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Uncovering the "Will of the People": Measuring Preference Polarization among voters

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Abstract

The present paper proposes and develops a way of conceptualizing and measuring preference polarization. A society is said to be polarized on preferences if there are significantly sized clusters of individuals with preference orders on alternatives that are *reverse* of each other. The main theoretical contribution of the paper is an easily implementable technique of extracting the weights of all such clusters embedded in a given preference profile. These weights are used to measure preference polarization. The technique is applied on ballot data from the city council elections for the City of Cambridge, Massachusetts. The analysis of the ballot data provide evidence of increasing preference polarization among the voters.

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1 Introduction

The paper attempts a conceptualization and measurement of preference polarization and applies the measure to ballot data from the city council elections for the City of Cambridge, Massachussetts. An individual's preference is represented by a strict transitive ranking of a set of available alternatives. A population's preferences are represented by a distribution of individuals on the set of all possible strict transitive rankings of these alternatives. Such a distribution is henceforth described as a *preference profile*. The population's preferences are said to be polarized if, broadly speaking, there are significantly sized clusters of individuals whose rank orders of the alternatives are directly reverse of each other. With three candidates, for example, the population's preferences are completely polarized if half the people rank the candidates as $A > B > C$ and the other half rank them as $C > B > A$.

A systematic study and measurement of polarization began in the 1990's motivated by its perceived link to social tension and unrest and by a desire to distinguish it from (wealth or income) inequality. Amongst the seminal papers are those by Esteban and Ray (1994), Wolfson (1994), Duclos, Esteban and Ray (2004) and Montalvo and Reynal-Querol (2008) amongst others. In this literature, individuals in society are identified by a measurable characteristic or attribute that may be continuous - such as income level - or discrete - such as membership of a specific ethnic group. The state of the society is captured by a population distribution over the attribute. Society is said to be polarized if the population distribution is characterized by clusters with significant inter-cluster heterogeneity and intra-cluster homogeneity. That is, a society is polarized if it is divided into significantly sized groups, such that members of the same group identify with or feel close to each other and simultaneously feel alienated or removed from members of the other groups. In this literature, the distance between two levels of the attribute - for example, the difference between two income levels - provides a natural measure of the degree of ”closeness” or ”alienation” two individuals associated with these levels feel towards each other.

To apply these existing measures of polarization, one must first define groups of individuals that are ”close” or ”far apart” on *rank orders* of alternatives. Traditional political economy based on location models commonly identify an individual with his first choice in a rank order. Thus, two individuals with rank orders $A > B > C$ and $A > C > B$ are ”close” under this definition, whilst two individuals with rank orders $A > B > C$ and $B > A > C$ are not. The polarization measures may then be applied on a distribution of the citizens' most preferred alternatives (equivalently, on a distribution of plurality tallies across candidates). A large body of recent literature studying issues such as partisan voting behavior, political posturing by candidates and various socioeconomic implications of political polarization, implicitly or explicitly adopt such an approach

- see, for example, Haimanko, Le Breton and Weber (2007), Krasa and Polborn (2014), Testa (2012), Woo (2005) amongst others. However, as we show with a series of motivating examples in Section 2, such an approach is confounding if the objective is to study preference polarization. More specifically, while a distribution of the citizens' most preferred alternatives may serve to study *voting behavior*, it is less than an adequate tool by itself to study *voters' preferences*.[1] The latter requires a different approach and additional tools. Our paper fills this methodological gap.

Moreover, many traditional models also assume all individual preferences over alternatives to be single peaked - an assumption of convenience that avoids problems related to preference aggregation[2]. The examples also show that this assumption is particularly restrictive for the study of preference polarization.

The present paper proposes and develops a way of conceptualizing and measuring preference polarization. By our approach, a society's preferences are polarized if there are significantly sized clusters of individuals with preference orders that are *reverse* of each other, thus defining groups that feel most alienated from each other. Characterization and measurement of such clusters serve a very important purpose. As explained later in the paper, when such clusters are significantly present in a populace, the outcomes of many standard voting procedures fail to capture the "will of the people" in the sense that they leave significant segments unhappy, despite free and fair elections. Example 3 in section 2 elaborates on this observation. In particular, plurality or any procedure that partially uses plurality is specially vulnerable to such criticism. As a perceived failure of a democratic process to reflect popular sentiments is often responsible for social tension or unrest, we feel it is specially important to be able to identify such clusters if only to ascertain the amount of confidence one can place on the outcome of a specific voting procedure.[3]

The main theoretical contribution of the paper is an easily implementable technique of identifying and measuring all preference clusters of the type described above. We build on the linear algebraic approach to voting theory, pioneered by Saari (1999; 2000a; 2000b) and sometimes described as geometric voting

[1]Many scholars discuss the need to distinguish between voters' preferences and voters' action or behaviour with respect to polarization. Krasa and Polborn (2014), for example, notes that direct analysis of voters' preferences on different policy issues rather than of voting behaviour may lead to contradictory conclusions about the phenomenon of polarization in the US. They cite several such studies.

[2]The assumption is primarily made to rule out existence of Condorcet cycles in the set of rankings and guarantee the existence of Condorcet winners.

[3]It is beyond the scope of the present work to discuss what type of procedure should be ideally used when such clusters are significantly present. There may be no good answer to this question as it takes us to the very heart of some of the well known aggregation or voting paradoxes in collective choice theory. Such paradoxes arise precisely because many types of "problem" profiles, including the ones discussed here, are embedded in a given preference profile. All standard procedures are vulnerable to these paradoxes in varying degrees. The paradoxes have been studied by scholars over the last three centuries, beginning with Borda (1781), Condorcet (1785), Dodgson (1884) and Nanson (1882). Arrow's seminal *impossibility theorem* (1951) brought renewed and formal attention to these problems, in recent times. Since then, a substantial body of work has grown up on comparative studies of procedures. For a comprehensive survey, see Balinski and Laraki (2010), Brahms (2002) and Nurmi (1996, 2002) amongst others.

theory.[4] A highlight of this approach and one of Saari's most important contributions is the idea that a given preference profile can be expressed as a sum of different types of *structured component profiles*. The profile decomposition approach can be adapted to perform different tasks. Saari uses it to provide a deep insight into aggregation paradoxes and why they occur. Specifically, he uses it to construct an orthogonal basis of component profiles that span the space of all possible profiles, such that specific basis (component) profiles influence specific voting procedures but not others (see Saari (2000b)). The purpose is to identify basis profiles responsible for specific aggregation anomalies.[5]

We, on the other hand, use Saari's approach to decompose a given profile into structured component profiles that also admit a *meaningful collective character*. Specifically, the type of component profiles described above that interests us most, is named *Reverse profiles* in the paper.[6] Their most notable feature is that they assign an equal number of voters to each ranking that has a specific candidate in the first and last places. The weights of these profiles are used to measure the extent of preference polarization amongst voters.

With a set of n candidates or alternatives, a preference profile is a $n!$ dimensional vector. The existing linear algebraic method of decomposing such a vector is to first characterize an appropriate set of $n!$ orthogonal basis vectors and then express the given preference profile as a linear sum of these basis profiles. By itself, this is a computationally intensive if not an impossible task, for a large n. For example, under this approach, analyzing the Cambridge City Council ballot data would require characterizing 18! orthogonal basis profiles on an average and as many as 25! profiles for some of the years, *based on the number of official candidates alone* (see Section 6 for more details). A bigger problem with this approach, however, is that the vast majority of these basis profiles so derived, may have little or no meaningful collective character.[7] Drawing upon an earlier work (Chandra and Roy, 2013), we therefore develop and use a different profile decomposition technique that is computationally also more easily implementable.

Our decomposition method requires plurality tallies of candidates and pairwise tallies for *all* candidate pairs as input. In other words, we need information on how many voters ranked a specific candidate in the first place and how many voters prefer a specific candidate over another when any two such candidates are compared - in essence, information of the preference profile. Thus the information requirement for

[4]Hodge and Klima (2005) and Balinski and Laraki (2010) provide some exposition of the linear algebraic framework, techniques and many of Saari's important results.

[5]For example, Saari identifies basis (component) profiles (well known as Condorcet profiles) that influence pairwise aggregation procedures but not positional or sum-scoring methods.

[6]Young (1975) and more recently, Nurmi (1996) also contain discussions of these types of profiles. Similar profiles have been termed *Symmetric profiles* in Saari (2000b). Our characterization is a little different from Saari's in keeping with our objectives and hence we use a different name for these profiles.

[7]For example, Saari (2000b) illustrates this approach for $n = 4$ candidates but notes that such an approach may not be useful for $n \geq 6$. Also see Sections 4 and 5 for more details.

the decomposition method is admittedly high and not easily available. There are however some political elections which require voters to rank all candidates and the ballot data provide full information of the voters' preference profile.[8]

The Cambridge City Election Commission requires voters to rank candidates and is one of the few in the US that requires its voters to do so. There are, however, specific problems with this ballot data that are discussed in the text, chief of which is that voters are not required to rank *all* candidates. They are required to rank at least one. Specific assumptions need to be made to tide over the difficulty that this poses for calculating the set of pairwise tallies and these assumptions affect our empirical findings. We nevertheless apply our techniques on this data for the period 1997-2013 as a first test of our techniques and present these results as the first fruit of the theory of measurement of preference polarization. The empirical findings for this period provide some evidence of recent increase in preference polarization amongst voters - an opinion voiced by other studies and frequently by the media.

The structure of the rest of the paper is as follows. Section 2 presents several 3-candidate examples to motivate and illustrate some of our main arguments. Section 3 presents the tools of geometric voting theory used in the paper and formally introduces Reverse profiles. Sections 4 and 5 present the connection between plurality, Reverse profiles and polarization and our main theoretical results. Finally, in Section 6, we test our method and measures on ballot data from the Cambridge City Council, elections.

2 Motivating Examples

2.1 Example 1

The first example shows that using the distribution of citizens according to their most preferred points could be confounding for the understanding of preference polarization. Consider the two preference profiles p and q each with 4 voters, shown in Table (1)

[8] See Balinski and Laraki, (2010) for a list of such elections.

Table 1:

	Profile p			Profile q	
	ranking	no. of voters		ranking	no. of voters
(1)	$A > B > C$	1	(1)	$A > B > C$	1
(2)	$A > C > B$	1	(2)	$A > C > B$	1
(3)	$B > A > C$	0	(3)	$B > A > C$	1
(4)	$C > A > B$	0	(4)	$C > A > B$	1
(5)	$B > C > A$	1	(5)	$B > C > A$	0
(6)	$C > B > A$	1	(6)	$C > B > A$	0

The distribution of the voters on the domain of candidates, $\{A, B, C\}$, according to the voters' most preferred points - or equivalently, the plurality tallies of the candidates - is the same for both profiles and given by $(2, 1, 1)$. It is however not reasonable to say that both electorates are equally polarized. An equal number of voters love and hate A in p. In q on the other hand, no voter hates A. Further, although not the first choice of two voters in the q-electorate, A is nevertheless their second best choice. Any voting procedure based on plurality would elect A under both profiles but the outcome would generate more animosity amongst the p-voters than amongst the q voters. The example also shows why the assumption of single-peakedness is problematic for the study of preference polarization. The assumption excludes profiles such as p, which are most interesting for such a study.

2.2 Example 2

Our approach to polarization consists of identifying and measuring certain structured cluster of rankings, instead of studying the distribution of citizens according to their most preferred alternatives. Accordingly, clusters such as p in Example 1 are characterized as "polarized". The next example illustrates a structured cluster that we consider to be "not polarized". Consider the profile q' which is similar to profile q in Table (1) above, except that two more voters with A as their first choice have been added.

Table 2:

	Profile q'				
	ranking	no. of voters		ranking	no. of voters
(1)	$A > B > C$	2	(2)	$A > C > B$	2
(3)	$B > A > C$	1	(4)	$C > A > B$	1
(5)	$B > C > A$	0	(6)	$C > B > A$	0

The profile q' has the special property that under *all* known standard voting procedures A always emerges as the winner and B and C are tied for second place (see Section 3 for further details). Moreover, the aggregate ranking remains consistent over subsets of candidates, implying that it cannot be manipulated by choice of a specific procedure or strategic participation/non-participation in the race by a candidate. For this reason, it is reasonable to argue that there is some amount of consensus amongst the voters as to who should be the winner. In other words, it is reasonable to use the term ”will of the people” for a cluster of rankings such as q', and reasonable to say that the ”will of the people” favors A. Note that even for such a cluster, A is clearly not everybody's first ranked candidate. The point is that A is nobody's last ranked. Should A be elected by some procedure, the level of discontent amongst these voters may be expected to be less compared to amongst the p voters, say, in Example 1.

