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Abstract

Let V be a congruence modular variety satisfying (C2) whose two generated free algebra is finite. If V has AP and RS then V has (R). Here (C2) and (R) are two properties defined in terms of the commutator in modular varieties.

Keywords

amalgamation property, abelian, subalgebra

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Comments

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ANOTHER CONSEQUENCE OF AP IN RESIDUALLY SMALL, CONGRUENCE MODULAR VARIETIES

CLIFFORD BERGMAN

Abstract. Let \mathcal{V} be a congruence modular variety satisfying (C2) whose two generated free algebra is finite. If \mathcal{V} has AP and RS then \mathcal{V} has (R). Here (C2) and (R) are two properties defined in terms of the commutator in modular varieties.

In this paper we continue the investigation of residually small, congruence modular varieties with the amalgamation property. In [1] we showed that $RS + AP \Rightarrow CEP$ in a congruence modular variety that satisfied 3 crucial properties: (C2), (R) and 4-finiteness (defined below). The purpose of this paper is to show that the assumption that the variety satisfies (R) follows from the remaining ones. Following the proof of Corollary 9, we present a detailed example that may aid in the comprehension of the arguments.

We use fairly standard notation of universal algebra. Algebras are denoted by boldface latin letters and their underlying sets by the lightface equivalent. $\text{Con}(\mathbf{A})$ denotes the lattice of congruences of the algebra \mathbf{A} , with smallest and largest congruences 0 and 1. Congruences are represented by lower case Greek letters. $\alpha \wedge \beta$, $\alpha \vee \beta$ and $[\alpha, \beta]$ are the meet, join and commutator of the congruences α and β .

In a congruence modular variety, the commutator is a binary operation with a number of powerful properties. See [2, Section 4] or [4, Section 6] for the details and proofs. We say that an algebra \mathbf{A} has property (C2) if for all $\alpha, \beta \in \text{Con}(\mathbf{A})$, $[\alpha, \beta] = \alpha \wedge \beta \wedge [1, 1]$, and has property (R) if for all subalgebras \mathbf{B} of \mathbf{A} , $[1_{\mathbf{A}}, 1_{\mathbf{A}}] \upharpoonright \mathbf{B} = [1_{\mathbf{B}}, 1_{\mathbf{B}}]$. A variety \mathcal{V} has (C2) or (R) just in case every member has the property. We comment that $[1_{\mathbf{A}}, 1_{\mathbf{A}}] \upharpoonright \mathbf{B} \geq [1_{\mathbf{B}}, 1_{\mathbf{B}}]$ holds for all algebras \mathbf{A} and subalgebras \mathbf{B} , thus (R) asserts the reverse inclusion.

We say a variety \mathcal{V} is n -finite if the \mathcal{V} -free algebra on n generators is finite, where n is a natural number. If $f : \mathbf{A} \rightarrow \mathbf{B}$ is a homomorphism

and $\alpha \in \text{Con}(\mathbf{B})$, then $f^{-1}(\alpha) = \{(a, b) \in A \times A : (f(a), f(b)) \in \alpha\}$ is a congruence on \mathbf{A} . We say f is an *essential extension* if f is injective and $f^{-1}(\alpha) = 0_{\mathbf{A}} \Rightarrow \alpha = 0_{\mathbf{B}}$ for all $\alpha \in \text{Con}(\mathbf{B})$. We write $\mathbf{A} \leq_E \mathbf{B}$ to indicate that there is an essential extension from \mathbf{A} to \mathbf{B} (or, informally, that \mathbf{B} is an essential extension of \mathbf{A}). If f is any embedding and $\alpha \in \text{Con}(\mathbf{B})$ then f/α denotes the induced embedding from $\mathbf{A}/f^{-1}(\alpha)$ to \mathbf{B}/α given by $f/\alpha(a/f^{-1}(\alpha)) = f(a)/\alpha$, for $a \in A$.

Assume throughout that \mathcal{V} is a congruence modular variety satisfying (C2), AP and RS. In [7] it was proved that for any $\mathbf{A} \in \mathcal{V}$, $[1_{\mathbf{A}}, 1_{\mathbf{A}}]$ is *neutral* in $\text{Con}(\mathbf{A})$, that is, for any $\alpha, \beta \in \text{Con}(\mathbf{A})$, $\{\alpha, \beta, [1, 1]\}$ generates a distributive sublattice.

Suppose \mathbf{A} is a member of \mathcal{V} , θ_0 and θ_1 congruences of \mathbf{A} such that $\theta_0 \wedge \theta_1 = 0$. Then for any congruence α , $\alpha = (\alpha \vee \theta_0) \wedge (\alpha \vee \theta_1) \wedge (\alpha \vee [1, 1])$. For, by modularity,

$$\begin{aligned} (\alpha \vee \theta_0) \wedge (\alpha \vee \theta_1) \wedge (\alpha \vee [1, 1]) &= \alpha \vee ((\alpha \vee \theta_0) \wedge (\alpha \vee \theta_1) \wedge [1, 1]) \\ &= \alpha \vee [\alpha \vee \theta_0, \alpha \vee \theta_1] \text{ (by (C2))} \\ &= \alpha \vee [\theta_0, \theta_1] \\ &= \alpha. \end{aligned}$$

If A/θ_1 is abelian, then $\theta_1 \geq [1, 1]$, so the above identity reduces to: $\alpha = (\alpha \vee \theta_0) \wedge (\alpha \vee [1, 1])$. Furthermore, if $\theta_0 \circ \theta_1 = 1$ then, since the commutator of a product is the product of the commutators [2, 4.6], $\alpha \geq (\alpha \vee \theta_0) \wedge (\alpha \vee \theta_1) \wedge [1, 1] = ((\alpha \vee \theta_0) \wedge \xi_0) \wedge ((\alpha \vee \theta_1) \wedge \xi_1)$ where $\xi_i/\theta_i = [1, 1]$ on \mathbf{A}/θ_i , for $i = 0, 1$.

We first recall two lemmas from [1]. The first is true in any congruence modular variety.

