING MAIN BODY KINEMATICS
W1W2=W1*W2
W1W3=W1*W3
W2W3=W2*W3
W1SQR=W1*W1
W2SQR=W2*W2
W3SQR=W3*W3
D1=W1W2+W12+W23+W33
D2=W1W2+W12+W23+W33
D3=W1W2+W12+W23+W33
SURFACE OF PENDULUM CONTRIBUTIONS TO EQUATIONS OF MOTION
DO 15 I=1,NPEN
  CSALPH=DCOS(ALPH(I))
  SNALPH=DSSIN(ALPH(I))
  CSBETA=DCOS(BETA(I))
  SNBETA=DSSIN(BETA(I))
  T11=CSALPH
  T12=SNALPH
  T21=-SNALPH=CSBETA
\[ T_{22} = \text{CSALPH} \text{CSBETA} \]
\[ T_{23} = \text{SNBETA} \]
\[ T_{12D} = \text{ALPHDT} \text{CSALPH} \]

**Body Fixed Angular Rates (RVM(I))**

\[ W_1 = W_1 + \text{BETADT(I) • T_{11}} \]
\[ W_{21} = W_2 + \text{BETADT(I) • T_{12}} \]
\[ W_3 = W_3 + \text{ALPHDT}(I) \]
\[ W_{12} = W_1 + W_2 \]
\[ W_{13} = W_1 + W_3 \]
\[ W_{23} = W_2 + W_3 \]
\[ W_{1SQR} = W_1 + W_2 \]
\[ W_{2SQR} = W_2 + W_2 \]
\[ W_{3SQR} = W_3 + W_3 \]

**Calculating Pendulum Position and Length in B Basis**

\[ L_1 = L(I) + T_{21} \]
\[ L_2 = L(I) + T_{22} \]
\[ L_3 = L(I) + T_{23} \]
\[ R_{11} = R_1(I) + L_1 \]
\[ R_{21} = R_2(I) + L_2 \]
\[ R_{31} = R_3(I) + L_3 \]

**Calculating Partial Rates of Change of Orientation**

\[ U(10, I) = 0.00 \]
\[ U(11, I) = -R_{31} \]
\[ U(12, I) = R_{31} \]
\[ U(13, I) = R_{31} \]
\[ U(14, I) = 0.00 \]
\[ U(15, I) = -R_{11} \]
\[ U(16, I) = -R_{21} \]
\[ U(17, I) = -R_{11} \]
\[ U(18, I) = 0.00 \]
\[ U(19, I) = L_3 + T_{12} \]
\[ U(20, I) = L_3 + T_{11} \]
\[ U(21, I) = L(I) + \text{CSBETA} \]
\[ U(22, I) = -L_2 \]
\[ U(23, I) = L_1 \]
\[ U(24, I) = 0.00 \]

**Assembling Pendulum Contributions to Equations of Motion**

\[ \Phi(1, I) = \text{L}^3(-\text{ALPHDT}(I) + W_1 + \text{BETADT}(I) + (T_{12DT} + T_{11W} + W_3)) - L_2 * \text{BETADT}(I) = (T_{11W} + W_2 + T_{12W} + W_1) - R_1(I) - W_{2SQR} + W_{3SQR} + W_{1W3} \]
\[ \Phi(3, I) = L_1 + W_{2SQR} - L_1 = (W_{2SQR} + W_{3SQR}) + W_{1W3} \]