Along with Reverse profiles, profiles of this type have an important place in this paper and are formally introduced in Section 3. We describe this type of profile as ”non-polarized” and belonging to the opposite end of the spectrum as profile p of Example 1, in our analysis. These profiles were introduced to the social choice literature by Saari who named them *Basic profiles*. He also shows that only Basic profiles have the strong property discussed in the previous paragraph, namely *procedure independency* of outcomes and *consistency* over subsets of candidates.

2.3 Example 3

The third example shows that profiles of the type p in Table (1), when significantly present in a given preference profile, may generate election outcomes under standard procedures that are hard to justify as the ”will of the people”. Consider the profile s in Table (3).

Table 3:

Profile s					
	ranking	no. of voters		ranking	no. of voters
(1)	$A > B > C$	7	(2)	$A > C > B$	6
(3)	$B > A > C$	2	(4)	$C > A > B$	0
(5)	$B > C > A$	8	(6)	$C > B > A$	5

Under plurality, A wins the election with 13 votes followed by B in second place with 10 votes and C

in third place with 5 votes. On the other hand, if Borda Count is used, B wins with 32 points, followed by A with 26 points and C last ranked with 24 points. Thus either A or B can be elected by an appropriately chosen procedure, implying that a procedure may be chosen to engineer a specific outcome. We cannot conclude what is the true ”will of the people” from the results of a voting procedure. A decomposition of the profile provides some insight into why this may be happening and into the electorate's preferences. The profile s is actually a direct sum of two profiles s_1 and s_2 shown in Table(4).

Table 4:

Profile s_1			Profile s_2		
	ranking	no. of voters		ranking	no. of voters
(1)	$A > B > C$	6	(1)	$A > B > C$	1
(2)	$A > C > B$	6	(2)	$A > C > B$	0
(3)	$B > A > C$	0	(3)	$B > A > C$	2
(4)	$C > A > B$	0	(4)	$C > A > B$	0
(5)	$B > C > A$	6	(5)	$B > C > A$	2
(6)	$C > B > A$	6	(6)	$C > B > A$	1

Profile s_1 is a scaled up version of the profile p in Table (1) - specifically, $s_1 = 2p$. That is, the size of the two groups who have preferences directly opposite of each others' is now twice as big as the size under p. Profile s_2 is similar to profile q' in Table (2) except that s_2 favors B as the winner under any standard procedure (instead of A, as under q'), and A and C are tied for the second place. Thus $s = 2p + q'$. The conflict between the plurality and the Borda rankings obtained from the profile s is caused by the greater weight of the component profile s_1 relative to the component profile s_2, in profile s. If the weight of s_1 is reduced and brought to a level sufficiently lower than the weight of s_2, then the voting outcome for the sum of the two profiles would be more aligned with the outcome for s_2. And as s_2 is a profile on which all standard procedures agree, we shall see more convergence of the procedures on the sum of the two profiles.

On $s = 2p + q'$, if plurality is used to determine the winner, A will win but a significant portion of the electorate will be unhappy with the result. A is clearly a polarizing candidate. It is natural to ask, under the circumstances, whether Borda Count (which favors candidates who are in the middle) rather than plurality or some other method should be used to determine the winner. The paper does not seek an answer to this question. In fact, a good answer may not exist as no procedure is clearly better than others under all circumstances.

Some scholars have argued that if all voters are treated *impartially*, the component profile s_1 should be interpreted as a complete tie across the candidates for the following reason. The rankings $A > B > C$ and $C > B > A$ offset one another, being reverse of each other and supported by the same number of voters. The rankings $A > C > B$ and $B > C > A$ offset one another for the same reason (see Balinski and Laraki, (2010) for more details of this argument). Further, if s_1 can be interpreted as a tie, the election winner should be determined by s_2, the tie breaker. As all procedures agree on s_2, any procedure is good and the election winner becomes B. We are not convinced about the reasonableness of this argument at present, for the following reason. If s_1 consists of most of the electorate and s_2 only a small part of it, such a step magnifies s_2's role in determining an election outcome. Clearly, a winner determined by a small part of an electorate cannot be described as the true "will of the people" as election outcomes are expected to be. In fact, it is not clear what the true will of the electorate of s is, for, they are not speaking with one voice.

The linear algebraic approach to voting theory attempts to express a given preference profile as a sum of multiple structured component profiles, including profiles of the type s_1 and s_2. Under the standard approach, a complete set of $n!$ orthogonal basis profiles (for a n candidate race) must first be constructed - a set that includes but is not limited to profiles of type s_1 and s_2. This is a computationally intensive if not impossible task. More importantly however, a large number of these component basis profiles have little collective charater - such as s_1 and s_2 have - and are capable of providing little insight into voters' preferences.

A most important result of the paper (Section 4) is that plurality tallies are determined by component profiles of type s_1 and s_2 only and by nothing else. As plurality and voting procedures partially based on plurality, are of special interest to this paper, our decomposition objective is served if we can extract all component profiles of these two types only. Othet types of component profiles are of little interest. The rest of the paper discusses an alternative decomposition technique therefore, that allows us to do precisely this and is easily implementable. The weights of the s_1 type profiles so extracted, provide measures of preference polarization amongst voters.

Section 3 formally defines the various types of component profiles of interest.

3 The algebra of geometric voting theory

3.1 Preference profiles and their components

Voters have strict transitive preferences over n candidates indexed $i = 1 \dots n$. Hence there are $n!$ different ways of ranking these candidates. Assume an electorate of a given and fixed size. A profile $p = (p_1 \dots p_{n!}) \in \mathbf{R}^{n!}_{+}$ is a distribution of voters across these rankings, with $p_j =$ the number of voters with preferences given by the jth ranking of the candidates. A profile differential $p' \in \mathbf{R}^{n!}$ is the difference between two different profiles for an electorate of a given size, implying that p' may have negative components and that its components add up to zero.

A preference aggregation procedure (voting procedure) attempts to obtain an aggregate or social ranking of the candidates based on a profile. Standard procedures fall into two classes, *pairwise procedures* and *positional* or *sum-scoring* procedures.

Pairwise procedures are based on pairwise comparision of the candidates. Condorcet's "successive reversal" and the "maximal agreement" procedures, Kemeny's method, Copeland's method are commonly used examples of such procedures. We begin by counting the number of voters in p, who rank i over j and call this, i's pairwise tally against j. The normalized pairwise score difference between i and j is defined as,

$$a_{ij} = \frac{(i\text{'s tally against } j - j\text{'s tally against } i)}{\text{total number of voters}} \tag{1}$$

Thus if $a_{ij} > 0$, then i beats j in a pairwise comparison and if $a_{ij} < 0$, then j beats i. Further, note that, $a_{ij} = -a_{ji}$ and as a_{ij} is normalized, $-1 \leq a_{ij} \leq 1$. A pairwise procedure constructs a social ranking of the candidates from the set of pairwise score differences $\{a_{ij}\}_{i \neq j, i<j}$.

Positional or sum-scoring methods assign fixed points to a candidate depending on his/her position in an individual's ranking. The aggregate or social ranking is obtained by the sum of these points across all individuals. Plurality and the Borda Count are commonly used examples. The plurality method involves a voter awarding one point to his/her first ranked candidate and zero to all other candidates placed in other positions in his/her ranking. Plurality tallies are the sum of all the points awarded by all the voters. The Borda Count (BC) assigns $n-1$ points to the first ranked candidate of each voter, $n-2$ points to the second ranked candidate and so on and 0 to the last ranked candidate. The aggregate ranking is based on the sum of all points awarded by the voters. Amongst the positional methods, Borda Count has the unique and interesting feature that it is a pairwise *as well as* a positional method as the Borda ranking is equivalent to ranking the candidates according to the sum of their normalized pairwise score differences against other candidates - that is, ranking the candidates by assigning the ith candidate a score of $\sum_{j \neq i} a_{ij}$.[9]

An analytically useful type of profile with a collective character is the $K^n \in \mathbf{R}_+^{n!}$ profile. A K^n profile has one voter for each possible ranking, thus representing an equitable or uniform distribution of voters across all possible rankings. A K^n profile yields a tied outcome across all candidates under any preference aggregation

[9] See Saari 2000b for details.

procedure. Moreover, the relative ranking of candidates are not affected by addition or subtraction of a scaler multiple of a K^n profile to any other given profile, under any procedure. Such an operation changes the aggregate tallies but not the differences in tallies between any two candidates.[10]

Let V denote the total number of individuals or voters. It is useful to view any given profile p as a perturbation from a $\frac{V}{n!}K^n$ profile - that is, as a perturbation from an initial uniform distribution of voters. Thus p can be expressed as $\frac{V}{n!}K^n + p'$ for some profile differential $p' \in \mathbf{R}^{n!}$. For example, consider the profile $s = (7,6,2,0,8,5)$ of 28 voters described in Table 3. s can be expressed as $\frac{28}{6}K^3 + s'$ where $s' = (\frac{14}{6}, \frac{8}{6}, -\frac{16}{6}, -\frac{28}{6}, \frac{20}{6}, \frac{2}{6})$.

To understand the usefulness of this approach, first note that as $\frac{V}{n!}K^n$ has completely tied outcomes for all candidates under any procedure, the profile differential p' yields the same social ranking of the candidates as p under any procedure. That is, under any procedure, the tallies of the candidates under p and p' may differ but the aggregate ranking of the candidates obtained are the same. Thus for analytical purposes, a profile p and an appropriate profile differential p' are equivalent.

Secondly, under this approach, a profile p of an electorate of size V is viewed as being obtained from the (uniformly distributed) $\frac{V}{n!}K^n$ profile by moving voters away from specific rankings and adding them to other specific rankings. The negative components of p' indicate which rankings have suffered depletion and the positive components of p' indicate which rankings have experienced gains. For example, the profile $s = \frac{28}{6}K^3 + s'$ of Table 3 is obtained from a $\frac{28}{6}K^3$ profile by moving voters away from the rankings (3) and (4) (under which A is middle ranked) and adding them to the other rankings (under which A is first or last ranked). Figuratively speaking, any profile p is a result of ”padding” and ”thinning” of specific rankings of an uniform distribution of voters. The various types of component profiles discussed in the paper that admit a collective character are *structured ”padding” and ”thinning” of a weighted K^n profile*.

In developing the decomposition methodology, we use the corresponding profile differential p' for any p, rather than p itself, as the analytical building block. The advantage of this approach is that as a profile differential is orthogonal to K^n, a decomposition of p' does not include neutral K^n effects (that merely changes tallies without affecting results). Alternatively, as a profile differential is a profile with the number of voters normalized to zero, the decomposition of p' (rather than p) allows us to focus on the pure act of padding and thinning that is responsible for the structure of p, rather than its mass.

[10]There are many other types of profiles with this feature, that are not discussed here. Saari(2000a,b) describes the set of all such profiles as the *Universal Kernel*.

3.2 Basic profiles

The next step is to understand the structure of the type of profile illustrated in Example 2 that, by our definition, represents an electorate that is non-polarized. To understand this structure, first fix a candidate, say i. Take a K^n profile and shift a voter from each ranking which has i last ranked and add the voter to a ranking which has i first ranked, taking care not to add more than one such voter to a ranking. The profile $p(i)$ thus obtained equals $K^n + p'(i)$ where the profile differential $p'(i)$ has one voter for each ranking that has i top ranked, (-1) voter for each ranking that has i bottom ranked and 0 voter for each ranking that has i ranked somewhere in the middle.

We describe the profile differential $p'(i)$ thus obtained as a *Basic profile* (after Saari) favoring candidate i and denote it by B_i^n. Following our earlier explanation, $p(i)$ and $p'(i) = B_i^n$ yield the same social ranking of the candidates and are hence analytically interchangeable. Thus the term Basic profile may be used to refer to either. A 4-candidate Basic profile favoring candidate A is illustrated in Table (5).

Table 5:

	Reference ranking	B_1^4		Reference ranking	R_1^4
1.	$A > B > C > D$	(1)	13.	$D > C > B > A$	(-1)
2.	$A > B > D > C$	(1)	14.	$C > D > B > A$	(-1)
3.	$A > C > B > D$	(1)	15.	$D > B > C > A$	(-1)
4.	$A > C > D > B$	(1)	16.	$B > D > C > A$	(-1)
5.	$A > D > C > B$	(1)	17.	$B > C > D > A$	(-1)
6.	$A > D > B > C$	(1)	18.	$C > B > D > A$	(-1)
7.	$B > A > C > D$	(0)	19.	$D > C > A > B$	(0)
8.	$B > A > D > C$	(0)	20.	$C > D > A > B$	(0)
9.	$C > A > B > D$	(0)	21.	$D > B > A > C$	(0)
10.	$C > A > D > B$	(0)	22.	$B > D > A > C$	(0)
11.	$D > A > B > C$	(0)	23.	$C > B > A > D$	(0)
12.	$D > A > C > B$	(0)	24.	$B > C > A > D$	(0)

A society represented by B_i^n has candidate i as some voter's first choice and nobody's last choice. Further, the number of voters who rank candidate i first is greater than the number of voters who rank any other candidate j, first. Thus under *any* preference aggregation procedure, the aggregate ranking for this profile will have the i-th candidate top ranked and everyone else tied in the second place. There are $(n-1)$ *linearly independent* such Basic profiles in a n candidate field.