LEMMA 1. *Let $f : \mathbf{B} \rightarrow \mathbf{A}$ be an essential estension. Suppose θ and ψ are congruences on \mathbf{A} such that θ is meet-irreducible, $\psi \neq 0$ and $\theta \wedge \psi = 0$. Then $f/\theta : \mathbf{B}/f^{-1}(\theta) \rightarrow \mathbf{A}/\theta$ is essential.*

LEMMA 2. *Let \mathbf{A} be subdirectly irreducible and suppose $\mathbf{B}_0 \times \mathbf{B}_1 \leq_E \mathbf{A}$. Then either \mathbf{B}_0 or \mathbf{B}_1 is trivial.*

We introduce some special notation for congruences on product algebras. If $(\mathbf{D}_i : i \in I)$ is a family of algebras and $\mathbf{D} = \Pi(\mathbf{D}_i : i \in I)$, then $p_i^{\mathbf{D}}$ is the canonical projection homomorphism from \mathbf{D} to \mathbf{D}_i . The kernel of $p_i^{\mathbf{D}}$

is η_i^D . The symbols p and η are reserved for this purpose. If γ is a congruence on \mathbf{D}_i , then γ_i^- denotes the congruence $p_i^{-1}(\gamma)$ on D . In particular, ξ_i denotes $[1, 1]_i^-$.

LEMMA 3. *Let $f_i : \mathbf{B}_i \rightarrow \mathbf{A}_i$ be an essential estension and \mathbf{A}_i subdirectly irreducible and non-abelian for $i = 0, 1$. Then $f_0 \times f_1 : \mathbf{B}_0 \times \mathbf{B}_1 \rightarrow \mathbf{A}_0 \times \mathbf{A}_1$ is essential.*

PROOF: Suppose $\psi \in \text{Con}(\mathbf{A}_0 \times \mathbf{A}_1)$ such that $(f_0 \times f_1)^{-1}(\psi) = 0$. By (C2), $\psi \geq (\psi \vee \eta_0^A) \wedge \xi_0 \wedge (\psi \vee \eta_1^A) \wedge \xi_1$. Let β_i be the congruence on \mathbf{A}_i corresponding to $\psi \vee \eta_i$ under the canonical projection, for $i = 0, 1$. Then $0 = (f_0 \times f_1)^{-1}(\psi) \geq f_0^{-1}(\beta_0 \wedge [1, 1]) \times f_1^{-1}(\beta_1 \wedge [1, 1])$, which implies that $f_i^{-1}(\beta_i \wedge [1, 1]) = 0$ for $i = 0, 1$. Since f_i is essential, $\beta_i \wedge [1, 1] = 0$ on \mathbf{A}_i and, since \mathbf{A}_i is subdirectly irreducible and non-abelian, $\beta_i = 0$. Thus $\psi \vee \eta_i^A = \eta_i^A$, so $\psi \leq \eta_i$ for $i = 0, 1$. Therefore $\psi \leq \eta_0 \wedge \eta_1 = 0$ as desired. ■

LEMMA 4. *Let $\mathbf{B} \leq_E \mathbf{A}$ with $\mathbf{A} \in \mathcal{V}$. Then for any $\mathbf{D} \in \mathcal{V}$, $\mathbf{B} \times \mathbf{D} \leq_E \mathbf{A} \times \mathbf{D}$.*

PROOF: Let $\psi \in \text{Con}(\mathbf{A} \times \mathbf{D})$ such that $\psi \upharpoonright (B \times D) = 0$. We first show $\psi \leq \eta_0^{A \times D}$. Suppose not. Then $\psi \vee \eta_0^{A \times D} > \eta_0^{A \times D}$ so, since \mathbf{A} is an essential extension of \mathbf{B} , there are $b_1, b_2 \in B, d \in D$ such that $b_1 \neq b_2$ but $(b_1, d) \equiv (b_2, d) \pmod{\psi \vee \eta_0^{A \times D}}$. By Gumm [4, 4.5], η_0 permutes with every congruence on $\mathbf{A} \times \mathbf{D}$, thus there is $(a, d') \in A \times D$ such that $(b_1, d)\eta_0^{A \times D}(a, d')\psi(b_2, d)$, which implies $(b_1, d')\psi(b_2, d)$. But $\psi \upharpoonright (B \times D) = 0$ implies $b_1 = b_2$ which is a contradiction.

Now, by modularity, $\psi = \eta_0^{A \times D} \wedge (\psi \vee \eta_1^{A \times D})$. Therefore $0 = \psi \upharpoonright (B \times D) = \eta_0^{B \times D} \wedge (\psi \vee \eta_1^{A \times D}) \upharpoonright (B \times D)$. But $(\psi \vee \eta_1^{A \times D}) \upharpoonright (B \times D) \geq \eta_1^{B \times D}$ and both are complements of $\eta_0^{B \times D}$, so $(\psi \vee \eta_1^{A \times D}) \upharpoonright (B \times D) = \eta_1^{B \times D}$. It follows that $\psi \vee \eta_1^{A \times D} = \eta_1^{A \times D}$, and therefore $\psi \leq \eta_0^{A \times D} \wedge \eta_1^{A \times D} = 0$. ■

LEMMA 5. *Let \mathbf{A} be subdirectly irreducible and non-abelian and let $\mathbf{B} \in \mathcal{V}$. Suppose $(\sigma_i : i \in I)$ is a family of congruences on $\mathbf{A} \times \mathbf{B}$ whose meet is 0. Then for some $j \in J, \sigma_j \leq \eta_0^{A \times B}$.*

PROOF: For each $i \in I, \sigma_i \geq (\sigma_i \vee \eta_0) \wedge \xi_0 \wedge (\sigma_i \vee \eta_1) \wedge \xi_1$. Thus $0 = \xi_0 \wedge \bigwedge (\sigma_i \vee \eta_0 : i \in I) \wedge \xi_1 \wedge \bigwedge (\sigma_i \vee \eta_1 : i \in I)$. Joining with η_0 and applying modularity, $\eta_0 = \xi_0 \wedge \bigwedge (\sigma_i \vee \eta_0) \wedge (\eta_0 \vee (\xi_1 \wedge \bigwedge (\sigma_i \vee \eta_1)))$. Now η_0 is completely meet-irreducible, so either $\eta_0 = \xi_0$ or $\eta_0 = (\sigma_j \vee \eta_0)$ some $j \in I$, or $\eta_0 = \eta_0 \vee (\xi_1 \wedge \bigwedge (\sigma_i \vee \eta_1))$. The first is impossible since A is non-abelian and the last because it implies that $\eta_0 \geq \eta_1$. Thus $\eta_0 \geq \sigma_j$ as desired. ■

LEMMA 6. *let $\mathbf{B} \leq_E \mathbf{A}$, \mathbf{A} subdirectly irreducible, \mathbf{B} finite and non-abelian. Then \mathbf{B} is subdirectly irreducible.*

PROOF: Suppose not. Write \mathbf{B} as a subdirect product of subdirectly irreducible algebras $(\mathbf{B}_i : i \in I)$. Since B is finite, we can assume that I is finite and furthermore that no proper subset of I yields a decomposition of \mathbf{B} . Since \mathbf{B} is non-abelian, there is a $k \in I$ such that \mathbf{B}_k is non-abelian. Let θ be the kernel of the projection of $\prod \mathbf{B}_i$ onto \mathbf{B}_k and θ' the kernel onto $\prod(\mathbf{B}_i : i \neq k)$. As \mathbf{B} is assumed to be subdirectly reducible, $\theta \upharpoonright B \neq 0$.

