Options and market information: a mean-variance portfolio approach

David William Bullock

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Options and market information: A mean-variance portfolio approach

Bullock, David William, Ph.D.

Iowa State University, 1989
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A mean-variance portfolio
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David William Bullock
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The major objective of this study was to extend the mean-variance portfolio model to incorporate option markets. Previous attempts in the literature have been only partially successful and this is due to an improper specification of the mean vector and variance-covariance matrix of returns. An option introduces special problems in modeling because of its truncated return structure. Conventional mean-variance analysis has not been developed to handle this problem. This study uses a statistical theorem, based upon the definition of a conditional moment, to arrive at a mean vector and variance-covariance matrix of returns for the portfolio containing options.

Using the modified portfolio model, several interesting observations were made regarding the informational role of option markets. The determination of the optimal portfolio positions was found to be related to the interaction of information incorporation into the speculative and hedging components of the option and futures demand equations. When an individual investor had information only regarding the variance of prices, the "straddle" position was used to capitalize upon this information.

Heterogeneous expectations, among traders, regarding the future spot price variance was found to be a possible justification of speculative trading in commodity options. These information differentials were sufficient to make the option market informationally inefficient in the Grossman-Stiglitz sense. However, information regarding the future spot
price mean was found to be a more significant factor in influencing the value-added to an investor's portfolio.

The equilibrium option price was found to be a weighted average of each of each investor's subjective valuation of the option, plus some extraneous terms representing the subjective valuations of the futures and a component representing the net hedging pressure upon the market. The introduction of an option market was found to significantly increase the volume of futures trading but had little impact upon other variables. An examination of the importance of factors influencing the bias of the equilibrium option price from an unweighted average of each investor's subjective value provided inconclusive results.
CHAPTER ONE
INTRODUCTION

Options are a versatile financial instrument in the toolbox of the modern investor. Different options can be combined with each other or with other instruments to tailor a customized pattern of returns. Options on securities were introduced in 1973 with the opening of the Chicago Board Options Exchange. In 1984, the Commodity Futures Trading Commission pilot program was inaugurated for options on commodity futures. Since 1984, options have been introduced on numerous commodity futures contracts (such as soybeans, live cattle, pork bellies, etc.). Most of these contracts have been successful using trading volume as a criterion. There have been instances, however, where a commodity option has not been successful (for example, options on Minneapolis wheat futures).

When zero-sum markets (futures and options) emerged as a popular investment vehicle, there arose a need to develop a theoretical framework to facilitate an understanding of these markets. The publishing of Debreu's *Theory of Value* (1959) and Arrow's *Essays in the Theory of Risk Bearing* (1971) introduced the concept of general equilibrium under uncertainty. This concept is also known as Arrow-Debreu contingent claims markets. It is upon the Arrow-Debreu framework that most modern financial investment economics are based [see Ingersoll (1987) and Ohlson (1987) for a review of literature in this area].
Because the framework had already been applied to the study of some other asset markets (for example, securities and insurance contracts), it was natural to extend the Arrow-Debreu framework to futures and options markets. The Capital Asset Pricing Model (CAPM), developed simultaneously by Lintner (1965), Mossin (1966), and Sharpe (1964), and the Arbitrage Pricing Theory which was introduced by Ross (1976) are among the theoretical frameworks that are based upon Arrow-Debreu markets.

From financial investment economics arose the theory of no-arbitrage pricing, the predominant school of thought regarding present day option theory. No-arbitrage pricing theory assumes the option market is in equilibrium if there is an absence of arbitrage opportunities where a riskless profit can be achieved. Thus, the option's value can be determined by deducing the price at which riskless profit opportunities are removed through arbitrage.

One of the advantages of no-arbitrage pricing theory is that the pricing relationships, using this construct, are not dependent upon any particular utility function. Thus, the problem of expressing investor preferences is not present when no-arbitrage pricing theory is used.

Another advantage of using no-arbitrage pricing theory is it uses the powerful, but easy to apply, tools of stochastic calculus. A student of no-arbitrage pricing theory, armed with these tools, can bring this knowledge to bear in solving a wide range of financial problems that an investor or investment firm faces.
Another school of thought, applied to the study of futures and options markets, is rooted in the principles of expected utility maximization. These principles are spelled out in the axioms of Von Neumann and Morgenstern (1947). The deduction of theoretical results using general expected utility functions is difficult. Because of this problem, it is often preferable to use a subclass of utility functions which are linear functions of the first two moments of the probability distribution (i.e., mean and variance).

The body of literature on mean-variance analysis is becoming quite developed with regards to futures markets. However, its application to option markets, in a couple of studies, has been without complete success. The problem with extending mean-variance analysis to options is that the distribution of returns to an option position is truncated at the strike price. The mean-variance framework is not able to handle a mixture of continuous and truncated variables.

This study has two major objectives:

1. To extend the mean-variance expected utility model to include option markets in a manner that is more consistent with the basic mathematical structure of these markets.
2. To use the mean-variance framework in the study of several questions. These questions are related to the process of information incorporation into option markets and the information aspect of the market equilibrium price of options.
The importance of extending the mean-variance model to include option markets is two-fold. First, the mean-variance model has the advantage of giving mathematically tractible results. Also, it is a reasonably accurate representation of how investors and producers make business decisions. Most market participants facing price risk are concerned about the following questions:

1. What is the most likely price going to be?
2. How far can the actual price deviate from the most likely price and with what probability?

These questions can be summarized by the information contained in the first two moments of the individual investor's probability distribution for the unknown price.

Another reason for extending the mean-variance model to include option markets is that the model is ideally suited for the representation of the individual trader's beliefs and preferences. The mean-variance model provides results that illustrate the impact of various investor beliefs and preferences upon the market equilibrium. While no-arbitrage pricing theory implicitly requires informationally efficient pricing equilibriums, the mean-variance framework allows for the study of informationally inefficient equilibriums. It also makes allowances for the study of disequilibrium behavior in zero-sum markets.

Examining the information role of option markets is important for two reasons. The first reason is the existence of an obvious gap in the
literature regarding the information role of options. Proponents of the no-arbitrage pricing school of thought view options as "derivate" assets which are devoid of information content and whose value can be solely determined by the underlying asset's market variables. There has, however, been a recent move on the part of economists to consider the information role of option markets.

Another reason for examining the information role of option markets is that option existence may provide an external benefit to society. This is conditional upon option prices providing useful information. This benefit is through the free provision of information that would otherwise be costly to collect by individuals.

In this study, a mean-variance framework will be used to examine several questions regarding option markets and information incorporation. Whenever possible, analytic results from the portfolio model will be used to illuminate answers to these questions. Analytic results are not always possible to derive, however, due to the mathematical complexity of the model. In those cases, a numerical simulation of the portfolio model will be used along with standard statistical procedures to provide possible answers to some of these questions.

Chapter Two contains a review of the major literature related to the overall subject matter of this study. Literature focused on the more specific topics of each chapter is reviewed in the introduction to each chapter.

Chapter Three addresses the extension of the mean-variance portfolio model to include option markets. The model developed in this
chapter is an improvement upon previous mean-variance option models. This is because the explicit form of the mean vector and the variance-covariance matrix of expected returns are derived using a statistical theorem that allows the moments to be a function of the subjective mean and subjective variance of the future spot price. Thus, the mean, variance, and covariances of the option return are endogenous to the model. This endogeneity provides results (analytic and numerical) with properties that are more representative of the actual mathematical structure of options.

Chapter Four examines how information about the moments of the future spot price distribution influences the optimal portfolio positions for an individual investor. These positions are derived using the analytic comparative statics of the portfolio model and by using simulation analysis.

The noisy rational expectations equilibrium was used by Grossman (1977) to explain the existence of a futures market from an information perspective. Chapter Five extends the Grossman framework to include options, and rationalizes option market existence from an information perspective.

In Chapter Six, the value of an option contract is examined for an individual investor when the contract is added to his/her portfolio. This analysis assumes that futures and physicals are already present in the investor's portfolio. This is done by using a numerical simulation model where a money metric is used to measure the value-added of the option.
Chapter Seven examines equilibrium price determination in futures and option markets. This analysis uses a two-trader partial equilibrium model which allows for information incorporation from each individual trader. Allowances are made for differing information sets among the traders.

A summary of the results of this study is presented in Chapter Eight. Also, a discussion of directions for future research in the subject matter will be included in this chapter.
A significant amount of research regarding option markets has been concerned with demonstrating the return-tailoring that options provide to an investor's portfolio. This is due to the return-truncating aspect that an option provides when included in the portfolio. Marshall (1989, Chapter 19) provides an excellent survey of the research in this area.

Another branch of options research is concerned with deriving option pricing relationships using riskless arbitrage and Arrow-Debreu contingent claims markets. Srenkle (1962) discussed using the price of warrants (options on stocks) to derive the investor's subjective expectations about the mean and variance of the future stock price. Black and Scholes (1973) derived an exact option pricing formula for European options on stocks. They assumed a lognormal price distribution and known variance for stock prices. Black (1976) extended the Black-Scholes option pricing model to include options on commodity futures contracts. An excellent survey of the research in the area of options valuation is contained in Rubenstein and Cox (1985).