Moreover, any preference profile that can be expressed as a *linear sum of Basic profiles only*, yield the same social ranking under any aggregation procedure. Consider a given profile p that can be expressed as,

$p = \frac{V}{n!}K^n + a_1 B_1^n + a_2 B_2^n + \ldots + a_n B_n^n$, where a_i's are non-negative constants.

Under a pairwise procedure, the social ranking on p is obtained from the pairwise scores, $\{a_{ij}\}_{i \neq j, i<j}$. On p, the difference a_{ij} can be shown to be equal to $a_i - a_j$, the difference in the weights or coefficients of the B_i^n and B_j^n profiles. Thus under any pairwise procedure, in the aggregate ranking, the relative rank of i and j is determined by the sign of $a_i - a_j$ alone. If $a_i - a_j > 0$, i is ranked above j in the aggregate or social ranking. If $a_i - a_j < 0$, i is ranked below j and if $a_i - a_j = 0$, i and j have the same place in the social ranking under any pairwise procedure. Thus, the aggregate ranking thus obtained has the same linear order as the set of coefficients $\{a_i\}$ and is therefore *transitive*. Moreover, under any pairwise procedure, the social ranking remains consistent over subsets of candidates.

Under a positional procedure, for the profile p, the difference in the positional tallies between any two candidates, can be shown to be equal to $k(a_i - a_j)$, where k is a constant and *uniform* for all candidate pairs, (i, j). Thus, under any positional procedure, in the aggregate ranking, the relative rank of two candidates will depend only on their relative Basic profile weights, a_i and a_j and very importantly, not on any other a_k. In other words, the aggregate ranking obtained is the same under all positional procedures and consistent over subsets of candidates. Moreover, the aggregate ranking is the same as that obtained under any pairwise procedure and hence the same as the linear order on the set of coefficients, $\{a_i\}$.

Profiles such as p pose the least challenges for democratic governance as the social ranking on p cannot be engineered by an appropriate choice of procedure or manipulated by candidates through strategic participation in races. Voters making up a B_i^n profile collectively like the i-th candidate more than any other candidate and are completely indifferent across the others. By a slight stretch of the imagination and perhaps abuse of terms, we may describe the Basic profile B_i^n as representing the preferences of the voters making up the "popular base" of candidate i. Thus, the social ranking on p reflects the relative sizes of the candidates' "popular bases".

3.3 Reverse profiles

The next step is to understand the structure of the type of profile p illustrated in Example 1 that, by our definition, represents an electorate that is polarized. We begin with a definition.

Definition 1 *Fix an integer k, such that* $2 \leq k \leq \frac{n+1}{2}$. *For* $k < \frac{n+1}{2}$, *a generic i-inclined Reverse profile,* R_i^n, *has 1 voter for each ranking in which the i-th candidate is first and last ranked, (-1) voter for each ranking in which he/she is k-th ranked and to the reversal of this ranking and 0 voters for all other rankings. If*

$k = \frac{n+1}{2}$, the profile has 1 voter for each ranking in which the candidate is first and last ranked, (-2) voters for each ranking in which the candidate is k-th ranked and 0 voters for all other rankings.

To understand the structure of the Reverse profile R_i^n, assume $n > 3$ and fix $k = 2$. R_i^n has the same structure and tallies as a K^n profile with a voter moved from each ranking in which i is either second or $(n-1)$th ranked and added to a ranking in which i is first or last ranked. In words, such a profile is obtained by padding the rankings in which i is placed at the two extremes and thinning the rankings in which i is placed somewhere in the middle.

REMARK 1: Note that the value of k determines which rankings with i in the middle are thinned and the voters moved to the extremes. If $k = 2$, voters are moved away from the rankings in which i is in the second or $(n-1)$th place. If $k = 3$ voters are moved away from rankings in which i is in the third or $(n-2)$th place. For R_i^n to be a proper profile differential, it is important that *some* designated rankings with i in the middle of the rankings have (-1) voters assigned to them. Hence these profiles are defined for a fixed k. However, the specific choice of k does not matter for our main results. Note for example, that the plurality ranking of the candidates under R_i^n is the same whatever be the choice of k. We use the adjective "generic" in the definition to indicate that we have a degree of freedom in the characterization of these profiles.

REMARK 2: Adding a K^n profile to a R_i^n profile shows that each ranking that has i first and last ranked are supported by the same number of voters. As a matter of fact the statement may be generalized to say that each ranking and its reverse in the $K^n + R_i^n$ profile is supported by the same number of voters. Hence we name these profiles, Reverse profiles. As Section 4 shows, the social ranking of the candidates for B_i^n and R_i^n are the same under plurality but R_i^n demonstrates extreme preference polarization amongst voters. Hence, the weights of the R_i^n profiles in a given profile may be used to construct measures of preference polarization.

REMARK 3: Generic Reverse profiles are similar but not identical in structure to *Symmetric profiles* defined by Saari (2000b) - hence, our use of a different name. Saari's construction is designed for his specific objective of explaining all possible voting paradoxes under all procedures. Specifically, in his case, positional tallies need to be expressed as deviations from the Borda Count (2000b, Proposition 1) as the weights of the Basic profiles are not directly available.[11] The approach introduces many different types of

[11] In the absence of direct information about the weights of the Basic profiles, the Borda Count can be used as a surrogate, because Borda scores are influenced *only* by Basic profiles.

component profiles that do not always have meaningful collective characters, partly to facilitate the task of profile decomposition in the absence of direct information about the weights of Basic profiles. Our approach includes a technique for extracting the weights of the Basic profiles directly and hence we are able to bypass such problems.

Table (6) illustrates the R_1^4 and R_2^4 profiles for a four candidate field. The next section describes the connection between plurality tallies of candidates and the different Reverse profiles and the first of our main results.

Table 6:

Reference ranking	R_1^4	R_2^4	Reference ranking	R_1^4	R_2^4
1. $A > B > C > D$	(1)	(-1)	13. $D > C > B > A$	(1)	(-1)
2. $A > B > D > C$	(1)	(-1)	14. $C > D > B > A$	(1)	(-1)
3. $A > C > B > D$	(1)	(-1)	15. $D > B > C > A$	(1)	(-1)
4. $A > C > D > B$	(1)	(1)	16. $B > D > C > A$	(1)	(1)
5. $A > D > C > B$	(1)	(1)	17. $B > C > D > A$	(1)	(1)
6. $A > D > B > C$	(1)	(-1)	18. $C > B > D > A$	(1)	(-1)
7. $B > A > C > D$	(-1)	(1)	19. $D > C > A > B$	(-1)	(1)
8. $B > A > D > C$	(-1)	(1)	20. $C > D > A > B$	(-1)	(1)
9. $C > A > B > D$	(-1)	(-1)	21. $D > B > A > C$	(-1)	(-1)
10. $C > A > D > B$	(-1)	(1)	22. $B > D > A > C$	(-1)	(1)
11. $D > A > B > C$	(-1)	(-1)	23. $C > B > A > D$	(-1)	(-1)
12. $D > A > C > B$	(-1)	(1)	24. $B > C > A > D$	(-1)	(1)

4 Plurality and Reverse profiles

For any fixed k, there are n (first place) Reverse profiles for a n-candidate field, related by $\sum_{i=1}^{n} R_i^n = 0$. The following proposition lays out the properties of these profiles, for the given k.

Proposition 1 *For a given k,*

1. *The set of $\{R_i^n\}_{i=1...n}$ profiles are not pairwise orthogonal to each other and span a $(n-1)$ dimensional subspace of the profile space.*

2. *The set of $\{R_i^n\}_{i=1...n}$ profiles are pairwise orthogonal to the set of $\{B_i^n\}_{i=1...n}$ profiles.*

3. *The plurality tallies of B_i^n and R_i^n profiles are identical, with candidate i receiving $(n-1)!$ points and every other candidate receiving $-(n-2)!$ points each. The pairwise scores for each candidate pair*

under a R_i^n profile is a complete tie.

Proof: See Appendix II.

The set of positional tallies of all candidates under any sum-scoring or positional aggregation method is a n dimensional vector in R_+^n, with each component of the vector representing the total points received by a candidate under the specific method. A main implication of Proposition 1 is that although the Reverse profiles are orthogonal to the Basic profiles, the plurality tallies for both profiles lie in an identical direction in R_+^n. This is not surprising given the tone of our discussions relating to Examples 1 and 2. In fact, the statement formally proves for n candidates our earlier observation that a distribution of plurality tallies is unable to identify which type of population it is obtained from - a non-polarized or a polarized one.

The proofs make clear that the specific choice of k does not matter for the plurality properties of R_i^n discussed in Proposition 1. Without loss of generality therefore, we shall characterize all Reverse profiles by fixing $k = 2$, for the rest of the paper.

The following theorem provides the foundation for our decomposition methodology and is the first main result of this paper.

Theorem 1 *Let p be any given profile. The difference in the plurality tallies of any two candidates, for p, can be fully explained by Basic and generic Reverse profile components.*

Proof: Differences in tallies under any specific procedure, including plurality, are not affected by neutral profile components such as K^n which influences these tallies uniformly for all candidates. As shown earlier, Basic profile components of p influence the difference in the plurality tallies of any two candidates. The residual difference in the plurality tallies can therefore be explained by profile components orthogonal to Basic profiles that influence plurality tallies. We want to characterize the set of such orthogonal profiles in the following way. Each such component profile (like a Basic profile) must affect the plurality tally of a specific candidate but not others - implying that for each such component profile, all candidates other than the specific one must be tied under plurality. Profiles orthogonal to Basic profiles that affect the plurality tally of a specific candidate must have an equal number of voters for each ranking with the candidate in the first and last places. The structure of the remaining rankings (which have the specific candidate in other positions) does not matter, so long as for each such component profile, (1) the other candidates are tied under plurality - implying a symmetric distribtuion of voters across rankings with other candidates in the first place and (2) the component proflie is a profile differential - the total number of voters is zero. Such profiles are therefore fully characterized by generic Reverse profiles. Δ.

A main implication of Theorem 1 is that differences in plurality tallies of all candidate pairs, for any profile p, can be explained by its Basic and Reverse profile components. No other type of structured component profile need be considered to explain these differences. Thus, suppose we have a given profile p and a vector of plurality tallies for the candidates based on p. The vector of plurality tallies may be *fully* explained by assuming that the given profile is a linear combination of a K^n profile, n Basic profiles and n Reverse profiles. By the theorem, the difference in the plurality tallies of any two candidates i and j is determined by the Basic and Reverse profile weights, a_i, a_j, r_i and r_j and by nothing else.

Related to the previous implication is that plurality tallies may be used to extract the weights $\{r_i\}_{i=1}^n$. However, these weights cannot be identified from the plurality tallies alone as the latter are also influenced by the weights of the Basic profiles $\{a_i\}_{i=1}^n$. It is therefore important to first chracterize the exact relationship between these tallies and the coefficients of the Basic and Reverse profiles. The next most important result of the paper does precisely this.

Let p be any given profile and let $\tau = (\tau_1 \dots \tau_n)$ denote the vector of plurality tallies of the candidates.

Corollary 1 *The plurality tally differences between any two candidates, $\tau_i - \tau_j$, can be uniquely decomposed into two components, one determined by Basic profiles and the other by Reverse profiles. Specifically, for any candidate pair (i, j), $\tau_i - \tau_j = n(n-2)!((a_i - a_j) + (r_i - r_j)$*

Proof: By Theorem 1, we can assume without loss of generality that the given profile can be expressed as, $p = \sum_{i=1}^n a_i B_i^n + \sum_{i=1}^n r_i R_i^n + \frac{V}{n!} K^n$ as no other type of component profile contributes to plurality tallies.

Note moreover that only $(n-1)$ of the Basic profiles and $(n-1)$ of the Reverse profiles are independent.

From Proposition 1, the tallies of B_i^n and R_i^n are in identical direction, for all i. Define $\mathbf{t_i}$ as a vector with $(n-1)!$ as its i-th component and $-(n-2)!$ as all the other components. Denote $\mathbf{1} = (1, 1, \dots 1)$, a n-component vector. Then the plurality tallies for p are given by

$$\tau = \sum_{i=1}^{n} (a_i + r_i)\mathbf{t_i} + \frac{V}{n}\mathbf{1} \tag{2}$$

Denote by $\alpha = \sum_{i=1}^n (a_i + r_i)(n-2)! - \frac{V}{n}$. A simple manipulation yields,

$$\tau = n(n-2)!\omega - \alpha\mathbf{1} \tag{3}$$

where the n-dimensional vector $\omega = \{a_i + r_i\}_{i=1}^n$. The difference in the plurality tallies for the (i, j) pair is therefore given by,

$$\tau_i - \tau_j = n(n-2)!((a_i - a_j) + (r_i - r_j) \quad \Delta \tag{4}$$

See Appendix for details of some of the specific steps of the proof. Corollary 1 shows that if the differences $(a_i - a_j)$ are known, then the coefficients $\{r_i\}_{i=1}^n$ can be obtained from plurality tallies, subject to a normalization.