Let ψ be a maximal member of $\{\gamma \in \text{Con}(\prod \mathbf{B}_i) : \gamma \upharpoonright B = 0\}$, which exists by Zorn's lemma. We claim that $\psi \leq \theta$. For, by (C2) $\psi \geq (\psi \vee \theta) \wedge (\psi \vee \theta') \wedge [1, 1]$. Therefore, $0 = \psi \upharpoonright B = (\psi \vee \theta) \upharpoonright B \wedge (\psi \vee \theta') \upharpoonright B \wedge [1_{\mathbf{B}}, 1_{\mathbf{B}}]$. Letting ν denote the kernel of the projection of \mathbf{B} onto \mathbf{B}_k , $\nu = \nu \vee 0 = (\psi \vee \theta) \upharpoonright B \wedge (\nu \vee (\psi \vee \theta') \upharpoonright B) \wedge (\nu \vee [1, 1])$ by modularity and the neutrality of $[1, 1]$. Now $\mathbf{B}/\nu \cong \mathbf{B}_k$ so ν is meet-irreducible. But $\nu \not\geq [1, 1]$ since \mathbf{B}_k is non-abelian, and $\nu \geq (\psi \vee \theta') \upharpoonright B \geq \theta' \upharpoonright B$ implies $\theta' \upharpoonright B = \theta' \upharpoonright B \wedge \nu = (\theta' \wedge \theta) \upharpoonright B = 0$, contradicting the minimality of I . Thus $\nu = (\psi \vee \theta) \upharpoonright B$. As there is a one-to-one correspondence between the congruences of \mathbf{B} above ν and those of $\prod \mathbf{B}_i$ above θ , we conclude that $\psi \vee \theta = \theta$ as claimed.

Therefore $\psi = \theta \wedge (\psi \vee \theta')$, so $(\prod \mathbf{B}_i)/\psi \cong \mathbf{B}_k \times \mathbf{C}$, where $\mathbf{C} = \prod \mathbf{B}_i / (\psi \vee \theta')$. Furthermore, by the maximality of ψ , $\mathbf{B}_k \times \mathbf{C}$ is an essential extension of \mathbf{B} . Now we apply *AP* to $(\mathbf{B}, \mathbf{A}, \mathbf{B}_k \times \mathbf{C}, f_1, g_1)$ where f_1 is the essential embedding of \mathbf{B} into \mathbf{A} , and g_1 the embedding established above. Let (\mathbf{D}, f_2, g_2) be the resulting completion of the diagram. Let δ be a maximal member of $\{\gamma \in \text{Con}(\mathbf{D}) : (f_2 \circ f_1)^{-1}(\gamma) = 0\}$, and let $q : \mathbf{D} \rightarrow \mathbf{D}/\delta$ be the canonical projection. Then $q \circ f_2 \circ f_1 = q \circ g_2 \circ g_1$ is an essential extension and since f_1 and g_1 are essential, we conclude that $q \circ f_2$ and $q \circ g_2$ are embeddings, and in fact are essential as well.

Now, \mathbf{A} is subdirectly irreducible and $q \circ f_2$ is an essential embedding of \mathbf{A} into \mathbf{D}/δ , so \mathbf{D}/δ must be subdirectly irreducible. But by $q \circ g_2$, \mathbf{D}/δ is an essential extension of $\mathbf{B}_k \times \mathbf{C}$, so by Lemma 2, either \mathbf{B}_k or \mathbf{C} is trivial. But \mathbf{B}_k is non-abelian, so is non-trivial, and $\mathbf{C} = \prod \mathbf{B}_i / (\theta' \vee \psi)$ trivial implies $\theta' \vee \psi = 1$, so that $\psi = \theta$. But then $\theta \upharpoonright B = 0$ which is a contradiction. ■

THEOREM 7. *Let \mathbf{B} be abelian, $\mathbf{B} \leq_E \mathbf{A}$ and \mathbf{A} subdirectly irreducible. Then \mathbf{A} is abelian.*

PROOF: Suppose not. Without loss of generality, \mathbf{A} is a maximal essential extension of \mathbf{B} . Let $f : \mathbf{B} \rightarrow \mathbf{A}$ be that extension. Recall from [2], that the congruence $\Delta_{1,1}$ of $\mathbf{B} \times \mathbf{B}$ is a common complement of the coordinate projection kernels, η_0 and η_1 . Hence, with $\mathbf{C} = \mathbf{B}^2/\Delta_{1,1}$, there is an isomorphism $h : \mathbf{B}^2 \rightarrow \mathbf{B} \times \mathbf{C}$ such that $h(x, y) = (x, y')$ for some $y' \in \mathbf{C}$. By Lemma 4, $g_1 = (f \times id_{\mathbf{C}}) \circ h$ is an essential embedding of \mathbf{B}^2 into $\mathbf{A} \times \mathbf{C}$, and by Lemma 3, $f_1 = f \times f$ is an essential embedding of \mathbf{B}^2 into \mathbf{A}^2 . We apply AP to $(\mathbf{B}^2, \mathbf{A}^2, \mathbf{A} \times \mathbf{C}, f_1, g_1)$. Arguing as we did in the previous lemma, the diagram is completed by (\mathbf{D}, f_2, g_2) in which f_2 and g_2 are essential embeddings of \mathbf{A}^2 and $\mathbf{A} \times \mathbf{C}$ respectively, into \mathbf{D} .

