Oscillation and nonoscillation of third order functional differential equations

Abdalla S. Tantawy

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Oscillation and nonoscillation of third order functional differential equations

Tantawy, Abdalla S., Ph.D.

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1 INTRODUCTION AND REVIEW

1.1 Introduction

This chapter is essentially introductory in nature. Its main purpose is to introduce some basic concepts from the theory of differential equations with deviating, advanced, or mixed type arguments, to sketch some important results from the theory of oscillation of ordinary differential equations, and to demonstrate some new problems in oscillation theory caused by deviating argument.

Section 1.2 is concerned with the general idea of functional differential equations and the classification according to the argument. Section 1.3 provides some examples in application. Section 1.4 gives an idea about the existence of solution of functional differential equation. Sections 1.5 and 1.6 sketch necessary important results and summarize certain main topics in oscillation theory. Finally, we introduce some definitions and theorems which are important tools in oscillation and nonoscillation analysis.
1.2 General idea about Functional Differential Equations

We know that in ordinary differential equations the unknown function and its derivatives are all evaluated at the same instant, \( t \). A more general type of differential equations, often called a functional differential equation, is one in which the unknown function occurs with various different arguments. For example, we might have

\[
\begin{align*}
    y'(t) &= -2y(t - 1), \\
    y'(t) &= y(t) - y(t/2) + y'(t - 1), \\
    y'(t) &= y(t)y(t - 1) + ty(t + 2), \\
    y''(t) &= -y'(t) - y'(t - 1) - 3\sin(y(t)) + \cos(t), \text{ or} \\
    y'''(t) &= y''(t) + y'(t + 1) - \cos(y(t - 1)).
\end{align*}
\]

The simplest and perhaps the most natural type of functional differential equation is a "Delay Differential Equation" (or "Differential Equation with retarded argument"). This means an equation expressing some derivative of \( y \) at time \( t \) in terms of \( y \) (and its lower derivatives if any) at \( t \) and at earlier instants. The first and the fourth examples above are of this type. The other two are not, because, the second example involves the highest order derivative at two different instants and this kind is called neutral differential equation. The third and the fifth examples contain an advanced argument.

Clearly, when working with functional differential equations, one must write out the argument of the unknown function. We cannot omit them as we have been
1.3 Examples from Applications

In this section, we give some examples of physical and biological system in which the present rate of change of some unknown function(s) depends upon past values of the same function(s).

1.3.1 Control system

Any system involving a feedback control will almost certainly involve time delays. This arises because a finite time is required to sense information and then act to it.

In the 1930s and 1940s Callender, Hartree, and Porter [8], Minorsky [33], [34], [35] and others began studies of certain problems in which delays become significant.

Consider, as a simple example, a system whose motion is governed by a second order, linear homogeneous differential equation with positive constant coefficients

\[ m y''(t) + b y'(t) + ky(t) = 0. \]  

(1.1)

The solution of this equation with arbitrary specified initial conditions, \( y(t_0) \) and \( y'(t_0) \), is a function which decays exponentially towards zero. We know that

...
equation (1.1) has been classified in terms of the magnitude of the damping coefficient.

The case $b^2 < 4mk$ is called "underdamped", and the solution oscillates as it decays.

The case $b^2 > 4mk$ is called "overdamped". The solution does not oscillate, but dies out exponentially at a slower rate when $b$ becomes larger.

The case $b^2 = 4mk$ is called "critically damped" because, in a sense, solutions approach zero most rapidly in this case.

Let us assume that the system is underdamped and we wish to somehow increase the damping coefficient to bring the system closer to critical damping, thereby diminishing the oscillations more rapidly. Perhaps we would even prefer a slight over damping so as to eliminate oscillations.

If our physical system is simple mass hanging from a spring in the laboratory then it is quite simple to increase $b$. We might simply immerse the whole system in molasses, or if that makes $b$ too big we could try motor oil instead.

However, if, as in Minorsky's case, our system is a ship rolling in the waves and $y$ is the angle of tilt from the normal upright position, we must be more ingenious in trying to increase $b$. We might, for example, introduce ballast tank, partially
filled with water, in each side of the ship. We would also have a servomechanism
designed to pump water from one tank to the other in an attempt to counteract the
roll of the ship. Hopefully, this would introduce another term proportional to \( y'(t) \)
in the equation, say \( qy'(t) \). Thus we consider

\[
my''(t) + by'(t) + qy'(t) + ky(t) = 0. \tag{1.2}
\]

But now one must recognize that the servomechanism cannot respond instantaneously. Thus instead of equation (1.2) we should consider

\[
my''(t) + by'(t) + gy'(t - r) + ky(t) = 0. \tag{1.3}
\]

The control takes time \( r > 0 \) to respond and thus the control term is propor­tional to the velocity at the earlier instant, \( t-r \). It seems possible that such a
time lag could result in the opposite direction to that which is desired. Thus it is
conceivable that rather than helping to stabilize the system, such a control might
make matters worse, and even cause undesired oscillations.

1.3.2 Prey-Predator population models

Let \( x(t) \) be the population at time \( t \) of some species of animals called prey and
\( y(t) \) be the population of a predator species which lives off these prey. We assume
that \( x(t) \) would increase at a rate proportional to \( x(t) \) if the prey were left alone, i.e.,
we would have \( x'(t) = a_1 x(t) \), where \( a_1 > 0 \). However, the predators are hungry,
and the rate at which each of them eats prey is limited only by this ability to find
prey. Thus we shall assume that the activities of predators reduce the growth rate of \( x(t) \) by an amount proportional to the product \( x(t)y(t) \), i.e.,

\[
x'(t) = a_1 x(t) - b_1 x(t)y(t),
\]

where \( b_1 \) is another positive constant.

Now let us also assume that the predators are completely dependent on the prey as their food supply. If there were no prey, we assume that

\[
y'(t) = -a_2 y(t),
\]

where \( a_2 > 0 \), i.e., the predator species would die out exponentially. However, given food the predators breed at the rate proportional to their number of food available to them. Thus we consider the pair of equations

\[
\begin{align*}
x'(t) &= a_1 x(t) - b_1 x(t)y(t) \\
y'(t) &= -a_2 y(t) + b_2 x(t)y(t),
\end{align*}
\]

where \( a_1, a_2, \) and \( b_2 \) are positive constants. This well-known model was invented and studied by Lotka [31], and Volterra [39], [40].

Vito Volterra was trying to understand the observed fluctuation in the sizes of populations \( x(t) \) of commercially desirable fish and \( y(t) \) of larger fish which fed on the smaller ones in the Adriatic Sea in the decade from 1914 to 1923.
Wangersky and Cunningham [41] proposed to modify system (1.6) so that the birth rate of prey has a further limitation, while the birth rate of predators responds to changes in the magnitudes of $x$ and $y$ only after a delay $r > 0$. Thus they replaced system (1.6) with

$$
\begin{align*}
    x'(t) &= a_1 [1 - x(t)/p] x(t) - b_1 y(t) x(t) \\
    y'(t) &= -a_2 y(t) + b_2 x(t - r) y(t - r).
\end{align*}
$$

(1.7)

The solution of (1.7) is, in principle, more elementary than the solution of (1.6). But the problem of obtaining useful information about the solutions of (1.7) much more difficult.

### 1.4 Theory of Existence

Let us consider the ordinary differential equation (ODE)

$$
y' = f(t, y), \text{ together with the initial condition } \quad y(t_0) = y_0. \tag{1.8}
$$

It is well known that under certain assumptions with respect to $f$ the initial value problem (1.8) and (1.9) has a unique solution and it is equivalent to the integral equation

$$
y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s)) ds, \quad t \geq t_0. \tag{1.10}
$$
Next, we consider the differential equation of the form

$$y'(t) = f(t, y(t), y(t - \tau)), \quad \tau > 0, \quad t \geq t_0$$  \hspace{1cm} (1.11)$$
in which the right hand side depends not only on the instantaneous position \(y(t)\), but also on \(y(t - \tau)\), the position at \(\tau\) units back, that is to say, the equation has past memory. Such an equation is called an ODE with a delay or retarded argument. Whenever necessary, we shall consider the integral equation

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s), y(s - \tau)) \, ds$$  \hspace{1cm} (1.12)$$
that is equivalent to equation (1.11). In order to define a solution of equation (1.11), we need to have a known function \(\phi(t)\) on the interval \([t_0 - \tau, t_0]\), instead of just the initial condition \(y(t_0) = y_0\).

The basic initial value problem for an ordinary differential equation with delayed argument is posed as follows:

On the interval \([t_0, T]\), \(T \leq \infty\), we seek a continuous function \(y\) that satisfies (1.11) and an initial condition

$$y(t) = \phi(t), \quad t \in E_{t_0}$$  \hspace{1cm} (1.13)$$
where \(t_0\) is an initial point, \(E_{t_0} = [t_0 - \tau, t_0]\) is the initial set; the known function \(\phi(t)\) on \(E_{t_0}\) is called the initial function. Usually, it is assumed that \(y(t+0) = \phi(t_0)\).

We always mean a one-sided derivative when we speak of the derivative at an end point of an interval.

Under general assumptions, the existence and uniqueness of solutions to the initial value problem (1.11) and (1.13) can be established. In the case of variable
9

delay \( \tau = \tau(t) > 0 \) in equation (1.11), it is also required to find a solution of this equation for \( t > t_0 \) such that on the initial set \( E_{t_0} = t_0 \cup \{ t - \tau(t) : t - \tau(t) < t_0, t \geq t_0 \} \), \( y(t) \) coincides with the given initial function \( \phi(t) \). If it is required to determine the solution on the interval \([t_0, T]\), then the initial set is

\[
E_{t_0T} = \{ t_0 \} \cup \{ t - \tau(t) : t - \tau(t) < t_0, t_0 \leq t \leq T \}.
\]

Now we consider the differential equation of nth order with \( l \) deviating arguments, of the form

\[
y^{(m_0)}(t) = f(t, y(t), \ldots, y^{(m_0-1)}(t), y(t-\tau_1(t)), \ldots, y^{(m_1-1)}(t-\tau_1(t)), \ldots, y^{(m_l-1)}(t-\tau_l(t)))
\]

where the deviations \( \tau_i > 0 \), and \( \max_{0 \leq i \leq l} m_i = n \).

In order to formulate the initial value problem with respect to (1.14), we require the following notation. Let the given initial point be \( t_0 \). Each deviation \( \tau_i(t) \) defines the initial set \( E_{t_0}^{(i)} \) given by

\[
E_{t_0}^{(i)} = \{ t_0 \} \cup \{ t - \tau_i(t) : t - \tau_i(t) < t_0, t \geq t_0 \}.
\]

We denote \( E_{t_0} = \bigcup_{i=1}^l E_{t_0}^{(i)} \), and on \( E_{t_0} \) we must be given continuous function \( \phi_k(t), k=0,1,\ldots,\mu \), with \( \mu = \max_{1 \leq i \leq l} m_i \). In applications, it is most natural to consider the case where on \( E_{t_0} \), \( \phi_k(t) = \phi_{0}^{(k)}(t), k=0,1,\ldots,\mu \) but it is not generally necessary.

The nth order differential equation should be given initial values

\[
y_0^{(k)}, k=0,1,\ldots,n-1.
\]
Now let \( y^{(k)}_0 = \phi_k(t_0), \ k = 0, 1, 2, ..., \mu. \)

If \( \mu < n - 1 \), then, in addition, the numbers \( y^{(\mu+1)}_0, ..., y^{(n-1)}_0 \) are given.

If the point \( t_0 \) is an isolated point of \( E_{t_0} \), then \( y^{(0)}_0, ..., y^{(n-1)}_0 \) are also given.

For equation (1.14), the basic initial value problem consists of the determination of an \((n-1)\) times continuously differentiable function \( y \) that satisfies equation (1.14) for \( t > t_0 \) and the conditions

\[
\begin{align*}
y^{(k)}(t_0 + 0) &= y^{(k)}_0, \ k = 0, 1, ..., n - 1 \\
y^{(k)}(t - \tau_i(t)) &= \phi_k(t - \tau_i(t)), \text{ if } t - \tau_i < t_0, \\
&\quad k = 0, 1, ..., \mu; \ i = 1, 2, ..., l.
\end{align*}
\]

At a point \( t_0 + (k - 1)r \) the derivative \( y^{(k)}(t) \), generally speaking, is discontinuous, but the derivatives of lower order are continuous.

### 1.5 Definition of Oscillation

Before we define oscillation of solutions, let us consider some simple examples.

**Example 1** The equation

\[
y'' + y = 0
\]

has periodic solutions \( y(t) = \cos(t) \) and \( y(t) = \sin(t) \).

**Example 2** Consider the equation

\[
y''(t) - (1/t)y'(t) + 4t^2y(t) = 0
\]
whose solution is $y(t) = \sin(t^2)$. This solution is not periodic but has an oscillatory property.

**Example 3** Consider the equation

$$y''(t) + (1/2)y(t) - (1/2)y(t - \pi) = 0, \quad t \geq 0$$

whose solution $y(t) = 1 - \sin(t)$ has an infinite sequence of multiple zeros. This solution also has an oscillatory property.

**Example 4** Consider the equation

$$y''(t) - y(-t) = 0$$

which has an oscillatory solution $y_1(t) = \sin(t)$ and a nonoscillatory solution $y_2(t) = e^t + e^{-t}$.

Let us now restrict our discussion to those solutions $y(t)$ of the equation

$$y''(t) + a(t)y(t - \tau(t)) = 0 \quad (1.15)$$

which exist on some ray $[T_y, \infty)$ and satisfy $\sup\{|y(t)|: t \geq T\} > 0$ for every $T \geq T_y$. In other words, $|y(t)| \not\equiv 0$ on any infinite interval $[T, \infty)$. Such a solution sometimes is said to be regular solution.

We usually assume that $a(t) \geq 0$ or $a(t) \leq 0$ in (1.15), and doing so we mean to imply that $a(t) \not\equiv 0$ on any infinite interval $[T, \infty)$.

There are various definitions for oscillation of solutions of ODE (with or without deviating arguments). In this section, we give two definitions of oscillation,
which are most frequently used in the literature.

As we see from the above examples, the definition of oscillation of a regular solutions can have two different forms:

**Definition 1** A nontrivial solution $y(t)$ is said to be oscillatory if and only if it has arbitrarily large zeros for $t \geq t_0$, that is, there exists a sequence of zeros $\{t_n\}$ ($y(t_n) = 0$) of $y(t)$ such that

$$\lim_{n \to \infty} t_n = \infty.$$ 

Otherwise, $y(t)$ is said to be nonoscillatory.

For nonoscillatory solutions there exists a $t_1$ such that $y(t) \neq 0$, for $t > t_1$.

**Definition 2** A nontrivial solution $y(t)$ is said to be oscillatory if it changes sign on $(T, \infty)$, where $T$ is any number.

When $\tau(t) \equiv 0$ and $a(t)$ is continuous in (1.15), the two definitions given above are equivalent. This is because of the fact that the uniqueness of solution makes multiple zeros impossible. However, as example 3 suggests, a differential equation with deviating arguments can have solutions with multiple zeros. These two definitions are different, especially for higher order ordinary differential equations which may have solutions with multiple zeros.

Definition 1 is more general than definition 2. The solution $y(t) = 1 - \sin(t)$ of example 3 is oscillatory according to Definition 1 and is nonoscil-
latory according to Definition 2.

In example 3, the possibility of multiple zeros of nontrivial solution is a consequence of the retardation, since if $\tau(t) \equiv 0$, the corresponding equation has no solutions with multiple zeros.

**Definition 3** For the system of first order differential equation with deviating arguments

\[
x'(t) = f_1(t, x(t), x(\tau_1(t)), y(t), y(\tau_2(t)))
\]
\[
y'(t) = f_2(t, x(t), x(\tau_1(t)), y(t), y(\tau_2(t)))
\]

the solution $(x(t), y(t))$ is said to be strongly (weakly) oscillatory if each (at least one) of its components is oscillatory.

1.6 Review of Oscillation Theory

Since (1836) Sturm introduced the concept of oscillation when he studied the problem of heat transmission, oscillation theory has been an important area of research in the qualitative theory of ODE. Oscillation theory of ordinary differential equations with deviating arguments (ODEWDA) is a natural extension of ODE, while certain known results in oscillation theory for ODE carry over to ODEWDA somewhat. Therefore some background in oscillation theory for ODE is essential for understanding oscillation theory of ODEWDA.

We shall recall only some facts concerning oscillation theory of ODE that are useful for our discussion.
We consider the second order ODE

\[ y''(t) + a(t)y = 0. \]  

(1.16)

Sturm’s comparison theorem for equation (1.16) is a very important result [30] in oscillation theory. Using this comparison theorem, it is easy to see the following conclusions:

(a) For the linear differential equation (1.16), solutions are either all oscillatory or all nonoscillatory. Equation (1.16) is said to be oscillatory if every solution of (1.16) is oscillatory and it is said to be nonoscillatory otherwise.

(b) We consider another second order linear ODE

\[ y''(t) + b(t)y(t) = 0. \]  

(1.17)

If \( a(t) \leq b(t) \) for all \( t \geq t_0 \), and (1.16) is oscillatory, then so is (1.17). Moreover, from (a), if (1.17) is oscillatory then so is (1.16).

Using Sturm’s comparison theorem, we can obtain the oscillatory property of an ODE from some other ODE with known oscillatory behavior. In fact, many good oscillation criteria have been obtained from Sturm’s comparison theorem. For example, consider the Euler equation

\[ y''(t) + \frac{a}{t^2}y(t) = 0. \]  

(1.18)

It is well known that (1.18) is nonoscillatory when \( a=1/4 \), and (1.18) is oscillatory when \( a = (1 + \epsilon)/4, \epsilon > 0 \). According to (b) we obtain the following oscillation
criteria:
t^2a(t) \leq 1/4 implies (1.18) is nonoscillatory, and
t^2a(t) > (1 + \epsilon)/4, \epsilon > 0 implies (1.18) is oscillatory.

(c) Assume that \( a(t) \leq 0 \). Then equation (1.16) is nonoscillatory. This follows from the conclusion in (b).

The comparison method is one of the important methods in the oscillation theory of second order linear ODE. There is much literature dealing with extension of the comparison method to nonlinear ODE and higher order ODE.

Now we consider a second order nonlinear ODE

\[
y''(t) + q(t)f(y(t)) = 0. \tag{1.19}
\]

Interest in nonlinear oscillation problems for equation of this type began with the publication of the pioneering work by Atkinson [3]. We would like to point out the fact that the nonlinearity in (1.19) may generate both oscillatory and nonoscillatory solutions.