The next section discusses an easily implementable technique of extracting the Basic profile weights $\{a_i\}_{i=1}^n$. The technique is based on an earlier result in Chandra and Roy, (2013).

5 Isolating Basic and Reverse component profiles

Our objective is to extract the weights, $\{a_i\}_{i=1}^n$, without constructing a set of $n!$ orthogonal basis profiles. A Theorem from Saari (2000a) provides a pathway to this. Saari's result says that pairwise score differences, $\{a_{ij}\}_{i \neq j, i<j}$, for all candidate pairs are fully determined by Basic profile and another class of component profiles known as Condorcet profiles. The result is reported below,

Saari (Proposition 5, 2000a): Pairwise score differences, a_{ij} are the sum of two orthogonal components - a component contributed by the set of Basic profiles and another component contributed by the set of all Condorcet profiles. Other types of profiles contribute nothing towards these values.

Although a comprehensive discussion of Condorcet profiles is beyond the scope of this paper, a brief introduction is in order, as Saari's theorem is the foundation for a result in Chandra and Roy (2013) that shows how Basic profile weights may be extracted from the pairwise score differences.

5.1 Condorcet profiles and pairwise scores

Begin by first specifying a *reference ranking* of the candidates, say $1 > 2 > 3 \ldots > n$, and index this ranking as (1). Consider the two sets of cyclic rankings generated by the reference ranking (1), listed in Table (7). The set of rankings in the first column of the table, $c_{(1)}^n$ is described as the *Condorcet n-tuple* generated by the reference ranking (1). The second set of rankings, listed in column 2 of the table, $\rho(c_{(1)})^n$, is another set of cyclic rankings that is the reverse of the first set. The sets $c_{(1)}^n$ and $\rho(c_{(1)})^n$ are thus two specific sets of cyclic rankings of the candidates with the feature that each ranking in the set $\rho(c_{(1)})^n$ is a reversal of a ranking in $c^n_{(1)}$.

Table 7:

$c^n_{(1)}$	$\rho(c_{(1)})^n$
$1 > 2 > 3 \ldots > n$	$n > n-1 > n-2 \ldots > 1$
$2 > 3 > 4 \ldots > 1$	$n-1 > n-2 > n-3 \ldots > n$
$3 > 4 > 5 \ldots > 2$	$n-2 > n-3 > n-4 \ldots > n-2$
$\ldots$	$\ldots$
$n > 1 > 2 \ldots > n-1$	$1 > n-1 > n-2 \ldots > 2$

A *Condorcet profile* $C^n_{(1)}$ associated with the reference ranking (1), is a profile that has one voter for each ranking in $c^n_{(1)}$ and (-1) voter for each ranking in $\rho(c^n_{(1)})$ and zero voter for each remaining ranking in the profile. The first or reference ranking of the set $c^n_{(1)}$ uniquely characterizes a Condorcet profile. A n candidate field has $\frac{(n-1)!}{2}$ distinct Condorcet profiles - a very large number. Condorcet profiles thus vastly outnumber the Basic and Reverse profiles. A reason why constructing a set of $n!$ orthogonal basis profiles is practically nearly impossible is the difficulty of characterizing these distinct Condorcet profiles for an arbitrary n.

The $C^n_{(1)}$ profile can be obtained from a K^n profile by moving a voter away from each of the $\rho(c^n_{(1)})$ rankings and adding it to each of the $c^n_{(1)}$ rankings. Alternatively, a Condorcet profile is obtained from a K^n profile by adding a voter to each ranking in the set of cyclic rankings that form a Condorcet n-tuple and taking a voter away from the reversal of this ranking.

A Condorcet profile has the feature that each candidate is placed in each position by exactly the same number of voters. Thus under any positional method, inlcuding plurality, such a profile produces a complete tie. However it generates an intransitive ranking of the candidates under pairwise methods.

Saari's (2000a) result says that the pairwise score difference, a_{ij}, for the candidate pair (i, j), is a sum of two components, a^T_{ij} and a^C_{ij}, such that the component a^T_{ij} is determined by the weights of the Basic profiles and the component a^C_{ij} is determined by the weights of all the Condorcet profiles. The Appendix provides additional explanations and illustration with 3 candidates.

The result shows that it may be possible to extract the Basic profile weights $\{a_i\}$ from the pairwise scores by using linear algebraic techniques. However, once again the curse of dimensionality makes a direct application of this result difficult for any arbitrary n as the first step would require characterizing all the *directional vectors* of the pairwise score differences generated by the $\frac{(n-1)!}{2}$ distinct Condorcet profiles.

There is no currently available computational algorithm to characterize these for any arbitrary n.

Our earlier paper, Chandra and Roy (2013), describes an alternative approach that avoids this curse of dimensionality. The next subsection describes this technique and its potential usefulness for the present work.

5.2 Extracting weights of Basic and Reverse profiles

Assume a given or initial set of pairwise scores, $\{a_{ij}^0\}$, for all candidate pairs. The Chandra and Roy (2013) method consists of a simple set of addition operations on the set of initial scores to form a set of revised pairwise scores. Denote by $\{a_{ij}^1\}$, the revised pairwise scores. The $\{a_{ij}^1\}$'s are obtained from the $\{a_{ij}^0\}$'s by the following algorithm,

$$a_{ij}^1 = a_{ij}^{(0)} + \frac{1}{2}\sum_{k\neq i,j}(a_{ik}^{(0)} + a_{kj}^{(0)}), \forall i, j \tag{5}$$

Take the given pairwise score difference, $a_{ij}^{(0)}$, between candidates i and j. Revise this score difference by a weighted average of all pairwise score differences between i and all other candidates (except j) and between j and all other candidates (except i). The main result of Chandra and Roy (2013) shows that the revised scores, $\{a_{ij}^1\}$, are a scaler multiple of the differences in the weights of the Basic profiles - in other words, the scores, $\{a_{ij}^1\}$, are free of the Condorcet components, $\{a_{ij}^C\}$. It is reported as the following proposition without the proof. The interested reader is referred to the original paper.

Proposition 2 *(Chandra and Roy (2013)) The revised pairwise scores, $\{a_{ij}^1\}$, for all candidate pairs satisfy the following property: $a_{ij}^1 = (1+\frac{1}{2}(n-2))(a_i - a_j)$ or alternatively, $a_i - a_j = \frac{a_{ij}^1}{(1+\frac{1}{2}(n-2))}$.*

Although a complete proof of this result is beyond the scope of the present paper, very briefly, the result follows from the strong additive transitivity property of Basic profiles described earlier and a specific property of the components, $\{a_{ij}^C\}$, generated by the Condorcet profiles- namely, $\sum_{j\neq i} a_{ij}^C = 0$ for all i. The last property is equivalent to an intransitive aggregate ranking over the candidate pairs.[12]

[12]The algorithm (5) is suggested by the following intuition: Condorcet profiles generate an intransitive social ranking over candidates because pairwise scores, by their nature, do not use the full information contained in the set of multilateral rankings that is the profile. Instead, they use partial information about the profile in the form of selective binary components of these rankings. In Saari's words, pairwise scores do not recognize or use the transitivity property of the individual rankings, for the notion of transitivity is *irrelevant* over two candidates. The algorithm (5) essentially attempts to restore some of these lost information contained in the original multilateral rankings. The pairwise score difference for a candidate pair (i, j) is revised by placing a positive weight on the scores of i against other candidates (the a_{ik} 's) and the scores of j against other candidates (the a_{kj}). In other words, the difference between i and j is *re-assessed* by using all possible indirect evidence concerning i and j against other candidates. If i won massively against j but lost against k, whereas j won massively against k, our method will reduce the margin

The computational advantage of this method and the reason for its use in the present work is that forming the revised scores, $\{a^1_{ij}\}$, require only simple addition operations, as opposed to characterizing the directional vectors of the pairwise score differences generated by the $\frac{(n-1)!}{2}$ distinct Condorcet profiles. The weights of the Basic proifles may be extracted by first forming the scores, $\{a^1_{ij}\}$, and then dividing them by the uniform constant $(1+\frac{1}{2}(n-2))$.

Normalized Basic profile weights

The algorithm (5) enables us to extract the differences $a_i - a_j$ for all (i, j) pairs from the given pairwise scores but not the coefficients a_i themselves. A normalization is needed in light of the Section 6 application (There are only $(n-1)$ independent Basic profiles).

In principle, any one of the coefficients a_i, $i = 1 \ldots n$ may be set to zero to obtain the remaining $(n-1)$ coefficients. From the point of view of constructing measures, however, it is convenient to choose the normalizing coefficient (the zero coefficient) in such a way that the weights of the remaining $(n-1)$ independent Basic profiles are non-negative. We use the following steps to select the normalizing coefficient.

As the differences $(a_m - a_n)$ are ordered, choose the (m, n) pair for which this difference is maximized. Suppose that $\max_{(m,n)}(a_m - a_n) = (a_i - a_j)$. Note that $(a_i - a_j) \geq 0$ and therefore $(a_j - a_i) = \min_{(m,n)}(a_m - a_n) \leq 0$. Select a_j to be the normalizing coefficient, that is, set $a_j = 0$. Obtain the values of all the remaining a_is from the differences.

Note that $a_j = 0$ implies $a_i \geq 0$. Note that $\forall m \neq j$, $(a_m - a_j) = (a_m - a_i) + (a_i - a_j)$. Since $(a_j - a_i) \leq (a_m - a_i) \leq (a_i - a_j)$, it follows that $(a_m - a_j) \geq 0$, implying $a_m \geq a_j = 0$. Thus all other coefficients are positive.

Normalized Reverse profile weights

Once the differences $a_i - a_j$ are obtained from pairwise scores using our technique, the differences $r_i - r_j$ can be obtained from equation (4) and plurality tally differences $\tau_i - \tau_j$. As with the Basic profile weights, any one of the r_i coefficients can be normalized to zero to obtain the other coefficients. To obtain a set of non-negative r_i's therefore, we use the same steps as above. Note however, that, in general, the value of j for which a_j is normalized to zero may not be the same as the value of k for which r_k is normalized to zero. In other words, the two normalizations are independent.

by which i won against j. These revisions are meant to recapture the spirit of the original multilateral rankings which contain information about how each candidate stands within the entire group of candidates in voters' preferences, rather than how each candidate stands relative to only a specific another.

REMARK 4: It is important to clarify what the weights, a_i, r_i exactly mean. In particular, such weights in general *cannot* be interpreted as a ”share or proportion” of the electorate. As described in Section 3, any given profile is viewed as a padded and thinned $\frac{V}{n!}K^n$ profile or a perturbation from a uniform distribution of voters, where V is the number of voters. A weight measures the ”thickness” of the padding and thinning performed relative to the $\frac{V}{n!}$ profile, to obtain the given profile. Suppose a profile can be expressed as $p = \frac{V}{n!}K^n + a_iB_i^n$. If a_i is equal to $\frac{V}{n!}$, we conclude that the entire profile has the same structure as a B_i^n profile. Thus, when a given profile is a linear combination of many Basic and Reverse profiles, the relative values of the a_i and r_i coefficients provide direct measures of the importance or strength of these component profiles relative to the others. The above normalizations are equivalent to setting the weakest of the Basic and Reverse profiles to have zero weights.

We illustrate our decomposition technique with two 3-candidate examples.

5.3 Example 4

The following preference-profile comes from the election of a president of the *Social Choice and Welfare Society* and reported in Balinski and Laraki (2010).

Table 8:

Rankings	No. of voters	Rankings	No.of voters
1. $A > B > C$	13	4. $C > B > A$	8
2. $A > C > B$	11	5. $C > A > B$	11
3. $B > C > A$	9	6. $B > A > C$	0

It is easy to check that C is the Condorcet winner, the majority rule ranking is $C > A > B$ and the Borda ranking is $A > C > B$. The pairwise scores are $a_{12} = 9/26$, $a_{13} = -1/13$ and $a_{23} = -2/13$. The plurality tallies are $A = 24, B = 9$ and $C = 19$ inducing the plurality ranking $A > C > B$.