We claim first that there is a congruence ψ on \mathbf{D} such that $f_2^{-1}(\psi) = \eta_1^{A \times A}$. Let $(\nu_i : i \in I)$ be a family of completely meet-irreducible congruences on \mathbf{D} with $\bigwedge \nu_i = 0$. Let $\sigma_i = f_2^{-1}(\nu_i)$ for all $i \in I$. Applying Lemma 5 twice, there are indices $j, k \in I$ such that $\sigma_j \leq \eta_0^{A \times A}$ and $\sigma_k \leq \eta_1^{A \times A}$. Therefore $0 = \sigma_j \wedge \sigma_k = f_2^{-1}(\nu_j \wedge \nu_k)$ which implies that $\nu_j \wedge \nu_k = 0$ on \mathbf{D} . Now if $\nu_j = 0$ then \mathbf{D} is subdirectly irreducible and f_2 is an essential extension of \mathbf{A}^2 into \mathbf{D} which is impossible as Lemma 2 would imply that \mathbf{A} is trivial (hence abelian). Therefore, by Lemma 1, $f_2/\nu_k : \mathbf{A} \times \mathbf{A}/\sigma_k \rightarrow \mathbf{D}/\nu_k$ is essential. But $\sigma_k \leq \eta_1^{A \times A}$ means that $\mathbf{A} \times \mathbf{A}/\sigma_k \cong \mathbf{A} \times \mathbf{A}/(\eta_0 \vee \sigma_k) \times \mathbf{A}$, and again by Lemma 2, $\eta_0 \vee \sigma_k = 1$, thus $\sigma_k = \eta_1$ by modularity. Thus $\psi = \nu_k$ is the desired congruence.

Now we take $\delta = g_2^{-1}(\psi)$ on $\mathbf{A} \times \mathbf{C}$. Observe that $g_1^{-1}(\delta) = (f_2 \circ f_1)^{-1}(\psi) = f_1^{-1}(\eta_1^{A \times A}) = \eta_1^{B \times B}$. Therefore on $\mathbf{A} \times \mathbf{C}$, $\eta_0^{A \times C} \wedge \delta = \eta_1^{A \times C} \wedge \delta = 0$ since $g_1^{-1}(\eta_0^{A \times C} \wedge \delta) = \eta_0^{B \times B} \wedge \eta_1^{B \times B} = 0$ and $g_1^{-1}(\eta_1^{A \times C} \wedge \delta) = \Delta_{1,1} \wedge \eta_1^{B \times B} = 0$, and g_1 is essential. Since \mathbf{C} is abelian, $[1, 1] \leq \eta_1^{A \times C}$. Thus, applying the neutrality of $[1, 1]$, $\eta_0^{A \times C} \leq (\eta_0^{A \times C} \vee \delta) \wedge (\eta_0^{A \times C} \vee [1, 1]) = \eta_0^{A \times C} \vee (\delta \wedge [1, 1]) \leq \eta_0^{A \times C} \vee (\delta \wedge \eta_1^{A \times C}) = \eta_0^{A \times C}$. But \mathbf{A} is subdirectly irreducible and non-abelian, so we conclude from this that $\eta_0 \geq \delta$. But then $0 = \eta_0^{A \times C} \wedge \delta = \delta$. This is impossible as $g_1^{-1}(\delta) = \eta_1^{B \times B} \neq 0$. ■

THEOREM 8. *Let \mathcal{V} be 2-finite. Then \mathcal{V} has (R).*

PROOF: Suppose to the contrary, that \mathcal{V} fails to satisfy (R).

CLAIM. *There are algebras $\mathbf{B}, \mathbf{A} \in \mathcal{V}_{SI}$, both non-abelian, \mathbf{B} finite, $\mathbf{B} \leq_E \mathbf{A}$ and $[1_{\mathbf{A}}, 1_{\mathbf{A}}] \uparrow B > [1_{\mathbf{B}}, 1_{\mathbf{B}}]$.*

PROOF OF CLAIM: Since (R) fails, there are algebras $\mathbf{B}'', \mathbf{A}'' \in \mathcal{V}$ such that $[1_{\mathbf{A}''}, 1_{\mathbf{A}''}] \uparrow B'' > [1_{\mathbf{B}''}, 1_{\mathbf{B}''}]$. Pick $(a, b) \in [1_{\mathbf{A}''}, 1_{\mathbf{A}''}] \uparrow B'' - [1_{\mathbf{B}''}, 1_{\mathbf{B}''}]$

and let \mathbf{B}' be the subalgebra of \mathbf{B}'' generated by $\{a, b\}$. Then B' is finite, $\mathbf{B}' \leq \mathbf{B}'' \leq \mathbf{A}''$, so $(a, b) \notin [1_{\mathbf{B}'}, 1_{\mathbf{B}'}]$.

Now let γ be a maximal member of $\{\delta \in \text{Con}(\mathbf{A}'') : \delta \upharpoonright B' = 0\}$ and let $\mathbf{A}' = \mathbf{A}''/\gamma$. Then $\mathbf{B}' \leq_E \mathbf{A}'$ and $[1_{\mathbf{A}'}, 1_{\mathbf{A}'}] \upharpoonright B' = ([1_{\mathbf{A}''}, 1_{\mathbf{A}''}] \vee \gamma) \upharpoonright B' > [1_{\mathbf{B}'}, 1_{\mathbf{B}'}]$ by the homomorphism property of the commutator [2, 4.4(1)] or [4, 6.9].

Let $(\pi_i : i \in I)$ be a family of completely meet-irreducible congruences on \mathbf{A}' whose meet is 0. As B' is finite, the set $\{\pi_i \upharpoonright B' : i \in I\}$ is finite. Let J be a minimal (finite) subset of I such that $\bigwedge(\pi_j \upharpoonright B' : j \in J) = 0$. Since $\mathbf{B}' \leq_E \mathbf{A}'$, $\bigwedge(\pi_j : j \in J) = 0$ on \mathbf{A}' as well. By the neutrality of $[1_{\mathbf{B}'}, 1_{\mathbf{B}'}]$, $[1_{\mathbf{B}'}, 1_{\mathbf{B}'}] = \bigwedge([1_{\mathbf{B}'}, 1_{\mathbf{B}'}] \vee \pi_j \upharpoonright B' : j \in J)$. Therefore, there is $k \in J$ such that $(a, b) \notin [1_{\mathbf{B}'}, 1_{\mathbf{B}'}] \vee \pi_k \upharpoonright B'$.

Now we apply Lemma 1 with $\theta = \pi_k$, $\psi = \bigwedge(\pi_j : j \neq k)$. ψ is nonzero by the minimality of J . Thus with $\mathbf{A} = \mathbf{A}'/\pi_k$ and $\mathbf{B} = \mathbf{B}'/\pi_k \upharpoonright B'$, we have $\mathbf{B} \leq_E \mathbf{A}$ (by the lemma), \mathbf{A} subdirectly irreducible (since π_k is completely meet-irreducible) and \mathbf{B} finite. Furthermore, $(a, b) \in [1_{\mathbf{A}'}, 1_{\mathbf{A}'}]$ implies $(a/\theta, b/\theta) \in [1_{\mathbf{A}}, 1_{\mathbf{A}}]$, and by the choice of k , $(a/\theta, b/\theta) \notin [1_{\mathbf{B}}, 1_{\mathbf{B}}]$. Thus, $a/\theta \neq b/\theta$, so \mathbf{A} is non-abelian. Therefore, by Theorem 7, \mathbf{B} is non-abelian and so by Lemma 6, \mathbf{B} is subdirectly irreducible. This proves the claim.