A special case of (1.19) is

\[
y''(t) + a(t)y^\alpha(t) = 0. \tag{1.20}
\]

Equation (1.20) is said to be superlinear if \( \alpha > 1 \), and sublinear if \( \alpha < 1 \). We usually need to distinguish between these cases in our study because of the difference in the type of results that are known. For example, consider

\[
y''(t) + a(t) |y(t)|^\alpha \text{sgn}y(t) = 0 \tag{1.21}
\]
where \( a(t) \in C(\mathbb{R}^+) \) and \( a(t) \geq 0 \). Then

For \( \alpha > 1 \) (superlinear), (1.21) is oscillatory iff

\[
\int_{s}^{\infty} a(s) \, ds = \infty.
\]

For \( \alpha < 1 \) (sublinear), (1.21) is oscillatory iff

\[
\int_{s}^{\infty} sa(s) \, ds = \infty.
\]

Finally, we would like to summarize the main topics discussed extensively in the literature of oscillation theory of ODE:

1. Establishing criteria for oscillation or nonoscillation of all solutions
2. Obtaining conditions such that an ODE has an oscillatory solution or a nonoscillatory solution with some asymptotic property
3. Discussing the distribution of zeros and the variability of amplitude of the oscillatory solutions
4. Investigating the oscillation and asymptotic property of nonoscillatory solutions of ODE with a forcing term
5. Finding the relation between oscillation and other qualitative properties, such as boundedness and convergence to zero.

We shall present some examples to show that the oscillation theory of differential equations with deviating arguments is very complex.

**Example 5** Consider the equation with delay

\[
y'(t) + y(t - \pi/2) = 0.
\]
It has oscillatory solutions $y = \sin(t)$ and $y = \cos(t)$. The equation

$$y'(t) + y(t + \pi/2) = 0$$

also has oscillatory solutions $y = \sin(t)$ and $y = \cos(t)$. However, the equation

$$y'(t) + y(t) = 0$$

has no oscillatory solutions.

**Example 6** Consider the second order equation with delay

$$y''(t) + y(\pi - t) = 0$$

**it has both an oscillatory solution** $y_1 = \sin(t)$ **and a nonoscillatory solution** $y_2 = e^t - e^{\pi - t}$.

As we mentioned before, for second order linear ODE either all solutions oscillate or all solutions are nonoscillatory. Thus we see that second order equations with delay create some new problems in oscillation theory. For example, consider

$$y''(t) + p(t)y(\tau(t)) = 0. \quad (1.22)$$

We need to establish various sets of conditions under which either:

(a) all solutions are oscillatory;
(b) all solutions are nonoscillatory;
(c) the equation has a nonoscillatory solution;
(d) the equation has an oscillatory solution; or
(e) the equation has both oscillatory and nonoscillatory solutions.
Example 7 The equation with delay given by

\[ y''(t) - y(t - \pi) = 0 \]

has the oscillatory solutions \( y(t) = \sin(t) \) and \( y(t) = \cos(t) \). But

\[ y''(t) - y(t) = 0 \]

has no oscillatory solution.

This example suggests that we need to find conditions for oscillatory solutions of (1.22), whenever \( p(t) \leq 0 \).

Example 8 Consider the system

\[
\begin{align*}
  x'(t) &= 2x(t) - y(t) \\
  y'(t) &= x(t) + y(t)
\end{align*}
\]

every solution \((x(t), y(t))\) oscillates. But the system

\[
\begin{align*}
  x'(t) &= 2x(t) - y(t - 1/3\ln 4) \\
  y'(t) &= x(t - 1/3\ln 4) + y(t)
\end{align*}
\]

has the nonoscillatory solution \( x(t) = \exp(3/2t) \), \( y(t) = \exp(3/2t) \).

Therefore we need to study the effect of deviating arguments on the oscillation of systems.

Deviating arguments can occur in many complex forms. For example, we will consider equations where the deviating arguments depend on the solution itself, like

\[
y''(t) + p(t)y(t - \tau(y(t))) = 0. \tag{1.23}
\]
Since the oscillation theory of ODEWDA presents some new problems that are not relevant for the corresponding ODE, a study of the oscillation and nonoscillation caused by deviating arguments is most interesting.

1.7 Basic Definitions and Theorems

We need to mention some of the basic definitions and theorems that we will relay on later.

**Definition 4** A subset $S$ of a normed space $X$ is called bounded if there is a number $M$ such that $\|x\| \leq M$ for all $x \in S$.

**Definition 5** A set $S$ in a vector space $X$ is called convex if for any $x, y \in S$, $ax + (1 - a)y \in S$ for all $a \in [0,1]$.

**Definition 6** Let $N,M$ be normed linear spaces, and $X$ be a subset of $N$. An operator $T : X \rightarrow M$ is continuous at a point $x \in X$ if and only if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\|Tx - Ty\| < \varepsilon$ for all $y \in X$ such that $\|x - y\| < \delta$. $T$ is continuous on $X$, or simply continuous, if it is continuous at all points of $X$.

**Theorem 1** Every continuous mapping of a closed convex set in $\mathbb{R}^n$ into itself has a fixed point.

**Definition 7** A subset $S$ of a normed space $B$ is compact if and only if every infinite sequence of elements of $S$ has a subsequence which converges to an element of $S$.

It is known that compact sets are closed and bounded, but not vice versa, in general.
Definition 8 A subset $S$ of a normed linear space $N$ is relatively compact if and only if every sequence in $S$ has a subsequence converging to an element of $N$.

Theorem 2 If $S$ is a convex, compact subset of a normed linear space, then every continuous mapping of $S$ into itself has a fixed point.

Theorem 3 If $S$ is a convex, closed subset of a normed linear space and $R$ a relatively compact subset of $S$, then every continuous mapping of $S$ into $R$ has a fixed point.

Theorem (3) is the more useful form for the theory of ODE.

Definition 9 Let $\{f_m\}$ be a sequence of real valued functions defined on a set $D \subseteq \mathbb{R}^n$.

(i) The sequence $\{f_m\}$ is called a uniform cauchy sequence if for any positive $\varepsilon$ there exists an integer $M(\varepsilon)$ such that when $m > k \geq M$ one has $\|f_k(x) - f_m(x)\| < \varepsilon$ for all $x \in D$.

(ii) The sequence $\{f_m\}$ is said to converge uniformly on $D$ to a function $f$ if for any $\varepsilon > 0$ there exists $M(\varepsilon)$ such that when $m > M$ one has $\|f_k(x) - f(x)\| < \varepsilon$ uniformly for all $x \in D$.

Theorem 4 Let $u_k$, $k=1,2,\ldots$, be given real valued functions defined on a set $D$ subset of $\mathbb{R}^n$. Suppose there exist nonnegative constants $M_k$ such that $\|u_k(x)\| \leq M_k$ for all $x$ in $D$ and

$$\sum_{i=1}^{\infty} M_k < \infty$$
then the sum

\[ \sum_{k=1}^{\infty} u_k(x) \]

converges uniformly on D.

**Definition 10** Let \( \mathcal{F} \) be a family of real valued functions defined on a set \( D \subseteq \mathbb{R}^n \).

Then

(i) \( \mathcal{F} \) is called uniformly bounded if there is a nonnegative constant \( M \) such that 
\[ \| f(x) \| \leq M \text{ for all } x \text{ in } D \text{ and for all } f \text{ in } \mathcal{F}. \]

(ii) \( \mathcal{F} \) is called equicontinuous on \( D \) if for any \( \epsilon > 0 \) there is a \( \delta \) (independent of \( x, y \) and \( f \)) such that 
\[ \| f(x) - f(y) \| < \epsilon \text{ whenever } \| x - y \| < \delta \text{ for all } x \text{ and } y \text{ in } D \text{ and for all } f \text{ in } \mathcal{F}. \]

**Theorem 5** Let \({D} \) be a closed, bounded subset of \( \mathbb{R}^n \) and let \( \{ f_m \} \) be a real valued sequence of functions in \( C(D) \). If \( \{ f_m \} \) is equicontinuous and uniformly bounded on \( D \), then there is a subsequence \( \{ f_{m_k} \} \) and a function \( f \) in \( C(D) \) such that \( \{ f_{m_k} \} \) converges to \( f \) uniformly on \( D \).

**Theorem 6** Let \( r, k \) and \( f \) be real and continuous functions which satisfy \( r(t) \geq 0 \), \( k(t) \geq 0 \), and

\[ r(t) \leq f(t) + \int_{a}^{t} k(s) r(s) ds, \quad a \leq t \leq b. \]
Then

\[ r(t) \leq \int_a^t f(s)k(s) \exp \int_s^t k(u)du \, ds + f(t), \quad a \leq t \leq b. \]

**Definition 11** Let \( \partial D \) denotes the boundary of a set \( D \subset \mathbb{R}^n \) and let \( \bar{D} = D \cup \partial D \) denotes the closure of \( D \). Given a sequence of real numbers \( \{a_m\} \), we define

\[
\lim_{m \to \infty} \sup a_m = \lim_{m \to \infty} a_m = \inf \left( \sup_{m \geq 1} a_k \right)
\]

and

\[
\lim_{m \to \infty} \inf a_m = \lim_{m \to \infty} a_m = \sup \left( \inf_{m \geq 1} a_k \right).
\]

It is easy to check that \( -\infty \leq \lim \inf a_m \leq \lim \sup a_m \leq +\infty \) and that the limsup and liminf of \( a_m \) are, respectively, the largest and smallest limit points of the sequence \( \{a_m\} \). Also, the limit of \( a_m \) exists if and only if the limsup and the liminf are equal. In this case the limit is their common value.

In the same way, if \( f \) is an extended real valued function on \( D \), then for any \( b \in \bar{D} \),

\[ \lim_{x \to b} \sup_{\epsilon > 0} f(x) = \inf \left( \sup \{f(y) : y \in D, \ 0 < |y - b| \leq \epsilon \} \right). \]

The limit inf is similarly defined.
1.8 Notes

In Chapter II, we study the properties of solutions of the third order linear homogeneous differential equation. In Section 2.1, we find the conditions for the equation

$$(a(t)(b(t)y')')' + (q_1y)' + q_1y' = 0$$  \hfill (1.24)

to be oscillatory. In Section 2.2, we study the effect of the delay and find the conditions for the equation

$$(a(t)(b(t)y')')' + q_1y(t) + q_2y(t - \tau(t)) = 0$$  \hfill (1.25)

to be oscillatory.

In Chapter III, we study the properties of the third order nonlinear, nonhomogeneous functional differential equation. In Section 3.1, we find the conditions for the equation

$$(a(t)(b(t)y')')' + qF(y(g(t))) = f(t)$$  \hfill (1.26)

to be oscillatory. In Section 3.2, we study the behavior of the nonoscillatory solutions of equation (1.26). In Section 3.3, we study a special case of equation (1.26) to find the asymptotic nature of the nonoscillatory solutions.

In Chapter IV, we study the properties of solutions of the neutral differential equation. In Section 4.1, we find the conditions for the equation

$$(y(t) + p(t)y(t - \tau))''' + f(t, y(t), y(t - \sigma)) = 0$$  \hfill (1.27)
to be oscillatory. In Section 4.2, we find the conditions for the equation

\[(y(t) + P(t)y(t - \tau))^\left(2^n\right) + Q(t)y(t - \sigma) = 0, \ n = 1, 2, \ldots\] (1.28)

to be oscillatory.

In Chapter V, we study the nonoscillatory and asymptotic nature of the nth order differential equations with delay. In Section 5.1, we find the asymptotic nature of the nonoscillatory solutions of equation

\[y^{(n)}(t) + a(t)y(t - \tau(t)) = f(t).\] (1.29)

In Section 5.2, we find the asymptotic nature of the nonoscillatory solutions of the equation

\[y^{(n)}(t) + p(t)f(y(t), y_{\sigma_1}(t), \ldots, y_{\sigma_{n-1}}(t)) = F(t),\] (1.30)

where,

\[y_{\sigma_i}(t) = y(t - \sigma_i).\]
2 PROPERTIES OF SOLUTIONS OF THIRD ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

2.1 Oscillation of Differential Equations without Delay

There is a large literature on the behavior of solutions of third order linear differential equations [5], [20], [26] and [29]. Many of these papers deal with the oscillatory and/or nonoscillatory properties of their solutions. V. S. Rao and R. S. Dahiya [36] discussed the behavior of the solutions of the equation
\[(r(y')')' + (q_1 y')' + q_2 y' = 0.\] (2.1)

In this section, we shall be interested in the generalized equation
\[(b(t)(a(t)y')')' + (q_1 y')' + q_2 y' = 0.\] (2.2)

A nontrivial solution of (2.2) is said to be oscillatory on the interval I if it has infinitely many zeros on I; otherwise, nonoscillatory. The equation (2.2) is said to be oscillatory or nonoscillatory, respectively, depending on the existence or nonexistence of an oscillatory solution.

Assume that
\[a, b \in C^2(t_0, \infty), \ a, b > 0 \quad and \quad \] (2.3)
\[q_i \in C^1(t_0, \infty), \ i = 1, 2 \quad (2.4)\]
Theorem 7 Let

(i) \((q_1 + q_2) \geq 0,\)

(ii) \((q_2 - q_1)' \leq 0,\)

(iii) \((a'b - ab') \geq 0,\) and

(iv) \(\lim_{t \to \infty} \frac{a'(t)}{a(t)} < \infty\)

the conditions (i), (ii), and (iii) hold on \([t_0, \infty)\) but not identically zero on any subinterval of \([t_0, \infty).\)

If \(\{A + B \int_{t_0}^{t} b^{-1}(s) \, ds - \int_{t_0}^{t} b^{-1}(s)q_1(s) \, ds\} < 0\) \(\quad (2.5)\)

for \(t\) sufficiently large and any constants \(A\) and \(B,\) then equation \((2.2)\) is oscillatory.

Proof

Let \(y\) is a nonoscillatory solution of equation \((2.2).\) Let \(t_1\) be the last zero of \(y,\) then without loss of generality we can assume that there exists \(t_2 \geq t_1\) such that \(y(t) > 0\) for \(t \in [t_2, \infty)\) and \(y'(t) > 0\) for \(t \in (t_2, \alpha)\) with some \(\alpha\) close to \(t_2.\) Let \(t_3 \in (t_2, \alpha).\)

Dividing equation \((2.2)\) by \(y\) and integrating from \(t_3\) to \(t,\) we get
\[\int_{t_3}^{t} \left( \frac{(b(ay')')'}{y} \right) (s) \, ds + \int_{t_3}^{t} \left( \frac{(q_1'y')}{y} \right) (s) \, ds + \int_{t_3}^{t} \left( \frac{(q_2'y')}{y} \right) (s) \, ds = 0\]

Now integrating by parts, we obtain

\[\left( \frac{b(ay')'}{y} \right) (t) - \left( \frac{b(ay')'}{y} \right) (t_3) + \int_{t_3}^{t} \left( \frac{(b(ay')')}{y^2} \right) (s) y'(s) \, ds + \int_{t_3}^{t} \left( \frac{q_1'y'}{y} \right) (s) \, ds + \int_{t_3}^{t} \left( \frac{q_2'y'}{y} \right) (s) \, ds = 0\]

This implies

\[\left( \frac{b(ay')'}{y} \right) (t) + \int_{t_3}^{t} \frac{(q_1 + q_2)y'}{y} \, ds + \int_{t_3}^{t} \frac{a'by}{y^2} y' \, ds + \int_{t_3}^{t} \frac{aby'y''}{y^2} \, ds = k_1 - q_1(t), \quad (2.6)\]

where

\[k_1 = \left( \frac{b(ay')'}{y} \right) (t_3) + q_1(t_3)\]

The last integral in (2.6) can be written as follows

\[\int_{t_3}^{t} \frac{ab(y'^2)}{2y^2} \, ds = \left( \frac{aby'^2}{2y^2} \right) (t) - \left( \frac{aby'^2}{2y^2} \right) (t_3) - \int_{t_3}^{t} \left( \frac{ab}{2y^2} \right) y'^2 ds,\]

or,

\[\int_{t_3}^{t} \left( \frac{ab}{2y^2} \right) y'^2 \, ds = \left( \frac{aby'^2}{2y^2} \right) (t) - \int_{t_3}^{t} \frac{(ab)y'^2}{2y^2} \, ds + \int_{t_3}^{t} \frac{aby'^3}{y^3} \, ds - k_2 \quad (2.7)\]
where
\[ k_2 = \left( \frac{aby'^2}{2y^2} \right) (t_3). \]

Substituting (2.7) in (2.6), we get
\[
\left( \frac{b(ay')'}{y} \right) (t) + \int_{t_3}^{t} \left( \frac{q_1 + q_2}{y} \right) y' ds + \int_{t_3}^{t} \frac{a'by'^2}{y^2} ds + \left( \frac{aby'^2}{2y^2} \right) (t) \\
- \int_{t_3}^{t} \frac{(ab'y')^2}{2y^2} ds + \int_{t_3}^{t} \frac{aby'^3}{y^3} ds = k_1 + k_2 - q_1(t)
\]

Dividing by \( b(t) \), it follows
\[
\left( \frac{(ay')'}{y} \right) (t) + \frac{1}{b(t)} \int_{t_3}^{t} \left( \frac{q_1 + q_2}{y} \right) y' ds + \frac{1}{b(t)} \int_{t_3}^{t} \frac{a'by'^2}{y^2} ds + \left( \frac{ay'^2}{2y^2} \right) (t) \\
- \frac{1}{b(t)} \int_{t_3}^{t} \frac{(ab'y')^2}{2y^2} ds + \frac{1}{b(t)} \int_{t_3}^{t} \frac{aby'^3}{y^3} ds = \frac{k}{b(t)} - \frac{q_1(t)}{b(t)},
\]
or,
\[
\left( \frac{(ay')'}{y} \right) (t) + b^{-1}(t) \int_{t_3}^{t} \left( \frac{q_1 + q_2}{y} \right) y' ds + b^{-1}(t) \int_{t_3}^{t} \left( a' - ab' \right) \frac{y'^2}{2y^2} ds \\
+ b^{-1}(t) \int_{t_3}^{t} \frac{aby'^3}{y^3} ds + \left( \frac{ay'^2}{2y^2} \right) (t) = kb^{-1}(t) - b^{-1}(t)q_1(t),
\]
where
\[ k = k_1 + k_2. \]
Now integrating (2.8) from $t_3$ to $t$, we obtain

$$
\int_{t_3}^{t} \frac{(ay')}{y} \, ds + \int_{t_3}^{t} b^{-1}(s) \int_{t_3}^{s} (q_1 + q_2) \frac{y'}{y} \, duds + \int_{t_3}^{t} b^{-1}(s) \int_{t_3}^{s} (a'b - ab') \frac{y'^2}{2y^2} \, duds
$$

$$
+ \int_{t_3}^{t} b^{-1}(s) \int_{t_3}^{s} ab'y' \frac{3}{y^3} \, duds + \int_{t_3}^{t} ay'^2 \frac{3}{y^2} \, ds = k \int_{t_3}^{t} b^{-1}(s) \, ds - \int_{t_3}^{t} b^{-1}(s)q_1(s) \, ds,
$$
or,

$$
\left( \frac{ay'}{y} \right)(t) + \frac{3}{2} \int_{t_3}^{t} \frac{ay'^2}{y^2} \, ds + \int_{t_3}^{t} b^{-1}(s) \int_{t_3}^{s} (q_1 + q_2) \frac{y'}{y} \, duds
$$

$$
+ \int_{t_3}^{t} b^{-1}(s) \int_{t_3}^{s} (a'b - ab') \frac{y'^2}{2y^2} \, duds + \int_{t_3}^{t} b^{-1}(s) \int_{t_3}^{s} ab'y' \frac{3}{y^3} \, duds
$$

$$
= k \int_{t_3}^{t} b^{-1}(s) \, ds - \int_{t_3}^{t} b^{-1}(s)q_1(s) \, ds + M,
$$

where

$$
M = \left( \frac{ay'}{y} \right)(t_3).
$$

Suppose $y'(t) > 0$ for $t \in [t_3, \infty)$, then by using (i), (iii) and (2.5) in (2.9), we obtain

$$
\frac{a(t)y'(t)}{y(t)} < 0.
$$

(2.10)
Since $a(t) > 0$ and $y(t) > 0$ then from (2.10) we have $y'(t) < 0$ for sufficiently large $t$ and this contradicts our assumption. So it is clear that there exists $t_4 \geq t_3$ such that $y'(t_4) = 0$.

