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Sharp bounds for decomposing graphs into edges and triangles

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Sharp bounds for decomposing graphs into edges and triangles

Abstract
Let $\pi_3(G)$ be the minimum of twice the number of edges plus three times the number of triangles over all edge-decompositions of $G$ into copies of $K_2$ and $K_3$. We are interested in the value of $\pi_3(n)$, the maximum of $\pi_3(G)$ over graphs $G$ with $n$ vertices. This specific extremal function was first studied by Gyori and Tuza [Decompositions of graphs into complete subgraphs of given order, Studia Sci. Math. Hungar. 22 (1987), 315--320], who showed that $\pi_3(n)<9n^2/16$.

In a recent advance on this problem, Kral, Lidicky, Martins and Pehova [arXiv:1710:08486] proved via flag algebras that $\pi_3(n)<(1/2+o(1))n^2$, which is tight up to the $o(1)$ term.

We extend their proof by giving the exact value of $\pi_3(n)$ for large $n$, and we show that $K_n$ and $K_{n/2,n/2}$ are the only extremal examples.

Disciplines
Algebra | Discrete Mathematics and Combinatorics | Mathematics

Comments

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SHARP BOUNDS FOR DECOMPOSING GRAPHS INTO EDGES AND TRIANGLES

A. BLUMENTHAL, B. LIDICKÝ, O. PIKHURKO, Y. PEHOVA, F. PFENDER AND J. VOLEC

Abstract. Let \( \pi_3(G) \) be the minimum of twice the number of edges plus three times the number of triangles over all edge-decompositions of \( G \) into copies of \( K_2 \) and \( K_3 \). We are interested in the value of \( \pi_3(n) \), the maximum of \( \pi_3(G) \) over graphs \( G \) with \( n \) vertices. This specific extremal function was first studied by Győri and Tuza [Decompositions of graphs into complete subgraphs of given order, Studia Sci. Math. Hungar. 22 (1987), 315–320], who showed that \( \pi_3(n) \leq 9n^2/16 \). In a recent advance on this problem, Král’, Lidický, Martins and Pehova [arXiv:1710:08486] proved via flag algebras that \( \pi_3(n) \leq (1/2 + o(1))n^2 \), which is tight up to the \( o(1) \) term. We extend their proof by giving the exact value of \( \pi_3(n) \) for large \( n \), and we show that \( K_n \) and \( K_{\lceil n/2 \rceil, \lceil n/2 \rceil} \) are the only extremal examples.

1. Introduction

In recent work of Král’, Lidický, Martins and Pehova [15], they proved using the flag algebra method (see [18, 3, 1, 4, 7, 8, 12, 16] for applications to other problems in extremal combinatorics) that the edges of any \( n \)-vertex graph can be decomposed into copies of \( K_2 \) and \( K_3 \) whose total number of vertices is at most \( (1/2 + o(1))n^2 \). This was a conjecture of Győri and Tuza [19], but the problem itself can be traced back to Erdős, Goodman and Pósa [6] who considered the problem of minimising the total number of cliques in an edge-decomposition of an arbitrary \( n \)-vertex graph. They showed the following:

\textbf{Theorem 1} (Erdős, Goodman, Pósa [6]). The edges of every \( n \)-vertex graph can be decomposed into at most \( n^2/4 \) complete graphs.

The only extremal example for this bound is the bipartite Turán graph \( T_2(n) \). Moreover, this result still holds if we restrict the sizes of the cliques used in the decomposition to 2 and 3 (that is, triangles and single edges). In a series of papers published independently by Chung [13], Győri and Kostochka [10], and Kahn [14],

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they proved that in fact something stronger than Theorem 1 is true, confirming a conjecture by Katona and Tarján:

**Theorem 2** (Chung [13], Győri and Kostochka [10], Kahn [14]). *Every n-vertex graph can be edge-decomposed into cliques whose total number of vertices is at most \( \frac{n^2}{2} \).*