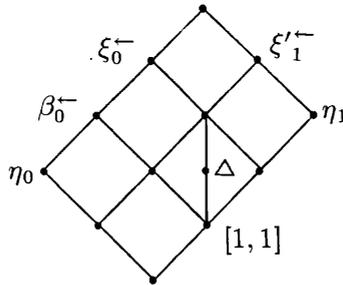


Figure 1

For the remainder of the proof, let $\bar{\xi} = [1_{\mathbf{A}}, 1_{\mathbf{A}}]$, $\xi = \bar{\xi} \upharpoonright B$ and $\beta = [1_{\mathbf{B}}, 1_{\mathbf{B}}]$. Consider the congruence $\Delta_{1, \xi}$ of \mathbf{B} . Recall from [2, 4.7(2)] that $\Delta_{1, \xi}$ is the congruence on \mathbf{B}^2 generated by all pairs $((\begin{smallmatrix} x \\ x \end{smallmatrix}) (\begin{smallmatrix} y \\ y \end{smallmatrix}))$ such that $x \xi y$ in \mathbf{B} . (Here we are employing the Freese-McKenzie convention of denoting the elements of B^2 as vertical pairs.) By [2, 4.7 — 4.11], $y[1, \xi]z \Leftrightarrow$

$(\begin{smallmatrix} x \\ y \end{smallmatrix}) \Delta_{1,\xi}(\begin{smallmatrix} x \\ z \end{smallmatrix}) \Leftrightarrow (\begin{smallmatrix} y \\ x \end{smallmatrix}) \Delta_{1,\xi}(\begin{smallmatrix} z \\ x \end{smallmatrix})$, some $x \in B$. By (C2), $[1_{\mathbf{B}}, \xi] = \xi \wedge [1_{\mathbf{B}}, 1_{\mathbf{B}}] = \xi \wedge \beta = \beta$ as $\xi = [1_{\mathbf{A}}, 1_{\mathbf{A}}] \upharpoonright B > \beta$. Therefore, $\eta_0 \wedge \Delta_{1,\xi} = \eta_0 \wedge \beta_1^-$ since $(\begin{smallmatrix} x \\ y \end{smallmatrix}) \Delta_{1,\xi}(\begin{smallmatrix} x \\ z \end{smallmatrix}) \Leftrightarrow y[1, \xi]z \Leftrightarrow y\beta z \Leftrightarrow (\begin{smallmatrix} x \\ y \end{smallmatrix}) \beta_1^-(\begin{smallmatrix} x \\ z \end{smallmatrix})$. Similarly, $\eta_1 \wedge \Delta_{1,\xi} = \eta_1 \wedge \beta_0^-$. Also $\eta_i^{B \times B} \vee \Delta_{1,\xi} = \xi_i^-$, $i = 0, 1$. To see this note that every generator of $\Delta_{1,\xi}$ is a member of ξ_i^- , thus $\eta_i \vee \Delta_{1,\xi} \leq \xi_i^-$. On the other hand, $(\begin{smallmatrix} x \\ y \end{smallmatrix}) \xi_0^-(\begin{smallmatrix} u \\ v \end{smallmatrix}) \Rightarrow x \xi u \Rightarrow (\begin{smallmatrix} x \\ y \end{smallmatrix}) \eta_0(\begin{smallmatrix} x \\ u \end{smallmatrix}) \Delta_{1,\xi}(\begin{smallmatrix} u \\ v \end{smallmatrix}) \eta_0(\begin{smallmatrix} u \\ v \end{smallmatrix})$.

Now, let $\mathbf{G} = \mathbf{B}/\beta$. Of course \mathbf{G} is finite and abelian. $(\mathbf{B} \times \mathbf{B})/(\eta_0 \wedge \beta_1^-) \cong \mathbf{B} \times \mathbf{G}$, and we define Δ to be the projection of $\Delta_{1,\xi}$ on $\mathbf{B} \times \mathbf{G}$ and ξ' the projection of ξ on \mathbf{G} . Then $\text{Con}(\mathbf{B} \times \mathbf{G})$ contains the lattice of Figure 1 as a sublattice. Observe that $[1, 1] = [1, 1]_0^- \wedge [1, 1]_1^- = \beta_0^- \wedge \eta_1$ on $\mathbf{B} \times \mathbf{G}$.

Let $f : \mathbf{B} \rightarrow \mathbf{A}$ be the inclusion map and i the identity of \mathbf{G} . By Lemma 4, $f \times i$ is an essential embedding. Let $\mathbf{C} = (\mathbf{B} \times \mathbf{G})/\Delta$. As $[1, 1] \leq \Delta$, \mathbf{C} is abelian. Let $g : \mathbf{B} \times \mathbf{G} \rightarrow \mathbf{A} \times \mathbf{C}$ given by $g(x, y) = (f(x), (x, y)/\Delta)$. Then $\ker(g) = \eta_0^{B \times G} \wedge \Delta = 0$, so g is an embedding. Note that $g^{-1}[1, 1] = g^{-1}(\xi_0^- \wedge \eta_1^{A \times C}) = \xi_0^- \wedge \Delta = \Delta$. We apply the amalgamation property to $(\mathbf{B} \times \mathbf{G}, \mathbf{A} \times \mathbf{G}, \mathbf{A} \times \mathbf{C}, f \times i, g)$ yielding $(\mathbf{E}, \bar{f}, \bar{g})$ (see Figure 2).

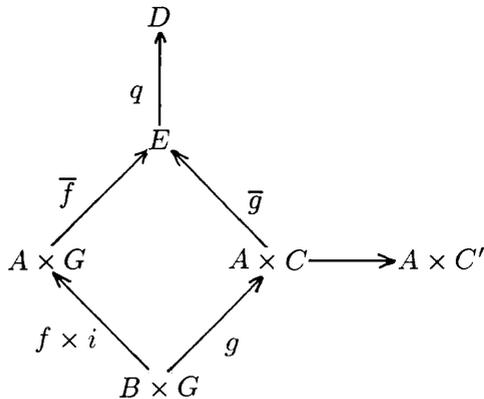


Figure 2

Let γ be a maximal member of $\{\delta \in \text{Con}(\mathbf{E}) : (\bar{g} \circ g)^{-1}(\delta) = 0\}$, and let $q : \mathbf{E} \rightarrow \mathbf{E}/\gamma = \mathbf{D}$ be the canonical projection. Since Figure 2 commutes

$q \circ \bar{f} \circ (f \times i)$ is essential, so $q \circ \bar{f}$ is an embedding and is essential. Let $\psi = \bar{g}^{-1}(\gamma)$ on $\mathbf{A} \times \mathbf{C}$. Thus, $g^{-1}(\psi) = 0$.