Now, we shall conclude the theorem by showing that

$$y(t_1) = y'(t_4) = 0$$

contradicts $y(t) > 0$ for $t \geq t_1$.

Multiplying equation (2.2) by $y$ and integrating from $t_1$ to $t$, it follows

$$\int_{t_1}^{t} (b(ay')'y).yds + \int_{t_1}^{t} (q_1y)'y.yds + \int_{t_1}^{t} q_2y'.yds = 0,$$

or,

$$(b(ay')'.y)(t) - (b(ay')'.y)(t_1) - \int_{t_1}^{t} b(ay')'.y'yds +$$

$$(q_1y^2)(t) - (q_1y^2)(t_1) - \int_{t_1}^{t} q_1y'yds + \int_{t_1}^{t} q_2y'yds = 0,$$

or,

$$(b(ay')'.y)(t) - (b(ay')'.y)(t_1) - \int_{t_1}^{t} (aby''y' + a'by'^2)ds$$

$$+(q_1y^2)(t) - (q_1y^2)(t_1) - \int_{t_1}^{t} q_1(y^2)'ds$$
\[ \frac{1}{2} t \int_{t_1}^{t} q_1(y^2)' ds + \frac{1}{2} t \int_{t_1}^{t} q_2(y^2)' ds = 0. \]  

(2.11)

Since \( y(t_1) = 0 \) then (2.11) becomes

\[(b(ay')', y)(t) - \int_{t_1}^{t} (aby'' + a'by'^2) ds + (q_1 y^2)(t) + \frac{1}{2} \int_{t_1}^{t} (q_2 - q_1)(y^2)' ds = 0,\]

or,

\[(aby'' + a'by')(t) - \int_{t_1}^{t} a'by'^2 ds - \frac{1}{2} \int_{t_1}^{t} ab(y'^2)' ds + \frac{1}{2} \int_{t_1}^{t} (q_2 - q_1)(y^2)' ds + (q_1 y^2)(t) = 0. \]  

(2.12)

Integrating by parts and using again \( y(t_1) = 0 \), we get

\[(aby'' + a'by')(t) - \int_{t_1}^{t} a'by'^2 ds - \frac{1}{2} ((aby^2)(t) - (aby^2)(t_1)) + \frac{1}{2} \int_{t_1}^{t} (ab)'y'^2 ds + \frac{1}{2} \int_{t_1}^{t} (q_2 - q_1)(y^2)' ds + (q_1 y^2)(t) = 0,\]

or,

\[(aby''(t) + (a'by')(t) - \frac{1}{2} (aby^2)(t) + \frac{1}{2} (aby^2)(t_1) + \frac{1}{2} ((q_2 - q_1)y^2)(t) - ((q_2 - q_1)y^2)(t_1)) - \frac{1}{2} \int_{t_1}^{t} (q_2 - q_1)y^2 ds + (q_1 y^2)(t) = 0,\]

or,

\[(aby''(t) + (a'by')(t) - \frac{1}{2} (aby^2)(t) + \frac{1}{2} (aby^2)(t_1) + \frac{1}{2} ((q_2 - q_1)y^2)(t)) + \frac{1}{2} \int_{t_1}^{t} (ab' - a'b)y'^2 ds - \frac{1}{2} \int_{t_1}^{t} (q_2 - q_1)y^2 ds = 0. \]  

(2.13)
Define

$$F(y(t)) = \frac{1}{2} aby'^2 - abyy'' - a'byy' - \frac{1}{2} (q_1 + q_2)y^2. \quad (2.14)$$

Using (2.14) in (2.13), we have

$$F(y(t)) = F(y(t_1)) + \frac{1}{2} \int_{t_1}^{t} (ab' - a'b)y'^2 ds - \frac{1}{2} \int_{t_1}^{t} (q_2 - q_1)'y'^2 ds, \quad (2.15)$$

where

$$F(y(t_1)) = \frac{1}{2} (aby'^2)(t_1) > 0. \quad (2.16)$$

Using (ii), (iii) and (2.16) in (2.15) we prove that $F(y(t))$ is an increasing function, which vanishes whenever $y$ has a double zero, i.e., $y = y' = 0$.

Since $y(t_4) > 0$ and $y'(t_4) = 0$ then, we have

$$F(y(t_4)) = -(aby'')(t_4) - \frac{1}{2} (q_1 + q_2)y^2(t_4) > 0. \quad (2.17)$$

From (2.17) we see that

$$y''(t_4) < 0. \quad (2.18)$$

It is clear that $y'(t)$ can not vanish more than once in $[t_4, \infty)$. Hence

$$y'(t) < 0 \quad \text{for} \quad t > t_4 \quad \text{and}$$

$$\lim_{t \to \infty} y(t) \quad \text{exists.}$$
Now we divide the rest of the argument into three cases depending on the sign of $y''(t)$:

**CASE1** Let $y''(t) \leq 0$ eventually. Then $y(t)$ becomes eventually negative and this contradicts

$$y(t) > 0 \quad \text{for all } t \geq t_1.$$ 

**CASE2** Let $y''(t) \geq 0$ eventually. Then, since $y'(t) < 0$ for $t > t_4$, we would have

$$\lim_{t \to \infty} y'(t) = 0$$

and consequently

$$\lim_{t \to \infty} \frac{F(y(t))}{a(t)b(t)} \leq \lim_{t \to \infty} \left( \frac{y'}{2}y'^2 - yy'' - \frac{a'}{a}yy' \right)$$

(2.19)

Since $y(t) > 0$, $y''(t) \geq 0$ eventually and $\lim_{t \to \infty} y(t)$ exists and by using (iv), we get from (2.19)

$$\lim_{t \to \infty} \frac{F(y(t))}{a(t)b(t)} \leq 0$$

which contradicts the fact that $F$ is increasing.

**CASE3** Suppose $y''$ changes its sign for arbitrary large $t$. Then for any $\epsilon > 0$, there exists a large $t$ such that

$$0 > y'(t) > -\epsilon$$
and a relative maximum of $y(t)$ at $\bar{t}$ such that
\[ 0 > y'(\bar{t}) > -\epsilon \quad \text{and} \quad y''(\bar{t}) = 0. \]

Then we have
\[ F(y(\bar{t})) \leq a(\bar{t})b(\bar{t})\frac{\epsilon^2}{2} + |a'(\bar{t})|b(\bar{t})y(\bar{t})\epsilon \]
for arbitrary large $\bar{t}$ and this implies that
\[ \lim_{t \to \infty} F(y(t)) \leq 0. \]

This is a contradiction to the fact that $F(y(t))$ is increasing. Therefore the proof is complete.

**Example 9** Consider the equation
\[ (e^{-t}(e^ty')')' + (ty)' + (1 - t)y' = 0 \quad (2.20) \]
where
\[ a(t) = e^t, \]
\[ b(t) = e^{-t}, \]
\[ q_1(t) = t \quad \text{and} \]
\[ q_2(t) = 1 - t. \]

It is clear that
\[ q_1 + q_2 = t + 1 - t = 1 > 0, \]
All conditions of theorem 7 are satisfied. Hence, all solutions of equation (2.20) are oscillatory.

In fact $y = \sin(t)$ is a solution of equation (2.20).

### 2.2 Oscillation of Differential Equations with Delay

In this section, we study the effect of delay on the properties of solutions of equation of the form

$$ (b(ay')')' + q_1y(t) + q_2y(\tau(t)) = 0, \quad (2.21) $$

where $a, b, q_1$ and $q_2$ are the same as in (2.3) and (2.4) and $\tau(t)$ satisfies the following:

$$ \tau(t) \leq t, \quad \tau'(t) > 0 \quad \text{and} \quad \tau(t) \to \infty \quad \text{as} \quad t \to \infty. \quad (2.22) $$

**Theorem 8** Assume that

\begin{align*}
q_1 & \geq 0 \quad \text{and} \quad q_2 \geq 0 \quad \text{for} \quad t \in [t_0, \infty), \quad (2.23) \\
|b'(t)| & \leq 0 \quad \text{for} \quad t \in [t_0, \infty), \quad \text{and} \quad (2.24)
\end{align*}
there exists a differentiable function

\[ \delta : [t_0, \infty) \rightarrow (0, \infty) \text{ such that } \]

\[ \delta' < 0 \quad \text{and} \quad \delta'' > 0 \quad \text{for} \quad t \geq t_0 \quad \text{and} \]

\[ \delta' \leq \frac{b'}{2b} \delta \quad \text{for} \quad t \geq t_0. \]  \hspace{1cm} (2.25)

If

\[ \int_0^\infty \left\{ (q_1 + q_2)\delta - \frac{a(t)b\delta'^2}{4B_t\tau_0^2} \right\} dt = \infty, \]  \hspace{1cm} (2.27)

\[ \int_0^\infty \frac{t}{a\delta} dt < \infty \]  \hspace{1cm} (2.28)

and

\[ \int_0^\infty \frac{1}{a\delta} \left( \int t \int \frac{q_1 + q_2}{b} \delta du \right) ds \ dt = \infty \]  \hspace{1cm} (2.29)

then every solution \( y(t) \) of equation (2.21) is either oscillatory or

\[ \lim_{t \to \infty} y(t) = 0. \]

Proof

Let \( y(t) \) be a nonoscillatory solution of equation (2.21). We may assume without loss of generality that \( y(t) > 0 \) for \( t \geq t_0 \), then there exists \( t_1 \geq t_0 \) such that \( y(\tau(t)) > 0 \) for \( t \geq t_1 \). From equation (2.21) we have
\[(b(ay')')' = -q_1 y(t) - q_2 y(\tau(t)).\] \hspace{1cm} (2.30)

It is clear that \(-(b(ay')')'\) is positive for \(t \geq t_1\), hence \(y(t)\) is monotone and one-signed. \(y'(t)\) and \(y''(t)\) are also monotone and one-signed for sufficiently large \(t\).

**Claim 1:**
\[(ay')' > 0 \text{ for } t \geq t_1.\] \hspace{1cm} (2.31)

From equation (2.30) we have
\[b(ay')'' + b'(ay')' \leq 0,
\]
or,
\[(ay')'' \leq -\frac{b'}{b} (ay')'.\] \hspace{1cm} (2.32)

If \((ay')' \leq 0\), then \(ay'\) is decreasing and concave down. Therefore \(ay'\) is eventually negative which is a contradiction. Thus (2.31) is true.

**Claim 2:**
\[y'(t) < 0 \text{ for } t \geq t_1.\] \hspace{1cm} (2.33)

If \(y' \geq 0\) for \(t \geq t_1\) then \(y(t)\) is increasing and positive.

Define the function
\[
\omega(t) = \frac{(b(ay')')\delta(t)}{y(\tau(t))}.
\] \hspace{1cm} (2.34)
By differentiating (2.34), we have

\[
\omega'(t) = \frac{y(\tau(t))}{y(\tau(t))} \cdot \left( b'(ay') y' \delta + y(\tau(t)) (b(ay') y') \delta' - (b(ay') y') y'(\tau) \delta' \right),
\]

or,

\[
\omega'(t) = (b(ay') y') \frac{\delta}{y(\tau(t))} + b(ay') \frac{\delta'}{y(\tau(t))} - \frac{(b(ay') y') y'(\tau)}{y(\tau)} \delta'.
\] (2.35)

Using (2.30) and (2.34) in (2.35), we obtain

\[
\omega'(t) = -q_1 \delta \frac{y(t)}{y(\tau)} - q_2 + \delta \omega - \omega y'(\tau) \delta'.
\] (2.36)

Since \( y' \geq 0 \) and \( (ay')' > 0 \), then by Kiguradze's lemma [27], we have

\[
ay' \geq Bt(ay')' \text{ for some } B > 0
\] (2.37)

and

\[
((ay')')'(t) \leq (a(\tau)y'(\tau))' \text{ for } t \geq t_2 \geq t_1.
\] (2.38)

Since \( y' \geq 0 \) for \( t \geq t_1 \) then \( y(\tau) \leq y(t) \),

or,

\[
\frac{y(t)}{y(\tau)} \geq 1.
\]
Now (2.36) can be written as

$$\omega'(t) \leq -(q_1 + q_2)\delta + \frac{\delta'}{\delta} \omega - \frac{a(\tau)y'(\tau)}{a(\tau)y(\tau)} \tau' \omega$$  \hspace{1cm} (2.39)$$

By using (2.37) in (2.39), it follows

$$\omega'(t) \leq -(q_1 + q_2)\delta + \frac{\delta'}{\delta} \omega - B \frac{(a(\tau)y'(\tau))'}{a(\tau)y(\tau)} \tau' \omega.$$  \hspace{1cm} (2.40)$$

Now from (2.38) and (2.40), we get

$$\omega'(t) \leq -(q_1 + q_2)\delta + \frac{\delta'}{\delta} \omega - B \frac{(a(\tau)y'(t))'}{a(\tau)y(\tau)} \tau' \omega.$$ $$

Using (2.34) again, we obtain

$$\omega'(t) \leq -(q_1 + q_2)\delta + \frac{\delta'}{\delta} \omega - B \frac{\tau' \omega}{a(\tau)b(t)\delta} \omega^2,$$

or,

$$\omega'(t) \leq -(q_1 + q_2)\delta - \left\{ \left( \frac{B\tau'}{a(\tau)b\delta} \right) \omega^2 - \frac{\delta'}{\delta} \omega \right\}.$$ $$

Now completing the square on the right hand side, it follows
\[
\omega'(t) \leq -(q_1 + q_2)\delta - \left( \frac{B\tau' I}{a(\tau)b\delta} \right)^{1/2} \omega - \frac{\delta'/2\delta}{\left( \frac{B\tau' I}{a(\tau)b\delta} \right)^{1/2}} + \frac{\delta'^2 a(\tau)b}{4B\tau' I\delta},
\]

or,

\[
\omega'(t) \leq -(q_1 + q_2)\delta + \delta'^2 \frac{a(\tau)b}{4B\tau' I\delta}. \tag{2.41}
\]

Integrating (2.41) from \(t_2\) to \(t\), we get

\[
\omega(t) - \omega(t_2) \leq - \int_{t_2}^{t} \left\{ (q_1 + q_2)\delta - \delta'^2 \frac{a(\tau)b}{4B\tau' I\delta} \right\} ds,
\]

or,

\[
\int_{t_2}^{t} \left\{ (q_1 + q_2)\delta - \delta'^2 \frac{a(\tau)b}{4B\tau' I\delta} \right\} ds \leq \omega(t) - \omega(t). \tag{2.42}
\]

Because of (2.31) and (2.34), we would claim that

\[
\omega(t) > 0 \quad \text{for } t \geq t_1
\]

and then (2.42) could be written as

\[
\int_{t_2}^{t} \left\{ (q_1 + q_2)\delta - \delta'^2 \frac{a(\tau)b}{4B\tau' I\delta} \right\} ds \leq \omega(t),
\]
or,
\[
\int_{t_2}^{t} \left\{ (q_1 + q_2) \delta - \delta \frac{a(\tau)b}{4B\tau}\right\} ds < \infty.
\]

This contradicts (2.27). Hence (2.33) is true.

From (2.33) and the fact that \( y(t) > 0 \) for \( t \geq t_1 \), there exists a constant \( c \geq 0 \) such that
\[
\lim_{t \to \infty} y(t) = c.
\]
We will prove that \( c = 0 \).