Stein (1986) proved that the Arrow-Debreu framework, and in particular the Capital Asset Pricing Model (CAPM), is not appropriate to the study of futures markets. In particular, the CAPM assumption that investors will hold risky assets in proportion to their availability in the market implies investors will not hold futures contracts. Stein also proved that the CAPM assumption of predetermined asset quantities
available in the market is inappropriate to the study of futures. This is because open interest and trading volume are endogenous to these markets.

As an alternate to the Arrow-Debreu assumptions, Stein proposed using expected utility theory and, in particular, a mean-variance utility function to derive the equilibrium futures price. To illustrate information differences between market participants, Stein decomposed the individual investor's forecast error for the future spot price into a Bayesian (avoidable) component and an unavoidable component. The investor can reduce his/her Bayesian component through information collection, but has no control over the unavoidable component.

Stein defined Muth Rational Expectations (MRE) as the situation where the Bayesian error is zero among the market participants. MRE is synonymous with the definition of rational expectations as contained in the seminal work by Muth (1961). Asymptotic Rational Expectations (ARE) was defined as the situation where the market converges to MRE as the number of traders converges to infinity.

Using this framework, Stein was able to make the following conclusions about futures markets:

1. The presence of futures markets has an impact upon the spot market equilibrium by changing the supply of output.
2. There is a ratio of amateur to professional speculators that maintains maximum trading volume. This ratio is equal to the
amateur speculator's forecast error variance over the forecast error variance of the professional speculator.

3. The entry of amateur speculators will not increase the variance of the market forecast error. This is conditional upon the variance of the amateur's forecast error being less than six times the variance of the professional's forecast error.

4. A large total volume of speculation can be expected to lower the spot price through risk sharing.

5. The speed of convergence to MRE depends on three factors: (a) the average price elasticity of current production and of current consumption, (b) the slope of the marginal carrying cost function, and (c) the risk premium.

6. A change in the time varying equilibrium futures price produces an immediate corresponding change in the cash price.

7. The variance of the change in futures price relative to the change in the cash price is negatively related to the distance to futures maturity.

Stein defined professional speculators as those having a zero Bayesian error and amateurs as having a nonzero Bayesian error.

The principles of mean-variance analysis, the ranking of uncertain "lotteries" by a function of mean and variance, were spelled out in the seminal work by Markowitz (1952). It was shown that mean-variance analysis is consistent with the axioms of expected utility theory if (a) the utility function of wealth is a quadratic or (b) the probability
distribution of returns is normal. Condition b is most often used because condition a implies that the investor regards the assumption of risk as an inferior good.

Levy and Markowitz (1979) modified conditions a and b by demonstrating that the conditions can be approximately satisfied while maintaining the consistency of mean-variance analysis with expected utility theory.

Meyer (1987) generalized a result of Tobin (1958) by proving that mean-variance analysis will be consistent with the axioms of expected utility theory if the distribution of returns satisfies a location and scale condition. This condition assumes that the distribution is invariant to linear transformations of the returns.

A summary of the usefulness of mean-variance analysis, as an analytical tool in firm risk problems, is contained in Robison and Barry (1987). Mean-variance analysis is justified for three reasons:

1. Expected utility itself is an approximation of the true unknown preference function and actual estimates of the true probability distribution are difficult to obtain. Thus, it is reasonable to use mean-variance analysis as an approximation to both.

2. Mean-variance analysis has better analytical qualities than expected utility theory for most problems.

3. Most risk-averse utility functions can be represented as a quadratic using a second-order Taylor series approximation.
Robison and Barry used the mean-variance framework as a tool to examine the various responses that a firm can take to risk.

The chapters, in Robison and Barry, on indirect and direct outcomes (Chapter 14) and on insurance contracts (Chapter 15) are of particular interest. These chapters illustrate problems associated with including truncated variables with continuous variables in the decision maker's objective function.

To handle the problem of truncation, Robison and Barry derived the mean and variance of the truncated variable as a function of a probability integral defined over the relevant range. From this formulation, some simple ordering relationships between the components of the variance-covariance matrix were derived. To simplify the problem, Robison and Barry derived a generic form of the variance-covariance matrix that incorporated these ordering relationships. However, this matrix is not fully endogenous regarding the mean and variance of the continuous variable(s).

The return structure of an option is a truncated function of the underlying asset price and poses special problems in modeling. Wolf (1987) used the Robison and Barry generic specification of the mean vector and variance-covariance matrix of returns in a portfolio model containing inventory, futures, and options. Using the probability integral functions, Wolf was able to intuitively derive ordering relationships between the components of the variance-covariance matrix.

A major conclusion of Wolf's study was that changes in the expected net return for futures result in larger changes in the optimal
options position as compared to the optimal futures position. Wolf also
found that changes in the expected net return on the option cause larger
changes in the optimal option position relative to the optimal futures
position. To support his findings, Wolf used a simulation model with a
hyperbolic absolute risk aversion (HARA) utility function to derive the
optimal market positions. These simulations assumed that the mean
return of the option can be represented by Black's formula.

Hanson (1988) used Wolf's model and simulation analysis to analyze
several questions regarding the inclusion of put options into a hedger's
portfolio. Hanson's model assumes a preharvest hedge where the individ­
ual investor has a fixed or random endowment of the cash commodity at
the moment when the hedge is to be lifted. Futures and/or put options
are used as the hedging instrument in the investor's portfolio.

In a simulation experiment, Hanson examined the consistency of
mean-variance and expected utility results for the portfolio containing
put options. There were negligible differences between the optimal
positions implied by mean-variance analysis and by expected utility
maximization. This is an important result because it provides an
empirical justification for using mean-variance analysis with portfolios
containing options.

Hanson also concluded that put options are of very little value to
a hedger and will only be used as a speculative medium. This conclusion
is based on the theoretical model and on simulation results where a
money metric between the expected utility of the portfolio containing
inventory and futures, and a portfolio containing inventory, futures,
and put options is calculated. The data set used by Hanson was generated using a partial factorial experimental design. An assumed maximum bias of four percent was used for deviations of market futures and option prices from their true values.

Some previous studies have examined option portfolios using objective functions that are a variation upon the mean variance model. Hauser and Anderson (1987) used a mean-semivariance model to empirically examine optimal option hedging strategies for a soybean producer. The hedger's return was found to be more sensitive to changes in variance than to changes in risk aversion.

Some other previous studies have exclusively used simulation analysis to examine optimal option portfolios. Schroeder et al. (1989) examined various risk response strategies for the case of a risk averse cattle feeder using cash, futures, put options, and call options. The strategies of outright cash position, one-to-one futures hedge, and purchased in-the-money put were found to dominate the option fence and option spread strategies.

A discussion of variants of the mean-variance model and simulation analysis is contained in Barry (1984). Some of these variants include the mean-semivariance model, the safety-first model, and the mean-absolute deviation model.
Consider a two-period model where investors determine their optimal portfolio positions in period one by maximizing a mean-variance portfolio containing physicals, futures, and a put option. Also assume that the positions will be offset in period two. The individual investor's profit function can be represented as

\[ \pi = (P_2 - P_1) \cdot I - h_2 \cdot I^2 + (P_2 - P_f) \cdot X + (P_R - \text{Max}[0, K - P_2]) \cdot R \]

where

- \( \pi \) = profit of holding portfolio from period one to period two,
- \( I \) = physical inventory held from period one to period two,
- \( X \) = amount of futures purchased in period one and held to period two,
- \( R \) = amount of put option written in period one,
- \( P_1 \) = physical price in period one,
- \( P_2 \) = physical price in period two,
- \( P_f \) = futures price in period one,
- \( P_R \) = put option premium in period one,
- \( g \) = storage cost coefficient,
- \( K \) = strike price of put option.
This model implicitly assumes that the futures and physical prices converge in period two (no basis risk). Also, the model assumes that the option expires in period two. The model assumes zero transactions costs, and a quadratic storage cost function. All of these assumptions were made to simplify the solution process. However, a relaxation of the preceding assumptions (such as the introduction of transactions costs) has the potential for producing results that will notably differ from this model's results.

The individual investor's objective is to maximize the following Von Neumann-Morgenstern expected utility function:

\[(3.2) \quad \mathbb{E}[U(\pi)] = \mathbb{E}[\pi] - \beta A \cdot \text{Var}[\pi],\]

with respect to \(I, X, \) and \(R,\) where

\[\mathbb{E}[\cdot] = \text{expectation operator},\]
\[\text{Var}[\cdot] = \text{variance operator},\]
\[A = \text{investor's risk aversion coefficient} \geq 0.\]

This model does not contain a call option because, as was pointed out by Wolf, the futures market would become redundant in that expanded framework. This redundancy is a result of the "synthetic futures" position that can be obtained with puts and calls. An asset will become redundant, as has been noted by Rubenstein and Cox, if an asset can be
replicated with existing assets. This redundancy causes the demand equation for futures to be indeterminate when puts, calls, and futures are included with physicals in the model.

