The Inverse Eigenvalue Problem of a Graph

Wayne Barrett
Brigham Young University

Steve Butler
Iowa State University, butler@iastate.edu

Shaun Fallat
University of Regina

H. Tracy Hall
Brigham Young University

Leslie Hogben
Iowa State University, hogben@iastate.edu

See next page for additional authors

Follow this and additional works at: http://lib.dr.iastate.edu/math_conf

Part of the Algebra Commons, and the Discrete Mathematics and Combinatorics Commons

Recommended Citation
Barrett, Wayne; Butler, Steve; Fallat, Shaun; Hall, H. Tracy; Hogben, Leslie; Lin, Jephian C. H.; Shader, Bryan; and Young, Michael, "The Inverse Eigenvalue Problem of a Graph" (2016). Mathematics Conference Papers, Posters and Presentations. 4.
http://lib.dr.iastate.edu/math_conf/4

This Report is brought to you for free and open access by the Mathematics at Iowa State University Digital Repository. It has been accepted for inclusion in Mathematics Conference Papers, Posters and Presentations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
1 Overview of the Field

Inverse eigenvalue problems appear in various contexts throughout mathematics and engineering, and refer
to determining all possible lists of eigenvalues (spectra) for matrices fitting some description. The inverse
eigenvalue problem of a graph refers to determining the possible spectra of real symmetric matrices whose
pattern of nonzero off-diagonal entries is described by the edges of a given graph (precise definitions of this
and other terms are given in the next paragraph). This problem and related variants have been of interest for
many years and were originally approached through the study of ordered multiplicity lists. It was thought
by many researchers in the field that at least for a tree $T$, determining the ordered multiplicity lists of $T$
would suffice to determine the spectra of matrices described by $T$. When it was shown in [2] that this was
not the case, the focus of much of the research in the area shifted to the narrower question of maximum
multiplicity, or equivalently maximum nullity or minimum rank of matrices described by the graph. While
the maximum multiplicity has been determined for many families of graphs, including all trees, in general it
remains an open question and active area of research (see [4, 5] for extensive bibliographies). More recently,
there has been progress on the related question of determining the minimum number of distinct eigenvalues
of matrices described by a given graph [1, 3]. Maximum nullity, minimum number of distinct eigenvalues,
and ordered multiplicity lists all provide information that can in some cases be used to solve the inverse
eigenvalue problem for a specific graph, but the question of the structures or properties that are necessary to
allow this to be done more generally is fundamental.

For a (simple, undirected) graph $G = (V, E)$ with vertex set $V = \{1, \ldots, n\}$ and edge set $E$, the set of
symmetric matrices described by $G$, $S(G)$, is the set of all real symmetric $n \times n$ matrices $A = [a_{ij}]$ such that
for $i \neq j$, $a_{ij} \neq 0$ if and only if $ij \in E$. The maximum multiplicity of $G$ is

$$M(G) = \max\{\text{null}_A(\lambda) : A \in S(G), \lambda \in \text{spec}(A)\}$$

and the minimum rank of $G$ is $\text{mr}(G) = \min\{\text{rank} A : A \in S(G)\}$. It is easily seen that $M(G) = \max\{\text{null} A : A \in S(G)\}$ and $\text{mr}(G) + M(G) = |G|$, where $|G|$ is the number of vertices of $G$. For a real
symmetric matrix $A$, $q(A)$ denotes the number of distinct eigenvalues of $A$ and for a graph $G$, the minimum
number of distinct eigenvalues of $G$ is $q(G) = \min\{q(A) : A \in S(G)\}$. Given a real symmetric matrix $A$ with distinct eigenvalues $\lambda_1 < \cdots < \lambda_r$ having multiplicity $m_i$ for $\lambda_i$, $i = 1, \ldots, r$, the ordered multiplicity list of $A$ is $(m_1, \ldots, m_r)$. For a graph $G$, the set of ordered multiplicity lists of $G$ is the set of all ordered multiplicity lists of matrices $A \in S(G)$. Observe that $q(G)$ is the minimum number of entries in an ordered multiplicity list of $G$ and $M(G)$ is the maximum value of an entry in an ordered multiplicity list of $G$.

2 Recent Developments

At a research group meeting in Iowa in July 2015 attended by most of the participants of this FRG, several new tools were developed to attack the inverse eigenvalue problem of a graph [3]. These include the Strong Spectral Property (SSP) and Strong Multiplicity Property (SMP), matrix properties that generalize the Strong Arnold Property (SAP) (see [3] for precise definitions of the SSP and the SMP, and [6] for the SAP). All graphs having $q(G) \geq |G| - 1$ were characterized in [3].

3 Scientific Progress Made

While at BIRS we established several additional tools for the inverse eigenvalue problem of a graph: We defined and proved precise forms of minor monotonicity of SMP and SSP that preserve the multiplicities or eigenvalues of the minor in the larger graph but have some restrictions on additional multiplicities/eigenvalues. For matrices with SSP, we developed a method to produce another matrix with the same spectrum except one of the multiple eigenvalues has been split into one or more distinct eigenvalues.

We used minor monotonicity and subgraph monotonicity established in [3] to solve the inverse eigenvalue problem for graphs of order 5 (4 or less was known previously), and to determine the forbidden minors for graphs having at most one multiple eigenvalue; the latter are shown in Figure 1.

![Figure 1: Any graph that has at least two multiple eigenvalues must have one of these 11 graphs as a minor.](image)

4 Outcome of the Meeting

We are in the process of preparing a paper for submission to a journal describing the research discussed in Section 3; we expect to post a draft on arxiv by the end of 2016. We also made plans to work on extensions
of SSP and SMP.

5 Acknowledgement

We thank BIRS for providing a wonderful environment in which to conduct mathematical research.

References