Let \( c > 0 \), then there exists \( t^* \geq t_1 \) such that
\[
y(\tau(t)) > \frac{c}{2} \text{ for } t \geq t^*. \tag{2.43}
\]
Define the function
\[
G(t) = ay' \delta \text{ for } t \geq t^*. \tag{2.44}
\]
Differentiating (2.44), we get
\[
G'(t) = (ay')' \delta + ay' \delta' \tag{2.45}
\]
Multiplying (2.45) by \( b(t) \) and differentiating again, we obtain
\[
bG'' + b'G' = (b(ay')')' \delta + 2b(ay')' \delta' + b'(ay')' \delta' + b(ay') \delta'' \tag{2.46}
\]
Using (2.45) in (2.46), we have
\[
G''(t) = \frac{b}{b} (b(ay')')' \delta + 2(ay')' \delta' - \frac{b'}{b} (ay')' \delta' + (ay') \delta'' \tag{2.47}
\]
Now if we use (2.30) in (2.47), we get

$$G''(t) = -\frac{\delta}{b}(q_1 y(t) + q_2 y(\tau)) + 2(ay')' \left( \delta' - \frac{b'}{2b}\delta \right) + (ay')\delta''$$  \hspace{1cm} (2.48)

Using (2.25) and (2.33) in (2.48), will give

$$G''(t) \leq -\frac{\delta}{b}(q_1 y(t) + q_2 y(\tau)) + 2(ay')' \left( \delta' - \frac{b'}{2b}\delta \right)$$ \hspace{1cm} (2.49)

By using (2.26) and (2.31) in (2.49), we obtain

$$G''(t) \leq -\frac{\delta}{b}(q_1 y(t) + q_2 y(\tau))$$ \hspace{1cm} (2.50)

Using (2.43) in (2.50), we deduce that

$$G''(t) \leq -\frac{c}{2b}(q_1 + q_2) \text{ for } t \geq t^*.$$ \hspace{1cm} (2.51)

Integrating (2.51) from $t^*$ to $t$, we get

$$G'(t) \leq G'(t^*) - \frac{c}{2} \int_{t^*}^{t} (q_1 + q_2) \frac{\delta}{b} \, ds$$

Integrating again, will lead to

$$G(t) \leq G(t^*) + G'(t^*)(t - t^*) - \frac{c}{2} \int_{t^*}^{t} \int_{t^*}^{s} (q_1 + q_2) \frac{\delta}{b} \, dus$$ \hspace{1cm} (2.52)
Now if we use (2.33) in (2.44), we conclude that

\[ G(t) < 0 \quad \text{for} \quad t \geq t^*. \]  \hfill (2.53)

Using (2.25), (2.31) and (2.33) in (2.45), we get

\[ G(t') > 0 \quad \text{for} \quad t \geq t^*. \]  \hfill (2.54)

Now using (2.53) and (2.54) in (2.52), we obtain

\[ G(t) \leq G'(t^*) t - \frac{c}{2} \int_{t^*}^{t} \int_{t^*}^{s} (q_1 + q_2) \frac{\delta}{b} \, du \, ds \]  \hfill (2.55)

From (2.44) and (2.55), it follows

\[ y'(t) \leq G'(t^*) \frac{t}{a \delta} - \frac{c}{2} \int_{t^*}^{t} \int_{t^*}^{s} (q_1 + q_2) \frac{\delta}{b} \, du \, ds \]  \hfill (2.56)

Integrating (2.56) also from \( t^* \) to \( t \), we have

\[ y(t) \leq y(t^*) + G'(t^*) \int_{t^*}^{t} \frac{s}{a \delta} \, ds - \frac{c}{2} \int_{t^*}^{t} \int_{t^*}^{s} \int_{t^*}^{u} (q_1 + q_2) \frac{\delta}{b} \, dv \, du \, ds \]  \hfill (2.57)

Taking the limit as \( t \to \infty \), we get
Now using (2.28) and (2.29) in (2.58), will give a contradiction.

Hence \( c = 0 \) and the proof is complete.

Example 10 Consider the equation

\[
\left( e^{-t/4} (e^{t/2} y')' \right)' + \frac{1}{8} e^{t/4} y(t) + \frac{1}{4} e^{(t/4 - \pi)} y(t - \pi) = 0 \tag{2.59}
\]

where

\[
\begin{align*}
a(t) &= e^{t/2}, \\
b(t) &= e^{-t/4}, \\
q_1(t) &= \frac{1}{8} e^{t/4}, \\
q_2(t) &= \frac{1}{4} e^{(t/4 - \pi)}, \\
\delta(t) &= e^{-t/8}, \text{ and} \\
\tau(t) &= t - \pi
\end{align*}
\]

It is clear that

\[
\tau(t) = t - \pi < t, \quad \tau' = 1 > 0,
\]

\[
\tau(t) \to \infty \text{ as } t \to \infty,
\]

\[
\delta' = -\frac{1}{8} e^{-t/8} < 0,
\]
\[ \delta'' = \frac{1}{64} e^{-t/8} > 0, \]
\[ \delta' = \frac{b'}{2b} \delta = -\frac{1}{8} e^{-t/8} < 0, \]

\[ \int_0^\infty \left( (q_1 + q_2) \delta - \frac{a(t)}{4B} \right) dt = \int_0^\infty \left( \frac{1}{8} e^{t/8} (1 + 2e^{-\pi}) - \frac{e^{-\pi/2} e^{t/8}}{256B(t - \pi)} \right) dt = \infty, \]

\[ \int_0^\infty \frac{t}{a\delta} dt = \int_0^\infty te^{t/8} dt = \infty \text{ and} \]

\[ \int_0^\infty \frac{1}{a\delta} \int_s^t (q_1 + q_2) \delta du ds dt = \]

\[ \int_0^\infty e^{3t/8} \int_s^t \frac{1}{8} e^{3u/8} (1 + 2e^{-\pi}) du ds dt = \infty. \]

All the conditions of theorem 8 are satisfied, then the conclusion of the theorem holds.

In fact \( y(t) = e^{-t} \) is a solution of equation (2.59).
3 PROPERTIES OF SOLUTIONS OF THIRD ORDER
NONHOMOGENEOUS FUNCTIONAL DIFFERENTIAL
EQUATIONS

3.1 Oscillation Properties

In this section, we are interested in the oscillatory behavior of the equation

\[(b(ay')')' + qF(y(g(t)))) = f(t)\]  

where

(i) \(a, b, q, g, f: [t_0, \infty) \rightarrow \mathbb{R}\) are continuous,

(ii) \(F: \mathbb{R} \rightarrow \mathbb{R}\) continuous,

(iii) \(a > 0, b > 0 \text{ and } b' \leq 0 \text{ for } t \in [t_0, \infty),\)

(iv) \(g(t) \geq 0 \text{ for } t \in [t_0, \infty)\)

and not identically zero for any ray of the form \([t^*, \infty) \text{ for some } t^* \geq t_0,\)

(v) \(g(t) \rightarrow \infty \text{ as } t \rightarrow \infty, \text{ and}\)

(vi) \(yF(y) > 0 \text{ for } y \neq 0.\)

S. R. Grace and B. S. Lalli [21] studied the oscillatory behavior of the equation

\[(ax')' + q f(x(g(t)))) = e(t).\]  

Thus in this section we extend the study to the third order and add number of new
results.

We consider only solutions of equation (3.1) which are defined for large $t$. The oscillatory solution of equation (3.1) is considered in the usual sense, i.e., a solution of equation (3.1) is called oscillatory if all its solutions are oscillatory. It is almost oscillatory if every solution $y(t)$ of equation (3.1) is either oscillatory or

$$\lim_{t \to \infty} y(t) = 0.$$  

**Theorem 9** Let

$$F'(y) \geq k > 0 \quad \text{for } y \neq 0. \quad (3.3)$$

Assume that there exists a function

$$\phi : [t_0, \infty) \to \mathbb{R}$$

and differentiable functions

$$\delta, \sigma : [t_0, \infty) \to (0, \infty)$$

such that

$$(b(a\phi')')' = f(t), \quad (3.4)$$

$$\phi(t), \phi'(t), \phi''(t) \to 0 \quad \text{as } t \to \infty, \quad (3.5)$$

$$\sigma(t) \leq \min\{ t, g(t) \}, \quad (3.6)$$

$$\sigma'(t) > 0, \quad (3.7)$$

$$\sigma(t) \to \infty \quad \text{as } t \to \infty, \quad (3.8)$$

$$\delta'(t) < 0 \quad \text{for } t \geq t_0, \quad (3.9)$$
\[ \delta''(t) > 0 \quad \text{for} \quad t \geq t_0, \quad \text{and} \quad \text{(3.10)} \\
\delta'(t) \leq \left( \frac{b'}{2b} \right) \delta(t) \quad \text{for} \quad t \geq t_0. \quad \text{(3.11)} \\
\]

If

\[ \int_{\infty}^{\infty} \left( \delta q - \frac{a(\sigma)b\delta''}{4B\lambda \sigma'\delta} \right) dt = +\infty, \quad 0 < \lambda < 1, \quad B > 0, \quad \text{(3.12)} \]

\[ \int_{\infty}^{\infty} \frac{1}{a\delta} \int_{t_0}^{t} \int_{t_0}^{u} \left( \frac{\delta q}{b} \right) dvdu dt = +\infty, \quad \text{and} \quad \text{(3.13)} \]

\[ \int_{\infty}^{\infty} \frac{1}{a\delta} dt < \infty \quad \text{(3.14)} \]

then every solution of equation (3.1) is either oscillatory or

\[ \lim_{t \to \infty} y(t) = 0. \]

**Proof**

Let \( y(t) \) be a nonoscillatory solution of equation (3.1). We may assume without loss of generality that \( y(t) > 0 \) for \( t \geq t_0 \) then there exists a \( t_1 \geq t_0 \) such that \( y(\sigma(t)) > 0 \) for \( t \geq t_1 \).

Consider the equation

\[ y(t) = x(t) + \phi(t) \quad \text{for} \quad t \geq t_1, \quad \text{(3.15)} \]

then, from equation (3.1), we have

\[ (b(a(x + \phi)'')' + q F(y(g(t))) = f(t), \]
or,

\[(b(ax')')' + q F(y(g(t))) + (b(a\phi')')' = f(t). \tag{3.16}\]

Using (3.4) in (3.16), we get

\[(b(ax')')' = -q F(y(g(t))) \text{ for } t \geq t_1. \tag{3.17}\]

It is clear that \(-(b(ax')')'\) is eventually positive for \(t \geq t_1\).

Hence \(x(t)\) is monotone and one-signed. \(x'(t)\) and \(x''(t)\) are also monotone and one-signed for sufficiently large \(t\).

If \(x(t) < 0\) for \(t \geq t_1\) then

\(y(t) < \phi(t)\) for \(t \geq t_1\) and by using (3.5), we get a contradiction to the assumption that \(y(t) > 0\).

Hence we must have

\[x(t) > 0 \text{ for } t \geq t_1. \tag{3.18}\]

Claim 1

\[(ax')' > 0 \text{ for } t \geq t_1. \tag{3.19}\]

Using (iv) and (vi) in (3.17), we obtain

\[(b(ax')')' \leq 0,\]

or,

\[(ax'')'' \leq -\frac{b'}{b} (ax')' \tag{3.20}\]

if

\[(ax')' \leq 0\]
then, by using (iii) in (3.20), we have

$$(ax')'' \leq 0.$$  

This implies that $ax'$ is decreasing and concave down, hence $a(t)x'(t)$ is eventually negative.

Therefore $x(t)$ is eventually negative which contradicts (3.18).

Claim 2

Now we claim that

$$x'(t) < 0 \text{ for } t \geq t_1. \quad (3.21)$$

If

$$x'(t) > 0 \text{ for } t \geq t_1$$

then, from (3.15), we get

$$y(g(t)) = x(g(t)) + \phi(g(t)) \text{ for } t \geq t_1. \quad (3.22)$$

Since $x(t)$ is increasing and positive and

$$\phi(t) \to 0 \text{ as } t \to \infty$$

then there exists $t_\lambda \geq t_1$ sufficiently large such that

$$y(g(t)) \geq \lambda x(g(t)) \text{ for } t \geq t_\lambda. \quad (3.23)$$

Using (3.3) in (3.23), we get

$$F(y(g(t))) \geq F(\lambda x(g(t))) \text{ for } t \geq t_\lambda. \quad (3.24)$$
Now, define the function
\[ \omega(t) = \left( \frac{b(ax')'}{F(\lambda x(\sigma))} \right)'(t). \] (3.25)

Differentiating (3.25), we get
\[
\omega'(t) = \frac{F(\lambda x(\sigma)) (b(ax')')' + F'(\lambda x(\sigma)) (b(ax')')' - (b(ax')')' F' (\lambda x(\sigma)) x'(\sigma)'}{F^2(\lambda x(\sigma))},
\]
or,
\[
\omega'(t) = \frac{(b(ax')')'}{F(\lambda x(\sigma))} + \frac{b(ax')'}{F(\lambda x(\sigma))} - \frac{(b(ax')')'}{F^2(\lambda x(\sigma))} \lambda F' x'(\sigma)'. \] (3.26)

Using (3.17) and (3.25) in (3.26), we get
\[
\omega'(t) = -q \frac{F'(y(g(t)))}{F'(\lambda x(\sigma))} \delta' + \omega \frac{\delta'}{\delta} - \omega \frac{F'(\lambda x(\sigma))}{F'(\lambda x(\sigma))} \lambda x'(\sigma)'. \] (3.27)

Since \( x'(t) \geq 0 \) and \( (ax')' > 0 \)
then by Kiguradz's lemma [27], we have
\[
(ax')'(t) \geq B_1 t ((ax')')'(t) \quad \text{for some } B_1 > 0 \] (3.28)
and
\[
(ax')'(t) \leq (a(\sigma)x'(\sigma))'(t) \quad \text{for } t \geq t_\lambda. \] (3.29)

Using (3.24) and (3.28) in (3.27), we get
By making use of (3.3), we obtain

$$\omega'(t) \leq -q\delta + \omega \frac{\delta'}{\delta} - \omega \lambda \frac{F'(\lambda x(\sigma))}{F(\lambda x(\sigma))} B_1 \sigma \frac{(a(\sigma)x'(\sigma))'}{a(\sigma)} \sigma'$$

Using (3.29) in (3.30), we get

$$\omega'(t) \leq -q\delta + \omega \frac{\delta'}{\delta} - \omega \lambda B \sigma \frac{(a(\sigma)x'(\sigma))'}{a(\sigma)F(\lambda x(\sigma))}, \quad B = kB_1 \quad (3.30)$$

Using (3.29) in (3.30), we get

$$\omega'(t) \leq -q\delta + \omega \frac{\delta'}{\delta} - \omega \lambda B \sigma \frac{(a(\sigma)x'(\sigma))'}{a(\sigma)F(\lambda x(\sigma))} - \omega \lambda \frac{b(ax'(\sigma))'}{ba(\sigma)F(\lambda x(\sigma))}$$ \quad (3.31)

Using (3.25) again in (3.31), we get

$$\omega'(t) \leq -q\delta + \omega \frac{\delta'}{\delta} - \omega \lambda B \frac{\sigma \sigma'}{a(\sigma)b \delta}, \quad (3.32)$$

or,

$$\omega'(t) \leq -q\delta - \lambda B \frac{\sigma \sigma'}{a(\sigma)b \delta} \left\{ \omega^2 - \frac{a(\sigma)b \delta'}{\lambda B \sigma \sigma'} \omega \right\} \quad (3.33)$$

Completing the square in (3.33), we get

$$\omega'(t) \leq -q\delta - \lambda B \frac{\sigma \sigma'}{a(\sigma)b \delta} \left\{ \omega - \frac{a(\sigma)b \delta'}{2\lambda B \sigma \sigma'} \right\}^2 + \frac{a(\sigma)b \delta'^2}{4\lambda B \sigma \sigma' \delta},$$
or,

$$\omega'(t) \leq -\left\{ q\delta - \frac{a(\sigma)b\delta^2}{4\lambda B\sigma\sigma'} \right\}$$ \hspace{1cm} (3.34)

Integrating (3.34) from $t_\lambda$ to $t$, we have

$$\omega(t) - \omega(t_\lambda) \leq \int_{t_\lambda}^{t} \left\{ (q\delta - \frac{a(\sigma)b\delta^2}{4\lambda B\sigma\sigma'})(s) \right\} \, ds,$$

or,

$$\int_{t_\lambda}^{t} \left\{ (q\delta - \frac{a(\sigma)b\delta^2}{4\lambda B\sigma\sigma'})(s) \right\} \, ds \leq \omega(t_\lambda) - \omega(t) \leq \omega(t_\lambda) < \infty. \hspace{1cm} (3.35)$$

From (3.12), it is clear that (3.35) is a contradiction.

Hence (3.21) is true.

From (3.18) and (3.21), there exists a constant $c \geq 0$ such that

$$\lim_{t \to \infty} x(t) = c.$$

We will prove that $c = 0$.

Assume that $c > 0$ and from (3.15), we have

$$\lim_{t \to \infty} y(g(t)) = \lim_{t \to \infty} x(g(t)) = c.$$
Hence there exists a $t_2 \geq t_1$ such that

$$y(g(t)) \geq \frac{c}{2} \text{ for } t \geq t_2$$  \hspace{1cm} (3.36)

Define the function

$$G(t) = a(t)x'(t)\delta(t) \text{ for } t \geq t_2$$  \hspace{1cm} (3.37)

Differentiating (3.37), we get

$$G'(t) = (a(t)x'(t))'\delta(t) + a(t)x'(t)\delta'(t)$$  \hspace{1cm} (3.38)

Multiplying (3.38) by $b(t)$ and differentiating the resulting equation, we have

$$b'(t)G'(t) + b(t)G''(t) = b(t)(a(t)x'(t))'\delta' + (b(ax')')'\delta$$
$$+ b(ax')'\delta' + b'ax'\delta' + b(ax')\delta''$$  \hspace{1cm} (3.39)

Using (3.38) in (3.39), we get

$$G''(t) = (ax')'\delta' + \frac{\delta}{b}(b(ax')')' + \delta'(ax')' + ax'\delta'' + \frac{b'}{b}(ax')\delta'$$
$$- \frac{b'}{b}(ax')'\delta - \frac{b'}{b}(ax')\delta'$$

or,

$$G''(t) = (b(ax')')\frac{\delta}{b} + 2(ax')'\delta' - \frac{b'}{b}(ax')'\delta + (ax')\delta''$$  \hspace{1cm} (3.40)
Substituting (3.17) in (3.40), we get

\[ G''(t) = -\frac{\delta}{b} q F(y(g(t))) + (ax')' \left( 2\delta' - \frac{b'}{b} \delta \right) + (ax')\delta'' \]  

(3.41)

Using (3.3) and (3.36) in (3.41), we obtain

\[ G''(t) \leq -\frac{\delta}{b} q F\left(\frac{c}{2}\right) + (ax')' \left( 2\delta' - \frac{b'}{b} \delta \right) + (ax')\delta'' \]  

(3.42)

Now, by using (3.10), (3.11), (3.19) and (3.21) in (3.42), we have

\[ G''(t) \leq -\frac{\delta}{b} q F\left(\frac{c}{2}\right) \text{ for } t \geq t_2. \]  

(3.43)

Integrating (3.43) from \( t_2 \) to \( t \), we get

\[ G'(t) \leq G'(t_2) - F\left(\frac{c}{2}\right) \int_{t_2}^{t} \left( q \frac{\delta}{b} \right)(s) \, ds \]

Integrating again from \( t_2 \) to \( t \), we get

\[ G(t) \leq G(t_2) + G'(t_2)(t - t_2) - F\left(\frac{c}{2}\right) \int_{t_2}^{t} \int_{t_2}^{s} \left( q \frac{\delta}{b} \right)(u) \, du \, ds. \]  

(3.44)

From (3.37) and (3.38), it is clear that
\[ G(t) < 0 \quad \text{for } t \geq t_2 \quad \text{and} \quad (3.45) \]

\[ G'(t) > 0 \quad \text{for } t \geq t_2. \quad (3.46) \]

Using (3.45) and (3.46) in (3.44), we get

\[ G(t) \leq G'(t_2)t - F\left(\frac{c}{2}\right) \int_{t_2}^{t} \int_{t_2}^{s} \left( q_{\theta} \right) (u) du ds. \quad (3.47) \]

Dividing by \( a \delta \), we conclude that

\[ x'(t) \leq G'(t_2) \frac{t}{a \delta} - F\left(\frac{c}{2}\right) \frac{1}{a \delta} \int_{t_2}^{t} \int_{t_2}^{s} \left( q_{\theta} \right) (u) du ds. \quad (3.48) \]

Integrating (3.48) from \( t_2 \) to \( t \), we get

\[ x(t) \leq x(t_2) + G'(t_2) \int_{t_2}^{t} \frac{s}{a \delta} ds - F\left(\frac{c}{2}\right) \int_{t_2}^{t} \frac{1}{a \delta} \int_{t_2}^{s} \int_{t_2}^{u} \left( q_{\theta} \right) (v) dv du ds. \quad (3.49) \]

Using (3.14), (3.18) and (3.46) in (3.49), we obtain

\[ x(t) \leq -F\left(\frac{c}{2}\right) \int_{t_2}^{t} \frac{1}{a \delta} \int_{t_2}^{s} \int_{t_2}^{u} \left( q_{\theta} \right) (v) dv du ds. \quad (3.50) \]

Taking the limit as \( t \to \infty \) in (3.50) and using (3.13), we get a contradiction to (3.18).