For a given graph \( G \) on \( n \) vertices, let \( \pi_k(G) \) be the minimum over all decompositions of the edges of \( G \) into cliques \( C_1,\ldots,C_\ell \) of size at most \( k \) of the sum \( |C_1| + |C_2| + \cdots + |C_\ell| \). With this notation, the conclusion of the above theorem is that \( \min_{k \in \mathbb{N}} \pi_k(G) \leq \frac{n^2}{2} \). In light of Theorem 2, Tuza [19] conjectured that \( \pi_3(G) \leq \frac{n^2}{2} + o(n^2) \), and in fact that \( \pi_3(G) \leq \frac{n^2}{2} + O(1) \). In [9] Győri and Tuza showed that \( \pi_3(G) \leq \frac{9n^2}{16} \). This was the best known bound until recently, when using the celebrated flag algebra method by Razborov [18], Kráľ, Lidický, Martins and Pehova [15] proved the asymptotic version of Tuza’s conjecture:

**Theorem 3** (Kráľ, Lidický, Martins and Pehova [15]). *Every n-vertex graph \( G \) satisfies \( \pi_3(G) \leq (1/2 + o(1))n^2 \).*

We show, by building upon the proof in [15], that in fact \( \pi_3(G) \leq \frac{n^2}{2} + 1 \), and that the extremal graphs \( G \) which maximise \( \pi_3(G) \) are the complete graph \( K_n \) and the bipartite Turán graph \( T_2(n) \). Which of these two graphs is extremal is a matter of divisibility of \( n \) by 6. In the case of the Turán graph, \( \pi_3(T_2(n)) = 2\lceil n/2 \rceil \lfloor n/2 \rfloor \), giving \( \frac{n^2}{2} \) for even \( n \) and \( \frac{n^2 - 1}{2} \) for odd \( n \). For graphs with minimum degree \( n - o(n) \), the following result shows that we can decompose them only into copies of \( K_3 \), as long as they are triangle-divisible; that is, if each vertex has even degree and the total number of edges is divisible by three.

**Theorem 4** (Barber, Kuhn, Lo, Osthus [2] and Dross [5]). *For every \( \varepsilon > 0 \), if \( G \) is a triangle-divisible graph of large order \( n \) and minimum degree at least \( (0.9 + \varepsilon)n \), then \( G \) has a triangle decomposition.*

In particular, for each residue class of \( n \) mod 6, the optimal triangle-edge decompositions of \( K_n \) are in Table 1.

<table>
<thead>
<tr>
<th>( n \mod 6 )</th>
<th>optimal decomposition of ( K_n )</th>
<th>( \pi_3(K_n) )</th>
<th>( \pi_3(T_2(n)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>triangle-divisible + perfect matching</td>
<td>( \frac{n^2}{2} )</td>
<td>( \frac{n^2}{2} )</td>
</tr>
<tr>
<td>1</td>
<td>triangle-divisible</td>
<td>( \binom{n}{2} )</td>
<td>( \frac{n^2 - 1}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>triangle-divisible + perfect matching</td>
<td>( \frac{n^2}{2} )</td>
<td>( \frac{n^2}{2} )</td>
</tr>
<tr>
<td>3</td>
<td>triangle-divisible</td>
<td>( \binom{n}{2} )</td>
<td>( \frac{n^2 - 1}{2} )</td>
</tr>
<tr>
<td>4</td>
<td>triangle-divisible + perfect matching + ( K_{1,3} )</td>
<td>( \frac{n^2}{2} + 1 )</td>
<td>( \frac{n^2}{2} )</td>
</tr>
<tr>
<td>5</td>
<td>triangle-divisible + ( C_4 )</td>
<td>( \binom{n}{2} + 4 )</td>
<td>( \frac{n^2 - 1}{2} )</td>
</tr>
</tbody>
</table>
Let us define