\[ S(2, I) = L(I) + \text{BETADT}(I) = (-T_{11W} + W_2 + T_{12W} - L_3) + \text{BETADT}(I) \]
\begin{align*}
1 &= (T_{11} DT-T_{12} W_3) + \text{ALPHDT}(I) \cdot W_2 + W_2 W_3 = R_3 (I) - R_2 (I) = (W_{1SQR+1SQR}) + W_1 W_3 = R_1 (I) - R_3 (I) \\
B (3, I) &= L_2 = (\text{ALPHDT}(I) \cdot W_2 + \text{BETADT}(I) = (T_{11} DT-T_{12} W_3) \\
1 &= \text{ALPHDT}(I) \cdot W_2 - \text{BETADT}(I) \\
1 &= (W_{1SQR+1SQR}) + W_2 W_3 = R_2 (I) + W_1 W_3 = L_1 - L_3 = (W_{1SQR+1SQR}) + W_2 W_3 = L_2 \\
k &= 2 + 2 \phi \\
K_{PLUS} &= K + 1 \\
Z (1, 1) &= Z (1, 1) - M (I) = (0 (11, I) \cdot 2 + 0 (12, I) \cdot 2) \\
Z (1, 2) &= Z (1, 2) - M (I) = (0 (12, I) \cdot 0 (15, I)) \\
Z (1, 3) &= Z (1, 3) - M (I) = (0 (11, I) \cdot 0 (17, I)) \\
Z (1, k) &= Z (I) = 0 (11, I) \cdot 0 (21, I)) \\
Z (1, KPLUS) &= Z (I) = 0 (11, I) \cdot 0 (23, I)) \\
RHS (1) &= RHS (1) - M (I) = (0 (11, I) \cdot 0 (15, I) \cdot 0 (21, I) \cdot 0 (3, I)) \\
Z (2, 2) &= Z (2, 2) - M (I) = (0 (13, I) \cdot 0 (15, I) \cdot 0 (22, I) \cdot 0 (3, I)) \\
Z (2, 3) &= Z (2, 3) - M (I) = (0 (13, I) \cdot 0 (16, I) \cdot 0 (21, I)) \\
Z (2, k) &= Z (I) = 0 (13, I) \cdot 0 (19, I) \cdot 0 (15, I) \cdot 0 (21, I)) \\
Z (2, KPLUS) &= Z (I) = 0 (13, I) \cdot 0 (22, I) \cdot 0 (22, I)) \\
RHS (2) &= RHS (2) - M (I) = (0 (13, I) \cdot 0 (15, I) \cdot 0 (21, I) \cdot 0 (3, I)) \\
Z (3, 3) &= Z (3, 3) - M (I) = (0 (16, I) \cdot 2 + 0 (17, I) \cdot 2) \\
Z (3, K) &= Z (I) = 0 (16, I) \cdot 0 (20, I) \cdot 0 (17, I) \cdot 0 (20, I) \cdot 0 (21, I) \cdot 0 (3, I) \\
RHS (3) &= RHS (3) - M (I) = (0 (16, I) \cdot 0 (17, I) \cdot 0 (21, I) \cdot 0 (3, I)) \\
Z (K, K) &= Z (I) = 0 (19, I) \cdot 0 (19, I) \cdot 0 (22, I) \cdot 0 (19, I) \cdot 0 (20, I) \cdot 0 (17, I) \cdot 0 (20, I) \cdot 0 (21, I) \cdot 0 (3, I)) \\
Z (KPLUS, KPLUS) &= Z (I) = 0 (22, I) \cdot 0 (23, I) \cdot 0 (23, I) \cdot 0 (23, I) \cdot 0 (23, I) \\
RHS (KPLUS) &= RHS (I) = (0 (22, I) \cdot 0 (23, I) \cdot 0 (23, I) \cdot 0 (23, I) \cdot 0 (23, I) \\
RIGHT HAND SIDE ADDITION OF GENERALIZED ACTIVE FORCES FOR ALPHA AND BETA \\
BETA_CB &= \text{BETA} (I) - \text{BETA0} (I) \cdot 0 (3) \\
ALPHA_CB &= \text{ALPHA} (I) - \text{ALPH0} (I) \cdot 0 (3) \\
KAAOCB &= K\text{SPHGA} (I) = \text{ALPHCB} \\
KBDBC &= K\text{SPHGB} (I) = \text{BETACB} \\
DPAADT &= \text{DAMP} (I) - \text{ALPHDT} (I) \\
DPBBDT &= \text{DAMP} (I) - \text{BETADT} (I) \\
ASUM &= (K\text{AAOCB} + \text{DPAADT}) \cdot \text{GC} \\
BSUM &= (K\text{BBDC} + \text{DPBBDT}) \cdot \text{GC} \\
RHS (K) &= (\text{RHS} (K) - \text{BSUM}) \\
RHS (KPLUS) &= (\text{RHS} (KPLUS) - \text{ASUM}) \\
CONTINUE \\
RIGID BODY CONTRIBUTION TO LEFT AND RIGHT SIDES OF EQUATIONS OF MOTION \\
Z (1, 1) &= Z (1, 1) - T_{11} \\
Z (1, 2) &= Z (1, 2) - T_{12} \\
Z (1, 3) &= Z (1, 3) - T_{13}
\[ Z(2,2) = Z(2,2) - 122 \]
\[ Z(2,3) = Z(2,3) - 123 \]
\[ Z(3,3) = Z(3,3) - 133 \]
\[ \text{RHS}(1) = (\text{RHS}(1) + 01 + R2\text{THST}(F(2))) \]
\[ \text{RHS}(2) = (\text{RHS}(2) + 02) \]
\[ \text{RHS}(3) = (\text{RHS}(3) + 03 + R2\text{THST}(F(1)) \]