Hence the proof is complete.
Remark

Note that condition (3.3) is quite restrictive since it cannot be satisfied by a function of the type

\[ F(x) = x^\gamma, \quad \gamma > 1 \]

where \( \gamma \) is a quotient of odd positive integers.

Therefore, we relax that condition in the next theorem.

**Theorem 10** Suppose that

\[ F'(y) \geq 0 \quad \text{for } y \neq 0 \]

(3.51)

and all conditions (3.4)-(3.11), (3.13) and (3.14) hold.

If

\[ \int_{-\infty}^{\infty} q(t)\delta(t) \, dt = +\infty \]

(3.52)

then the conclusion of theorem 9 holds.

**Proof**

Let \( y(t) \) be a nonoscillatory solution of equation (3.1), say \( y(t) > 0 \) and \( y(\sigma(t)) > 0 \) for every \( t \geq t_1 \geq t_0 \). We proceed as in the proof of theorem 9 till we obtain

\[ \omega'(t) = -q \frac{F(y(g(t)))}{F(\lambda x(\sigma))} \delta + \omega \frac{\delta'}{\delta} - \omega \lambda \frac{F'(\lambda x(\sigma))}{F(\lambda x(\sigma))} x'(\sigma)\sigma' \]

(3.53)
Using (3.51) in (3.53), we get

\[ \omega'(t) \leq -q \frac{F(y(y(t)))}{F(\lambda x(\sigma))} \delta + \omega \frac{\delta'}{\delta} \quad (3.54) \]

Using (3.9) and (3.24) in (3.54), we obtain

\[ \omega'(t) \leq -q(t) \delta(t). \quad (3.55) \]

Integrating (3.55) from \( t_2 \) to \( t \), we get

\[ \omega(t) - \omega(t_2) \leq - \int_{t_2}^{t} q(s) \delta(s) \, ds, \]

or,

\[ \int_{t_2}^{t} q(s) \delta(s) \, ds \leq \omega(t_2) - \omega(t) < \omega(t_2) < \infty. \quad (3.56) \]

Taking the limit as \( t \to \infty \) in (3.56) and using (3.52), we get a contradiction.

Thus (3.21) is true and the rest of the proof is the same as in theorem 9.

Now, we relax a condition on \( \sigma \) and state the following theorem

**Theorem 11** Let conditions (3.4)-(3.6), (3.8)-(3.11) and (3.51) hold and suppose that

\[ \sigma'(t) \geq 0 \quad \text{for} \quad t \geq t_0. \quad (3.57) \]
If

\[ \int_{-\infty}^{\infty} \delta(\sigma(s)) q(s) \, ds = +\infty, \quad (3.58) \]

\[ \int_{-\infty}^{\infty} \frac{1}{a(s)\delta(\sigma(s))} \int_{t_0}^{t} \int_{t_0}^{s} \frac{\delta(\sigma(u))}{b(u)} q(u) \, du \, ds \, dt = +\infty \quad (3.59) \]

and

\[ \int_{-\infty}^{\infty} \frac{t}{a(t)\delta(\sigma(t))} \, dt < \infty \quad (3.60) \]

then the conclusion of theorem 9 holds.

**Proof**

To prove theorem 11, we again define

\[ \omega(t) = \frac{b(ax)'}{F(\lambda x(\sigma))} \delta(\sigma) \quad (3.61) \]

By differentiation, we have

\[ \omega'(t) = \frac{(b(ax)')'}{F(\lambda x(\sigma))} \delta(\sigma) + \frac{b(ax)'}{F(\lambda x(\sigma))} \delta'(\sigma)\sigma' \]

\[ -\frac{b(ax)'}{F^2(\lambda x(\sigma))} \delta(\sigma)\lambda F'x'(\sigma)\sigma' \quad (3.62) \]
Using (3.51) and (3.57) in (3.62), we obtain

\[ \omega'(t) \leq -q(t)\delta(\sigma). \]  

(3.63)

Integrating (3.63) and then using (3.58), we get a contradiction.

Thus (3.21) is true.

Proceeding as in theorem 9 and defining the function

\[ G(t) = a(t)x'(t)\delta(\sigma(t)). \]  

(3.64)

Differentiating (3.64), we get

\[ G'(t) = (a(t)x'(t))'\delta(\sigma(t)) + a(t)x'(t)\delta'(\sigma(t))\sigma'. \]  

(3.65)

Multiplying (3.65) by \( b(t) \) and differentiating the resulting equation, we obtain

\[ b(t)G''(t) + b'G'(t) = (b(a(t)x'(t))')'\delta(\sigma(t)) \]

\[ + b(a(t)x'(t))'\delta'(\sigma(t))\sigma' + b'(a(t)x'(t))'\delta'(\sigma(t))\sigma' \]

\[ + b(a(t)x'(t))\delta''(\sigma(t))\sigma'^2 + b(a(t)x'(t))\delta'(\sigma(t))\sigma'' \]  

(3.66)

Using (3.65) in (3.66), we have

\[ G''(t) = \frac{(b(a(t)x'(t))')'}{b}\delta(\sigma). \]
Following the same steps as in proof of theorem 9 and using (3.59) and (3.60), the proof could be completed.

**Example 11** Consider the equation

\[
(e^t y')'' + e^{t-\pi} y(t - \pi) = -\cos(t),
\]

where,

\[
\begin{align*}
a(t) &= e^t, \\
b(t) &= 1, \\
g(t) &= t - \pi, \\
q(t) &= e^{t-\pi},
\end{align*}
\]
\[ F(y) = y, \]
\[ \delta(t) = e^{-t/2}, \text{ and} \]
\[ \phi(t) = \frac{1}{2} e^{-t}(\sin(t) - \cos(t)). \]

It is clear that all conditions of theorem 9 are satisfied, because,

\[
\int_{t_0}^{\infty} \left( \delta q - \frac{a(\sigma)b\delta^2}{4\lambda B\sigma^2} \right) dt = \int_{t_0}^{\infty} \left( e^{t/2 - \pi} - \frac{e^{t-\pi} \cdot \frac{1}{4} e^{-t}}{4\lambda B(t - \pi)e^{-t/2}} \right) dt
\]

\[
= e^{-\pi} \int_{t_0}^{\infty} e^{t/2} \left( 1 - \frac{1}{16\lambda B(t - \pi)} \right) dt = +\infty,
\]

\[
\int_{t_0}^{\infty} \frac{1}{a^2} \int_{t_0}^{t} \int_{t_0}^{u} \frac{\delta q}{b} dvdu dt = \int_{t_0}^{\infty} e^{-t/2} \int_{t_0}^{t} \int_{t_0}^{u} e^{v/2 - \pi} dvdu dt
\]

\[
= 4e^{-\pi} e^{-t_0} \int_{t_0}^{\infty} e^{-t/2} e^{t/2} dt = +\infty,
\]

and

\[
\int_{t_0}^{\infty} \frac{1}{a^2} = \int_{t_0}^{\infty} e^{-t/2} dt < \infty
\]

Thus the conclusion of theorem 9 holds.

In fact \( y(t) = e^{-t} \cos(t) \) is a solution of equation (3.67).
Example 12 Consider the equation

\[ (e^{-t}e^{y}e^t)' + e^{4t}y^{5/3}(3t) = e^{-t}, \]  

(3.68)

where,

\[ a(t) = e^t, \]
\[ b(t) = e^{-t}, \]
\[ g(t) = 3t, \]
\[ q(t) = e^{4t}, \]
\[ F(y) = y^{5/3}, \]
\[ \delta(t) = e^{-t/2}, \text{ and} \]
\[ \phi(t) = te^{-t} + e^{-t}. \]

Since

\[ \int_{-\infty}^{\infty} q \delta \, dt = \int_{-\infty}^{\infty} e^{7t/2} \, dt = +\infty, \]

\[ \frac{1}{a\delta} \int_{t_0}^{t} \int_{t_0}^{u} \frac{\delta q}{b} \, dv \, du \, dt = \]

\[ e^{-t/2} \int_{t_0}^{t} \int_{t_0}^{u} e^{9v/2} \, dv \, du \, dt = \int_{-\infty}^{\infty} e^{4t} \, dt = +\infty. \]
and

\[ \int_{t_o}^{\infty} \frac{1}{a^\delta} \, dt = \int_{t_o}^{\infty} e^{-t/2} \, dt < \infty, \]

therefore all conditions of theorem 10 are satisfied.

Hence, the conclusion holds.

In fact \( y(t) = e^{-t} \) is a solution of equation (3.68).

## 3.2 Nonoscillation Properties

Hammett [25] studied the equation

\[ (r(t)x(t))' + p(t)x(g(t)) = f(t) \quad (3.69) \]

and proved that if

- \( p(t) \) and \( r(t) \) are positive, continuous and bounded away from zero and if
- \( f(t) \) is integrable on the positive half real line,

then, all nonoscillatory solutions of equation (3.69) approach zero.

In this section we extend Hammett-type study for the equation

\[ (b(ax')')' + qy(\sigma) = f(t), \quad (3.70) \]

where,

\[ a, b, q, \sigma \text{ and } f: [t_o, \infty) \to \mathbb{R} \text{ are continuous}, \quad (3.71) \]
0 < a \leq M_1, \quad 0 < b \leq M_2, \quad M_i > 0, \quad i = 1, 2 \quad \text{for } t \in [t_0, \infty), \quad (3.72)

q \geq 0 \quad \text{for } t \in [t_0, \infty) \quad \text{and not identically zero} \quad (3.73)

for any ray of the form [t^*, \infty) for some \ t^* \geq t_0,

\sigma \leq t, \quad \sigma(t) \to \infty \quad \text{as } t \to \infty \quad \text{and} \quad (3.74)

0 \leq \sigma' \leq M \quad \text{for } t \in [t_0, \infty), \quad M > 0. \quad (3.75)

We call a function \( x \in C(A, \infty) \) oscillatory if \( x(t) \) has arbitrary large zeros. Otherwise, we call \( x(t) \) nonoscillatory on \([A, \infty)\).

In what follows, only nontrivial continuously differentiable solutions on \([A, \infty)\), of equations under consideration. The term “solution” applies only to such solutions.

Hammett’s method was based on the theorem of Bhatia [7] which, as observed by Travis [38] does not apply to delay equations of the type (3.70).

**Theorem 12** Suppose that

\[
\int_{t_0}^{\infty} \frac{1}{b(t)} \, dt < \infty, \quad (3.76)
\]

\[
\int_{t_0}^{\infty} q(t) \, dt = +\infty, \quad \text{and} \quad (3.77)
\]

\[
\int_{t_0}^{\infty} |f(t)| \, dt < \infty. \quad (3.78)
\]

Let \( y(t) \) be a nonoscillatory solution of equation (3.70), then

\[
y'(t) \to 0 \quad \text{as } t \to \infty.
\]
Proof

Without any loss of generality we can assume that $y(t)$ is eventually positive. The case $y(t) < 0$ can be treated similarly.

Let $T > t_o$ be sufficiently large so that $y(t) > 0$ and $y(\sigma) > 0$ for $t \geq T$.

Integrating (3.70) between $T$ and $t$, we get

$$
(b(ay')')(t) - (b(ay')')(T) + \int_T^t q(s)y(\sigma(s)) \, ds = \int_T^t f(s) \, ds,
$$

or,

$$
(b(ay')')(t) - (b(ay')')(T) + \int_T^t q(s)y(\sigma(s)) \, ds \leq \int_T^t |f(s)| \, ds
$$

the right hand side of (3.79) remains finite as $t \to \infty$ due to (3.78).

Claim

$$
\int_T^\infty q(s)y(\sigma(s)) \, ds < \infty. \quad (3.80)
$$

If

$$
\int_T^\infty q(s)y(\sigma(s)) \, ds = +\infty
$$

then, it follows from (3.79) that

$$
(b(ay')) \to -\infty \text{ as } t \to \infty. \quad (3.81)
$$
Using (3.72) in (3.81), we obtain

\[(ay')' \rightarrow -\infty \text{ as } t \rightarrow \infty,\]

and by using (3.72) again, it follows

\[y' \rightarrow -\infty \text{ as } t \rightarrow \infty.\]

This forces \(y(t)\) to be negative, which is a contradiction to \(y(t) > 0\) for \(t \geq T\).

Hence (3.80) is true.

Now, from (3.77), we have

\[\int_{T}^{\infty} q(s) \, ds = +\infty \quad (3.82)\]

From (3.80) and (3.82), we get

\[\lim_{t \rightarrow \infty} \inf y(t) = \lim_{t \rightarrow \infty} \inf y(\sigma(t)) = 0. \quad (3.83)\]

In fact, if

\[\lim_{t \rightarrow \infty} \inf y(\sigma(t)) \geq \alpha > 0 \quad (3.84)\]

then condition (3.77) implies

\[\lim_{t \rightarrow \infty} \int_{T}^{t} q(s)y(\sigma(s)) \, ds = \infty\]
which contradicts (3.80). Hence (3.82) holds.

Now, there are two cases arise

**Case 1** Let \( y'(t) \) be nonoscillatory.

This implies \( y(t) \) is monotonic and from (3.83), it follows that

\[
\lim_{t \to \infty} y(t) = 0. \quad (3.85)
\]

**Case 2** Let \( y'(t) \) be oscillatory.

This will imply that

\[
\lim_{t \to \infty} \inf |y'(t)| = 0. \quad (3.86)
\]

If

\[ y'(t) \neq 0 \text{ as } t \to \infty \]

then, assume that,

\[
\lim_{t \to \infty} \sup |y'(t)| > \beta > 0. \quad (3.87)
\]

Now, from (3.85) and (3.86), there exists a sequence of points \( \{t_k\} \) in \([T, \infty)\) such that

\[
\int_{t_1}^\infty |f(t)| \, dt < \epsilon \text{ for some arbitrary } \epsilon > 0, \quad (3.88)
\]
\[ t_k \to \infty \text{ as } k \to \infty , \quad t_k \geq T \text{ for all } k , \quad (3.89) \]

\[ |y'(t_k)| < \frac{\beta}{4} \text{ for } k \geq 1 , \text{ and} \quad (3.90) \]

\[ \beta_k \geq 3/4 \beta \quad (3.91) \]

where \( \beta_k \) is true maxima of \( |y'(t)| \) in \([t_{k-1}, t_k]\) for all \( k \geq 1 \).

Let \( \gamma_k \in [t_{k-1}, t_k] \) such that

\[ |y'(\gamma_k)| = \beta_k \text{ and} \quad (3.92) \]

\[ a(\gamma_k) \geq 3/4 M_1 \quad (3.93) \]

Let \( \delta_k > t_{k-1} \) be the largest point less than \( \gamma_k \) such that

\[ |y'(\delta_k)| = \frac{\beta_k}{2} \text{ for some } k \geq 1 , \quad (3.94) \]

Also, let \( \eta_k < t_k \) be the smallest point greater than \( \gamma_k \) such that

\[ |y'(\eta_k)| = \frac{\beta_k}{2} \text{ for some } k \geq 1 . \quad (3.95) \]
The choice of $\delta_k$ and $\eta_k$ implies

$$|y'(t)| > \frac{\beta_k}{2} \text{ in } (\delta_k, \eta_k). \quad (3.96)$$

Integrating the expression $(ay')'$ from $\delta_k$ to $\gamma_k$, we get

$$a(\gamma_k)y'(\gamma_k) = a(\delta_k)y'(\delta_k) + \int_{\delta_k}^{\gamma_k} (ay')' \, dt,$$

or,

$$a(\gamma_k)|y'(\gamma_k)| \leq a(\delta_k)|y'(\delta_k)| + \int_{\delta_k}^{\gamma_k} |(ay')'| \, dt \quad (3.97)$$

Using (3.72) and (3.93) in (3.97), we obtain

$$\frac{3}{4}M_1 |y'(\gamma_k)| \leq M_1 |y'(\delta_k)| + \int_{\delta_k}^{\gamma_k} |(ay')'| \, dt \quad (3.98)$$

Using (3.92) and (3.94) in (3.98), we get

$$\frac{3}{4}M_1 \beta_k \leq M_1 \frac{\beta_k}{2} + \int_{\delta_k}^{\gamma_k} |(ay')'| \, dt,$$

or,

$$\frac{M_1}{4} \beta_k \leq \int_{\delta_k}^{\gamma_k} |(ay')'| \, dt. \quad (3.99)$$
Similarly, we have

\[ -a(\gamma_k)y'(\gamma_k) = -a(\eta_k)y'(\eta_k) + \int_{\gamma_k}^{\eta_k} (ay')' \, dt, \]

or,

\[ a(\gamma_k) | y'(\gamma_k) | \leq a(\eta_k) | y'(\eta_k) | + \int_{\gamma_k}^{\eta_k} (ay')' \, dt. \tag{3.100} \]

Again by using (3.72) and (3.93), we get

\[ \frac{3M_1}{4} | y'(\gamma_k) | \leq M_1 | y'(\eta_k) | + \int_{\gamma_k}^{\eta_k} (ay')' \, dt. \tag{3.101} \]

Using (3.92) and (3.95) in (3.101), we have

\[ \frac{M_1}{4} | y'(\gamma_k) | \leq \int_{\gamma_k}^{\eta_k} (ay')' \, dt. \tag{3.102} \]

Adding (3.99) and (3.102), we obtain

\[ \frac{M_1}{2} | y'(\gamma_k) | \leq \int_{\delta_k}^{\eta_k} (ay')' \, dt. \tag{3.103} \]