The call option is also omitted because it is possible to replicate a call by buying futures and purchasing puts. This position is often called a "synthetic call" and is illustrated, using a position diagram, in Figure 1.

![Position diagram for "synthetic call" position.](image-url)
Note that equation (3.1) is nonsmooth but continuous at the point where $K$ and $P_2$ are equal. A theorem that is contained in Appendix A was used to derive the mean and variance of $\pi$. This theorem is a generalization of results found in most standard statistical texts. It is useful because it provides a solution to the problem of deriving the analytic form of the mean vector and variance-covariance matrix of returns for physicals, futures, and options.

Using the theorem contained in Appendix A, the expectation of $\pi$ can be written as

\begin{equation}
E[\pi] = (E[P_2] - P_1) \cdot I - \log I^2 + (E[P_2] - P_2) \cdot X + (P_R - \alpha B_2) \cdot R,
\end{equation}

where

\begin{align*}
\alpha &= \text{Prob}(P_2 \leq K) = F(K), \\
B_2 &= K - E[P_2|P_2 \leq K] \geq 0,
\end{align*}

and the variance of $\pi$ as

\begin{equation}
\text{Var}[\pi] = \alpha \cdot \text{Var}[P_2|P_2 \leq K] \cdot (I+X+R)^2 \\
&\quad + (1-\alpha) \cdot \text{Var}[P_2|P_2 > K] \cdot (I+X)^2 \\
&\quad + \alpha (1-\alpha) \cdot (B_1(I+X) + B_2R)^2,
\end{equation}

where
\[ B_1 = E[P_2 | P_2 > K] - E[P_2 | P_2 \leq K] \geq 0. \]

Note that \( \alpha \) can be interpreted as the probability that the put option will be exercised, \( B_2 \) can be interpreted as the expected exercise value of the put option, and \( B_1 \) can be interpreted as a measure of the dispersion of the probability distribution of the future spot price around the strike price \((K)\).

For determining the optimal portfolio, equations (3.3) and (3.4) are substituted into equation (3.2) which yields the following first-order conditions for expected utility maximization:

\[
\begin{bmatrix}
\frac{\partial E[U(\pi)]}{\partial I} & 0 & - E[P_2] - P_1 - gI \\
\frac{\partial E[U(\pi)]}{\partial x} & 0 & - E[P_2] - P_f \\
\frac{\partial E[U(\pi)]}{\partial R} & 0 & P_R - \alpha B_2
\end{bmatrix}
\begin{bmatrix}
V_{11} & V_{11} & V_{12}
V_{11} & V_{12} & V_{22}
\end{bmatrix}
\begin{bmatrix}
I_d \\
X_d \\
R_d
\end{bmatrix}
\]

where

\[ V_{11} = \text{Var}[P_2] = \alpha \text{Var}[P_2 | P_2 \leq K] + (1 - \alpha) \text{Var}[P_2 | P_2 > K] + \alpha(1 - \alpha)B_1^2, \]

\[ V_{12} = \alpha \text{Var}[P_2 | P_2 \leq K] + \alpha(1 - \alpha)B_1 B_2, \]

\[ V_{22} = \alpha \text{Var}[P_2 | P_2 \leq K] + \alpha(1 - \alpha)B_2^2. \]
The d superscript is used to indicate the optimized portfolio demands. The variance-covariance terms ($V_{ij}$) are derived in Appendix B. The third vector in equation (3.5) is the vector of mean returns to the portfolio. Note that $\alpha B_2$ can be interpreted as the investor's subjective value of the put option since it is equal to the subjective probability of option exercise ($\alpha$) multiplied by the subjective expected value of the option if it is exercised ($B_2$).

The matrix of $V$ terms is the variance-covariance matrix of returns to the portfolio. Note that the returns to inventory and futures have identical variances and covariances since they have the same settlement price ($P_2$) when basis risk is ignored. Also note that the mean, variance, and covariances of the put option position are functions of the spot price mean and spot price variance. This is because they are determined by the conditional moments of the underlying spot price distribution.

The second-order conditions can be expressed by the following Hessian matrix:

$$
\begin{pmatrix}
-(g + AV_{11}) & -AV_{11} & -AV_{12} \\
-AV_{11} & -AV_{11} & -AV_{12} \\
-AV_{12} & -AV_{12} & -AV_{22}
\end{pmatrix}
$$

which is a negative definite matrix. This implies that the solution to (3.5) will provide a global maximum to the problem stated in equations
(3.1) and (3.2). Also, the matrix becomes negative semi-definite if $g$ equals zero. Also note that $R$ must represent the amount of written put options, as opposed to purchased, in order for the Hessian matrix to be negative definite.

Equation (3.5) yields the following portfolio demand functions:

\begin{align*}
(3.6) \quad r^d &= \frac{P_f - P_1}{g} \\
(3.7) \quad \chi^d &= \frac{E[P_2] - P_f}{A V_{11}} - \frac{V_{12}}{V_{11}} R^d - r^d, \\
(3.8) \quad R^d &= \frac{P_R - a B_2}{A V_{22}} - \frac{V_{12}}{V_{22}} (I+X)^d.
\end{align*}

Equation (3.6) indicates that inventory demand is a function of the market marginal return to storage $(P_f - P_1)$ divided by the second derivative of the storage cost function ($g$). Note that the futures price is used as a proxy for the expected spot price in determining how much to store. This is consistent with the mean-variance results of Holthausen (1979) and the expected utility results of Feder, Just, and Schmitz (1980). Note that the addition of the put option has no impact upon the mathematical form of the inventory demand equation.

Equation (3.7) shows futures demand to be a function of a speculative component $\left(\frac{E[P_2] - P_f}{A V_{11}}\right)$ and a hedging component $\left(\frac{V_{12}}{V_{11}} \cdot R^d + I^d\right)$. The form of the speculative component implies that the investor will buy (sell) futures if he/she believes that the spot
price mean is greater (less) than the current futures price (ignoring hedging demand). The size of speculative position is tempered by risk aversion, the expected spot price variance, and the absolute size of the difference in the numerator.

Note that the hedging component coefficient for put options \((- V_{12}/V_{11})\) can be interpreted as the marginal rate of substitution in variance between futures and put options. Thus, the hedging component represents the number of futures contracts needed (in opposite position to the option) for every corresponding option contract to maintain a riskless portfolio. Note that the hedging coefficient between futures and inventory is equal to -1 since inventory and futures are both settled on the same price \((P_2)\).

In traditional cross-hedging analysis (for instance, using live hog futures to hedge hams), the hedging component between the two instruments (live hog futures and cash hams) is estimated using ordinary least squares (OLS) or a variant thereof. Note that this is not necessary for estimating the hedging components of equations (3.7) and (3.8). This is because the relationship between returns for futures and for options is nonstochastic (since there is no basis risk or spread risk). Thus, the hedge ratios can be estimated simply by incorporating forecasts of the future spot price mean and variance into the hedge ratio formulas.

Equation (3.8) shows that the structure of the option demand equation is analogous to the structure of the futures demand equation. The speculative component is equal to \((P_\text{R} - \alpha B_2)/(A^*V_{22})\) and the hedging
The hedging component in this equation represents the number of option contracts needed (in opposite position) to every corresponding contract held in futures to maintain a riskless portfolio.

In this chapter, the analytic form of the portfolio demand equations was derived. An advantage of having the analytical form of the demand equations is that it facilitates computer modeling of the portfolio without the need to use iterative Monte Carlo techniques in arriving at solutions. Numerical simulations, using models incorporating the portfolio demand equations, are used in some of the remaining chapters to derive the more complex comparative statics results.

This model lends itself easily to computer application. An individual investor can input forecasts of the future spot price mean and variance, along with his/her preference for risk, to arrive at the optimal mean-variance portfolio. Thus, this model should be useful for market participants. However, as with any computer model, such a program should only be used as a marketing decision aid and not as a "black box" for making marketing decisions. This model is based on assumptions that are an extrapolation of reality. Thus, its results should be tempered with other information to arrive at the proper marketing plan.
In this chapter, the mean-variance portfolio model derived in Chapter Three is used to examine the conditions under which an investor will hold different positions in physical inventory, futures, and put options. Also, the effects, on the investor's portfolio demands, of changes in mean, variance, and risk aversion will be studied.

The interaction of information and opinion formation with preference for risk will be examined. This is to determine how a utility maximizing investor will adjust his/her market portfolio. For any market participant, the ability to gain and conduct a proper interpretation of information is of paramount importance to the profitability of market operations.

In this chapter, the portfolio model is used for the case of an individual investor. The chapter will particularly examine how an investor will incorporate information about the future spot price distribution in arriving at the optimal portfolio positions. In Chapter Six, the same portfolio model will be used to examine how the information availability influences the investor's private value of the option.