**DELET WALELATION FOR PEDNEUL RATES AND EULER PARAMETER RATES**

\[
\text{DO 20 I = 1, NPEN} \\
\text{KRT} = 2 + \text{NPEN} + 2 * I \\
\text{DERY(KRT)} = \text{BETADT}(I) \\
\text{DERY(KRT + 1)} = \text{ALPMHTD}(I) \\
20 \text{CONTINUE} \\
\]

\[
\begin{align*}
E1 &= \text{E}(4 + \text{NPENTQ}) \\
E2 &= \text{E}(5 + \text{NPENTQ}) \\
E3 &= \text{E}(6 + \text{NPENTQ}) \\
E4 &= \text{E}(7 + \text{NPENTQ}) \\
\text{DERY}(4 + \text{NPENTQ}) &= 0.5000 * (\text{E4} + \text{E1} - \text{E3} + 2\text{E2} + 2\text{E3} - 3\text{E4}) \\
\text{DERY}(5 + \text{NPENTQ}) &= 0.5000 * (\text{E2} + \text{E3} + 2\text{E4} - 2\text{E1} + 2\text{E2} - 3\text{E3}) \\
\text{DERY}(6 + \text{NPENTQ}) &= 0.5000 * (-\text{E2} + \text{E4} + 2\text{E1} - 2\text{E3} + 2\text{E2} - 3\text{E4}) \\
\end{align*}
\]

**SOLVING SIMULTANEOUS LINEAR EQUATIONS (ZDERY=RHS) FOR DERY**

\[
\text{JCNT} = 0 \\
\text{DO 25 J = 1, NEQS} \\
\text{DO 25 I = 1, J} \\
\text{JCNT} = \text{JCNT} + 1 \\
25 \text{CONTINUE} \\
\text{JJOB} = 0 \\
\text{IVAL} = 1 \\
\text{CALL LE2S(ZVEC, NEQS, RHS, IVAL, NEQS, IJOB, IWK, WKAREA, IER)} \\
\text{IF (IER EQ 130) WRITE(6,1)} \\
1 \text{FORMAT(1X,'MATRIX Z IS ILL-CONDITIONED')} \\
\text{IF (IER EQ 129) WRITE(6,2)} \\
2 \text{FORMAT(1X,'Z IS SINGULAR')} \\
\text{DO 30 I = 1, NEQS} \\
\text{DERY(I)} = \text{RHS}(I) \\
30 \text{CONTINUE} \\
\text{RETURN} \\
\text{END} \\
\]

**SUBROUTINE FCJ (NSTATE, TIME, Y, PD)**

\[
\text{IMPLICIT REAL*8(A-H,O-Z)} \\
\text{RETURN} \\
\text{END} \\
\]