Squaring (3.103) and applying Schwarz's inequality, we get

\[ \frac{M_1^2}{4} \beta_k^2 \leq \int_{\delta_k}^{\eta_k} b(t) \, dt \cdot \int_{\delta_k}^{\eta_k} (b(ay')',(ay')')' \, dt, \]
or,

\[
\frac{M^2}{4} \beta_k^2 \int_{\delta_k}^{\eta_k} \frac{1}{b(t)} \, dt \leq \int_{\delta_k}^{\eta_k} (b'(ay')'(ay')) \, dt. \tag{3.104}
\]

From (3.104), on integration by parts, we obtain

\[
\frac{M^2}{4} \beta_k^2 \int_{\delta_k}^{\eta_k} \frac{1}{b(t)} \, dt \leq \left\{ [b(ay')'(ay')](\eta_k) - [b(ay')'(ay')](\delta_k) \right\} \\
- \int_{\delta_k}^{\eta_k} (b(ay')')(ay') \, dt. \tag{3.105}
\]

If \( y'(t) > 0 \) in \([t_{k-1}, t_k]\)

then, the choice of \( \delta_k \) and \( \eta_k \) implies

\[
(ay')'(\eta_k) \leq 0 \quad \text{and} \quad (ay')'(\delta_k) \geq 0.
\]

Similarly, if \( y'(t) < 0 \) in \([t_{k-1}, t_k]\)

then, the choice of \( \delta_k \) and \( \eta_k \) implies

\[
(ay')'(\eta_k) \geq 0 \quad \text{and} \quad (ay')'(\delta_k) \leq 0.
\]

Therefore the expression

\[
\left\{ [b(ay')'(ay')](\eta_k) - [b(ay')'(ay')](\delta_k) \right\} \leq 0.
\]
Using this result in (3.105), we have

\[ \frac{M_1^2}{4} \beta_k^2 \frac{1}{\eta_k} \left( \frac{1}{b(t)} \right) dt \leq - \int_{\delta_k}^{\eta_k} (b(ay')')'(ay') dt. \tag{3.106} \]

Making use of (3.70) in (3.106), we get

\[ \frac{M_1^2}{4} \beta_k^2 \frac{1}{\eta_k} \left( \frac{1}{b(t)} \right) dt \leq \int_{\delta_k}^{\eta_k} q(t) y(\sigma)(ay')(t) dt - \int_{\delta_k}^{\eta_k} f(t)(ay')(t) dt, \]

or,

\[ \frac{M_1^2}{4} \beta_k^2 \frac{1}{\eta_k} \left( \frac{1}{b(t)} \right) dt \leq \int_{\delta_k}^{\eta_k} q(t) y(\sigma) |y'| dt + \int_{\delta_k}^{\eta_k} f(t) |a||y'| dt. \tag{3.107} \]

Since \(|y'(t)| \leq \beta_k\) in \([\delta_k, \eta_k]\), it follows from (3.107) that

\[ \frac{M_1^2}{4} \beta_k \frac{1}{\eta_k} \left( \frac{1}{b(t)} \right) dt \leq \int_{\delta_k}^{\eta_k} a q(t) y(\sigma) dt + \int_{\delta_k}^{\eta_k} a |f(t)| dt. \tag{3.108} \]

Using (3.72) in (3.108), we obtain

\[ \frac{M_1}{4} \beta_k \frac{1}{\eta_k} \left( \frac{1}{b(t)} \right) dt \leq \int_{\delta_k}^{\eta_k} a q(t) y(\sigma) dt + \int_{\delta_k}^{\eta_k} |f(t)| dt. \tag{3.109} \]
Now, since we have

\[ \int_T^{\infty} q(t)y(\sigma(t)) \, dt \geq \sum_{k=1}^{\infty} \int_{\delta_k}^{\eta_k} q(t)y(\sigma) \, dt \]  \hspace{1cm} (3.110) \]

From (3.109) and (3.110), we have

\[ \int_T^{\infty} q(t)y(\sigma(t)) \, dt \geq \sum_{k=1}^{\infty} \left\{ \frac{M_1}{4} \beta_k \left[ \int_{\delta_k}^{\eta_k} \frac{1}{b(t)} \, dt \right] - \int_{\delta_k}^{\eta_k} |f(t)| \, dt \right\} \]  \hspace{1cm} (3.111) \]

Using (3.111), we get

\[ \int_T^{\infty} q(t)y(\sigma(t)) \, dt \geq \sum_{k=1}^{\infty} \left\{ \frac{M_1}{4} \beta_k \right\} - \epsilon \]  \hspace{1cm} (3.112) \]

From (3.91) and (3.112), we obtain

\[ \int_T^{\infty} q(t)y(\sigma(t)) \, dt \geq \frac{3}{16} M_1 \beta \sum_{k=1}^{\infty} \left\{ \int_{\delta_k}^{\eta_k} \frac{1}{b(t)} \, dt \right\}^{-1} - \epsilon \]  \hspace{1cm} (3.113) \]

From condition (3.76), we have

\[ \lim_{k \to \infty} \int_{\delta_k}^{\eta_k} \frac{1}{b(t)} \, dt = 0, \]

because,

\[ \delta_k \to \infty \text{ and } \eta_k \to \infty \text{ as } k \to \infty. \]

The right hand side of (3.113) tends to \( \infty \) as \( t \to \infty \), which is a contradiction to
Thus as long as

\[ \lim_{k \to \infty} \sup |y'(t)| \]

remains greater than any positive number \( \beta \), we will run into the above contradiction. Hence

\[ \lim_{k \to \infty} \sup |y'(t)| = 0 \]

and the proof is complete.

**Example 13** Consider the equation

\[
\{(2 - e^{-t})(1 + e^{-t})y')' + ty(t - \pi) \right.
\]

\[ = te^{-t+\pi} + 6e^{-3t} - 6e^{-2t} - 2e^{-t}, \tag{3.114}
\]

where,

\[ a(t) = 1 + e^{-t}, \]

\[ b(t) = 2 - e^{-t}, \]

\[ q(t) = t, \]

\[ \sigma(t) = t - \pi, \; \text{and} \]

\[ f(t) = te^{-t+\pi} + 6e^{-3t} - 6e^{-2t} - 2e^{-t}. \]
It is clear that the conditions (3.71)-(3.75) hold and since

\[ \int_{\infty}^{\infty} \frac{1}{b(t)} \, dt = \int_{\infty}^{\infty} \frac{1}{2 - e^{-t}} \, dt \]

\[ = \left\{ 2 \ln(2 - e^{-t}) + e^{-t} - 2 \right\} (\infty) \]

\[ \left\{ 2 \ln(2 - e^{-t}) + e^{-t} - 2 \right\} (t_0) < \infty, \]

\[ \int_{\infty}^{\infty} q(t) \, dt = \int_{\infty}^{\infty} t \, dt = +\infty, \text{ and} \]

\[ \int_{\infty}^{\infty} |f(t)| \, dt = \int_{\infty}^{\infty} (te^{-t} + \pi + 6e^{-3t} - 6e^{-2t} - 2e^{-t}) \, dt < +\infty. \]

Then, all conditions of theorem 12 are satisfied. Hence the conclusion of the theorem holds.

In fact \( y(t) = e^{-t} \) is a solution of equation (3.114).

**Theorem 13** Suppose that the conditions of theorem 12 hold with the following change in condition (3.77) of that theorem:
For $\beta_k > \alpha_k$, 

$$\lim_{k \to \infty} (\beta_k - \alpha_k) = \infty, \text{ and}$$

$$\lim_{k \to \infty} \int_{\alpha_k}^{\beta_k} q(t) \, dt = \infty. \quad (3.115)$$

Let $y(t)$ be a nonoscillatory solution of equation (3.70), then

$$y(t) \to 0 \text{ as } t \to \infty.$$ 

Proof 

We proceed as in theorem 12 and arrive to the conclusion (3.80).

If

$$\lim_{t \to \infty} y(t) \neq 0$$

then

$$\lim_{t \to \infty} \sup y(\sigma(t)) > \phi > 0. \quad (3.116)$$

Again in the same manner of Hammett [25] we can find a sequence $\{t_k\}$ such that:

$$t_k \to \infty \text{ as } k \to \infty, \quad t_k \geq T \text{ for } k \geq 0, \quad (3.117)$$
for each \( k \geq 1 \) there exists \( t_k' \) such that

\[
t_{k-1} < t_k' < t_k \text{ and } y(\sigma(t_k')) < \phi. \tag{3.119}
\]

Let \([a_k, b_k]\) be the largest interval around \( t_k \) such that for \( k \geq 1 \)

\[
y(\sigma(a_k)) = y(\sigma(b_k)) = \phi \text{ and } y(\sigma(t)) > \phi \text{ for } t \in (a_k, b_k). \tag{3.121}
\]

Now, in the interval \([a_k, b_k]\) there exists a number \( \delta_k \) such that :

\[
y'(\sigma(\delta_k))\sigma'(\delta_k) = \frac{y(\sigma(t_k)) - y(\sigma(a_k))}{t_k - a_k}. \tag{3.122}
\]

Using (3.118) and (3.120) in (3.122), we get

\[
y'(\sigma(\delta_k))\sigma'(\delta_k) > \frac{2\phi - \phi}{b_k - a_k} = \frac{\phi}{b_k - a_k}. \tag{3.123}
\]

Since \( \sigma'(\delta_k) \) is bounded and nonnegative, then the left hand side of (3.123) tends to zero as \( t \to \infty \) by theorem 12. This leads to

\[
\lim_{k \to \infty} (b_k - a_k) = \infty. \tag{3.124}
\]
From (3.80), we have

\[ \infty > \int_T^\infty q(t)y(\sigma(t)) \, dt \]

\[ \geq \sum_{k=1}^\infty \int_{a_k}^{b_k} q(t)y(\sigma(t)) \, dt \]

\[ > \phi \sum_{k=1}^\infty \int_{a_k}^{b_k} q(t) \, dt \]

By using (3.115), we obtain a contradiction unless \( \phi = 0 \) and this essentially proves the theorem.

**Remark:**

The example 13 fulfills all the conditions of theorem 13 and the solution

\[ y(t) = e^{-t} \to 0 \text{ as } t \to \infty. \]

Next theorem gives conditions under which nonoscillatory solutions are integrable.

**Theorem 14** Suppose that conditions (3.76) and (3.78) of theorem 12 hold. Further, suppose that there exists a constant \( L \) such that

\[ q(t) \geq L > 0 \]
and \( y(t) \) is a nonoscillatory solution of equation (3.70), then

\[
y(t) \to 0 \text{ as } t \to \infty \text{ and } \int_\infty^{\infty} |y(t)| \, dt < \infty.
\]

**Proof**

Since \( q(t) \geq L > 0 \), then condition (3.77) of theorem 12 and (3.115) hold. Without loss of generality suppose \( y(t) > 0 \) and \( y(\sigma(t)) > 0 \) for \( t \geq T \). By theorem 13, we have

\[
y(t) \to 0 \text{ as } t \to \infty.
\]

Hence, from (3.80), it follows

\[
\infty > M \int_T^{\infty} q(t)y(\sigma(t)) \, dt \quad (3.125)
\]

Using (3.75) in (3.125), we get

\[
\infty > \int_T^{\infty} q(t)y(\sigma(t))\sigma'(t) \, dt
\]

Using the condition \( q(t) \geq L \), we obtain

\[
\infty > L \int_T^{\infty} y(\sigma(t))\sigma'(t) \, dt \quad (3.126)
\]
Using the substitution:

\[ u = \sigma(t) \]
\[ du = \sigma'(t)dt \]

in (3.126) shows that

\[ \infty > L \int_{\sigma^{-1}(t)}^{\infty} y(u) \, du \]

with \( u_0 = \sigma^{-1}(t) \).

This completes the proof of theorem 14.

By theorem 14, all nonoscillatory solutions of equation (3.114) are integrable on \([A, \infty)\).

### 3.3 Asymptotic Nature

Dahiya and Singh [15] studied the asymptotic nature of the equation

\[ y''' + a(t)y(t - \tau(t)) = f(t) \]  \hspace{1cm} (3.127)

under the two conditions

\[ \int_{\infty}^{\infty} t^2 \left| f(t) \right| \, dt < \infty \]

and

\[ \int_{\infty}^{\infty} t^2 \left| a(t) \right| \, dt < \infty. \]
Also see [1], [9] and [11]

In this section we study the asymptotic nature of the equation

\[ y''' + q(t)F(y(\sigma(t))) = f(t) \]  \hspace{1cm} (3.128)

which is a special case of equation (3.1) in section 3.1 when \( a=b=1 \). Equation (3.128) is more general than equation (3.127) and the required conditions are more relaxed than those of equation (3.127)

**Theorem 15** If

\[
\int_{0}^\infty |f(t)| \, dt < \infty, \quad (3.129)
\]

\[
\int_{0}^\infty t^2 q(t) \, dt < \infty \quad \text{and} \quad (3.130)
\]

\[
|F(y)| \leq |y(t)|. \quad (3.131)
\]

Then equation (3.128) has nonoscillatory solutions asymptotic to

\[ a_0 + a_1 t + a_2 t^2 \]

where \( a_2 \neq 0 \).
Proof

Integrating equation (3.128) from $t_0$ to $t$, we obtain

$$y''(t) = y''(t_0) - \int_{t_0}^{t} q(s)F(y(\sigma(s))) \, ds + \int_{t_0}^{t} f(s) \, ds \quad (3.132)$$

Integrating again, we get

$$y'(t) = y'(t_0) + y''(t_0)(t - t_0) - \int_{t_0}^{t} \int_{t_0}^{s} q(r)F(y(\sigma(r))) \, dr \, ds + \int_{t_0}^{t} \int_{t_0}^{s} f(r) \, dr \, ds. \quad (3.133)$$

Interchanging the integral signs in (3.133), we get

$$y'(t) = y'(t_0) + y''(t_0)(t - t_0) - \int_{t_0}^{t} \int_{s}^{t} q(r)F(y(\sigma(r))) \, ds \, dr + \int_{t_0}^{t} \int_{s}^{t} f(r) \, ds \, dr,$$

or,

$$y'(t) = y'(t_0) + y''(t_0)(t - t_0) -$$
\[
\int_{t_0}^{t} (t-r)q(r)F(y(\sigma(r))) \, dr + \int_{t_0}^{t} (t-r)f(r) \, dr,
\]
or,
\[
y'(t) = c_0 + c_1 t - \int_{t_0}^{t} (t-r)q(r)F(y(\sigma(r))) \, dr + \int_{t_0}^{t} (t-r)f(r) \, dr \tag{3.134}
\]
where,
\[
c_0 \text{ and } c_1 \text{ are appropriate constants.}
\]
Integrating (3.134) from \(t_0\) to \(\sigma(t)\), where
\[
\sigma(t) > t_0 \text{ for large } t,
\]
we get
\[
y(\sigma(t)) = y(t_0) + c_0(\sigma(t) - t_0) + \frac{c_1}{2}(\sigma^2(t) - t_0^2)
\]
\[
- \int_{t_0}^{\sigma} \int_{t_0}^{s} (s-r)q(r)F(y(\sigma)) \, dr \, ds + \int_{t_0}^{\sigma} \int_{t_0}^{s} (s-r)f(r) \, dr \, ds \tag{3.135}
\]
Again interchanging the integral signs in (3.135) and integrating, we have
\[
y(\sigma(t)) = y(t_0) + c_0(\sigma(t) - t_0) + \frac{c_1}{2}(\sigma^2(t) - t_0^2)
\]
\[
- \int_{t_0}^{\sigma} \frac{1}{2}(t-s)^2 q(s)F(y(\sigma)) \, ds + \int_{t_0}^{\sigma} \frac{1}{2}(t-s)^2 f(s) \, ds. \tag{3.136}
\]
Since
\[
0 < \sigma(t) - t_0 < t \text{ for large } t
\]
then from (3.136), we obtain

\[ |y(\sigma(t))| \leq |y(t_0)| + c_0 |t| + \frac{|c_1|}{2} t^2 \]

\[ + \int_{t_0}^{t} \frac{1}{2} (t-s)^2 q(s) \, |F| \, ds + \int_{t_0}^{t} \frac{1}{2} (t-s)^2 \, |f(s)| \, ds, \]

or,

\[ |y(\sigma(t))| \leq c_2 + c_3 t + c_4 t^2 + \]

\[ t^2 \int_{t_0}^{t} q(s) \, |F| \, ds + t^2 \int_{t_0}^{t} |f(s)| \, ds, \]  

(3.137)

where \( t > 1 \) large.

Using (3.131) in (3.137), we get

\[ |y(\sigma(t))| \leq c_2 + c_3 t + c_4 t^2 + \]

\[ t^2 \int_{t_0}^{t} q(s) \, |y(\sigma(s))| \, ds + t^2 \int_{t_0}^{t} |f(s)| \, ds, \]

\[ \frac{1}{t^2} \int_{t_0}^{t} q(s) \, |y(\sigma(s))| \, ds + \int_{t_0}^{t} |f(s)| \, ds, \]  

or,

\[ |y(\sigma(t))| \leq (c_2 + c_3 + c_4) t^2 + \]

\[ t^2 \int_{t_0}^{t} q(s) \, |y(\sigma(s))| \, ds + t^2 \int_{t_0}^{t} |f(s)| \, ds, \]

or,

\[ \frac{|y(\sigma(t))|}{t^2} \leq c_5 + \int_{t_0}^{t} s^2 q(s) |y(\sigma(s))| \, ds + \int_{t_0}^{t} |f(s)| \, ds, \]  

(3.138)
where
\[ c_5 = c_2 + c_3 + c_4 \]

From (3.129), we have
\[ \int_{t_0}^{t} |f(s)| \, ds \leq L, \quad (3.139) \]

where \( L \) is a constant.

Using (3.139) in (3.138), we obtain
\[ I < c + \int_{t_0}^{t} \frac{y(\sigma(s))}{s^2} \, ds, \quad (3.140) \]

where
\[ c = c_5 + L. \]

Applying Gronwall's inequality [6, p. 107] on (3.140), we get
\[ \frac{|y(\sigma(t))|}{t^2} \leq k \exp \left( \int_{t_0}^{t} s^2 q(s) \, ds \right), \quad (3.141) \]

From (3.130), we have
\[ \int_{t_0}^{t} s^2 q(s) \, ds \leq M, \quad (3.142) \]

where \( M \) is a positive constant.

Using (3.142) in (3.141), we get
where \( k_0 \) is a positive constant.