Applying the Chandra-Roy algorithm, the revised pairwise scores are $a_{12}^{(1)} = \frac{10}{26}$, $a_{13}^{(1)} = \frac{1}{52}$, and $a_{23}^{(1)} = -\frac{19}{52}$. The weights of the Basic profiles are obtained from the differences, $\hat{a}_{12} = a_1 - a_2 = \frac{20}{78}$, $\hat{a}_{13} = a_1 - a_3 = \frac{1}{78}$, $\hat{a}_{23} = a_2 - a_3 = -\frac{19}{78}$. As $\max(a_i - a_j) = a_1 - a_2$, we normalize $a_2 = 0$, implying $a_1 = \frac{20}{78}$ and $a_3 = \frac{19}{78}$. The Basic profile favoring A has a slightly greater weight than the one favoring C. To obtain the coefficients of the reverse profiles, note that $r_1 - r_2 = 370/78$, $r_1 - r_3 = 129/78$ and $r_2 - r_3 = -241/78$ using the formula of Theorem 3. As the maximum difference is $r_1 - r_2$, we set $r_2 = 0$ and obtain $r_1 = 370/78$ and $r_3 = 241/78$.

Note that for the profile of voters depicted in Table(10), the plurality and the Borda rankings of the candidates are the *same*. Further note that the differences $a_1 - a_3$ and $r_1 - r_3$ have the same sign. These two observations are by no means unconnected. The next section, Section 6, provides more insight and details into the connection between the differences $r_i - r_j$, $a_i - a_j$ and potential conflicts between plurality and Borda rankings of candidates. So far as this example is concerned, it suffices to point out that although the weight r_1 is greater than the weight r_3 and that this favors A under plurality, the extent of polarization present amongst the voters is not significant. In particular, polarization does *not* pose a problem for the democratic election of the President as either method elects the same candidate.

A more striking feature of this profile and a bigger problem than polarization is the fact that the Condorcet winner is different from the Borda/plurality winner. Although a detailed discussion of this feature is beyond the scope of this paper, the decomposition results above show that a significant presence of Condorcet profiles is responsible for this. The Condorcet components obtained from the pairwise scores and the revised pairwise scores are, $a^c_{12} = a_{12} - a^{(1)}_{12} = 9/26 - 10/26 = -1/26$; $a^c_{13} = a_{13} - a^{(1)}_{13} = -1/13 - 1/52 = -5/52$ and $a^c_{23} = a_{23} - a^{(1)}_{23} = -2/13 + 19/52 = 11/52$. These account for the difference between the Borda ranking and the majority rule ranking, specifically the switch from A to C as the winning candidate.

5.4 Example 5

We use our decomposition method to gain insight into a scenario described in the opening paragraph of the paper. Under this scenario, half the population has ranking $A > B > C$ and the other half has the reverse ranking $C > B > A$. Consider the following preference profile.

Table 9:

Rankings	No. of voters	Rankings	No.of voters
1. $A > B > C$	10	4. $C > B > A$	10
2. $A > C > B$	0	5. $C > A > B$	0
3. $B > C > A$	0	6. $B > A > C$	0

Note that the pairwise score differences are all zero. Thus using our technique the weights of the Basic profiles are all zero - that is $a_1 = a_2 = a_3 = 0$. The plurality scores have A and C tied and B losing to both by 10 votes. Applying equation (4) and associated normalization, we have $r_2 = 0$ and $r_1 = r_3 = 10/3$.

Thus the above profile is a sum of a R^3_1 and a R^3_3 profile of equal weights.

6 Measuring preference polarization

Remark 4 explains that the weights $\{r_i\}$ and $\{a_i\}$ measure the ”thickness” of the padding and thinning required relative to a uniform distribution of voters, to obtain a given set of plurality tallies from a given profile. The weights of the Reverse profiles, $\{r_i\}$, therefore may be used to construct measures of polarization. First of these measures is the average r_i coefficient over all candidates, denoted $\bar{r}$, and its variant, $\bar{r}_w$, which denotes this average over the winning candidates in an election (The Cambridge City Council has nine members - see below). A second useful measure is the ratio of $\bar{r}$ to the average Basic profile coefficient, $\bar{a}$. The ratio $\frac{\bar{r}}{\bar{a}}$ provides an aggregate measure of how strong the Reverse profiles are relative to Basic profiles and thus to what extent polarized preferences play a part in determining the plurality outcome. We use two variants of the ratio measure. The ratio $\frac{\bar{r}}{\bar{a}}$ measures this influence across all the candidates and the variant, $\frac{\bar{r}_w}{\bar{a}_w}$ measures it across the winning candidates.

Finally, Theorem 2 establishes the following relationship between the plurality tally differences and the differences between the Basic and Reverse profile coefficients.

$$(\tau_i - \tau_j) = n(n-2)!((a_i - a_j) + (r_i - r_j)) \tag{6}$$

Thus, the plurality tally difference $(\tau_i - \tau_j)$ for the candidate pair (i, j) and $(a_i - a_j)$ have the same sign if $\frac{r_i - r_j}{a_i - a_j} > -1$ and opposite signs if $\frac{r_i - r_j}{a_i - a_j} < -1$, when $(a_i - a_j) \neq 0$. Therefore, when $(a_i - a_j) \neq 0$ and $\frac{r_i - r_j}{a_i - a_j} < -1$, the relative plurality ranking of the (i, j) pair is a strong reversal of the relative ranking according to the pair's Basic profiles. Since the Borda ranking of the candidates is equivalent to ranking them according to their a_i's, when $\frac{r_i - r_j}{a_i - a_j} < -1$, the relative plurality ranking of the pair is a reversal of their relative ranking under the Borda Count. Specifically, when this is the situation, the candidate that is higher ranked according to plurality must be significantly more polarizing than the other, because the higher rank under plurality is due to the weight of the Reverse profile.

When $(a_i - a_j) = 0$ but $(r_i - r_j)$ is strictly positive or negative, the relative plurality ranking of the (i, j) pair is a weak reversal of the relative ranking according to their Basic profiles. In this situation, the two candidates have the same Borda rank but one of them is higher plurality ranked because of the stronger Reverse profile weight. Thus the ratios $\frac{r_i - r_j}{a_i - a_j}$ or the quantities $(r_i - r_j)$ (when $a_i - a_j = 0$) provide information about how polarizing specific candidates are relative to others. We count the number of distinct pairs whose relative rank under plurality is a strong or weak reversal of their relative rank under Borda and denote by Ψ the proportion of such candidate pairs in the total number of pairs. Ψ provides an aggregate measure of the extent to which the plurality method is vulnerable to preference polarization.

6.1 Results from the Cambridge City Council Elections

In this section, we test our method and measures on ballot data from the Cambridge (Massachussetts) City Council elections over the period 1997-2013. Elections are held every two years providing us with nine years of data.

The nine members of the City Council are elected under a proportional representation (PR) method over several counts of the ballots. Under this method a candidate is elected if he/she wins a certain proportion of the votes, called a quota. The quota is determined by dividing the total number of valid ballots by ten (the number of candidates to be elected plus one) and adding one to the result. The first count entails determining the plurality tallies of all the candidates. All candidates who reach the quota after the first count are declared elected. Any votes they receive beyond the quota are denoted surplus votes. Surplus votes are transferred to the second choice candidates on the surplus ballots. A formula determines which ballots are selected as surplus ballots. After surplus votes are transferred, candidates who have fewer than fifty tallies are eliminated and their votes are transferred to the next in preference. A new ranking is established of the continuing candidates, after this. The candidate with the lowest number of tallies after the two transfers is declared defeated and his/her ballots are transferred to the next continuing candidate marked on each ballot. Once a candidate reaches the quota, no more ballots are transferred to him/her. The process continues till all nine members are elected.

The present paper does not attempt a comprehensive critique of this specific voting procedure. Our specific objective is to uncover the weights of the Basic and Reverse profiles from the ballot data which consists of individual voters' rankings of the candidates. Our analysis may, however, provide some insight into the specific PR method as well. Reverse profiles influence the plurality tallies of the first count, thereby influencing the rest of the PR process and the final outcome. Significant coefficients of these profiles suggest polarization and casts doubt on any method based on plurlaity, for determining winners.[13]

The data set has certain limitations. The traditional model of social choice assumes voters to have strict preferences over *all* candidates. The Cambridge City electoral laws do not require voters to rank all candidates. Voters must rank at least one of them for the first place and are free to rank as many of the others as they like. On an average there are 18 or19 *official* candidates, on the ballot, out of which 9 city council members are elected. Most voters rank about only 4 or 5 candidates. Thus the major limitation of the data set is that many of the official candidates are *not* ranked by many of the voters.

[13] Although not the main focus of this paper, Condorcet profiles are known to influence the aggregate rankings of the continuing candidates when candidates who have already fulfilled the quota or the bottom ranked candidates are dropped and their votes are transferred to the next ranked candidate on the ballots. Our method of extraction of the Reverse profile coefficients, also provide a measure of the size of the Condorcet profiles as a by-product and to that extent is useful.

A second limitation of the data set is that under the electoral laws, voters also have the right to vote for unofficial candidates of their choice by writing their names in a designated space on the ballot. These candidates, described as *write-in* candidates, appear in most cases to be people well known within the very small group of voters who have ranked them but not widely known outside this circle. There is, however, one exception to this observation that happened in the year 2009. In the year 2009, a popular candidate who was successfully elected multiple times previously, failed to file her nomination papers on time and hence was not included in the official list of candidates. The candidate participated as a write-in candidate and ended up being elected. For our analysis for the year 2009, we treated this candidate as an official rather than as a write-in candidate. With the exception of 2009, every election year, there are typically 7-9 write-in candidates who are ranked (anywhere on the ballot) only by a very small set of individual voters.

The presence of write in candidates who are ranked by only very few and the fact that most voters rank only about 4 or 5 official candidates out of 18 or 19 cause the set of individual voters' rankings to be incomplete. As our decomposition method requires pairwise score differences for *all* candidate pairs, assumptions need to be made about how the candidates who are not ranked by a specific voter, stand relative to each other in that voter's preferences. We make the following two assumptions about the *official* candidates that we consider reasonable to overcome this difficulty with the data set. First, we assume that if a voter has not ranked a specific candidate, A (say), then the voter strictly prefers all the candidates that he or she has actually ranked to the candidate A. In other words, unranked candidates are ranked *below* the ranked candidates for any voter. Secondly, if a voter has not ranked two candidates A and B, we assume that the voter prefers A to B with probability half and B to A with probability half. So far as calculating pairwise score differences are concerned, this is equivalent to distributing all voters who have not ranked a specific pair (A,B), equally between A and B. These assumptions do reduce the accuracy of the pairwise score differences that we use to extract the Basic profile weights and to that extent affect the estimates of the Reverse profile weights. However, we consider these assumptions to be the most reasonable under the circumstances and present our results as a first attempt to apply the methods and measures discussed in the paper.

Instances of a write-in candidate, rather than an official candidate, being *ranked first* on the ballot, were very few for all the years, with the exception of the 2009 elections discussed above. We exclude the ballots where a write-in candidate is ranked first from our analysis[14], but retain the ballots where a write-in candidate is placed in between two official candidates.

A third but minor limitation of the data set is that, for many of the elections prior to 2005, we found

[14]We essentially do this to reduce the number of candidates, so that the ratio $\frac{\tau_i - \tau_j)}{n(n-2)!}$ does not vanish at the specified place after decimal, in our numerical calculations. This does not affect our qualitative results.

several ballots with multiple candidates ranked in the same position ("overvotes"). The problem of overvotes is significantly less beginning with 2005, because of a new practice put in place by the Election Commission that automatically ejects all such ballots and gives the voters another chance to redo their ballots. In keeping with the traditional model of social choice (that assumes strict transitive ranking), we excluded all ballots with multiple candidates placed in the same position. Thus, for the years 1997-2003, on an average about 8-9% of the total ballots were discarded. For the years 2005-2011, this percentage is about 1-2%. The combined discards (for all of the reasons explained above) account for some slight differences between our plurality tallies for the candidates and the official plurality tallies of the candidates, after the first count. Specific instances of such discrepancies are noted in the tables for each election year.

Tables 11-20 presents the numerical results of our analysis. Table 11 presents the values obtained for the various measures of polarization discussed in Section 5, for all the years. Figure 1 plots these values. Tables 12-20 provide the Basic and Reverse profile coefficients for all the candidates along with their aggregate rankings based on plurality, the Borda Count and the weights of their Basic profiles, for specific years. The main findings are summarized below.

6.2 Summary results for the period 1997-2013

With the exception of 1997, the values of $\bar{r}$, $\frac{\bar{r}}{\bar{a}}$, $\bar{r_w}$ and $\frac{\bar{r_w}}{\bar{a_w}}$ are generally higher for the period 2005-2013 than for the period 1999-2003. The ratio, $\frac{\bar{r}}{\bar{a}}$, $\bar{r_w}$, in particular shows steady increase from 2001. Moreover, after 2005, polarized preferences seem to have played a generally bigger role in determining the set of winners, compared to before 2005, as evidenced by the values of $\bar{r}_w$ and the ratio $\frac{\bar{r_w}}{\bar{a_w}}$ - although the ratio seems to have dropped off after 2009.