\[ \mathcal{E}_n = \begin{cases} 
\{T_2(n)\} & \text{if } n \equiv 1, 3, 5 \pmod{6}, \\
\{K_n\} & \text{if } n \equiv 4 \pmod{6}, \\
\{T_2(n), K_n\} & \text{if } n \equiv 0, 2 \pmod{6}, 
\end{cases} \]

the graphs in \{T_2(n), K_n\} which maximise \( \pi_3 \) in each residue class mod 6. For \( n \in \mathbb{N} \), let \( L_n \) be any member of \( \mathcal{E}_n \) and define \( \ell(n) := \pi_3(L_n) \). Clearly, \( \ell(n) \) is a lower bound on \( \pi_3(n) \), the maximum over all \( n \)-vertex graphs \( G \) of \( \pi_3(G) \).

Then, our main result is the following:

**Theorem 5.** There exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), we have \( \pi_3(n) = \ell(n) \) and the set of \( \pi_3(n) \)-extremal graphs is exactly \( \mathcal{E}_n \). This gives

\[
\pi_3(n) = \begin{cases} 
\frac{n^2}{2} & \text{for } n \equiv 0, 2 \pmod{6} \quad \text{attained only by } T_2(n) \text{ and } K_n, \\
\frac{(n^2 - 1)/2}{2} & \text{for } n \equiv 1, 3, 5 \pmod{6} \quad \text{attained only by } T_2(n), \\
\frac{n^2}{2} + 1 & \text{for } n \equiv 4 \pmod{6} \quad \text{attained only by } K_n.
\end{cases}
\]

A simple corollary of Theorem 5 is an affirmative answer to a question of Pyber [17], see also Problem 45 [19], for sufficiently large \( n \). In a decomposition of a graph, every edge is used exactly once. In a covering, every edge is used at least once.

**Corollary 6.** For sufficiently large \( n \), an edge set of every \( n \)-vertex graph be covered with triangles of weight 3 and edges of weight 2 such that their total weight is at most \( \lfloor n^2/2 \rfloor \).

### 2. Proof of Theorem 5

By analysing the dual solution to the optimisation problem considered in [15], we may obtain the following:

**Proposition 7.** For every \( \delta > 0 \) there exists \( n_1 \in \mathbb{N} \) such that if \( G \) is a graph of order \( n \geq n_1 \) with \( \pi_3(G) \geq \ell(n) - n^2/n_1 \), then \( G \) is \( \delta n^2 \)-close in edit distance to \( K_n \) or to \( T_2(n) \).

In the case when \( G \) is \( \delta n^2 \) edges away from \( T_2(n) \), a result by Győri [11] that a graph with \( n \) vertices and \( e(T_2(n)) + k \) edges where \( k = o(n^2) \) has at least \( k - O(k^2/n^2) \) edge-disjoint triangles almost immediately implies the desired result. More specifically, for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for large \( n \) every \( n \)-vertex graph with \( t_2(n) + k \) edges where \( k \leq \delta n^2 \) has at least \( k - \varepsilon k^2/n^2 \) edge-disjoint triangles. Since \( G \) is \( \delta n^2 \)-close to \( T_2(n) \), then it must have at least \( t_2(n) + \delta n^2 \) edges. From this we have that \( 3(2t_2(n) + k) - 3(2t_2(n) - k(1 - 3\varepsilon k/n^2)) \leq 2t_2(n) - k(1 - 3\varepsilon k/n^2) \leq 2t_2(n) \) for \( \delta \ll \varepsilon \ll 1 \). Equality is achieved only if \( k = 0 \), that is, if \( G \cong T_2(n) \).

---

[1] Two graphs \( G_1 \) and \( G_2 \) on the same vertex set are said to be \( k \)-close in edit distance (or simply \( k \)-close) if \( |E(G_1) \triangle E(G_2)| \leq k \).
In the case when $G$ is $\delta n^2$ edges away from $K_n$, by iteratively removing vertices of small degree, we may assume that $\delta(G) \geq (\ell(n) - \ell(n-1))/2$.