From the first integral of equation (3.134), it follows

\[
| \int_{t_0}^{t} (t - s)q(s)F(y(\sigma(s))) \, ds | \leq \int_{t_0}^{t} (t - s)q(s) | F | \, ds
\]

By using (3.131), we obtain

\[
| \int_{t_0}^{t} (t - s)q(s)F(y(\sigma(s))) \, ds | \leq t \int_{t_0}^{t} q(s) | y(\sigma(s)) | \, ds,
\]

or,

\[
| \int_{t_0}^{t} (t - s)q(s)F(y(\sigma(s))) \, ds | \leq t \int_{t_0}^{t} s^2 q(s) \frac{| y(\sigma(s)) |}{s^2} \, ds.
\]

Using (3.143), it follows

\[
| \int_{t_0}^{t} (t - s)q(s)F(y(\sigma(s))) \, ds | \leq k_0 Mt.
\]

Similarly, the second integral of equation (3.134) could be

\[
| \int_{t_0}^{t} (t - s)f(s) \, ds | \leq t \int_{t_0}^{t} | f(s) | \, ds
\]
By using (3.139), we have

$$\left| \int_{t_0}^{t} (t - s)f(s) \, ds \right| \leq Lt. \quad (3.145)$$

Using (3.144) and (3.145) in (3.134), we get

$$| y'(t) | \leq | c_0 | + | c_1 | t + k_0 Mt + Lt \leq | c_0 | + (| c_1 | + k_0 M + L)t = c'_0 + c'_1 t, \quad (3.146)$$

where $c'_0$ and $c'_1$ are positive constants.

Therefore equation (3.146) can be written in the form

$$y'(t) \rightarrow c'_0 + c'_1 t \quad \text{as} \quad t \rightarrow \infty,$$

or,

$$y(t) \rightarrow a_0 + a_1 t + a_2 t^2 \quad \text{as} \quad t \rightarrow \infty,$$

where,

$a_0$, $a_1$, and $a_2$ are appropriate constants and $a_2 \neq 0$. 
4 PROPERTIES OF SOLUTIONS OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS

Although the oscillatory theory of delay differential equations has been extensively developed during the last few years, there is no much work at this time dealing with the oscillatory behavior of solutions of neutral delay differential equations. The problem of asymptotic and oscillatory behavior or solutions of neutral delay differential equations is of both theoretical and practical interest. It suffices to note that equations of this type appear in the study of networks containing lossless transmission lines. Such networks arise, for example, in high-speed computers where lossless transmission lines are used to interconnect switching circuits.

E. A. Grove and G. Ladas [3] studied the equation

\[(y(t) + py(t - \tau))' + Q(t)y(t - \sigma) = 0\]  \hspace{1cm} (4.1)

where \(p, \tau, \) and \(\sigma\) are continuous and \(Q \in C([t_0, \infty), \mathbb{R}^+),\)

\[(y(t) + P(t)y(t - \tau))'' + Q(t)f(y(t - \sigma)) = 0\] (4.2)

where

\[P, Q : [t_0, \infty) \to \mathbb{R} \text{ are continuous,}\]

\[\tau \text{ and } \sigma \text{ are nonnegative constants,}\]

\[\text{and } f : \mathbb{R} \to \mathbb{R} \text{ is continuous.}\]

and E. A. Grove, G. Ladas and J. Schinas [24] studied the equation

\[(y(t) - py(t - \tau))^{(n)} + qy(t - \sigma) = 0\] (4.3)

where \(p, q, \tau \text{ and } \sigma\) are nonnegative constants and \(n\) is an odd natural number.

Also see [10], [12], [17] and [16].

In this chapter we study the behavior of the bounded solutions of the equation

\[(y(t) + p(t)y(t - \tau))''' + f(t, y(t), y(t - \sigma)) = 0\] (4.4)

and the oscillatory behavior of the solutions of the equation

\[(y(t) + p(t)y(t - \tau))^{(2n)} + Q(t)y(t - \sigma) = 0\] (4.5)
4.1 Oscillation of Third Order Nonlinear Neutral Delay Differential Equations

Consider the equation (4.4) where

\[ f \in C([0, \infty) \times \mathbb{R}^2, \mathbb{R}) \quad \text{and} \quad f(t, x, y)x > 0 \quad \text{for} \quad xy > 0, \tag{4.6} \]

\[ \tau, \sigma \text{ are nonnegative constants and} \quad 0 < p(t) \leq M \tag{4.7} \]

where \( M \) is a positive constant.

**Theorem 16** If

\[ \int_{t_0}^{\infty} t^2 f(t, a, a) \, dt = \infty \quad \text{for any} \quad a \neq 0 \tag{4.8} \]

then every bounded solution of equation (4.4) is either oscillatory or

\[ \lim_{t \to \infty} y^{(k)}(t) = 0 \quad \text{for} \quad k = 0, 1, 2. \]

**Proof**

Assume that there exists a nonoscillatory bounded solution \( y(t) \). Without loss of generality let \( y(t) > 0 \) for \( t \geq t_0 \), then there exists \( t_1 \geq t_0 \) such that

\[ y(t - \tau) > 0 \quad \text{and} \quad y(t - \sigma) > 0 \quad \text{for} \quad t \geq t_1 \tag{4.9} \]

Let

\[ z(t) = y(t) + p(t)y(t - \tau) \tag{4.10} \]
From (4.6) and (4.10), we have

\[ z'''(t) = -f(t, y(t), y(t - \sigma)) \]  \hspace{1cm} (4.11)

Using (4.6) and (4.9) in (4.11), we get

\[ z'''(t) < 0 \text{ for } t \geq t_1 \]  \hspace{1cm} (4.12)

Integrating (4.4) from \( s \) to \( t \), we obtain

\[ z''(t) - z''(s) + \int_s^t f(r, y(r), y(r - \sigma)) \, dr = 0 \]  \hspace{1cm} (4.13)

Now, we discuss two different possible cases:

**Case 1** Let \( z''(t) > 0 \)

then, we might have either, \( z'(t) > 0 \) which leads to \( z(t) \) is unbounded and from (4.7) and (4.10), we conclude that \( y(t) \) is also unbounded which contradicts the assumption, or, \( z'(t) < 0 \) for \( t \geq t_2 \geq t_1 \) which leads to \( z(t) \) is a decreasing function and bounded, i.e.,

\[ \lim_{t \to \infty} z(t) = c \text{ where } c \geq 0 \]

or,

\[ \lim_{t \to \infty} y(t) = \frac{c}{1 + M} \]

It is clear that

\[ \lim_{t \to \infty} y'(t) = \lim_{t \to \infty} y''(t) = 0. \]
We want to prove that $c = 0$.

Assume $c > 0$

Then from (4.13), we have

$$z''(t) \geq \int_t^\infty f(r, y(r), y(r - \sigma)) \, dr$$

(4.14)

Integrating (4.14), we get

$$-z'(t) \geq \int_t^\infty \int_{t_1}^\infty f(r, y(r), y(r - \sigma)) \, dr \, dt_1$$

(4.15)

Interchanging the integral signs in (4.15), we obtain

$$-z'(t) \geq \int_t^\infty (r - t) f(r, y(r), y(r - \sigma)) \, dr$$

Integrating again from $T$ to $t$, $T \geq t_2$, we get

$$z(T) - z(t) \geq \int_T^t \int_{t_1}^\infty (r - t_1) f(r, y(r), y(r - \sigma)) \, dr \, dt_1$$

(4.16)

Also, by interchanging the integral signs in (4.16), we have

$$z(T) - z(t) \geq \int_T^t \frac{(r - T)^2}{2} f(r, y(r), y(r - \sigma)) \, dr$$

(4.17)
Let \( t \to \infty \) in (4.17) and use (4.8), we obtain a contradiction. Hence \( c = 0 \).

Therefore

\[
\lim_{t \to \infty} y^{(k)}(t) = 0 \quad \text{for} \quad k = 0, 1, 2.
\]

**Case 2** Let \( z'''(t) < 0 \) for \( t \geq t_3 \)

this implies that for sufficiently large \( t \), \( z(t) < 0 \), which is a contradiction. This completes the proof.

**Example 14** Consider the equation

\[
(y(t) + e^{-t}y(t - \pi))''' + e^t y^2(t) + 8y^2(t - \pi/2) = 0 \quad (4.18)
\]

where,

\[
p(t) = e^{-t}, \quad \tau = \pi, \quad \sigma = \pi/2 \quad \text{and} \quad f(t, y(t), y(t - \sigma)) = e^t y^2(t) + 8y^2(t - \pi/2).
\]

Since

\[
\int_{-\infty}^{\infty} t^2 f(t, a, a) \, dt = \int_{-\infty}^{\infty} t^2(a^2 e^t + 8a^2) \, dt = \infty
\]

then all conditions of theorem 16 are satisfied, therefore the conclusion holds. In fact \( y(t) = e^{-t} \) is a solution of equation (4.18).
4.2 Oscillation of Even Order Neutral Delay Differential Equations

Consider the equation (4.5) where

\[ P, Q \in C([t_0, \infty), \mathbb{R}), \]
\[ 0 \leq P(t) < 1 \quad \text{and} \quad Q(t) \geq 0 \quad \text{for} \quad t \geq t_0. \]  \hfill (4.19)

Lemma 1 \textit{See} [28, p. 193]

Let 
\[ z(t) \] be an \( n \) times differentiable function on \( \mathbb{R}^+ \) of constant sign, \( z^{(n)}(t) \) be of constant sign and not identically zero in any interval \([t, \infty)\) and 
\[ z^{(n)}(t)z(t) \leq 0 \] \hfill (4.20)

then

(i) there exists \( t_2 \geq t_1 \) such that the function \( z^{(i)}(t) \), \( i = 1, 2, \ldots, n - 1 \), are of a constant sign on \([t_2, \infty)\)

(ii) there exists a number \( k \in \{1, 3, 5, \ldots, n - 1\} \) where \( n \) is even such that 
\[ z^{(i)}(t)z(t) \geq 0 \quad \text{for} \quad i = 0, 1, 2, \ldots, k, \quad t \geq t_2. \]  \hfill (4.21)

Proof

From (4.20), without loss of generality we assume that \( z(t) > 0 \), \( z^{(n)}(t) \leq 0 \) for \( t \geq t_1 \geq T \). Then \( z^{(n-1)}(t) \) is a nonincreasing function for \( t \geq t_1 \) and is not
constant on any \((T, \infty)\) for large \(T\). This implies that exactly one of the following is true:

(a) \(z^{(n-1)}(t) > 0 \) for \( t \geq t_1 \)

(b) \(z^{(n-1)}(t) < 0 \) for \( t \geq T_{n-1}^{(1)} \geq t_1 \)

From (b) together with \(z(t) > 0\), \(z(n)(t) \leq 0\), it follows that there exists a number \(T_{n-2}^{(1)} \geq T_{n-1}^{(1)}\) such that \(z^{(n-2)}(t) \leq 0\) for \( t > T_{n-2}^{(1)} \). Likewise, we have \(z^{(n-3)}(t) < 0\) for \( t \geq T_{n-3}^{(1)} \geq T_{n-2}^{(1)} \geq \ldots\), and hence \(z(t) < 0\) for \( t \geq T_{n-2}^{(1)} \), which is a contradiction.

Since \(z(t) > 0\) for \( t \geq t_1\), then (a) holds. Now we know that \(z^{(n-2)}\) is increasing and concave for \( t \geq t_1\). Therefore exactly one of the following possibilities holds true:

(c) \(z^{(n-2)}(t) \geq 0 \) for \( t \geq T_{n-2}^{(2)} \geq t_1 \)

(d) \(z^{(n-2)}(t) < 0 \) for \( t \geq t_1 \)

From (c) and (a), we obtain
\[ z^{(n-3)}(t) > 0 \text{ for } t \geq T_{n-3}^{(2)} \geq T_{n-2}^{(2)}. \]

Analogously, we get
\[ z^{(n-4)}(t) > 0 \text{ for } t \geq T_{n-4}^{(2)} \geq T_{n-3}^{(2)} \geq \ldots, \text{ and hence } z(t) > 0 \text{ for } t \geq T_{n-2}^{(2)} \geq T_{n-1}^{(2)}. \] Thus the functions \(z(j)\) \((j = 1, \ldots, n-1)\) are of constant sign for \( t \) sufficiently large.
If (d) holds, then \( z^{(n-3)} \) is decreasing and convex for \( t \geq t_1 \). Then exactly one of the following is true:

(e) \( z^{(n-3)}(t) > 0 \) for \( t \geq t_1 \)

(f) \( z^{(n-3)}(t) < 0 \) for \( t \geq T^{(3)}_{n-3} \geq t_1 \)

Thus we can repeat the above argument and show that the functions

\[ z(j) \ (j = 1, ..., n - 1) \]

are of constant sign for sufficiently large \( t \). This proves (i) and (ii) of Lemma 1.

**Theorem 17** If

\[
\int_{t_0}^{\infty} Q(s) \{1 - P(s - \sigma)\} \, ds = \infty
\]

(4.22)

then every solution of equation (4.5) is oscillatory.

**Proof**

Let \( y(t) \) is an eventually positive solution of (4.5), then there exists \( t_1 \geq t_0 \) such that

\[
y(t - \tau) > 0 \text{ and } y(t - \sigma) > 0.
\]

(4.23)

Assume that

\[
z(t) = y(t) + P(t)y(t - \tau)
\]

(4.24)
From (4.19), (4.23) and (4.24), we obtain
\[ z(t) > 0 \quad \text{for } t \geq t_1 \] (4.25)

Using (4.24) in (4.6), we have
\[ z^{(2n)}(t) = -Q(t)y(t - \tau) \] (4.26)

From (4.19), (4.23) and (4.26), we get
\[ z^{(2n)}(t) \leq 0 \quad \text{for } t \geq t_1 \] (4.27)

Thus
\[ z(t)z^{(2n)}(t) \leq 0 \quad \text{for } t \geq t_1 \] (4.28)

From (4.25), (4.27) and (4.28), we can apply Lemma 1.

For \( i = 1 \) in (4.21), we obtain
\[ z(t)z'(t) > 0 \quad \text{for } t \geq t_2 \]

or,
\[ z'(t) \geq 0 \quad \text{for } t \geq t_2 \] (4.29)

Also from (4.21), we get
\[ z^{(2n-1)}(t) \geq 0 \quad \text{for } t \geq t_2 \] (4.30)

By using (4.24) and (4.26), we have
\[ z^{(2n)}(t) + Q(t) \{ z(t - \sigma) - P(t - \sigma)y(t - \sigma - \tau) \} = 0. \] (4.31)
From (4.24) and in view of (4.19) and (4.23), we obtain

\[ z(t) \geq y(t) \quad (4.32) \]

By using (4.32) in (4.31), we get

\[ z^{(2n)}(t) + Q(t) \{z(t - \sigma) - P(t - \sigma)z(t - \sigma - \tau)\} \leq 0. \quad (4.33) \]

From (4.25) and (4.29), it follows that \( z(t) \) is an increasing function. Thus (4.33) can be written as

\[ z^{(2n)}(t) + Q(t) \{1 - P(t - \sigma)\} z(t - \sigma) \leq 0. \quad (4.34) \]

Integrating (4.34) from \( t_2 \) to \( t \), we have

\[ z^{(2n-1)}(t) - z^{(2n-1)}(t_2) + z(t_2 - \sigma) \int_{t_2}^{t} Q(t) \{1 - P(t - \sigma)\} \, ds \leq 0, \]

or,

\[ z^{(2n-1)}(t) \leq z^{(2n-1)}(t_2) - z(t_2 - \sigma) \int_{t_2}^{t} Q(t) \{1 - P(t - \sigma)\} \, ds. \quad (4.35) \]

Taking the limit as \( t \rightarrow \infty \) in (4.35) and using (4.22), we get a contradiction to
This completes the proof.

**Example 15** Consider the equation

\[(y(t) + 1/4e^{-t}y(t-2\pi))''' + (1-e^{-t})y(t-\pi) = 0, \quad t \geq t_0, \quad (4.36)\]

where

\[n = 2,\]
\[\tau = 2\pi,\]
\[\sigma = \pi,\]
\[P(t) = 1/4e^{-t}, \text{and}\]
\[Q(t) = 1 - e^{-t}.\]

Since

\[\int_{\infty}^{\infty} Q(s) \{1 - P(s - \sigma)\} \, ds =\]

\[\int_{\infty}^{\infty} (1 - e^{-s})(1 - 1/4e^{-(s-\pi)}) \, ds = \infty\]

then all conditions of theorem 17 are satisfied. Hence the conclusion holds.

In fact \(y(t) = \sin(t)\) and \(y(t) = \cos(t)\) are solutions of equation (4.36).
5 ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF Nth ORDER
DELAY DIFFERENTIAL EQUATIONS

5.1 Linear Equations

In this section, we study the asymptotic behavior of solutions of the linear delay differential equation

\[ y^{(n)}(t) + a(t)y(t - \tau(t)) = f(t) \]  \hspace{1cm} (5.1)

where

(i) \(a(t), f(t), \) and \(\tau(t)\) are assumed to be continuous on the whole real line \(\mathbb{R}\)

(ii) \(\tau(t) \geq 0, \) \(\tau(t)\) is bounded as \(t \to \infty\) and \(0 \leq \tau'(t) < 1\)

(iii) \(\int_{-\infty}^{\infty} |f(t)| \, dt < \infty\)

A solution \(y(t)\) of equation (5.1) which is continuous and defined on some half line \([t_0, \infty)\) is said to be oscillatory if it does not have last zero, i.e., if \(y(t_1) = 0, \) \(t_1 > t_0\) then there exists \(t_2 > t_1\) such that \(y(t_2) = 0\), otherwise, nonoscillatory. Equation (5.1) is said to be oscillatory if all its nontrivial continuous solutions are
oscillatory; otherwise, it is called nonoscillatory.

**Theorem 18** Under the above conditions and

\[ \int_{0}^{\infty} t^{n-1} |a(t)| \, dt < \infty \]  

(5.2)

equation (5.1) has nonoscillatory solution asymptotic to

\[ a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}, \quad a_{n-1} \neq 0 \]

Proof

Integrating (5.1) from \( t_0 \) to \( t \) \((n-1)\) times, we obtain

\[ y'(t) = b_0 + b_1 t + \cdots + b_{n-2} t^{n-2} + \]

\[ \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} f(s) \, ds - \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} a(s) y_\tau(s) \, ds \]  

(5.3)

where \( b_0, b_1, \ldots, b_{n-2} \) are appropriate constants and \( y_\tau(s) = y(s - \tau) \).

Let \( t \) be large such that \( t - \tau(t) > t_0 \).

Integrating (5.3) between \( t_0 \) and \( t - \tau \), we have

\[ y_\tau(t) = b'_0 + b'_1 t + \cdots + b'_{n-1} t^{n-1} + \]
where \( b'_0, b'_1, \cdots, b'_{n-2} \) are another set of appropriate constants.