**Comparative statics results**

Equations (3.7) and (3.8) can be graphed in two-dimensional space by plotting the desired put position \( R^d \) against the desired sum of inventory and futures positions \([(I + X)^d]\). Graphical analysis was used
for the initial comparative statics results. This was to provide some intuition as to how the speculative and hedging components can interact to determine the optimal portfolio positions.

For analytical purposes, equations (3.7) and (3.8) can be rewritten as

\[ \begin{align*}
(4.1) \quad R_d &= \sigma_1 + \beta_1(I+X)^d, \\
(4.2) \quad R_d &= \sigma_2 + \beta_2(I+X)^d, \\
\end{align*} \]

where

\[ \begin{align*}
\sigma_1 &= \frac{E[P_2] - P_f}{AV_{12}}, \\
\sigma_2 &= \frac{P_R - \alpha B_2}{AV_{22}}, \\
\beta_1 &= \frac{V_{11}}{V_{12}}, \\
\beta_2 &= \frac{V_{12}}{V_{22}}, \\
\end{align*} \]

\( \sigma_1 \) is equal to the inverse speculative component of the futures demand equation multiplied by the inverse of its hedging component. This will be positive if the investor believes that the futures price is a downwardly biased predictor of the future spot price. Also, \( \sigma_2 \) is
equal to the speculative component of the option demand equation. If 
the investor believes that the option market price is an overestimate of 
the true option value, then $\sigma_2$ will be positive.

$\beta_1$ is equal to the inverse hedging component of the futures demand 
equation and $\beta_2$ is equal to the hedging component of the option demand 
equation. Hereinafter, (4.1) will be represented as $D_1$ and (4.2) will 
be represented as $D_2$ in the graphical presentations.

For future purposes, the following definitions are used:

\[
B_3 = E[P_2|P_2>K] - K > 0,
\]
\[
V_{13} = (1-\alpha)\text{Var}[P_2|P_2>K] + \alpha(1-\alpha)B_1B_3 > 0,
\]
\[
V_{33} = (1-\alpha)\text{Var}[P_2|P_2>K] + \alpha(1-\alpha)B_3^2 > 0,
\]
\[
V_{23} = \alpha(1-\alpha)B_2B_3 > 0.
\]

$B_3$ is equivalent to the expected value of an exercised "synthetic" call 
position with strike price equal to $K$. $V_{13}$ is the covariance between 
the returns on the "synthetic" call position and inventory and/or 
futures. $V_{33}$ is the variance of the "synthetic" call position. $V_{23}$ is 
the covariance between the put and "synthetic" call positions. The 
following relationships can be defined from the definitions of the mean 
and variance terms:

R1. $B_1 = B_2 + B_3$,
R2. $V_{11} = V_{12} + V_{13}$,
R3. $V_{12} = V_{22} + V_{23}$,
Using R1 through R6, the following propositions can be proven:

**Proposition 1.** The value of the inverse hedging component of the futures demand equation ($\beta_1$) will always be greater than one. Also, the value of the hedging component of the option demand equation ($\beta_2$) will always be greater than one.

**Proof.**

Using R2, $\beta_1 = \frac{V_{11}}{V_{12}} = \frac{V_{12} + V_{13}}{V_{12}}$, which is equal to $1 + \frac{V_{13}}{V_{12}}$.

which is greater than one because $\frac{V_{13}}{V_{12}} > 0$.

Using R3, $\beta_2 = \frac{V_{12}}{V_{22}} = \frac{V_{22} + V_{23}}{V_{22}}$, which is equal to $1 + \frac{V_{23}}{V_{22}}$.

which is greater than one because $\frac{V_{23}}{V_{22}} > 0$.

Proposition 1 illustrates that the hedging component of the futures demand equation will be less than one and the hedging component of the option demand equation will be greater than one. This is logical since the variance of returns on the futures position is always greater
than the variance of returns on the corresponding option position (see relations R2, R3, and R4). Thus, a lesser amount of futures is needed to cover the corresponding variability of the option position. This implies that futures are the superior instrument to use if the investor's objective function is to maintain a riskless portfolio.

**Proposition 2.** The inverse hedging component of the futures demand equation ($\beta_1$) will always be greater than the hedging component of the option demand equation ($\beta_2$).

**Proof.**

\[
\beta_1 - \beta_2 = \frac{V_{11}V_{22} - V_{12}^2}{V_{12}V_{22}}, \text{ which is equal to } \frac{(V_{12}+V_{13})(V_{12}-V_{23})-V_{12}^2}{V_{12}V_{22}}
\]

using R2 and R3, which is equal to \[
\frac{V_{12}^2+V_{12}V_{13}-(V_{12}+V_{13})V_{23}-V_{12}^2}{V_{12}V_{22}}
\]

by multiplication. The preceding expression can be represented, using R2, as \[
\frac{V_{12}V_{13}V_{11}V_{23}}{V_{12}V_{22}}, \text{ which is equal to } \frac{\Gamma+V_{11}V_{23}V_{11}V_{23}}{V_{12}V_{22}} (\text{using R6}),
\]

which is equal to \[
\frac{\Gamma}{V_{12}V_{22}} > 0.
\]

Note that equations (4.1) and (4.2), along with Propositions 1 and 2, can be used to derive the optimal positions in options and futures.
Proposition 2 guarantees the existence and uniqueness of the optimal portfolio equilibrium (intersection of $D_1$ and $D_2$).

Figure 2 illustrates the four quadrants in which portfolio positions can be established. Quadrants I and II correspond to hedging positions in which $R^d$ and $(I + X)^d$ are opposite in nature. Quadrants III and IV correspond to double speculative positions in which $R^d$ and $(I + X)^d$ are on the same side of the market. Points on the $R^d$ or $(I + X)^d$
axis correspond to pure speculative positions in which the investor speculates only in \( R^d \) or \( (I + X)^d \).

The optimal physical inventory position \( (I^d) \) is exogenously determined outside this framework. This is because its functional form does not include any of the investor information variables [see equation (3.6)].

Figure 3 illustrates the position in which the investor chooses the "classic" hedge by matching inventory with an equal and opposite position in futures \( (I^d = -X^d) \). This is matched with a zero net position in the put option. The following proposition lists the conditions under which the "classic" hedge position will be taken.

**Proposition 3.** An investor will only use the "classic" hedge \( (I^d = -X^d, R^d = 0) \) if one of the following three conditions holds:

A. The investor believes that both market prices are unbiased 
\( (E[P_2] = P_F \) and \( P_R = \alpha B_2) \).

B. The investor is extremely risk averse \( (A \to \infty) \).

C. The investor has an extremely high estimate of \( \text{Var}[P_2] \)
\( (\text{Var}[P_2] \to \infty) \).

**Proof.** The situation illustrated in Figure 3 will only occur if \( \sigma_1 = \sigma_2 = 0 \) (both speculative components are zero). If \( E[P_2] = P_F \) and \( P_R = \alpha B_2 \), then the numerators of \( \sigma_1 \) and \( \sigma_2 \) will be equal to zero. If \( A \to \infty \) or \( \text{Var}[P_2] \to \infty \), then \( \sigma_1 \) and \( \sigma_2 \) will converge to zero. Note that the classic hedge will be taken if the discrete nature of futures and
options contracts is taken into account and if $\sigma_1$ and $\sigma_2$ are sufficiently close to zero.

Solving equations (4.1) and (4.2) gives the following demand functions for options and the sum of futures plus inventory:

\[(4.3)\quad (1+X)^d = \frac{\sigma_1 - \sigma_2}{\beta_1 - \beta_2},\]

\[(4.4)\quad R^d = \frac{\sigma_2 \beta_1 - \sigma_1 \beta_2}{\beta_1 - \beta_2}.\]

Figure 3 Graphical representation of the "classic" hedge.
If the speculative components of the futures and option demand equations are equal \( \sigma_1 = \sigma_2 \), then equation (4.3) implies that \((1 + X)^d\) will be equal to zero and the investor will either buy or sell puts. This will depend upon whether the intersection is below or above the origin. This corresponds to a situation where the investor hedges his/her inventory position with futures and speculates with the option. Let

\[
\theta = \frac{E[P_2] - P_F}{P_R - \alpha B_2}
\]

which is the value of the ratio of expected marginal speculative returns to futures divided by returns to put options. If \( \sigma_1 = \sigma_2 \), it can be shown that

\[
\frac{E[P_2] - P_F}{AV_{12}} = \frac{P_R - \alpha B_2}{AV_{22}} \quad \text{or} \quad \frac{E[P_2] - P_F}{P_R - \alpha B_2} = \frac{V_{12}}{V_{22}} = \beta_2 \quad \text{or} \quad \theta = \beta_2,
\]

which implies that the absolute value of the ratio of expected marginal speculative returns to futures divided by returns to options is equal to the hedging component of the put option demand curve.