We claim $\psi \leq \eta_0^{A \times C}$.

PROOF: Suppose not. Then $\eta_0^{A \times C} < \psi \vee \eta_0^{A \times C} = \alpha_0^-$ for some $\alpha \in \text{Con}(\mathbf{A})$. Since f is essential, $f^{-1}(\alpha) \neq 0$. Therefore $g^{-1}(\alpha_0^-) \neq 0$ on $\mathbf{B} \times \mathbf{G}$. By (C2) and the abelianness of \mathbf{C} , $\psi = (\psi \vee \eta_0^{A \times C}) \wedge (\psi \vee [1, 1]) \geq \alpha_0^- \wedge [1, 1]$. Therefore, $0 = g^{-1}(\psi) \geq g^{-1}(\alpha_0^-) \wedge \Delta$. By modularity, $\eta_0^{B \times G} = \eta_0^{B \times G} \vee (g^{-1}(\alpha_0^-) \wedge \Delta) = g^{-1}(\alpha_0^-) \wedge (\eta_0^{B \times G} \vee \Delta)$. Since \mathbf{B} is subdirectly irreducible, $\eta_0^{B \times G}$ is meet-irreducible, so $\eta_0^{B \times G} = g^{-1}(\alpha_0^-)$ or $\eta_0^{B \times G} \geq \Delta$. Both conclusions are false, so we have a contradiction.

By modularity, $\psi = \eta_0^{A \times C} \wedge (\psi \vee \eta_1^{A \times C}) = \eta_1 \vee \delta_1^-$ for some $\delta \in \text{Con} \mathbf{C}$. Therefore $(\mathbf{A} \times \mathbf{C})/\psi \cong \mathbf{A} \times \mathbf{C}'$ with $\mathbf{C}' = \mathbf{C}/\delta$. We now have a new commuting diagram (Figure 3) in which $h_0 = g/\psi$ is the composition $\mathbf{B} \times \mathbf{G} \rightarrow \mathbf{A} \times \mathbf{C} \rightarrow \mathbf{A} \times \mathbf{C}'$ and $h_1 = \bar{g}/\gamma : \mathbf{A} \times \mathbf{C}' \rightarrow \mathbf{D}$. Furthermore, since $h_1 \circ h_0 = q \circ \bar{f} \circ (f \times i)$ is essential, h_1 is essential. By the homomorphism property $h_0^{-1}[1, 1]_{\mathbf{A} \times \mathbf{C}'} = g^{-1}([1, 1]_{\mathbf{A} \times \mathbf{C}} \vee \psi) \geq g^{-1}([1, 1]_{\mathbf{A} \times \mathbf{C}}) = \Delta$.

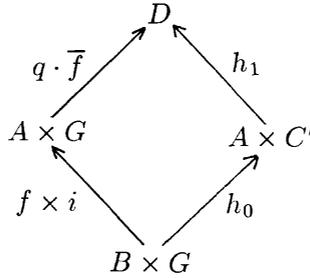


Figure 3

Let $(\pi_j : j \in J)$ be a family of completely meet-irreducible congruences of \mathbf{D} whose meet is 0. Now $\mathbf{B} \times \mathbf{G} \leq_E \mathbf{D}$ and $\mathbf{B} \times \mathbf{G}$ is finite. By arguing as we did in an earlier claim, we may assume without loss of generality that J is finite and, for no proper subset J' of J is $\bigwedge(\pi_j : j \in J') = 0$. For $j \in J$, let $\sigma_j = h_1^{-1}(\pi_j)$. By applying Lemma 5 to $\mathbf{A} \times \mathbf{C}'$, there is $k \in J$ such that $\sigma_k \leq \eta_0^{A \times C'}$. Thus $\sigma_k = \eta_0 \wedge (\sigma_k \vee \eta_1) = \eta_0 \wedge \alpha_1^-$ for some $\alpha \in \text{Con}(\mathbf{C}')$. Now let $\theta = \pi_k$ and $\theta' = \bigwedge(\pi_j : j \in J - \{k\})$. By Lemma 1, the map $h_1/\theta : (\mathbf{A} \times \mathbf{C}')/\sigma_k \rightarrow \mathbf{D}/\theta$ is essential. But \mathbf{D}/θ is

subdirectly irreducible and $\mathbf{A} \times \mathbf{C}'/\sigma_k \cong \mathbf{A} \times (\mathbf{C}'/\alpha)$. Then by Lemma 2, \mathbf{C}'/α is trivial, so $\sigma_k = \eta_0^{A \times C'}$. Retaining the definitions of θ and θ' , $(h_1 \circ h_0)^{-1}(\theta) = h_0^{-1}(\eta_0^{A \times C'}) = \eta_0^{B \times G}$.

Now consider any $j \in J - \{k\}$. Let $\delta = (h_1 \circ h_0)^{-1}(\pi_j)$. By Lemma 1 again, $(\mathbf{B} \times \mathbf{G})/\delta \leq_E \mathbf{D}/\pi_j$. By Lemma 6, either $(\mathbf{B} \times \mathbf{G})/\delta$ is abelian, or it is subdirectly irreducible. In other words, either $\delta \geq [1, 1]$ or δ is completely meet-irreducible. But by (C2), $\delta = (\delta \vee \eta_0^{B \times G}) \wedge (\delta \vee [1, 1])$. If $\delta \not\geq [1, 1]$, then $\delta \geq \eta_0^{B \times G}$. But then

$$\bigwedge ((h_1 \circ h_0)^{-1}(\pi_n) : n \in J - \{j\}) = \bigwedge ((h_1 \circ h_0)^{-1}(\pi_n) : n \in J) = 0$$

since $(h_1 \circ h_0)^{-1}(\pi_k) = \eta_0 \leq (h_1 \circ h_0)^{-1}(\pi_j)$. But this contradicts the minimality of J . Thus $\delta \geq [1, 1]$, so $(\mathbf{B} \times \mathbf{G})/\delta$ is abelian. Therefore, by Theorem 7, \mathbf{D}/π_j is abelian. Since this holds for all $j \neq k$, $\theta' \geq [1, 1]$ on \mathbf{D} .

