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Eventually Nonnegative Matrices and their Sign Patterns

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Abstract

A matrix A ∈ R n×n is eventually nonnegative (respectively, eventually positive) if there exists a positive integer k0 such that for all k ≥ k0, Ak ≥ 0 (respectively, Ak > 0). Here inequalities are entrywise and all matrices are real and square. An eigenvalue of A is dominant if its magnitude is equal to the spectral radius of A. A matrix A has the strong Perron-Frobenius property if A has a unique dominant eigenvalue that is positive, simple, and has a positive eigenvector. It is well known (see, e.g., [10]) that the set of matrices for which both A and AT have the strong Perron-Frobenius property coincides with the set of eventually positive matrices. Eventually nonnegative matrices and eventually positive matrices have applications to positive control theory (see, e.g., [13]).

Disciplines

Algebra | Discrete Mathematics and Combinatorics

Comments

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Eventually Nonnegative Matrices and their Sign Patterns

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1 Overview of the Field

A matrix $A \in \mathbb{R}^{n\times n}$ is *eventually nonnegative* (respectively, *eventually positive*) if there exists a positive integer k_0 such that for all $k \geq k_0$, $A^k \geq 0$ (respectively, $A^k > 0$). Here inequalities are entrywise and all matrices are real and square. An eigenvalue of A is *dominant* if its magnitude is equal to the spectral radius of A. A matrix A has the *strong Perron-Frobenius property* if A has a unique dominant eigenvalue that is positive, simple, and has a positive eigenvector. It is well known (see, e.g., [10]) that the set of matrices for which both A and A^T have the strong Perron-Frobenius property coincides with the set of eventually positive matrices. Eventually nonnegative matrices and eventually positive matrices have applications to positive control theory (see, e.g., [13]).

A *sign pattern (matrix)* is a matrix having entries in $\{+, -, 0\}$. For a real matrix A, $\mathrm{sgn}(A)$ is the sign pattern having entries that are the signs of the corresponding entries in A. The idea of studying sign patterns was introduced by the economist Paul Samuelson to model certain problems in economics for which the signs (but not the magnitudes) of the matrix entries are known. If $\mathcal{A}$ is an $n \times n$ sign pattern, the *sign pattern class* of $\mathcal{A}$, denoted $\mathcal{Q}(\mathcal{A})$, is the set of all $A \in \mathbb{R}^{n\times n}$ such that $\mathrm{sgn}(A) = \mathcal{A}$. If P is a property of a real matrix, then a sign pattern $\mathcal{A}$ *requires* P if every real matrix $A \in \mathcal{Q}(\mathcal{A})$ has property P, and $\mathcal{A}$ *allows* P or is *potentially* P if there is some $A \in \mathcal{Q}(\mathcal{A})$ that has property P. Numerous properties have been investigated from the point of view of characterizing sign patterns that require or allow a particular property (see, e.g, [5, 9] and the references therein).

Sign patterns that require eventual positivity or eventual nonnegativity are characterized in [7]. Potentially eventually positive (PEP) sign patterns are studied in [1], where several necessary or sufficient conditions are given for a sign pattern to be PEP, and PEP sign patterns of order at most three are characterized. Much less is known about whether a sign pattern is potentially eventually nonnegative (PEN) as compared with whether it is PEP, although there have been numerous papers on eventually nonnegative matrices (see for example [2, 3, 6, 8, 11, 12, 13, 14]).

2 Recent Developments and Problems Investigated

The study of PEP sign patterns utilizes the Perron-Frobenius eigenstructure of a positive matrix, and irreducible nonnegative matrices retain significant Perron-Frobenius properties. Motivated by this, we define a matrix to be *strongly eventually nonnegative* (SEN) if it is an eventually nonnegative matrix that has an irreducible nonnegative power. We also define a matrix to have the *semi-strong Perron Frobenius property* if its dominant eigenvalues are simple and nonzero, and its spectral radius has positive left and right eigenvectors. The class SSPF is those matrices A such that both A and A^T have the semi-strong Perron Frobenius property, so an SEN matrix is in SSPF.

Investigation of the classes of matrices SEN and SSPF led to consideration of additional classes, such as r-cyclic matrices. Our primary goal for the week at BIRS was to investigate sign patterns that allow matrices in SEN and/or SSPF and related classes. The requires problems for SEN and SSPF were not addressed because they have already been solved: It is shown in [7] that if an irreducible sign pattern $\mathcal{A}$ requires eventual nonnegativity, then $\mathcal{A}$ is nonnegative. Since an SSPF matrix must be irreducible, a sign pattern requires SSPF (or SEN) if and only if it is an irreducible nonnegative sign pattern.

3 Scientific Progress Made

Numerous results about PSEN and PSSPF sign patterns were established. The paper [4], which has been submitted, contains the details. The main results of our work are summarized below.

Theorem *If $\mathcal{A}$ is PSEN, then $\mathcal{A}$ is either PEP or r-cyclic.*

Theorem For $n \leq 3$, an $n \times n$ sign pattern $\mathcal{A}$ is PSEN if and only if $\mathcal{A}$ is PSSPF, and these sign patterns are characterized.

We identified the following question as a significant open problem for PSEN and PSSPF sign patterns.

Question *Is every $n \times n$ PSSPF sign pattern PSEN?*

Examples of matrices in SSPF that are not eventually nonnegative are known, but in each case the sign pattern of the matrix is PSEN (and in some cases PEP). If there is a PSSPF sign pattern that is not PSEN, then it must have order at least 4 (by the theorem above).