The proportion of relative ranking reversals under plurality, Ψ does not show a specific trend over the years but is generally significant at an average of 36% across all years.

6.3 Specific results for each election year

This subsection points out specific oddities in the election outcome for each year - oddities that may be attributed to the presence of strong Reverse profile components that favor or disfavor specific candidates.

1997

About 9% of the ballots were discarded. Candidate $C12$, Borda and a_i ranked 2nd, was edged out by Candidates $C06$, $C13$, $C14$ and $C15$, all of whom were Borda and a_i ranked below $C12$, in the first round of counting, because the plurality tallies of the latter were boosted by stronger Reverse profiles, as evidenced by the higher r_i values relative to r_{12}. $C12$ got elected in the second round after a transfer of surplus votes from the first round. $C14$, Borda and a_i ranked 6th, edged out $C01$, Borda and a_i ranked 5th, in the first round of counting because of a stronger Reverse component profile. $C01$ eventually got elected in the third round of counting. $C06$, Borda and a_i ranked 7th, is plurality ranked above $C04$, Borda and a_i ranked 1st, because $r_6 > r_4$.

1999

About 9.3% of the ballots were discarded. Our findings include: Borda and a_i 6th ranked Candidate $C08$ ($r_8 = 0.068$ approx), edged out Candidate $C18$ who was Borda and a_i 1st ranked ($r_{18} = 0$) in the first count. $C18$ eventually was elected in the 13th round. $C05$ who was Borda and a_i ranked 4th and $C19$ who was Borda and a_i ranked 5th similarly trailed behind $C08$ in the first count and eventually got elected in the 14th and 13th rounds respectively. $C20$ who was Borda and a_i 2nd ranked did *not* get elected.

2001

About 6.3% of the ballots were discarded. We find that $C12$ who was Borda and a_i ranked 7th was elected in the first count, whilst $C16$ who was Borda and a_i ranked 2nd was elected in the 13th count. $C17$, $C03$ and $C18$ who were Borda and a_i ranked 4th, 5th and 6th respectively (that is ranked before $C12$) were elected in the 7th, 9th and 14th counts. In contrast to what happened in 1999, however, *al*l of the first nine Borda and a_i ranked candidates were eventually elected to the Council. (Explain further)

2003

About 8.8% of the ballots were discarded. As it happened in 2001, there is a remarkable consistency between the first nine Borda and a_i rankings and the set of candidates who eventually got elected. A noticeable fact is that candidate $C16$ who is Borda and a_i ranked 3rd got elected in the 13th round, after candidates $C02$, $C04$, $C13$ and $C20$, all of them Borda and a_i ranked lower than $C16$, got elected in earlier rounds. $C16$ has a lower r_i coefficient compared to all of them. Candidate $C06$ was plurality, Borda and a_i first ranked and also the candidate with the lowest r_i coefficient.

2005

About 3.4% of the ballots cast this year were discarded. Amongst the most noticeable findings are: $C05$ who was Borda and a_i ranked 6th got elected in the first round, whereas, $C16$ who was Borda and a_i ranked 1st got elected in the 11th round. $C16$ got elected in later rounds than $C17$, $C03$, $C04$, $C18$ and $C13$, all of whom were Borda and a_i ranked lower than him/her. Note that $r_{16} = 0.$ whereas the r_i of all these candidates are higher and significantly so in case of $C05$. Very remarkably, $C12$ who was Borda and a_i ranked 7th did not get elected, whereas, $C10$ who was Borda and a_i ranked 11th got elected. $C10$ has a significantly higher r_i coefficient compared to $C12$. Even more interestingly, $C12$ ran and was elected in 1999, 2001 and 2003. In all these three years he/she showed remarkable consistency in the Borda/a_i rankings relative to the other candidates, being always placed 7th or 8th. $C10$ ran but lost in 2003 and interestingly enough also had a significantly high r_i coefficient in 2003.

2007

About 1.03% of the ballots cast this year were discarded. Some interesting findings are: $C13$ who was Borda and a_i ranked 1st got elected in the 9th round whereas Candidates $C01$, $C15$ $C06$ and $C11$, all of whom were Borda and a_i ranked lower but had higher r_i coefficients (significantly so, for $C11$, $C06$ and $C15$), got elected in earlier rounds. Also notable was that Candidate $C13$ who ran in 2001, 2003, 2005 and 2007, had low r_i coefficients in 2001 and 2003 and had $r_i = 0$ in 2005 and 2007. $C16$ who was Borda and a_i ranked 9th was defeated but $C11$ who was Borda and a_i ranked 10th but had a higher r_i coefficient, was elected. Two candidates $C03$ and $C14$, who have been elected in 1997, 1999, 2001, 2003 and 2005 were defeated in 2007. Further both had low, sometimes 0, r_i coefficients in all the years they were elected. In 2007 when they were defeated, their r_i coefficients were significantly higher compared to the earlier years, specially so for $C03$. Thus both candidates became significantly more polarizing figures in 2007 compared to what they were earlier. The year 2007 also marks the beginning of a period during which a significant number of candidates appear with high r_i coefficients, some of them amongst the winners.

2009

Only 0.8% of the ballots cast this year were discarded. Write-in candidate, $WI01$ is designated the 21st official candidate in our table. Amongst the findings are: A significant number of candidates have high r_i coefficients, some of them amongst the winners...Candidate $C19$ who was Borda and a_i ranked 4th and Candidate $C18$ who was Borda and a_i ranked 9th were not elected. Candidates $WI01$ who was Borda and a_i ranked 21st and Candidate $C02$ who was Borda and a_i ranked 10th were elected. Candidate $WI01$ had won as an "official" candidate in 1999, 2001, 2003, 2005 and 2007, had r_i coefficients generally less than 0.1 during these years. In 2009 his/her r_i was the highest at 0.5 approx. $WI01$ also had the lowest a_i coefficient at 0. Four out of the nine elected candidates had r_i coefficients that were significantly higher compared to pre-2007 norms.

2011

About 1.3% of the ballots cast this year were discarded. A significantly large number of candidates had significantly higher r_i coefficients compared to pre-2007 norms. Only three out of eighteen candidates had r_i coefficients less than 0.1 and four candidates had r_i coefficients higher than 0.4. Six out of the nine winners had significant r_i coefficients. Other interesting findings are, Candidate $C12$ who was Borda and a_i ranked 4th and Candidate $C17$ who was Borda and a_i ranked 9th were not elected. Instead Candidate $C04$ who was Borda and a_i ranked 11th and Candidate $C16$ who was Borda and a_i ranked 10th were elected. Candidate $C13$ who was Borda and a_i ranked 2nd was elected in the 13th round whereas $C15$ and $CC05$ who were Borda and a_i ranked lower but had higher r_i coefficients were elected in earlier rounds.

2013

Only 0.05% of the ballots were discarded. As in 2011, a significantly large number of candidates had significantly higher r_i coefficients compared to pre-2007 norms including amongst the winners. $C08$ edged out $C21$ in the plurality tallies because of a stronger Reverse proifle component favoring him/her. $C21$ did not get elected despite being Borda higher ranked than $C08$.

7 Appendix

7.1 Condorcet profiles, pairwise scores with 3 candidates

In a 3-candidate election, there are $3! = 6$ possible rankings of the candidates A, B and C. Let the rankings be numbered as follows:

Table 10:

1. $A > B > C$	4. $C > B > A$
2. $A > C > B$	5. $B > C > A$
3. $C > A > B$	6. $B > A > C$

Denote A as candidate 1, B as candidate 2 and C as candidate 3. The Basic profiles are given by, $B^3_1 = (1,1,0,-1,-1,0)$, $B^3_2 = (0,-1,-1,0,1,1)$, and $B^3_3 = (-1,0,1,1,0,-1)$ and note, $B^3_1 + B^3_2 + B^3_3 = 0$.

Each B^3_i has the same election outcomes as $B^3_i + K^3$ where $K^3 = (1,1,1,1,1,1)$. As K^3 has one voter favoring each ranking, it does not influence any election outcome. Note that $B^3_1 + K^3 = (2,2,1,0,0,1)$. Under the profile $B^3_1 + K^3 = (2,2,1,0,0,1)$, A wins over B or C under any pairwise or postional voting procedure. B and C are tied under any procedure. Thus everyone in this group of voters likes A best and is indifferent between the others.

There is a unique reference ranking and only one distinct Condorcet profile for a 3-candidate field. The Condorcet 3-tuple $c^3_{(1)}$ and its reversal set $\rho(c_{(1)})^3$ are the set of rankings in the table below.

Table 11:

$c^3_{(1)}$	$\rho(c_{(1)})^3$
$A > B > C$	$C > B > A$
$B > C > A$	$A > C > B$
$C > A > B$	$B > A > C$

The Condorcet profile for a 3-candidate field is described by the vector, $C^3 = (1,-1,1,-1,1,-1)$.

As the number of candidates increases, the dimension of the voters' profile and the number of distinct Condorcet profiles gets large very quickly. In a 4-candidate field the number of all possible rankings of the candidates is 24, implying that the Basic and the Condorcet profiles are 24-dimensional vectors. There are 4 Basic profiles (three of which are independent) and 3 distinct Condorcet profiles. In a 6-candidate field there are 6 Basic profiles and 60 distinct Condorcet profiles, each being a 6!-dimensional vector.

There are three possible pairwise scores differences in a 3-candidate field and hence the set of pairwise score differences is a 3-dimensional cube with each side given by the interval $[-1,1]$. A vector of pairwise score differences in this cube is represented as $a = (a_{12}, a_{13}, a_{23})$. The set of pairwise score differences with 4 candidates is a 6-dimensional cube.

The vector of pairwise score differences, a, generated by the three Basic profiles are $T^3_1 = (1,1,0)$, $T^3_2 = (-1,0,1)$, and $T^3_3 = (0,-1,-1)$. Under B^3_1, A unanimously beats B and C who are tied. Hence in T^3_1, $a_{12} = a_{13} = 1$ and $a_{23} = 0$. It is easy to check that these three vectors are linearly dependent and hence span a 2-dimensional subspace of the 3-dimensional cube $[-1,1]$.

With three candidates, the unique Condorcet profile generates the pairwise score differences vector $q = (a_{12}, a_{13}, a_{23}) = (1,-1,1)$. q illustrates the well known Condorcet triplet that A unanimously beats B, B unanimously beats C and C unanimously beats A. In a 4-candidate field there are three distinct Condorcet profiles and hence three such 6-dimensional directional vectors (see Saari 2000a for description). In a 5-candidate field there are twelve 10-dimensional directional vectors. As the number of distinct Condorcet profiles increase very rapidly with the candidates, characterizing the directional vectors for these profile becomes a long and involved process. This is one reason why direct profile decomposition is difficult to implement if $n > 4$.

Suppose that an electorate can be described by a combination of Basic and Condorcet profiles, $p = aB_1^3 + bB_2^3 + cB_3^3 + dC^3$, where a, b, c and d are any constants and the vectors B_1^3, B_2^3, B_3^3, and C^3 are as defined above.

It is easy to check that the number of voters favoring each possible ranking within the profile are as follows.

Table 12:

ranking	no.of voters	ranking	no. of voters
$A > B > C$	$(a-c+d)$	$C > B > A$	$(-a+c-d)$
$A > C > B$	$(a-b-d)$	$B > C > A$	$(-a+b+d)$
$C > A > B$	$(-b+c+d)$	$B > A > C$	$(b-c-d)$

The pairwise election tallies are calculated to be $(A:B) = ((2a-2b+d):(2b-2a-d))$, $(A:C) = ((2a-2c-d):(2c-2a+d))$, and $(B:C) = ((2b-2c+d):(2c-2b-d))$. Note that the pairwise tallies depend on the relative weights of the two relevant Basic profiles and the weight of the Condorcet profile. Further each pairwise score difference is a direct sum of two components - one attributable to the Basic profiles and the other to the Condorcet profile. For example, the pairwise score difference for the (A,B) pair is $(4a-4b+2d)$ (A's tally minus B's tally). The component attributable to the Basic profiles is $4a-4b$ and the component attributable to the Condorcet profile is $2d$.

7.2 Proof of Proposition 1

For this proof, we index the candidates by the lower case letters, $i = 1\dots n$, as we do in the text and name the candidates with upper case letters, $A, B \dots N$, whenever necessary for exposition. Thus the ith candidate is named I and the jth candidate J.