Now $(q \circ \bar{f})^{-1}[1_{\mathbf{D}}, 1_{\mathbf{D}}] \geq [1, 1] = \bar{\xi}_0^- \vee \eta_1$ on $\mathbf{A} \times \mathbf{G}$. Therefore

$$(q \circ \bar{f} \circ (f \times i))^{-1}[1_{\mathbf{D}}, 1_{\mathbf{D}}] \geq \xi_0^- \wedge \eta_1^{B \times G}.$$

On the other hand $(h_1 \circ h_0)^{-1}[1_{\mathbf{D}}, 1_{\mathbf{D}}] \geq h_0^{-1}[1, 1] \geq \Delta$. Putting these two together,

$$[1_{\mathbf{D}}, 1_{\mathbf{D}}] \upharpoonright B \times G \geq (\xi_0^- \vee \eta_1^{B \times G}) \vee \Delta = \xi_0^- \vee \xi_0'^-.$$

But $0 = \theta \wedge \theta' \geq \theta \wedge [1_{\mathbf{D}}, 1_{\mathbf{D}}]$, so $0 \geq (\theta \upharpoonright B \times G) \wedge ([1, 1] \upharpoonright B \times G) \geq \eta_0^{B \times G} \wedge \xi_0'^- \wedge \xi_0'^-$ which is false. Thus the theorem is proved. ■

Combining Theorem 8 with the main result of [1], we have the following.

COROLLARY 9. *Let \mathcal{V} be congruence modular, 4-finite and satisfy (C2). If \mathcal{V} has AP and RS then \mathcal{V} has CEP, in fact, \mathcal{V} has enough injectives.*

We present an example that we hope will illuminate the arguments used above. In addition, the example refutes a conjecture made while this work was in progress: Observe that (R) implies that every essential extension of an abelian algebra is abelian. Is the converse true, in the presence of (C2)? The answer is no, as the construction shows.

Let \mathcal{L} be the language $\langle \cdot, ^{-1}, 1, * \rangle$ of type $\langle 2, 1, 0, 1 \rangle$. All the algebras we construct here will satisfy the usual laws of group theory, plus the

identity $1^* = 1$. Thus 1 will be an idempotent element and there will be a one-to-one correspondence between the congruences of an algebra and “certain” subalgebras; following Kurosh [6], we call a subalgebra I an *ideal* if:

$$\forall x, y: \quad x^{-1} \cdot I \cdot x = I \text{ and } y \in I \Rightarrow (x \cdot y)^* \cdot x^{*-1} \in I.$$

Let \mathbf{C}_2^i and \mathbf{C}_4^i be algebras in this language whose groups are cyclic of orders 2 and 4 respectively and satisfying $x^* = x$. Let \mathbf{C}_2^t be the algebra with cyclic group of order 2 and satisfying $x^* = 1$. Observe that all three algebras are term equivalent to cyclic groups, so they are abelian.

Let \mathbf{B} be the algebra whose underlying group is

$$\langle a, b : a^3 = b^4 = 1, \quad ba = a^2b \rangle,$$

the dicyclic group of order 12, such that $(a^j b^k)^* = a^2 b^k$ if $a^j b^k \neq 1$. \mathbf{B} has two proper, non-trivial ideals:

$$T = \{1, ab^2, a^2, b^2, a, a^2 b^2\}$$

and

$$K = \{1, a, a^2\}.$$

(To see this, check that the map $a^j b^k \mapsto g^k$ is a homomorphism from \mathbf{B} to \mathbf{C}_4^i (g a generator of the group) with kernel K . There are no other ideals since there are no other subalgebras.) We have $\mathbf{B}/T \cong \mathbf{C}_2^i$ and $\mathbf{B}/K \cong \mathbf{C}_4^i$, therefore $[B, B] \leq K$. On the other hand, consider the term $\tau(x, y) = x^* \cdot y^* \cdot (x \cdot y)^{*-1}$. Computing in \mathbf{B} : $\tau(1, 1) = 1 = \tau(a, 1)$, but $\tau(1, a^2) = 1 \neq \tau(a, a^2) = a$. Since $1, a, a^2 \in K$, this inequality shows that $\{1\}$ does not have the “ $K - K$ term condition,” (see [2, 3.2] or [4, 6.8]), and so $[K, K] > \{1\}$. Putting these two observations together, $K \geq [B, B] \geq [K, K] \geq K$. Thus $[B, B] = [K, K] = K$, and \mathbf{B} has (C2).

Now let \mathbf{A} be the algebra with group structure:

$$\langle B, c : c^2 = 1, \quad ac = ca, \quad bc = cb \rangle$$

such that for $x \in B$, x^* is defined as it was for \mathbf{B} , $(xc)^* = x^*$ if $x \neq 1$, and $c^* = b^2$. Thus \mathbf{B} is a subalgebra of \mathbf{A} . The ideal lattice of \mathbf{A} is shown in Figure 4, where $J = T \cup \{xc : x \in T\}$. To see this, note that

$\mathbf{A}/T \cong \mathbf{C}_2^t \times \mathbf{C}_2^i$. As \mathbf{C}_2^t and \mathbf{C}_2^i are non-isomorphic (and therefore, *non-isotopic*), \mathbf{A}/T has no skew congruences. Thus the ideal lattice of \mathbf{A} above T is as claimed. Also, K is not an ideal of \mathbf{A} since $(ac) \cdot c^{-1} \in K$ but $(ac)^* \cdot c^{-1*} = a^* \cdot c^* = a^2 \cdot b^2 \notin K$. Thus, if I is any other ideal, not in the figure, then $I \cap B = \{1\}$. But for any $y \in A$, if $y \neq 1$ then $1 \neq y^* \in B$, so no such I can exist. Arguing as before $[A, A] = [T, T] = T$ since \mathbf{A}/T is abelian but $\{1\}$ does not have the “ $T - T$ term condition,” (use the same term). Thus \mathbf{A} satisfies (C2).

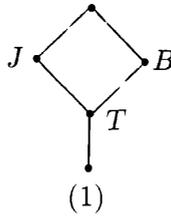


Figure 4.