If the ratios of the speculative components over the hedging components are equivalent for future and options \( \sigma_1/\beta_1 = \sigma_2/\beta_2 \), then equation (4.4) implies that option demand will be equal to zero and the
investor will either underhedge or overhedge in futures. This is
dependent upon whether the intersection is to the left or right of the
origin. This situation will occur when

\[
(4.6) \quad \frac{E[P_2] - P_f}{AV_{11}} = \frac{P_R - \alpha B_2}{AV_{12}} \quad \text{or} \quad \frac{E[P_2] - P_f}{P_R - \alpha B_2} = \frac{V_{11}}{V_{12}} = \beta_1 \text{ or } \theta = \beta_1.
\]

This implies that the ratio of the absolute expected marginal specula­tive returns to futures divided by returns to options is equal to the
inverse hedging component of the futures demand equation.

Table 1 illustrates the optimal portfolio positions given the
relationships between \( \theta \), \( \beta_1 \), and \( \beta_2 \). A graphical derivation of these
positions is contained in Appendix C.

The partial derivatives of the portfolio variance-covariance
matrix components with respect to \( A \), \( E[P_2] \), and \( \text{Var}[P_2] \) are derived in
Appendix D. Also, the partial derivatives of the speculative and
hedging components \( (\sigma_1, \sigma_2, \beta_1, \text{and } \beta_2) \) with respect to \( A \), \( E[P_2] \), and
\( \text{Var}[P_2] \) are derived in Appendix D.

The signs of the partial derivatives of the speculative and
hedging components with respect to risk aversion are as follows:

\[
(4.7) \quad \frac{\partial \sigma_1}{\partial A} \geq (<) 0 \text{ for } E[P_2] \leq (<) P_f,
\]

\[
(4.8) \quad \frac{\partial \sigma_2}{\partial A} \geq (<) 0 \text{ for } P_R \leq (<) \alpha B_2.
\]
Table 1 Optimal Portfolio Positions for Different Relationships Between $\theta$, $\beta_1$, and $\beta_2$

<table>
<thead>
<tr>
<th>Value of $\theta$</th>
<th>Optimal Positions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta &lt; \beta_2$</td>
<td>$R &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$I+X &lt; 0$</td>
</tr>
<tr>
<td>$\theta - \beta_2$</td>
<td>$R &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$I+X = 0$</td>
</tr>
<tr>
<td>$\beta_2 &lt; \theta &lt; \beta_1$</td>
<td>$R &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$I+X &gt; 0$</td>
</tr>
<tr>
<td>$\theta - \beta_1$</td>
<td>$R = 0$</td>
</tr>
<tr>
<td></td>
<td>$I+X &gt; 0$</td>
</tr>
<tr>
<td>$\theta &gt; \beta_1$</td>
<td>$R &lt; 0$</td>
</tr>
<tr>
<td></td>
<td>$I+X &gt; 0$</td>
</tr>
</tbody>
</table>

(4.9) \[ \frac{\partial \beta_1}{\partial A} = \frac{\partial \beta_2}{\partial A} = 0. \]

Relations (4.7) and (4.8) imply that increases in risk aversion ($A$) will always drive the speculative components ($\sigma_1$ and $\sigma_2$) toward the origin. Thus, the investor will reduce the absolute size of his/her speculative positions as the aversion to price risk becomes greater. Relation (4.9) implies that risk aversion has no effect upon the slope of the demand curves. Thus, the hedging components of the futures and option demand will be independent of the level of risk aversion, which has no impact upon the marginal rate of substitution in variance of returns between futures and options.

The partial derivatives of the speculative and hedging components with respect to the expected future spot price mean are as follows:
Equation (4.10) will be positive for \( E[P_2] \geq P_f \) and will remain positive for some \( E[P_2] < P_f \). Thus, increases in the predicted spot price mean will usually induce the investor to hold speculative futures positions that are more long or less short. This holds true unless the price prediction is sufficiently lower than the futures price. Also, equation (4.11) will be positive for \( P_R \geq \alpha B_2 \) and will remain positive for some \( P_R < \alpha B_2 \). Thus, increases in the predicted spot price mean will usually induce the investor to hold more short or less long speculative positions in the put option, unless the option is sufficiently undervalued by the market.

Equation (4.12) will always be positive. This implies that the inverse hedging component of the futures demand equation will always increase with increases in the predicted spot price mean. Thus, the
actual hedging component of the futures demand equation will decrease with increases in the predicted spot price mean. Equation (4.13) implies that the hedging component of the option demand equation will decrease with increases in the future spot price prediction. This is conditional upon the predicted price being greater than or equal to the strike price of the option.

The partial derivatives of the speculative and hedging components with respect to the predicted spot price variance are as follows:

\begin{equation}
\frac{\partial \sigma_1}{\partial \text{Var}[P_2]} \geq (\ <) \ 0 \text{ for } E[P_2] \leq (\ >) \ P_F^f.
\end{equation}

\begin{equation}
\frac{\partial \sigma_2}{\partial \text{Var}[P_2]} \geq (\ <) \ 0 \text{ for } \ -V_{22}(\partial \alpha B_2/\partial \text{Var}[P_2]) - (P_R - \alpha B_2)(\partial V_{22}/\partial \text{Var}[P_2]) \geq (\ <) \ 0.
\end{equation}

\begin{equation}
\frac{\partial \beta_1}{\partial \text{Var}[P_2]} \geq (\ <) \ 0 \text{ for } \ V_{12}(\partial V_{13}/\partial \text{Var}[P_2]) - V_{13}(\partial V_{12}/\partial \text{Var}[P_2]) \geq (\ <) \ 0.
\end{equation}

\begin{equation}
\frac{\partial \beta_2}{\partial \text{Var}[P_2]} \geq (\ <) \ 0 \text{ for } \ V_{22}(\partial V_{23}/\partial \text{Var}[P_2]) - V_{23}(\partial V_{22}/\partial \text{Var}[P_2]) \geq (\ <) \ 0.
\end{equation}

Equation (4.14) implies that increases in the predicted spot price variance will cause the amount of futures speculation to decline. Also, equation (4.15) will be negative for \( P_R \geq \alpha B_2 \) and will remain negative.
for some $P_R < \alpha B_2$. The limit of $\sigma_2$ as $\text{Var}[P_2]$ approaches infinity is zero because $B_2$ is in the numerator and $B_2^2$ is in the denominator (in $V_{22}$). Thus, increases in the predicted spot price variance will induce the investor to become more long or less short in the put option unless $P_R - \alpha B_2$ is negative by an amount such that the overall sign of equation (4.15) is positive.

The signs of equations (4.16) and (4.17) cannot be derived as functions of price relationships. Thus, no determinant relationship can be derived between the predicted spot price variance and the hedging components of the futures and options demand equations.

The partial derivatives of the futures plus inventory demand and option demand with respect to the speculative and hedging components are as follows:

\[
\frac{\partial (I + X)^d}{\partial (\sigma_1 - \sigma_2)} = \frac{1}{\beta_1 - \beta_2} > 0, \tag{4.18}
\]

\[
\frac{\partial (I + X)^d}{\partial (\beta_1 - \beta_2)} = \frac{\sigma_1 - \sigma_2}{(\beta_1 - \beta_2)^2} \geq (<) 0 \text{ as } \sigma_1 \geq (<) \sigma_2, \tag{4.19}
\]

\[
\frac{\partial R^d}{\partial \sigma_1} = \frac{-\beta_2}{\beta_1 - \beta_2} < 0, \tag{4.20}
\]

\[
\frac{\partial R^d}{\partial \sigma_2} = \frac{\beta_1}{\beta_1 - \beta_2} > 0, \tag{4.21}
\]

\[
\frac{\partial R^d}{\partial \beta_1} = \frac{(\sigma_1 - \sigma_2)\beta_2}{(\beta_1 - \beta_2)^2} \geq (<) 0 \text{ as } \sigma_1 \geq (<) \sigma_2, \tag{4.22}
\]
Upon examination of equations (4.7) through (4.23), it is not possible to derive global analytic partial derivatives of the demand equations with respect to the moments of the predicted spot price distribution. This is because the signs of the derivatives are dependent upon the prior positions held in the portfolio. However, a trader who has a position in these markets, can use a graphical analysis such as is illustrated in Appendix C to derive his/her new positions. The reception of new information about the future spot price moments will shift the futures ($D_1$) and options ($D_2$) demand curves based upon the partial derivative equations [(4.7) through (4.17)].

**Numerical simulation methods**

Since the global analytic partial derivatives are unobtainable (as was pointed out in the previous section), numerical simulation was used to examine the comparative static properties of the portfolio model. All of the simulations in this study were done using the Regression Analysis of Time Series (RATS) econometric package.

The data sets for all of the simulation models in this study used actual prices (spot, futures, and put options) for corn, soybeans, live cattle, and live hogs. These estimates were used as center points in the artificial generation of data. The prices were obtained from the
The basis for choosing these commodities was their importance in the midwest farming economy.