Part 1: Assume $k = 2$ to start with. Also, without loss of generality, consider the pair (R_1^n, R_2^n). R_1^n has non-zero voters for A in the 1-st, 2-nd, $(n-1)$-th and n-th places. R_2^n has non-zero voters for B in the 1-st, 2-nd, $(n-1)$-th and n-th places. The inner product of $(R_1^n)^T$ and R_2^n have non-zero components for all rankings in which (1) A is in the 1-st place and B is in the 2-nd, $(n-1)$-th or n-th place (2) A is in the 2-nd place and B is in the 1-st, $(n-1)$-th or n-th place (3) A is in the $(n-1)$-th place and B is in the 1-st, 2-nd or n-th place and (4) A is in the n-th place and B is in the 1-st, 2-nd or $(n-1)$-th place. In each of these cases (a total of twelve cases), A and B can be placed in their positions in $(n-2)!$ ways. The relevant components of R_1^n and R_2^n belong to the set $\{1,-1\}$. The non-zero components of the inner product equal

$$-(n-2)!-(n-2)!+(n-2)!-(n-2)!+(n-2)!-(n-2)!-(n-2)!+(n-2)!$$
$$-(n-2)!+(n-2)!-(n-2)!-(n-2)! = -4(n-2)!$$

Hence R_1^n and R_2^n are not orthogonal. By way of illustration, for $n=3$ and $n=4$, $(R_1^3)^T R_2^3 = -6$ and $(R_1^4)^T R_2^4 = -8$. The argument extends to all pairs of R_i^n profiles for $k=2$.

Next note that all the previous steps of the proof apply directly without any changes to any $k < \frac{n+1}{2}$. Now suppose we choose $k = \frac{n+1}{2}$ which can only happen if n is odd. Note that the candidate, I, can be in the k-th place in $(n-1)!$ rankings and that half of these rankings are reversals of the other half. Each such ranking has (-2) voters by construction. The inner product of $(R_1^n)^T$ and R_2^n have non-zero components for all rankings in which (1) A is in the 1-st place and B is in the $\frac{n+1}{2}$-th or n-th place (2) A is in the $\frac{n+1}{2}$-th place and B is in the 1-st, or n-th place (3) A is in the n-th place and B is in the 1-st, $\frac{n+1}{2}$-th place. The non-zero components of the inner product equal

$$-2(n-2)!+(n-2)!-2(n-2)!-2(n-2)!+(n-2)!-2(n-2)! \quad = \quad -6(n-2)!$$

which is not 0. Hence the non-orthogonality claim is true for any k and for all pairs of generic Reverse profiles.

Consider the sum $\sum_{i=1}^{n} R_i^n$ for $k=2$. Only four out of these n profiles at a time contribute non-zero voters for each ranking. Two of the profiles contribute (1) voter each for the first and last places. The other two profiles contribute (-1) each for the 2-nd and $(n-1)$-th places. Hence the sum is 0. Using similar argument, it is clear that the sum of any $(n-1)$ profiles out of the n profiles is not 0. Hence the set spans a $(n-1)$ dimensional subspace, for $k=2$. The steps apply directly without any changes to any $k < \frac{n+1}{2}$. When $k = \frac{n+1}{2}$, three out of these profiles contribute non-zero voters for each ranking at a time. Two of the profiles contribute (1) voter each for the first and last places. The profile contributes (-2) each for the $\frac{n+1}{2}$-th place. Hence the sum is 0

Part 2: Consider the inner product of $(R_i^n)^T$ and B_i^n, for any given k. This has non-zero components for all rankings in which (1) candidate I is in the 1-st place and (2) candidate I is in the n-th or last place. As there are $(n-1)!$ rankings in which candidate I is 1-st ranked and another $(n-1)!$ rankings in which he/she is last ranked, the non-zero components equal $(n-1)!.(1).(1)-(n-1)!.(1).(-1) = 0$. Hence this pair is orthogonal to each other.

Next assume that $k = 2$ and consider the inner product of $(R_i^n)^T$ and B_j^n, where $i \neq j$. This has non-zero components for all rankings in which (1) candidate J is in the 1-st place and I is in the 2-nd place (2) candidate J is in the 1-st place and I is in the $(n-1)$-th place (3) candidate J is in the n-th place and I is in the 2-nd place and (4) candidate J is in the n-th place and I is in the $(n-1)$-th place. The non-zero components equal $-(n-2)! - (n-2)! + (n-2)! + (n-2)! = 0$. Hence these two vectors are orthogonal and the claim is true.

Again, the arguments extend directly without any changes for any $k < \frac{n+1}{2}$. When $k = \frac{n+1}{2}$, the inner product has non-zero components for all rankings in which (1) candidate J is in the 1-st place and I is in the $\frac{n+1}{2}$-th place (2) candidate J is in the n-th place and I is in the $\frac{n+1}{2}$-th place. The non-zero components equal $-2(n-2)! + 2(n-2)! = 0$. Hence claim is true for any given k.

Part 3: Under a B_i^n profile, candidate I is ranked first $(n-1)!$ times and hence receives as many points. Candidate J receives non-zero votes only for rankings in which he/she is ranked first and candidate I is ranked last. There are $(n-2)!$ such rankings each with (-1) voter. Thus every other candidate receives $-(n-2)!$ points. Under a R_i^n profile, with $k = 2$, candidate I is ranked first $(n-1)!$ times and receives as many points. Candidate J receives non-zero votes for every ranking in which (1) J is first ranked and I is second ranked (2) J is first ranked and I is $(n-1)$-th ranked (3) J is first ranked and I is n-th ranked. There are $(n-2)!$ rankings in each category. J receives (-1) for each ranking in the first two categories and (1) for each ranking in the last category. Hence J receives $-(n-2)!$ points.

These tallies remain unchanged for any $k < \frac{n+1}{2}$. For $k = \frac{n+1}{2}$, candidate J receives non-zero votes for every ranking in which (1) J is first ranked and I is $\frac{n+1}{2}$-th ranked (2) J is first ranked and I is n-th ranked. There are $(n-2)!$ rankings in each category. J receives (-2) for each ranking in the first category and (1) for each ranking in the last category. Hence J receives $-(n-2)!$ points.

The total number of voters in a $B_i^n + K^n$ profile is $2(n-1)! + (n-2)(n-1)! = n!$. The total number of voters in a $R_i^n + K^n$ profile is $2(n-1)! + 2(n-1)! + (n-4)(n-1)! = n!$ for $n > 3$. The normalized plurality scores can be derived using the previous steps. Under a R_i^n profile, each ranking and its reversal has the same number of voters. Hence pairwise scores are a complete tie for each candidate pair.

7.3 Step details of Corollary 1

The profile is given by $p = \sum_{i=1}^n a_i B_i^n + \sum_{i=1}^n r_i R_i^n + \frac{V}{n!} K^n$. The plurality tally of candidate i, is $(a_i + r_i)(n-1)! - \sum_{j \neq i, j=1}^n (a_j + r_j)(n-2)! + \frac{V}{n}$. Arranging terms and using the defintions of $\mathbf{t_i}$ and $\mathbf{1}$, we then get equation(2).

Next note that,

$$(a_i+r_i)(n-1)!-\sum_{j\neq i,j=1}^{n}(a_j+r_j)(n-2)!+\frac{V}{n}$$

$$=(n-1)(n-2)!(a_i+r_i)-\sum_{j\neq i,j=1}^{n}(a_j+r_j)(n-2)!+\frac{V}{n}$$

$$=n(n-2)!-(\sum_{j=1}^{n}(a_j+r_j)(n-2)!-\frac{V}{n})$$

The definitions of ω and α then yields equation (3).

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Table 13: Summary results for the period 1997-2013

Years	$\bar{r}$	Std(r_i)	$\frac{\bar{r}}{\bar{a}}$	$\bar{r_w}$	$\frac{\bar{r_w}}{\bar{a_w}}$	Ψ
1997	0.203	0.153	0.94	0.070	0.20	0.40
1999	0.173	0.110	0.88	0.071	0.24	0.61
2001	0.193	0.146	0.77	0.061	0.16	0.23
2003	0.153	0.125	0.78	0.029	0.09	0.34
2005	0.224	0.171	0.85	0.089	0.22	0.34
2007	0.215	0.145	0.89	0.111	0.32	0.38
2009	0.240	0.146	0.92	0.150	0.43	0.32
2011	0.240	0.151	1.06	0.126	0.38	0.26
2013	0.253	0.118	1.38	0.172	0.26	0.24

Figure 1: Summary results for the period 1997-2013

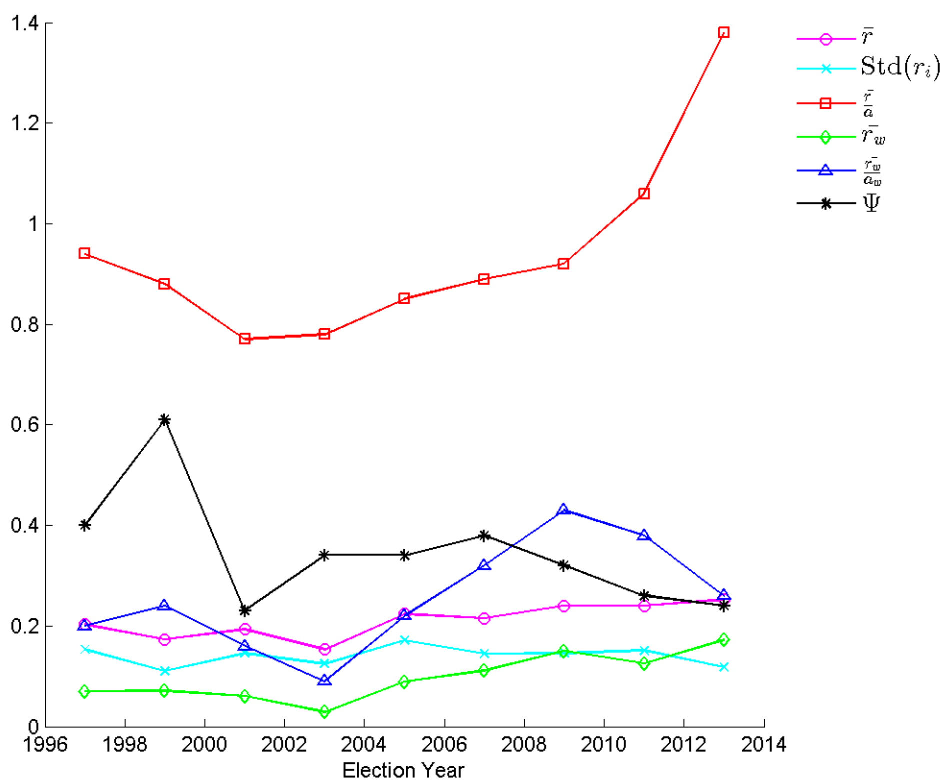

Table 14: 1997 elections: $a_2 = 0$, $r_4 = 0$

Status	Candidates	Plurality rank	Borda rank	a_i	Rank by a_i	r_i	Rank by r_i
E, 3rd	C01	6	5	0.378680243	5	0.040926482	15
D	C02	10	19	0	19	0.419606726	1
E, 14th	C03	9	8	0.33295654	8	0.086650186	12
E, 1st	C04	2	1	0.419606726	1	0	19
D	C05	15	18	0.001354005	18	0.418252721	2
E, 1st	C06	1	7	0.3532176	7	0.066389126	13
E, 11th	C07	11	11	0.136258229	11	0.283348497	9
D	C08	18*	17	0.041575915	17	0.378030811	3
D	C09	14	14	0.094198304	14	0.325408422	6
D	C10	16	16	0.066836377	16	0.352770349	4
D	C11	8	9	0.265489111	9	0.154117615	11
E, 2nd	C12	7	2	0.401894381	2	0.017712344	18
E, 1st	C13	5	3	0.387539479	3	0.032067247	17
E, 1s	C14	4	6	0.354014073	6	0.065592653	14
E, 1st	C15	3	4	0.384610907	4	0.034995819	16
D	C16	12	13	0.108161096	13	0.31144563	7
D	C17	17*	12	0.07116184	12	0.348444886	5
D	C18	19*	15	0.126075622	15	0.293531104	8
D	C19	13	10	0.188120905	10	0.231485821	10
	Average			0.216407966		0.20319876	

E, . = elected, count; D = defeated
C08 is ranked 17th, C18 is ranked 18th and C17 is ranked19th officially, after the first count.