Now we expand the language \mathcal{L} to include an additional constant symbol, and consider the algebra (\mathbf{A}, b) , which we continue to call \mathbf{A} . As $(b^* \cdot b^{-1})^2 = a$, \mathbf{A} has only one proper subalgebra, (\mathbf{B}, b) (which we call \mathbf{B}). The addition of a constant does not affect the ideal lattice. Let \mathcal{V} be the variety generated by this new algebra \mathbf{A} . By Kiss [5, Proposition I.5.4] \mathcal{V} has (C2) since every subalgebra of \mathbf{A} has (C2). Now in Freese-McKenzie [3], it is proved that a finitely generated congruence modular variety is residually small if and only if it satisfies another commutator condition:

$$\forall \alpha, \beta \quad \alpha \leq [\beta, \beta] \Rightarrow \alpha = [\alpha, \beta].$$

However a variety satisfying (C2) trivially satisfies this condition. Thus our variety \mathcal{V} is residually small.

On the other hand, \mathcal{V} does not satisfy (R) since $[A, A] \upharpoonright B = T \upharpoonright B = T \neq [B, B] = K$. Thus we can conclude from Theorem 8 that \mathcal{V} does not have AP.

Let us examine this variety a little more closely to see why Theorem 8 works. First, what are the subdirectly irreducible algebras? If \mathbf{S} is subdirectly irreducible and non-abelian, then by (C2), \mathbf{S} has non-abelian monolith, so by the Jónsson-Hagemann-Herrmann theorem [2, 10.1], \mathbf{S} is a homomorphic image of a subalgebra of \mathbf{A} , thus \mathbf{S} is isomorphic to \mathbf{A} or to \mathbf{B} . If \mathbf{S} is abelian, then by [3, Theorem 8], \mathbf{S} is finite. $\mathbf{S} = \mathbf{D}/L$, where \mathbf{D} is a subalgebra of a finite power of \mathbf{A} and $L \geq [D, D]$. Then \mathbf{D} is a subdirect product of algebras $(\mathbf{D}_i : i \in I)$ in which I is finite and every \mathbf{D}_i is isomorphic to \mathbf{A} or to \mathbf{B} . Therefore by (C2), $\mathbf{D}/[D, D]$ is a subdirect product of $(\mathbf{D}_i/[D_i, D_i] : i \in I)$. However, $\mathbf{D}_i/[D_i, D_i]$ is isomorphic to \mathbf{C}_4^i or to $\mathbf{C}_2^t \times \mathbf{C}_2^i$. Therefore \mathbf{S} lies in the subvariety generated by \mathbf{C}_4^i and \mathbf{C}_2^t . This subvariety is the varietal product (see [8]) $\mathcal{W}_1 \otimes_\sigma \mathcal{W}_2$ in which \mathcal{W}_1 is the variety generated by \mathbf{C}_4^i , \mathcal{W}_2 is generated by \mathbf{C}_2^t and $\sigma(x, y) = y \cdot (y^{-1} \cdot x)^*$. Therefore \mathbf{S} is isomorphic to one of $\mathbf{C}_4^i, \mathbf{C}_{4\nabla}^i, \mathbf{C}_2^i, \mathbf{C}_{2\nabla}^i, \mathbf{C}_2^t, \mathbf{C}_{2\nabla}^t$. (Here \mathbf{C}_∇ is the *linearization* of \mathbf{C} , recall we added a constant symbol to the language.)

Now we take another look at the situation in Theorem 8. The algebra \mathbf{G} is $\mathbf{B}/K \cong \mathbf{C}_4^i$ and the congruence $\Delta_{1,\alpha}$ corresponds to a skew ideal, N , lying between $K \times (K/K)$ and $T \times (T/K)$ on $\mathbf{B} \times \mathbf{G}$. There is another skew ideal, M , lying between $T \times (K/K)$ and $B \times (T/K)$ on the same algebra (see Figure 5). The algebra \mathbf{C} is $(\mathbf{B} \times \mathbf{G})/N$, which happens to be isomorphic to $\mathbf{C}_{4\nabla}^i \times \mathbf{C}_2^i$.

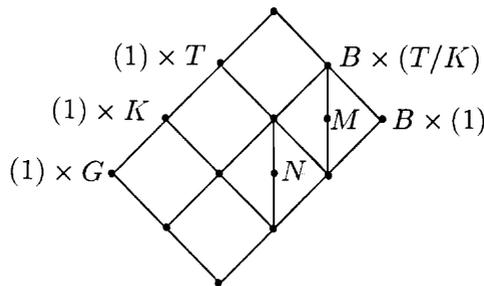


Figure 5

Let us suppose that the diagram $(\mathbf{B} \times \mathbf{G}, \mathbf{A} \times \mathbf{G}, \mathbf{A} \times \mathbf{C})$ can be amalgamated. Using (C2) and the fact that \mathbf{A} is a maximal, non-abelian,

subdirectly irreducible algebra, one can show that any algebra amalgamating this has a subdirect factor isomorphic to \mathbf{A} . Thus, without loss of generality, the diagram is amalgamated by $\mathbf{A} \times \mathbf{E}$ for some \mathbf{E} , and $(\{1\} \times E) \upharpoonright B \times G = \{1\} \times G$. The question is, what is $(A \times \{1\}) \upharpoonright B \times G$, which we call I ?

Observe that there are two maximal ideals disjoint from $\{1\} \times G$ on $\mathbf{A} \times G : A \times \{1\}$ and \overline{M} , a skew ideal lying between $T \times (K/K)$ and $B \times (T/K)$. Thus, one of these restricts to I , which is to say, I is either $B \times \{1\}$ or M on $\mathbf{B} \times \mathbf{G}$. In either case, observe that

$$\begin{aligned}
 K \times (K/K) &= I \cap N = ((A \times \{1\}) \upharpoonright A \times C \cap (A \times \{1_C\})) \upharpoonright B \times G \\
 &= \overline{K} \times \{1_C\}
 \end{aligned}$$

for some ideal \overline{K} on \mathbf{A} . But then $\overline{K} \upharpoonright B = K$, and no such ideal exists on \mathbf{A} .

Finally, observe that the variety \mathcal{V} refutes the conjecture mentioned earlier: every essential extension of an abelian algebra is abelian, but \mathcal{V} fails to have (R). For suppose there was an abelian algebra \mathbf{D}' and a non-abelian, essential extension \mathbf{C}' . Arguing as we did in the claim at the beginning of Theorem 8, there are algebras \mathbf{D} and \mathbf{C} with $\mathbf{D} \leq_E \mathbf{C}$, \mathbf{D} abelian and \mathbf{C} subdirectly irreducible and non-abelian. However, the only candidates for \mathbf{C} in \mathcal{V} are \mathbf{A} and \mathbf{B} which have no abelian subalgebras.

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