The spot market prices used were as follows: central Illinois no. 1 yellow (corn and soybeans), Iowa-Southern Minnesota (live hogs), and Texas-Oklahoma steer (live cattle). These prices were chosen because of their availability in the Wall Street Journal. The futures contract month was chosen on the basis of which had the largest open interest of the three nearby months. This criterion was used because prices for options are only provided for the three nearby futures contract months (in the Wall Street Journal). Also, the futures contract with the most open interest is usually the most actively traded.

The option strike price was chosen on the basis of which strike price was closest to at-the-money. This selection criterion was used because the option closest to at-the-money is usually the most actively traded.

The option implied volatilities were derived with the obtained prices and the valuation formula from the portfolio model \((aB_2)\). Random selection between January 1, 1989 and March 30, 1989 was the criterion upon which the sample dates were determined.

Table 2 illustrates the price data used to generate the data sets for the numerical simulations in this dissertation. The small number of actual data samples collected can be justified for two reasons. First, these data points serve only as center points in the artificial generation of much larger data sets. Thus, for each observation in Table 2, there are at least 50 to 200 additional observations (depending upon the
Table 2 Center Points for Data Set used in Simulation Model

<table>
<thead>
<tr>
<th>Date</th>
<th>Commodity</th>
<th>Futures Mnth.</th>
<th>Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$L_1$</td>
</tr>
<tr>
<td>1/03/89</td>
<td>CRN</td>
<td>Mar.</td>
<td>2.65</td>
</tr>
<tr>
<td>&quot;</td>
<td>SB</td>
<td>Mar.</td>
<td>7.87</td>
</tr>
<tr>
<td>&quot;</td>
<td>LC</td>
<td>Feb.</td>
<td>74.75</td>
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<tr>
<td>&quot;</td>
<td>LH</td>
<td>Feb.</td>
<td>42.50</td>
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<tr>
<td>2/15/89</td>
<td>LC</td>
<td>Apr.</td>
<td>75.25</td>
</tr>
<tr>
<td>&quot;</td>
<td>LH</td>
<td>Apr.</td>
<td>41.75</td>
</tr>
<tr>
<td>3/08/89</td>
<td>CRN</td>
<td>May</td>
<td>2.65</td>
</tr>
<tr>
<td>&quot;</td>
<td>SB</td>
<td>May</td>
<td>7.54</td>
</tr>
<tr>
<td>&quot;</td>
<td>LC</td>
<td>Apr.</td>
<td>78.75</td>
</tr>
<tr>
<td>&quot;</td>
<td>LH</td>
<td>Apr.</td>
<td>41.25</td>
</tr>
<tr>
<td>3/20/89</td>
<td>CRN</td>
<td>May</td>
<td>2.70</td>
</tr>
<tr>
<td>&quot;</td>
<td>SB</td>
<td>May</td>
<td>7.77</td>
</tr>
</tbody>
</table>

^aCRN represents corn, SB represents soybeans, LC represents live cattle, and LH represents live hogs.

^bPrices for corn and soybeans are in dollars per bushel, and are in dollars per hundredweight for live cattle and live hogs.

Second, the purpose of the simulations is to derive information from the mathematical model, not from the data itself. Therefore, even nonsensical numbers could serve this purpose; however, actual data were used for two reasons. These reasons are:
1. To provide an objective method for selection of the simulation data set. Using ad hoc numbers would be less defendable against the contention that the data were tampered with to produce specific results.

2. To demonstrate how actual data can be used with the portfolio model in arriving at the optimal positions.

In this chapter, two simulation experiments are presented that used the first observation of Table 2 (the corn data for January 3, 1989) to arrive at input values of the spot price, futures price, option price, implied volatility, and strike price for all data points. The storage cost coefficient (g) was set at .05 for all simulations. Changes in g have little impact upon the results except for changing the size of the inventory position.

If all of the observations from Table 2 were used, the amount of tabular and graphical results produced would be exorbitant (would produce 24 tables and 72 graphs instead of two tables and 6 graphs). Thus, the reason for using only one observation in this chapter. In the simulation experiments of Chapters Six and Seven, the full set of observations from Table 2 were used since the amount of produced output per observation was much smaller.

In the first simulation experiment, the optimal portfolio positions were obtained for various combinations of the input variables (expected spot price moments, risk aversion coefficient). These positions are summarized in tabular form for analysis. The objective of
this experiment was to determine the option user's positions for different combinations of the input variables.

The input values for the price distribution moments were generated using a full factorial model where the center points in the factorial design were the market estimates of the moments (futures price for expected spot price mean, implied volatility squared for the expected spot price variance). The additional points in the factorial design were generated as a percentage of the center points. These percentages were minus 12, minus six, plus six, and plus 12 (or 88, 94, 106, and 112 percent of the center points). These percentages were chosen subjectively from observation, for some live cattle and live hog cash price forecasts, of the percent deviation from the futures price.

Two values of the risk aversion coefficient were used. These values were .0001 (moderate risk aversion) and .000001 (low risk aversion). These coefficient values fall within the range found by King and Robison (1981) as the range in which most individuals fall.

The computer simulation model, that was used for both experiments, consisted of equations representing the portfolio demands [equations (3.6), (3.7), and (3.8)]. Formulas, assuming a normal distribution for the period two spot price, were included for the conditional moments in the computer model.

The second simulation experiment examined the comparative statics of changes in the spot price distribution moments and changes in risk aversion upon the portfolio demands. The major objective of this experiment was to derive a more complete picture of the impact upon the
portfolio demands for changes in the moments of the predicted spot price distribution. The data set used as an input to this model is different and reflects a wider range of possible values that the one used in the first simulation experiment. Instead of a tabular approach, the simulation results were examined through a series of graphs where the portfolio demands are plotted against the value of the input variables. This provides a more continuous picture of how changes in the input variables will influence the positions held in the portfolio.

For examining the comparative statics of changes in the expected spot price mean, the following assumptions were used in setting up the data set:

1. The risk aversion coefficient was fixed at .0001, which is the moderate risk aversion level.
2. The expected spot price variance was equivalent to the squared market implied volatility.
3. The spot price mean was assumed to take on values ranging from 70 to 130 percent, with increments of 0.3 percent, of the futures price (which gives 200 observations).

The first two assumptions were made so the other inputs (risk aversion, expected spot price variance) would not introduce biases in the results. The values of the other inputs represent approximate midpoints in the possible data ranges. The range of percentages for the expected mean were chosen through a process where the range was subjectively cut off
if the optimal positions appeared (on the graphs of the positions) to approach asymptotic limits.

For examining the comparative statics results of changes in the future spot price variance, the following assumptions were used in constructing the data set:

1. The risk aversion coefficient was fixed at .0001 (moderate risk aversion).
2. The expected spot price mean was set equal to the futures price.
3. The expected spot price variance was generated as the square of standard deviations ranging in value from one to 200 percent, in increments of one percent, of the futures price (200 observations).

These assumptions were made with the same reasoning that was used in choosing the data set for the future spot price mean.

The assumptions used in selecting the data set for examining the impact of changing risk aversion are as follows:

1. The expected spot price mean was set equal to 90 percent of the futures price.
2. The spot price variance was set equal to 110 percent of the squared market implied volatility.
3. The risk aversion coefficients were chosen ranging from .000001 to .02, in increments of .0001 (200 observations).

The rationale for the first two assumptions is if the spot price mean and the variance were set to their actual market values, then the positions will degenerate to the "perfect" hedge position illustrated in Proposition 3. Thus, the percentages were arbitrarily selected to give non-zero positions. The range used for the risk aversion coefficient was chosen to cover the entire spectrum of risk preference from very low risk aversion (.000001) to extremely high risk aversion (.02).

The assumptions, with regards to the selection of data for the second simulation experiment, will bias the results pertaining to the absolute size of the positions and the location of slope changes in the graphs. However, the overall shape of each graphical relationship should remain approximately the same for changes in the data selection assumptions.