We now proceed to decompose the edges of $G$ into edges and triangles in 3 stages:

**Stage 1:** Denote by $U$ the set of all vertices which have degree less than $(1 - c)n$ for some $c \ll 1$, and let $W = V(G) \setminus U$. By a double-counting argument $|U| = u \leq (2\delta/c)n^2$. For each vertex $u \in U$ in turn, remove a maximum family of edge-disjoint triangles, each containing $u$ and two vertices from $W$. Denote the resulting graph induced on $W$ by $G'$. Through a simple neighbourhood-chasing argument, we can show that $|\Gamma_{G'}(u) \cap W| \leq 1$ for all $u \in U$, that is, up to parity, the triangles removed during Stage 1 cover all $[U,W]$-edges in $G$.

**Stage 2:** Remove a maximum collection of edge-disjoint triangles from $G'$.

Using Theorem 4 we may consider this as a problem of setting aside a set $X$ of edges (which induce a bounded-degree graph) such that $G' - X$ is triangle-divisible. By considering the even- and odd-degree vertices in $G'$ separately, we can construct a set $X$ of size $|X| = p \leq (n-u)/2 + 2$ and maximum degree 2.

**Stage 3:** Decompose $X$ trivially into copies of $K_2$.

Let $t_1$ and $t_2$ denote the number of triangles removed respectively in Stages 1 and 2. By counting pairs of vertices inside $W$, we conclude that $t_1 + 3t_2 + p \leq \binom{n-u}{2}$. Moreover, since each vertex of $U$ has degree at most $(1 - c)n$ in $G$, we also have that $t_1 \leq u(1-c)n/2$.

Thus we obtain

$$
\pi_3(G) \leq 3t_1 + 3t_2 + 2 \binom{u}{2} + 2p + 2u
\leq 3t_1 + \left( \binom{n-u}{2} - p - t_1 \right) + 2 \binom{u}{2} + 2p + 2u
\leq u(1 - c)n + \binom{n-u}{2} + 2 \binom{u}{2} + p + 2u
= \binom{n}{2} + \frac{3u^2}{2} + \frac{3u}{2} - cnu + p.
$$

We now compare this bound with the conjectured maxima presented in Table 1.

First, suppose that $n$ is even. Here the larger value is achieved by $K_n$ and it is at least $\pi_3(K_n) \geq n^2/2 = \binom{n}{2} + \frac{n}{2}$. Since $u \leq (2\delta/c)n$, we have that $3u^2/2 + \frac{3u}{2} \leq cnu/2$ and so

$$
\pi_3(G) - \pi_3(K_n) \leq -cnu/2 + (n-u)/2 + 2 - (n+u)/2,
$$

which is non-negative only if $u = 0$, and since $G$ is extremal, all inequalities we used in upper-bounding $\pi_3(G)$ are tight. In particular, we get that $t_1 = u(1-c)n/2 = 0$, and hence $e(G) = 3t_2 + p = \binom{n-u}{2} = \binom{n}{2}$, meaning that $G \cong K_n$.

Now, suppose that $n$ is odd. In this case we have $\pi_3(T_2(n)) - n/2 \geq \pi_3(K_n) - O(1) = \binom{n}{2} - O(1)$. Similarly to the previous case, $\pi_3(G) - \pi_3(T_2(n)) \leq -cnu/2 + \frac{3u^2}{2} + \frac{3u}{2} - cnu/2$. We proceed similarly to the previous case.
which again is non-negative only if \( u = 0 \) and \( e(G) = \binom{n}{2} \). But since \( n \) is odd, this means that \( G \) has no odd-degree vertices and in fact \( \pi_3(G) \leq \binom{n}{2} + 2 < \pi_3(T_2(n)) \), a contradiction to the extremality of \( G \).

REFERENCES

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