**SUBROUTINE OUTP (TIME, Y)**

\[
\text{IMPLICIT REAL*8(A-H,O-Z)} \\
\text{REAL*8 TVEC, WVEC, WEVEC, W3VEC, T1VEC, T2VEC, T3VEC} \\
\]
DIMENSION TVEC(500), W1VEC(500), W2VEC(500), W3VEC(500),
    W7VEC(500), Z2VEC(500), T3VEC(500),
DIMENSION I (15), TMON (15), TSTEP(500), POFT (5, 15, 500), VOFT (5, 500),
    JSTEP(5, 500)
COMMON/BK3/NPENN, NPT2, NPTNT4, NSTATE, NEQS, NCTRL, IFLAG, JINC
COMMON/BK5/TMON
COMMON/BK6/TMON, TH2MON, TH3MON, TH1REP, TH2REP, TH3REP
COMMON/BK7/DGPRD, TVEC, W1VEC, W2VEC, W3VEC, T1VEC, T2VEC, T3VEC,
    T7VEC, JSTEP
COMMON/BK8/TSTEP, POFT, HOFT
JSTEP=JSTEP+1
TVEC(JSTEP)=TIME
W1VEC(JSTEP)=Y(1)=DGPRD
W2VEC(JSTEP)=Y(2)=DGPRD
W3VEC(JSTEP)=Y(3)=DGPRD
E1=Y(4+NPT4)
E2=5(5+NPT4)
E3=Y(6+NPT4)
E4=Y(7+NPT4)
E1SQR=E1**2
E2SQR=E2**2
E3SQR=E3**2
E4SQR=E4**2
C12=2.DO*(E1*E2-E3*E4)
C22=1.DO-2.DO*(E3SQR*E1SQR)
C31=2.DO*(E1*E3-E2*E4)
C32=2.DO*(E2*E3+E1*E4)
C33=1.DO-2.DO*(E1SQR+E2SQR)
E1=1.0-2.0*(E1SQR*E2SQR)
E2=Y(2)*E3
E3=Y(3)*E2
E4=Y(4)*E1
C12=2.DO*(E1*E2-E3*E4)
C22=1.DO-2.DO*(E3SQR*E1SQR)
C31=2.DO*(E1*E3-E2*E4)
C32=2.DO*(E2*E3+E1*E4)
C33=1.DO-2.DO*(E1SQR+E2SQR)

EULER ANGLE CALCULATION

IARG=-C12
IARG=C22
TVEC(JSTEP)=DGPRD=DAM2(YARG, IARG)
YARG=C32
IARG=DSQRT(1.DO-C32**2)
TVEC(JSTEP)=DGPRD=DAM2(YARG, IARG)
IF (JSTEP .NE. 1) GOTO 1
T3VEC(JSTEP)=0.0
GOTO 2
1 CONTINUE
T3VEC(JSTEP)=T3VEC(JSTEP-1) +W2VEC(JSTEP) *
(1.0-TVEC(JSTEP)-TVEC(JSTEP-1))
2 CONTINUE
TH3NOM=YNOM(2) @TIME
CALL EULER(TH1NOM, TH2NOM, TH3NOM, E1NOM, E2NOM, E3NOM, E4NOM)
YNOM(4+NPEST4) = E1NOM
YNOM(S+NPEST4) = E2NOM
YNOM(6+NPEST4) = E3NOM
YNOM(7+NPEST4) = E4NOM

CONTROL FORCE CALCULATION

DO 5 I=1, NCONTROL
FSTORE(I, JSTEP) = 0. DO
DO 10 J=1, NSTATE
DELTAY=Y(J) - YNON(J)
FSTORE(I, JSTEP) = FSTORE(I, JSTEP) + FORT(I, J, JINC) * DELTAY
CONTINUE
FSTORE(I, JSTEP) = FSTORE(I, JSTEP) + FORT(I, JINC)
5 CONTINUE
RETURN
END

SUBROUTINE EULER(TH1, TH2, TH3, E1, E2, E3, E4)
IMPLICIT REAL*(8)(A-H, O-Z)
CS1DV2=DCOS(TH1/2.00)
CS2DV2=DCOS(TH2/2.00)
CS3DV2=DCOS(TH3/2.00)
SN1DV2=DSIN(TH1/2.00)
SN2DV2=DSIN(TH2/2.00)
SN3DV2=DSIN(TH3/2.00)
E1=CS1DV2*SN2DV2*CS3DV2-SN1DV2*CS2DV2*SN3DV2
E2=CS1DV2*CS2DV2*SN3DV2+SN1DV2*SN2DV2*CS3DV2
E3=SN1DV2*CS2DV2*CS3DV2+CS1DV2*SN2DV2*SN3DV2
E4=CS1DV2*CS2DV2*CS3DV2-SN1DV2*SN2DV2*SN3DV2
RETURN
END
BIBLIOGRAPHY