Numerical simulation results

Tables 3 and 4 contain the results from the first simulation experiment. In both tables, inventory demand was constant and was equal to 31,000 bushels. The tabular results indicate that increases in the expected spot price mean will induce the investor to hold less short or more long futures positions. This result was expected since increases in the spot price mean will either decrease expected returns to a short futures position or increase expected returns to a long futures posi-
Table 3 Optimal Portfolio Positions For a Low Level of Risk Aversion$^a$

<table>
<thead>
<tr>
<th>Percent Deviations From Market Values</th>
<th>Net Desired Futures+Inventory$^b$</th>
<th>Desired Option Position (bu.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Frcst. Variance Frcst.</td>
<td>Position (bu.)</td>
<td></td>
</tr>
<tr>
<td>-12 -12</td>
<td>-419,108</td>
<td>296,079</td>
</tr>
<tr>
<td>-12 -06</td>
<td>-368,486</td>
<td>253,720</td>
</tr>
<tr>
<td>-12 00</td>
<td>-328,670</td>
<td>221,174</td>
</tr>
<tr>
<td>-12 +06</td>
<td>-296,579</td>
<td>195,525</td>
</tr>
<tr>
<td>-12 +12</td>
<td>-270,171</td>
<td>174,866</td>
</tr>
<tr>
<td>-06 -12</td>
<td>-169,954</td>
<td>125,371</td>
</tr>
<tr>
<td>-06 -06</td>
<td>-154,214</td>
<td>112,617</td>
</tr>
<tr>
<td>-06 00</td>
<td>-140,440</td>
<td>101,338</td>
</tr>
<tr>
<td>-06 +06</td>
<td>-128,287</td>
<td>91,283</td>
</tr>
<tr>
<td>-06 +12</td>
<td>-117,490</td>
<td>82,257</td>
</tr>
<tr>
<td>00 -12</td>
<td>-8,754</td>
<td>18,182</td>
</tr>
<tr>
<td>00 -06</td>
<td>-4,101</td>
<td>8,506</td>
</tr>
<tr>
<td>00 00</td>
<td>-116</td>
<td>241</td>
</tr>
<tr>
<td>00 +06</td>
<td>3,321</td>
<td>-6,873</td>
</tr>
<tr>
<td>00 +12</td>
<td>6,306</td>
<td>-13,040</td>
</tr>
<tr>
<td>+06 -12</td>
<td>48,144</td>
<td>119,107</td>
</tr>
<tr>
<td>+06 -06</td>
<td>44,807</td>
<td>107,246</td>
</tr>
<tr>
<td>+06 00</td>
<td>41,983</td>
<td>96,870</td>
</tr>
<tr>
<td>+06 +06</td>
<td>39,580</td>
<td>87,854</td>
</tr>
<tr>
<td>+06 +12</td>
<td>37,524</td>
<td>79,092</td>
</tr>
<tr>
<td>+12 -12</td>
<td>124,141</td>
<td>304,205</td>
</tr>
<tr>
<td>+12 -06</td>
<td>115,881</td>
<td>257,447</td>
</tr>
<tr>
<td>+12 00</td>
<td>108,611</td>
<td>222,030</td>
</tr>
<tr>
<td>+12 +06</td>
<td>102,166</td>
<td>194,480</td>
</tr>
<tr>
<td>+12 +12</td>
<td>96,413</td>
<td>172,551</td>
</tr>
</tbody>
</table>

$^a$ Risk aversion coefficient is equal to .000001 for the investor. Also note that the positions in futures and options may not be divisible by a standard size contract (i.e., 5,000 bushels).

$^b$ Net inventory equals 31,500 bushels.
Increases in the absolute difference between the futures price and the spot price mean will induce the investor to hold larger absolute positions in the put option. For situations where the spot price mean is less than the futures price, written put options are used to hedge the speculative overhang in the inventory and futures markets. For situations where the spot price mean is greater than the futures price, written put options are used to increase the long speculative position in futures and inventory.

For increases in the future spot price variance, the futures position converges to the opposite of the inventory position. This is because the investor will desire to hold a more completely hedged position in inventory and futures (no speculative overhang) during periods of increasing price volatility. Increases in the future spot price variance also induce the investor to hold less short or greater long positions in the put option. This is because increases in the expected variance will increase the value of holding the put option. Note, from the analytical model, that the option value \( \alpha B_2 \) is an increasing function of the variance.

By comparing the results of Table 3 to those of Table 4, we can examine the effects of increasing risk aversion upon the portfolio positions. An increase in risk aversion induces the investor to hold a futures position that converges to the opposite of the inventory position. This is the same result as for increases in the future spot price variance since both risk aversion and variance appear in the denominator.
Table 4 Optimal Portfolio Positions For a Moderate Level of Risk Aversion*

<table>
<thead>
<tr>
<th>Percent Deviations From Market Values</th>
<th>Net Desired Futures+Inventory&lt;sup&gt;b&lt;/sup&gt;</th>
<th>Desired Option Position (bu.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Forecast Variance Forecast</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-12 -12</td>
<td>-4,191</td>
<td>2,961</td>
</tr>
<tr>
<td>-12 -06</td>
<td>-3,685</td>
<td>2,537</td>
</tr>
<tr>
<td>-12 00</td>
<td>-3,287</td>
<td>2,212</td>
</tr>
<tr>
<td>-12 +06</td>
<td>-2,966</td>
<td>1,955</td>
</tr>
<tr>
<td>-12 +12</td>
<td>-2,702</td>
<td>1,749</td>
</tr>
<tr>
<td>-06 -12</td>
<td>-1,700</td>
<td>1,254</td>
</tr>
<tr>
<td>-06 -06</td>
<td>-1,542</td>
<td>1,126</td>
</tr>
<tr>
<td>-06 00</td>
<td>-1,404</td>
<td>1,013</td>
</tr>
<tr>
<td>-06 +06</td>
<td>-1,283</td>
<td>913</td>
</tr>
<tr>
<td>-06 +12</td>
<td>-1,175</td>
<td>823</td>
</tr>
<tr>
<td>00 -12</td>
<td>-88</td>
<td>182</td>
</tr>
<tr>
<td>00 -06</td>
<td>-41</td>
<td>85</td>
</tr>
<tr>
<td>00 00</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>00 +06</td>
<td>33</td>
<td>-69</td>
</tr>
<tr>
<td>00 +12</td>
<td>63</td>
<td>-130</td>
</tr>
<tr>
<td>+06 -12</td>
<td>481</td>
<td>1,191</td>
</tr>
<tr>
<td>+06 -06</td>
<td>448</td>
<td>1,072</td>
</tr>
<tr>
<td>+06 00</td>
<td>420</td>
<td>968</td>
</tr>
<tr>
<td>+06 +06</td>
<td>396</td>
<td>875</td>
</tr>
<tr>
<td>+06 +12</td>
<td>375</td>
<td>791</td>
</tr>
<tr>
<td>+12 -12</td>
<td>1,241</td>
<td>3,042</td>
</tr>
<tr>
<td>+12 -06</td>
<td>1,159</td>
<td>2,574</td>
</tr>
<tr>
<td>+12 00</td>
<td>1,086</td>
<td>2,220</td>
</tr>
<tr>
<td>+12 +06</td>
<td>1,022</td>
<td>1,945</td>
</tr>
<tr>
<td>+12 +12</td>
<td>964</td>
<td>1,726</td>
</tr>
</tbody>
</table>

<sup>a</sup> Risk aversion coefficient is equal to .0001 for the investor. Also note that the futures and options positions may not equal amounts divisible by a full contract.

<sup>b</sup> Net inventory equals 31,500 bushels.
of the speculative component of the futures demand equation. Increases in risk aversion also result in smaller option positions (in absolute value). Again, this is due to changes in the speculative component of the option demand equation.

One of the more interesting results occurs when the individual has no deviation in his/her estimate of the spot price mean from the mar-

![Figure 4 Position diagram for agent who has a -12 percent bias for corn price variance and a bias of zero percent for the corn price mean. (Note: The option position is less than one full contract).]
ket's, and has a deviation in his/her estimate of the spot price variance. This situation illustrates how investors, when they believe they have a better estimate of the true price variance than implied by the option price, will use options as a speculative medium. This situation will occur if an individual has information indicating large movements in the price, but there is a lack of knowledge as to which direction prices will move.

The optimal portfolio can best be demonstrated by use of a standard position diagram (see Figure 4). First, the inventory position is subtracted from the futures position to find the net exposure in this market. Then, by noticing that the exposure in the absolute size of the put option position is always approximately double the exposure in the futures plus inventory position, the position diagrams for -12 percent bias (deviation) in the variance can be drawn.

When price changes are small, this position will always return a profit; when price changes are large, a loss will be incurred. An individual, believing that the market was overestimating price variance, would find this position to be advantageous. To see this effect, consider Figure 5 in which a normal price distribution is overlaid upon the final position diagram from Figure 4.

The area under the price distribution to the left of \( \gamma \) and to the right of \( \tau \) represents the probability that this portfolio will return a loss. Note that the expected value of this position is positive, if this area multiplied by the corresponding losses is less in absolute value than the area under the curve between \( \gamma \) and \( \tau \) multiplied by the
corresponding profits. The expected profitability of this position will only depend on the difference between the price volatility implied by the market and the trader's perception of the true volatility.

One could replicate the discussion outlined above for alternative values of bias in the variance. Figure 6 provides a summary of these results. In reality, the trader's position will be limited by the

Figure 5  Overlay of normal price distribution with position diagram for an individual who believes the option market price overestimates the true price variance.
standard size of options and futures contracts; nevertheless, Figure 6 demonstrates how options can be used to profit from information that would be useless if only futures markets existed. This suggests that option markets can provide an important role in compensating those who collect and make available information about price variance, just as futures markets compensate those who collect information about the mean level.