Table 15: 1999 elections: $a_3 = 0$, $r_{18} = 0$

Status	Candidates	Plurality rank	Borda rank	a_i	Rank by a_i	r_i	Rank by r_i
E, 11th	*C*01	2	3	0.344874753	3	0.02540773	22
E, 14th	*C*02	4*	10	0.24075288	10	0.129529603	15
D	*C*03	19	24	0	24	0.370282483	1
D	*C*04	24	23	0.05359921	23	0.316683273	2
E, 14th	*C*05	5*	4	0.336589054	4	0.033693429	21
E, 13th	*C*06	3	8	0.266562658	8	0.103719824	17
D	*C*07	20	18	0.098987012	18	0.271295471	7
E, 1st	*C*08	1	6	0.302314402	6	0.067968081	19
D	*C*09	18	19	0.097422518	19	0.272859965	6
D	*C*10	11	16	0.134397888	16	0.235884595	9
D	*C*11	14	20	0.091334976	20	0.278947506	5
D	*C*12	22	22	0.064651179	22	0.305631304	3
E, 14th	*C*13	10	7	0.279454437	7	0.090828046	18
D	*C*14	21	21	0.087428112	21	0.282854371	4
D	*C*15	16*	15	0.151139721	15	0.219142762	10
E, 14th	*C*16	7	11	0.217337913	11	0.15294457	14
D	*C*17	13	12	0.189360568	12	0.180921915	13
E, 13th	*C*18	8	1	0.370282483	1	0	24
E, 13th	*C*19	6*	5	0.332297621	5	0.037984862	20
D	*C*20	9	2	0.349856661	2	0.020425822	23
D	*C*21	12	14	0.165788278	14	0.204494205	11
D	*C*22	17	13	0.187533868	13	0.182748615	12
D	*C*23	15*	9	0.255204783	9	0.1150777	16
D	*C*24	23	17	0.108824095	17	0.261458388	8
	Average			0.196916461		0.173366022	

E, . = elected, count; D = defeated
*C*02 is ranked 5th, *C*05 is ranked 6th and *C*19 is ranked 4th, officially, after the first count. *C*15 is ranked 15th and *C*23 is ranked 16th, officially, after the first count.

Table 16: 2001 elections: $a_1 = 0$, $r_5 = 0$

Status	Candidates	Plurality rank	Borda rank	a_i	Rank by a_i	r_i	Rank by r_i
D	$C01$	18	19	0	19	0.443075497	1
E, 1st	$C02$	3	3	0.414859265	3	0.028216232	17
E, 9th	$C03$	4	5	0.381725656	5	0.061349841	15
D	$C04$	16	18	0.028673663	18	0.414401835	2
E, 1st	$C05$	1	1	0.443075497	1	0	19
D	$C06$	15*	17	0.093947775	17	0.349127722	3
D	$C07$	14*	12	0.202280531	12	0.240794966	8
D	$C08$	12	13	0.127292042	13	0.315783455	7
D	$C09$	13	15	0.105527986	15	0.337547511	5
D	$C010$	11	11	0.268463485	11	0.174612012	9
E, 15th	$C011$	10	8	0.3472017	8	0.095873797	12
E, 1st	$C012$	2	7	0.350806973	7	0.092268524	13
D	$C013$	17	16	0.095946023	16	0.347129474	4
D	$C014$	9	10	0.275782371	10	0.167293126	10
E, 15th	$C015$	8	9	0.319238258	9	0.123837239	11
E, 13th	$C016$	6	2	0.417958958	2	0.025116539	18
E, 7th	$C017$	7	4	0.388129682	4	0.054945816	16
E, 14th	$C018$	5	6	0.373479872	6	0.069595626	14
D	$C019$	19	14	0.120370398	14	0.322705099	6
	Average			0.250250533		0.192824964	

E, . = elected, count; D = defeated

$C06$ is ranked 14th, $C07$ is ranked 15th, officially, after the first count.

Table 17: 2003 elections: $a_5 = 0$, $r_6 = 0$

Status	Candidates	Plurality rank	Borda rank	a_i	Rank by a_i	r_i	Rank by r_i
D	$C01$	13	13	0.142250572	13	0.204323903	8
E, 9th	$C02$	2	5	0.322525198	5	0.024049277	16
D	$C03$	8	16	0.074460729	16	0.272113746	5
E, 12th	$C04$	7*	7	0.30908604	7	0.037488436	14
D	$C05$	19	20	0	2	0.346574475	1
E, 1st	$C06$	1	1	0.346574475	1	0	20
D	$C07$	20	18	0.02376686	18	0.322807616	3
D	$C08$	18	17	0.056103618	17	0.290470857	4
D	$C09$	12	14	0.125982373	14	0.220592102	7
D	$C10$	15	12	0.161708137	12	0.184866339	9
D	$C11$	17	19	0.018420412	19	0.328154063	2
E, 13th	$C12$	9	8	0.299381604	12	0.047192871	13
E, 12th	$C13$	6*	4	0.325076691	4	0.021497784	17
D	$C14$	11	11	0.165160442	11	0.181414033	10
E, 13th	$C15$	5	9	0.254813264	9	0.091761211	12
E, 13th	$C16$	10	3	0.337468958	3	0.009105517	18
D	$C17$	14	15	0.074738277	15	271836198	6
E, 10th	$C18$	3	2	0.344553732	2	0.002020743	19
D	$C19$	16	10	0.181930175	10	0.164644301	11
E, 12th	$C20$	4	6	0.317334567	6	0.029239908	15
	Average			0.194066806		0.152507669	

E, . = elected, count; D = defeated
$C06$ is ranked 6th and $C13$ is ranked 7th, officially, after the first count.

Table 18: 2005 elections: $a_2 = 0$, $r_{16} = 0$

Status	Candidates	Plurality rank	Borda rank	a_i	Rank by a_i	r_i	Rank by r_i
D	C01	13	17	0.027062433	17	0.459551143	2
D	C02	18	18	0	18	0.486613576	1
E, 9th	C03	3	3*	0.434377909	3	0.052235667	16
E, 9th	C04	2	5	0.409046092	5	0.077567484	14
E, 1st	C05	1	6	0.405791705	6	0.080821871	13
D	C06	12	12	0.150867032	12	0.335746544	7
D	C07	15	16	0.066018547	16	0.420595029	3
D	C08	17	15	0.071293136	15	0.41532044	4
D	C09	14	13	0.094715069	13	0.391898507	6
E, 11th	C10	9	11	0.230661564	11	0.255952012	8
D	C11	16	14	0.09189506	14	0.394718516	5
D	C12	11	7	0.393973868	7	0.092639708	12
E, 10th	C13	7	4	0.424111421	4	0.062502155	15
E, 11th	C14	8	8	0.37030372	8	0.116309856	11
D	C15	10	10	0.257144827	10	0.229468749	9
E, 11th	C16	6	1	0.486613576	1	0	18
E, 5th	C17	4*	2	0.467411315	2	0.019202262	17
E, 10th	C18	5	9	0.35467301	9	0.131940566	10
	Average			0.263108905		0.223504671	

E, . = elected, count; D = defeated
C03 is ranked 4th and C17 is ranked 3rd, officially, after the first count.

Table 19: 2007 elections: $a_8 = 0$, $r_{13} = 0$

Status	Candidates	Plurality rank	Borda rank	a_i	Rank by a_i	r_i	Rank by r_i
E, 1st	$C01$	1	2	0.423140965	2	0.033877992	15
E, 9th	$C02$	7	4	0.364579521	4	0.092439435	13
D	$C03$	12	14	0.051708995	14	0.405309961	3
D	$C04$	13	13	0.07408695	13	0.382932006	4
E, 10th	$C05$	6	7	0.268489717	7	0.18852924	10
E, 7th	$C06$	3	6	0.339173315	6	0.117845642	11
D	$C07$	14	12	0.123069604	12	0.333949352	5
D	$C08$	15	16	0	16	0.457018956	1
E, 9th	$C09$	5	3	0.39536522	3	0.061653736	14
D	$C10$	16	15	0.051553466	15	0.40546549	2
E, 8th	$C11$	4	10	0.242232672	10	0.214786284	7
E, 10th	$C12$	8	8	0.265616995	8	0.191401962	9
E, 9th	$C13$	9	1	0.457018956	1	0	16
D	$C14$	10	11	0.19453085	11	0.262488106	6
E, 6th	$C15$	2	5	0.355915612	5	0.101103345	12
D	$C16$	11	9	0.262945546	9	0.19407341	8
	Average			0.241839274		0.215179682	

E, . = elected, count; D = defeated

Table 20: 2009 elections: $a_{21} = 0$, $r_{14} = 0$

Status	Candidates	Plurality rank	Borda rank	a_i	Rank by a_i	r_i	Rank by r_i
D	$C01$	18	19	0.100249059	19	0.399637763	3
E, 17th	$C02$	10	10	0.264162972	10	0.235723819	12
E, 1st	$C03$	1	2	0.470059345	2	0.029827446	20
D	$C04$	17	18	0.109561943	18	0.390324849	4
D	$C05$	15	15	0.159057867	15	0.340828924	7
E, 15th	$C06$	6	7	0.331563707	7	0.168323085	15
D	$C07$	16	16	0.153278278	16	0.346608513	6
E, 16th	$C08$	4	6	0.422708423	6	0.077178369	16
D	$C09$	13	14	0.171308213	14	0.328578579	8
D	$C10$	21	20	0.094439678	20	0.405447114	2
D	$C11$	20	17	0.141528433	17	0.358358358	5
E, 16th	$C12$	7	8	0.284016159	8	0.215870633	14
E, 17th	$C13$	8	5	0.42332809	5	0.076558702	17
E, 1st	$C14$	2	1	0.499886792	1	0	21
D	$C15$	14	12	0.219237094	12	0.280649697	10
D	$C16$	9	11	0.250363459	11	0.249523333	11
E, 1st	$C17$	3	3	0.451838744	3	0.048048048	19
D	$C18$	12	9	0.283706325	9	0.216180466	13
D	$C19$	11	4	0.445796987	4	0.054089804	18
D	$C20$	19	13	0.189624148	13	0.310262644	9
E, 17th	$WI01$	5	21	0	21	0.499886792	1
	Average			0.260272177		0.239614615	

E, . = elected, count; D = defeated

Table 21: 2011 elections: $a_8 = 0$, $r_1 = 0$

Status	Candidates	Plurality rank	Borda rank	a_i	Rank by a_i	r_i	Rank by r_i
E, 1st	$C01$	1	1	0.460430347	1	0	18
E, 9th	$C02$	4	3	0.393626618	3	0.066803729	16
E, 14th	$C03$	6	8	0.280711324	8	0.179719023	11
E, 13th	$C04$	7	11	0.230369018	11	0.230061328	8
E, 1st	$C05$	3	6	0.334550108	6	0.125880239	13
D	$C06$	13	15	0.051566073	15	0.408864274	4
D	$C07$	16	16	0.027006804	16	0.433423542	3
D	$C08$	18	18	0	18	0.460430374	1
D	$C09$	12	12	0.115530661	12	0.344899686	7
D	$C10$	17	17	0.02006979	17	0.440360557	2
E, 14th	$C11$	9	7	0.300004895	7	0.160425452	12
D	$C12$	11	4	0.354312208	4	0.106118139	15
E, 13th	$C13$	5	2	0.41154957	2	0.048880777	17
D	$C14$	14	13	0.112446766	13	0.347983581	6
E, 1st	$C15$	2	5	0.342165439	5	0.118264907	14
E, 14th	$C16$	8	10	0.258557632	10	0.201872714	9
D	$C17$	10	9	0.270571534	9	0.189858812	10
D	$C18$	15	14	0.061579989	14	0.398850358	5
	Average			0.223613821		0.236816526	

E, . = elected, count; D = defeated

Table 22: 2013 elections: $a_7 = 0$, $r_3 = 0$

Status	Candidates	Plurality rank	Borda rank	a_i	Rank by a_i	r_i	Rank by r_i
E, 16th	$C01$	4	12	0.190746034	12	0.24559919	14
E, 17th	$C02$	7	19	0.086814448	19	0.349530775	7
E, 1st	$C03$	1	1	0.436345223	1	0	25
D	$C04$	16	15	0.15585912	15	0.280486103	11
E, 17th	$C05$	8	10	0.236453246	10	0.199891977	16
D	$C06$	21	23	0.031583211	23	0.404762012	3
D	$C07$	14	25	0	25	0.436345223	1
E, 15th	$C08$	2	7	0283429729	7	0.152915495	19
E, 17th	$C09$	9	11	0.213169799	11	0.223175425	15
E, 17th	$C10$	5	5	0.291589963	5	0.14475526	21
D,	$C11$	19	22	0.040900191	22	0.395445032	4
D	$C12$	17	17	0.099520648	17	0.336824575	9
D	$C13$	25	24	0.028738607	24	0.407606616	2
D	$C14$	24	20	0.079603916	20	0.356741308	6
D	$C15$	22	21	0.076115675	21	0.360229549	5
D	$C16$	10	9	0.239360864	9	0.196984359	17
D	$C17$	12	4	0.294452571	4	0.141892652	22
E, 16th	$C18$	6	3	0.351547204	3	0.08479802	23
D	$C19$	13	13	0.190313942	13	0.246031282	13
E, 16th	$C20$	3	6	0.284865534	6	0.151479689	20
D	$C21$	11	2	0.369384494	2	0.066960729	24
D	$C22$	18	14	0.159792956	14	0.276552267	12
D	$C23$	15	8	0.242038933	8	0.19430629	18
D	$C24$	23	18	0.087332058	18	0.349013165	8
D	$C25$	20	16	0.112577923	16	0.323767301	10
	Average			0.183301		0.253043772	

E, . = elected, count; D = defeated