The diagram in Figure 1 below shows the relationship between classes of sign patterns studied, including 1) potentially eventually positive sign patterns (PEP), 2) potentially strongly eventually nonnegative sign patterns (PSEN), 3) sign patterns that have a realization A that has a simple positive dominant eigenvalue with positive right and left eigenvectors (PSSPF), 4) irreducible sign patterns (irreducible), 5) potentially eventually nonnegative sign patterns (PEN), 6) r-cyclic sign patterns (r-cyclic), 7) potentially nilpotent sign patterns (PN), and 8) nonnegative sign patterns (nonnegative). The regions marked with ?? would be empty if every PSSPF sign pattern were PSEN, i.e., if the answer to the Question were yes. For each of the other regions in the diagram, an example of a matrix in that region is provided in [4].

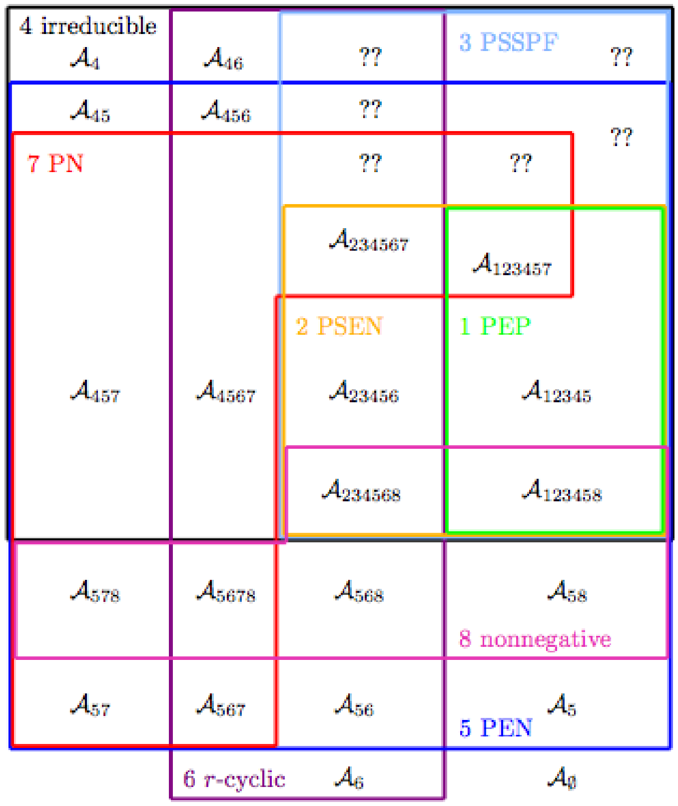

Figure 1: An Euler diagram of potentially eventually nonnegative sign patterns and related classes for patterns of order at least 2. The symbol $\mathcal{A}_{xyz}$ appearing in a region signifies we have an example of a pattern in this region. We do not have examples for the regions with a ?? in them. Those regions are empty if PSEN = PSSPF.

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References

[1] A. Berman, M. Catral, L.M. DeAlba, A. Elhashash, F. Hall, L. Hogben, I.-J. Kim, D.D. Olesky, P. Tarazaga, M.J. Tsatsomeros, P. van den Driessche, Sign patterns that allow eventual positivity, *Electronic Journal of Linear Algebra*, **19** (2010), 108–120.

[2] S. Carnochan Naqvi and J. J. McDonald, The combinatorial structure of eventually nonnegative matrices, *Electronic Journal of Linear Algebra*, **9** (2002), 255–269.

[3] S. Carnochan Naqvi and J. J. McDonald. Eventually nonnegative matrices are similar to seminonnegative matrices, *Linear Algebra and its Applications*, **281** (2004), 245–258.

[4] M. Catral, C. Erickson, L. Hogben, D. D. Olesky, P. van den Driessche, Sign patterns that allow eventual nonnegativity, under review.

[5] M. Catral, D.D. Olesky, P. van den Driessche, Allow problems concerning spectral properties of sign pattern matrices: A survey, *Linear Algebra and its Applications*, **430** (2009), 3080–3094.

[6] A. Elhashhash and D. Szyld, On general matrices having the Perron-Frobenius property, *Electronic Journal of Linear Algebra*, **17** (2008), 389–413.

[7] E. M. Ellison, L. Hogben, M. J. Tsatsomeros, Sign patterns that require eventual positivity or require eventual nonnegativity, *Electronic Journal of Linear Algebra*, **19** (2010), 98–107.

[8] S. Friedland, On an inverse problem for nonnegative and eventually nonnegative matrices, *Israel Journal of Mathematics*, **29** (1978), 43–60.

[9] F. J. Hall and Z. Li. Sign Pattern Matrices. In *Handbook of Linear Algebra, (L. Hogben, ed.)*, Chapman & Hall/CRC Press, Boca Raton, 2007.

[10] D. Handelman, Positive matrices and dimension groups affiliated to C^*-algebras and topological Markov chains, *Journal of Operator Theory* **6** (1981), 55–74.

[11] L. Hogben, Eventually cyclic matrices and a test for strong eventual nonnegativity, *Electronic Journal of Linear Algebra*, **19** (2010), 129–140.

[12] D. Noutsos and M. Tsatsomeros, Reachability and holdability of nonnegative states, *SIAM Journal on Matrix Analysis and Applications*, **30** (2008): 700–712.

[13] B. G. Zaslavsky and J. J. McDonald, A characterization of Jordan canonical forms which are similar to eventually nonnegative matrices with the properties of nonnegative matrices, *Linear Algebra and its Applications*, **372** (2003), 253–285.

[14] B. G. Zaslavsky and B.-S. Tam, On the Jordan form of an irreducible matrix with eventually nonnegative powers, *Linear Algebra and its Applications*, **302-303** (1999), 303–330.