From (5.4), we have

\[
|y_T(t)| \leq c_0 + c_1 t + \cdots c_{n-1} t^{n-1} + 
\int_t^s \int_t^s \frac{(s-r)^{n-2}}{(n-2)!} f(r) \, dr \, ds
\]

\[
+ \int_t^s \int_t^s \frac{(s-r)^{n-2}}{(n-2)!} a(r) |y_T(r)| \, dr \, ds
\]

where \( c_0, c_1, \cdots, c_{n-1} \) are positive constants.

By changing the signs of integration in (5.5), we get

\[
|y_T(t)| \leq c_0 + c_1 t + \cdots c_{n-1} t^{n-1} + 
\int_t^t \frac{(t-r)^{n-1}}{(n-1)!} |f(r)| \, dr + \int_t^t \frac{(t-r)^{n-1}}{(n-1)!} |a(r)| |y_T(r)| \, dr,
\]
or,

\[ |y_\tau(t)| \leq c_0 + c_1 t + \cdots + c_{n-1} t^{n-1} + \]

\[ t^{n-1} \int_{t_0}^t |f(r)| \, dr + t^{n-1} \int_{t_0}^t |a(r)| |y_\tau(r)| \, dr, \]

or,

\[ |y_\tau(t)| \leq (c_0 + c_1 + \cdots + c_{n-1}) t^{n-1} + \]

\[ t^{n-1} \int_{t_0}^t |f(r)| \, dr + t^{n-1} \int_{t_0}^t |a(r)| |y_\tau(r)| \, dr. \]

Dividing by \( t^{n-1} \), we get

\[ \frac{|y_\tau(t)|}{t^{n-1}} \leq c + \int_{t_0}^t |f(r)| \, dr \]

\[ + \int_{t_0}^t r^{n-1} |a(r)| \frac{|y_\tau(r)|}{r^{n-1}} \, dr \quad (5.6) \]

where \( c = c_0 + c_1 + \cdots + c_{n-1} \) is a positive constant.
From condition (iii), we have

\[ \int_{t_0}^{t} |f(r)| \, dr \leq L \text{ where } L > 0 \text{ constant.} \]

Thus, we can write (5.6) as

\[ \frac{|y(t)|}{t^{n-1}} \leq c + L + \int_{t_0}^{t} r^{n-1} |a(r)| \frac{|y(r)|}{r^{n-1}} \, dr, \]

or,

\[ \frac{|y(t)|}{t^{n-1}} \leq c' + \int_{t_0}^{t} r^{n-1} |a(r)| \frac{|y(r)|}{r^{n-1}} \, dr \]

where \( c' = c + L > 0 \)

Applying Gronwall's inequality [6, p. 107] on (5.7), we obtain

\[ \frac{|y(t)|}{t^{n-1}} \leq k \exp \left( \int_{t_0}^{t} r^{n-1} |a(r)| \, dr \right) \]

(5.8)

Using (5.2) in (5.8), we get

\[ \frac{|y(t)|}{t^{n-1}} \leq k_o \]

(5.9)

where \( k_o \) is a positive constant.

From the first integral of equation (5.3), we have
Similarly, the second integral can be written as

\[
\left| \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} \alpha(s) y_T(s) \, ds \right| \leq t^{n-2} \int_{t_0}^{t} |\alpha(s)| |y_T(s)| \, ds,
\]

or,

\[
\left| \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} \alpha(s) y_T(s) \, ds \right| \leq t^{n-2} \int_{t_0}^{t} s^{n-1} |\alpha(s)| \frac{|y_T(s)|}{s^{n-1}} \, ds,
\]

or, by using (5.2) and (5.9), we get

\[
\left| \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} \alpha(s) y_T(s) \, ds \right| \leq k_0 L' t^{n-2}
\]

where

\[L' = \int_{t_0}^{t} s^{n-1} |\alpha(s)| \, ds \text{ is a positive constant.}\]

Using (5.10) and (5.11) in (5.3), we obtain

\[
y'(t) \to a'_0 + a'_1 t + \cdots + a'_{n-2} t^{n-2} \text{ as } t \to \infty
\]

or, by integration, we could write

\[
y(t) \to a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} \text{ as } t \to \infty, \quad a_{n-1} \neq 0.
\]
This completes the proof.

**Remark**

R. S. Dahiya and B. Singh [15] studied the asymptotic behavior of the solutions of the same equation under the condition that

\[ \int_0^\infty t^{n-1} |f(t)| \, dt < \infty \quad (5.14) \]

But for to sufficiently large the condition (iii) is much better because, for example, when \( n = 4 \) condition (5.14) becomes

\[ \int_0^\infty t^3 |f(t)| \, dt < \infty \]

which is not satisfied in the cases when

\[ f(t) = 1/t^4, 1/t^3 \text{ and } 1/t^2 \]

but the condition (iii) is satisfied in these cases.

### 5.2 Nonlinear Equations

Consider the equation

\[ y^{(n)}(t) + p(t)f(y(\tau(t), y_1'(t), \ldots, y_{n-1}^{(n-1)}(t))) = F(t) \quad (5.15) \]
where

\[ y_\tau(t) = y(t - \tau(t)) \] and \[ y^{(i)}_{\sigma_i}(t) = y^{(i)}(t - \sigma_i(t)), \ i = 1, 2, \ldots, n - 1 \]

The delay terms \( \tau \) and \( \sigma \) are assumed to be real valued, continuous, nonnegative, nondecreasing and bounded by a common constant \( M \) on the half line \([t_0, \infty)\) for some \( t_0 \geq 0 \).

It is also assumed that \( p(t) \) and \( F(t) \) are real valued and continuous on \([t_0, \infty)\).

Now, we assume the following:

(i) \( f : \mathbb{R}^n \to \mathbb{R} \) is continuous and sufficiently smooth to generate the existence of solution on \([t_0, \infty)\).

(ii) \(|f(x_1, x_2, \ldots, x_n)| \leq |x_1|\) on \([t_0, \infty)\).

(iii) \( \lim_{t \to \infty} (t - \tau(t)) = \infty \)

**Theorem 19** Let equation (5.15) satisfies conditions (i)-(iii) and if

\[ \int_{t_0}^{\infty} |F(t)| \ dt < \infty \] (5.16)

\[ \int_{t_0}^{\infty} t^{n-1} |p(t)| \ dt < \infty \] (5.17)
then equation (5.15) has nonoscillatory solutions asymptotic to

\[ b_0 + b_1 t + b_2 t^2 + \cdots + b_{n-1} t^{n-1}, \quad b_{n-1} \neq 0. \]
Proof

Let $t$ be so large that $t - \tau(t) > t_0 > 0$.

Integrating (5.15) $(n-1)$-times between $t_0$ and $t$, we have

$$y'(t) = \sum_{i=0}^{n-2} \frac{y^{(i)}(t_0)}{i!}(t-t_0)^i - \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} p(s)f(\tau(s), y_1'(s), \cdots) ds + \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} F(s) ds$$

or,

$$y'(t) = a_0 + a_1 t + \cdots + a_{n-2} t^{n-2} + \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} F(s) ds - \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} p(s)f(\tau(s), y_1'(s), \cdots, y_{\sigma_{n-1}}'(s)) ds$$

where $a_0, a_1, \cdots, a_{n-2}$ are appropriate constants.

Integrating (5.19) between $t_0$ and $t - \tau(t)$, we get

$$y_{\tau}(t) = a'_0 + a'_1 t + \cdots + a'_{n-1} t^{n-1} + \int_{t_0}^{t-\tau} \int_{t_0}^{s} \frac{(s-r)^{n-2}}{(n-2)!} F(r) dr ds$$
\[ -\int_{t_0}^{t-\tau} \int_{t_0}^{s} \frac{(t-s)^{n-2}}{(n-2)!} p(r)f(y(r),y'_{\sigma_1}(r),\ldots,y'_{\sigma_{n-1}}(r)) \, dr \, ds \quad (5.20) \]

where \( a'_0, a'_1, \ldots, a'_{n-1} \) are another set of appropriate constants.

Taking the absolute value of both sides in (5.20), we have

\[ |y(\tau(t))| \leq |a'_0| + |a'_1|t + \cdots + |a'_{n-1}|t^{n-1} \]

\[ + \int_{t_0}^{t-\tau} \int_{t_0}^{s} \frac{(s-r)^{n-2}}{(n-2)!} |F(r)| \, dr \, ds + \]

\[ \int_{t_0}^{t-\tau} \int_{t_0}^{s} \frac{(t-s)^{n-2}}{(n-2)!} |p(r)||f(y(r),y'_{\sigma_1}(r),\ldots,y'_{\sigma_{n-1}}(r))| \, dr \, ds \quad (5.21) \]

Since \( t \) is large, \( t - \tau(t) < t \) and \( t > 1 \), then

\[ |y(\tau(t))| \leq (|a'_0| + |a'_1| + \cdots + |a'_{n-1}|)t^{n-1} \]

\[ + \int_{t_0}^{t} \int_{t_0}^{s} \frac{(s-r)^{n-2}}{(n-2)!} |F(r)| \, dr \, ds + \]

\[ \int_{t_0}^{t} \int_{t_0}^{s} \frac{(t-s)^{n-2}}{(n-2)!} |p(r)||f(y(r),y'_{\sigma_1}(r),\ldots,y'_{\sigma_{n-1}}(r))| \, dr \, ds \quad (5.22) \]

Interchanging integral signs in (5.22) and integrating, we obtain
\[ |y_r(t)| \leq c \, t^{n-1} + \frac{\int_0^t (t-r)^{n-1}}{(n-1)!} |F(r)| \, dr \]
\[ + \int_0^t \frac{(t-r)^{n-1}}{(n-1)!} |p(r)||f(y_r(r), y_{\sigma_1}^{(n-1)}(r), \ldots, y_{\sigma_{n-1}}^{(n-1)}(r))| \, dr \]  

(5.23)

where

\[ c = |a'_0| + |a'_1| + \cdots + |a'_{n-1}| > 0. \]

Equation (5.23) could be written as

\[ |y_r(t)| \leq c \, t^{n-1} + t^{n-1} \int_0^t |F(r)| \, dr \]
\[ + t^{n-1} \int_0^t |p(r)||f(y_r(r), y_{\sigma_1}^{(n-1)}(r), \ldots, y_{\sigma_{n-1}}^{(n-1)}(r))| \, dr \]  

(5.24)

Using (ii) in (5.24) and dividing by \( t^{n-1} \), we get

\[ \frac{|y_r(t)|}{t^{n-1}} \leq c + \int_0^t |F(r)| \, dr + \int_0^t |p(r)||y_r(r)| \, dr \]  

(5.25)

Using (5.16) in (5.25), we obtain

\[ \frac{|y_r(t)|}{t^{n-1}} \leq c + L + \int_0^t |p(r)||y_r(r)| \, dr \]  

(5.26)
where
\[ \int_{t_0}^{t} |F(r)| \, dr \leq L, \quad L \text{ is a positive constant} \quad (5.27) \]

From (5.26), we have
\[ \frac{|y(t)|}{t^{n-1}} \leq c' + \int_{t_0}^{t} r^{n-1} |p(r)| \left| \frac{y(r)}{r^{n-1}} \right| \, dr \quad (5.28) \]

where \( c' = c + L \)

Applying Gronwall's inequality [6, p. 107] on (5.28), we obtain
\[ \frac{|y(t)|}{t^{n-1}} \leq k \exp \left( \int_{t_0}^{t} r^{n-1} |p(r)| \, dr \right) \quad (5.29) \]

Using (5.17) in (5.29), we get
\[ \frac{|y(t)|}{t^{n-1}} \leq k_o \quad (5.30) \]

where \( k_o = k \cdot e^M \) is a positive constant and
\[ \int_{t_0}^{t} r^{n-1} |p(r)| \, dr \leq M \quad (5.31) \]
From the first integral of equation (5.19), we have

\[ \left| \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} F(s) \, ds \right| \leq t^{n-2} \int_{t_0}^{t} |F(s)| \, ds \leq L t^{n-2} \quad (5.32) \]

Similarly, the second integral can be written as

\[ \left| \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} p(s)f(y_\tau(s), y_{\sigma_1}(s), \cdots, y_{\sigma_{n-1}}(s)) \, ds \right| \]

\[ \leq t^{n-2} \int_{t_0}^{t} |p(s)||y_\tau(s)| \, ds, \]

or,

\[ \left| \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} p(s)f(y_\tau(s), y_{\sigma_1}(s), \cdots, y_{\sigma_{n-1}}(s)) \, ds \right| \]

\[ \leq t^{n-2} \int_{t_0}^{t} s^{n-1} |p(s)| \frac{|y_\tau(s)|}{s^{n-1}} \, ds, \]

or, by using (5.30), we get

\[ \left| \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} p(s)f(y_\tau(s), y_{\sigma_1}(s), \cdots, y_{\sigma_{n-1}}(s)) \, ds \right| \]

\[ \leq k_0 t^{n-2} \int_{t_0}^{t} s^{n-1} |p(s)| \, ds \quad (5.33) \]
Using (5.31) in (5.33), we get

\[
\left| \int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} p(s) f(y_{\sigma}(s), y_{\sigma_1}'(s), \ldots, y_{\sigma_{n-1}}^{(n-1)}(s)) \, ds \right| \leq k_0 M t^{n-2}
\]

Thus

\[
\int_{t_0}^{t} \frac{(t-s)^{n-2}}{(n-2)!} p(s) f(y_{\sigma}(s), y_{\sigma_1}'(s), \ldots, y_{\sigma_{n-1}}^{(n-1)}(s)) \, ds \to d t^{n-2}
\]

as \( t \to \infty \)

where \( d \) is an appropriate constant.

Using (5.32) and (5.34) in (5.19), we obtain

\[
y'(t) \to b'_0 + b'_1 t + \ldots + b'_{n-2} t^{n-2} \quad \text{as} \quad t \to \infty
\]

or, by integration, we could write

\[
y(t) \to b_0 + b_1 t + \ldots + b_{n-1} t^{n-1} \quad \text{as} \quad t \to \infty , \quad b_{n-1} \neq 0.
\]

The proof is complete.

**Theorem 20** Let the function \( f \) in equation (5.15) satisfies the conditions (i) and (ii). In addition suppose that all solutions of equation (5.15) are oscillatory and

\[ f(x_1, x_2, \ldots, x_n) \text{ is nondecreasing} \]
with respect to each variable \(x_1, x_2, \ldots, x_n\). \hfill (5.37)

Also let

\[ F(t) = 0 \quad \text{for} \quad t \in [t_0, \infty) \] \hfill (5.38)

and

\[ p(t) > 0 \quad \text{for} \quad t \in [t_0, \infty) \] \hfill (5.39)

then

\[ \int_{\infty} s^{n-1} p(s) \, ds = \infty. \]

Proof

Suppose that the theorem is false and

\[ \int_{\infty} s^{n-1} p(s) \, ds < \infty. \]

Choose \(t_0\) sufficiently large, so that

\[ \int_{t_0}^{\infty} s^{n-1} p(s) \, ds < \frac{k}{2}. \] \hfill (5.40)

Construct a solution \(y(t)\) of equation (5.15) satisfying the initial conditions

\[
\begin{align*}
y(t) &= 0 \quad \text{for} \quad t \leq t_0, \\
y^{(i)}(t_0) &= 0, \quad i = 1, 2, 3, \ldots, n-2, \text{and} \\
y^{(n-1)}(t_0) &= k
\end{align*}
\] \hfill (5.41)
We claim that this solution of (5.15) is nonoscillatory contradicting the last hypothesis. Otherwise, let $t_1$ be the first zero of $y(t)$ in $(t_0, \infty)$. Then without loss of any generality $y(t) \geq 0$, $y_T(t) \geq 0$ and $y^{(j)}_{\sigma}(t) \geq 0$ for $t \leq t_1$.

In view of (5.37), we have

$$f(y_T(t), y^{(1)}_{\sigma}(t), ..., y^{(n-1)}_{\sigma}(t)) \geq 0 \quad \text{for } t \leq t_1 \quad (5.42)$$

By making use of (5.39) and (5.42) in (5.15), we obtain

$$y^{(n)}(t) \leq 0 \quad \text{for } t \leq t_1 \quad (5.43)$$

Integrating (5.43) n-times from $t_0$ to $t$, $t_0 \leq t \leq t_1$, we get

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + ... + \frac{y^{(n-1)}(t_0)}{(n-1)!}(t - t_0)^{n-1}$$

$$- \int_{t_0}^{t} \frac{(t - s)^{n-1}}{(n-1)!} p(s)f(y_T(s), y^{(1)}_{\sigma}(s), ..., y^{(n-1)}_{\sigma}(s)) \, ds \quad (5.44)$$

Using (5.41) in (5.44), it follows

$$y(t) \leq k \frac{(t - t_0)^{n-1}}{(n-1)!} \quad \text{for } t_0 \leq t \leq t_1$$

and

$$y(t) = 0 \quad \text{for } t \leq t_0$$

Therefore

$$y(t) \leq k \, t^{n-1}$$

$$y_T(t) \leq k \, t^{n-1}$$

$$y^{(j)}_{\sigma}(t) \leq k_j \, t^{n-j-1} \quad (5.45)$$
for $t \leq t_1$ and $j = 1, 2, ..., n - 1$.

Now integrating (5.15) again from $t_0$ to $t$, we obtain

$$y^{(n-1)}(t) = y^{(n-1)}(t_0) - \int_{t_0}^{t} p(s)f(y_T(s), y_{\sigma_1}(s), ..., y_{\sigma_{n-1}}(s)) \, ds \quad (5.46)$$

By using (5.41), we have

$$y^{(n-1)}(t) = k - \int_{t_0}^{t} p(s)f(y_T(s), y_{\sigma_1}(s), ..., y_{\sigma_{n-1}}(s)) \, ds, \quad t_0 \leq t \leq t_1. \quad (5.47)$$

From (5.37), (5.45) and (5.47), we get

$$y^{(n-1)}(t) \geq k - \int_{t_0}^{t} p(s)f(k s^{n-1}, k_1 s^{n-2}, ..., k_{n-1}) \, ds \quad (5.48)$$

Using (5.39) in (5.48), it follows

$$y^{(n-1)}(t) \geq k - \int_{t_0}^{t} p(s)s^{n-1} \, ds \quad (5.49)$$

Again by using (5.40) in (5.49), we get

$$y^{(n-1)}(t) \geq k - \frac{k}{2} = \frac{k}{2} > 0 \quad (5.50)$$

Since $y^{(n-1)}(t) > 0$ for $t_0 \leq t \leq t_1$ then the function $y(t)$ which has a zero at $t_0$ cannot have other zeros for $t > t_0$. Otherwise by Rolle's theorem and by using the
initial conditions (5.41) \( y^{(n-1)}(t) \) should have a zero in \([t_0, t_1]\).

Therefore the proof is complete.
6 BIBLIOGRAPHY