The results of the second simulation experiment are summarized in Figures 7, 8, and 9. Figure 7 illustrates the comparative static effects, upon the portfolio positions, of changes in the expected spot price mean. For the futures plus inventory position, a monotonically increasing relationship exists between the position and the expected spot price. Thus, as the expected spot price increases, the investor will be induced to hold less short or more long positions in the sum of futures plus inventory. Note that as the expected spot price falls below a particular point (approximately 87 percent on the graph) the slope of the relationship becomes steeper. It is at this point where the investor becomes almost completely certain of a decrease in the futures price (he/she is in the left tail of the market price distribution) and will place an increasing amount of short futures positions in response to this belief.

The relationship between the spot price mean and the desired option position is nonmonotonic. For deviations away from the futures price in either direction, the investor will take larger short positions
in the put option. The inflection points in the relationship occur at approximately 87 and 113 percent of the futures price.

As with the futures position, these points represent areas where the investor's expected spot price mean is in the tails of the actual price distribution. On the lower side of the futures price, written put options are used to hedge the speculative short positions in inventory plus futures. On the upper side of the futures price, written put options are used to hedge the speculative long positions in inventory plus futures.
Figure 7 Comparative statics of changes in the expected spot price upon the inventory plus futures position and upon the option position.
options are used to add to the speculative long position held in the inventory and futures markets.

Figure 8 illustrates the comparative static results for changes in the future spot price variance. For futures plus inventory, increases in the future spot price variance will induce the investor to hold less short and more long positions up to a particular point (standard deviation of approximately 15 percent of the futures price). Then the futures position converges to the opposite of the inventory position.

The relationship between the option position and the spot price variance follows an almost opposite position to futures plus inventory. However, the option position is larger than the futures plus inventory position. The rationale behind this is that the futures plus inventory position is used as a partial hedge against the speculative position in the option and is used to complete the "straddle" position discussed earlier.

The point at which the inventory plus futures and the option positions start to slowly converge to zero (approximately 13 percent of the futures price) is the point at which the risk effect overtakes the return effect with regards to increasing variance in the speculative component of the option demand equation.

The risk effect is directly due to the interaction of variance and risk aversion in the denominator of the speculative component of the option demand equation \((A \cdot V_{11})\). The return effect is through the option valuation formula \((\alpha \beta_2)\) in the numerator of the speculative component of
the option demand equation. As variance increases, the difference between the market option price and the investor's option value \( (F_R - aB_2) \) becomes more negative since the option value increases.

Figure 9 illustrates the relationship between the risk aversion coefficient and the portfolio positions. Both relationships (futures plus inventory and put option) show that the absolute size of the net positions converges to zero as risk aversion increases. This corresponds with the results of the analytical model.

Conclusions

The objective of this chapter was to examine how an individual investor will incorporate information about the future spot price mean and variance along with risk preference to determine the optimal mean-variance portfolio positions. To facilitate this examination, analytic results were derived from the theoretical form of the model. Where analytic results were not possible, a numerical simulation of the model was used.

The solution to the analytic form of the portfolio model indicates that investors will adjust their portfolio positions depending upon the relationship between the ratio of expected marginal speculative returns and the hedging components of the futures and option demand equations. Thus, the optimal portfolio positions will be adjusted such that an equilibrium is maintained between the returns to speculation and the reduction in risk due to hedging. This result is a natural extension of the mean-variance objective function where a desired balance is main-
Figure 8 Comparative statics of changes in the expected spot price variance upon the inventory plus futures position and upon the option position.
tained between maximizing the mean return and minimizing the variance of returns.

A significant result from the analytic form of the model is that an investor will maintain a "perfect" hedge position if he/she agrees with the market estimates (futures price and implied volatility) of the future spot price moments. The "perfect" hedge position is where the investor matches one-to-one opposite positions in inventory and futures with a zero net position in the option. Thus, the investor will not participate in the option market unless he/she has information leading to a disagreement with at least one of the market estimates of the price moments.

According to the simulation results, futures will be used as the primary speculative medium for changes in the spot price mean. An increase in the expected spot price mean will induce the investor to hold more long or less short positions in the futures contract. An increase in the absolute difference between the futures price and the expected spot price mean will induce the investor to hold a larger absolute position in the put option.

When the investor is speculating on the mean using a net short position in the sum of futures and inventory, the option is used to partially hedge the speculative position. When the investor is speculating on the mean using a net long position in the sum of futures and inventory, the option is used to add to the speculative position.

The benefits of this type of position are quite obvious when the conditional nature of the option position is taken into account. The
Figure 9 Comparative statics of changes in risk aversion upon the inventory plus futures position and upon the option position.
option hedge allows the investor to capture speculative price gains without the offsetting position (option is allowed to expire) and covers speculative losses with an offsetting position (option is exercised).

For changes in the future spot price variance, the investor will use options as the primary speculative medium. An increase in price variance will induce the investor to hold less short or more long positions in the put option up to a particular point where the return effect of increasing variance upon the option is overwhelmed by the risk effect. After this point is reached, the absolute size of the option position converges to zero. For changes in variance, the futures position is used as a partial hedge of the speculative option position and to complete the "straddle" position.

The importance of options as a variance speculation instrument is particularly apparent for the case where the investor agrees with the futures price prediction of the spot price mean but disagrees with the market implied volatility prediction of the future spot price variance. In this case, the investor will combine the portfolio positions into a "straddle" position which can be used to profit from changes in actual price volatility.
CHAPTER FIVE
NOISY RATIONAL EXPECTATIONS AND THE
INFORMATIONAL ROLE OF OPTION MARKETS

Speculation has long been recognized as a fundamental component of zero-sum markets. As Malinvaud (1985, p. 359) states, "Any relevant study of speculation must probably combine the exchange of risk and the exchange of information." Despite this, most studies on futures markets ignore a fundamental feature of information exchange by assuming that traders with rational expectations share a common information set. An important exception, offered by Grossman, shows that information differentials may be invoked to explain the very existence of futures markets. Thus, futures markets can be viewed as places where information is exchanged and where investors earn a return for collecting useful information.

In this chapter, Grossman's model is extended to include an option market. The treatment of options as "derivate assets" (Rubinstein, 1987) and the derivation of option pricing formulae under assumptions such as perfect markets with complete information availability have neglected the possible information content of option prices. By casting option trading within Grossman's framework, however, option prices can be shown to have an information role comparable to that of futures prices.
Previous research

The no-arbitrage pricing approach to option market equilibrium is based upon Arrow-Debreu contingent claims markets. The Arrow-Debreu framework requires a complete set of markets that cover every possible contingency. This requirement is necessary for the market equilibrium to be Pareto efficient.

The completeness assumption has been modified by assuming that there are markets covering every contingency except those with a zero probability measure. This concept is known as dynamically complete markets, developed by Breeden (1984) and Duffie and Shafer (1985).

The rational expectations hypothesis was defined by Muth as a situation where "expectations of firms (or, more generally, the subjective probability of outcomes) tend to be distributed, for the same information set, about the prediction of the theory (or the "objective" probability distribution of outcomes)." There are three assertions that can be made from this hypothesis: (a) the economic system does not waste information, (b) expectations formation is dependent upon the structure of the the system describing the economy, and (c) publicly announced predictions or policies will have no substantial effect upon the operation of the economic system.

One possible criticism of the rational expectations hypothesis is that it assumes that economic agents must be able to consciously solve equations. There are two rebuttals to this contention:
1. The behavior of the economic agents is already summarized in the structure of the economic system. Thus, the agent is not required to consciously "solve equations" but rather he/she subconsciously solves these equations which are already implicit in his/her behavior.

2. Muth's definition does not imply that each individual agent solves the same set of equations, but rather a group of agents, with the same information set, implicitly solves these equations in the aggregate.

When the assumptions of rational expectations and dynamically complete markets are combined, it is possible for an uninformed trader to observe market prices and gather all of the informed traders' information. Thus, markets will become Pareto efficient by converging to a rational expectations equilibrium that is informationally efficient. Galai (1977) tested the efficiency of options on the Chicago Board Options Exchange by comparing the actual market prices with those generated by the Black-Scholes option pricing model. If consistent profits could be made using the Black-Scholes formula, then the market was considered informationally inefficient.

A major problem with rational expectations equilibrium is there are no incentives for individuals to collect information. This is because all of their information will be signaled to others through the price system. Thus, collectors of information are providing an external benefit which cannot be internalized. The presence of viable zero-sum
markets (futures and options) provides a direct contrast to rational expectations equilibrium. This is because individuals (speculators) in these markets do make returns to information collection. Grossman introduced the concept of "noisy" rational expectations equilibrium to explain this phenomenon.

Noisy rational expectations equilibrium is merely the assumption of rational expectations without dynamically complete markets. Thus, there are certain contingent claims ("noise") with a nonzero probability measure and no direct market. Grossman and Stiglitz (1976, 1980) proved that a Pareto efficient, noisy rational expectations equilibrium occurs when the proportion of informed and uninformed traders is such that the marginal returns and marginal costs of information collection are equivalent.

Fieger (1978) offered a counter-argument to noisy rational expectations equilibrium through the introduction of capital constraints on traders. The consistent losses of the uninformed traders will force them out of the market and cause the market to converge to a rational expectations equilibrium in which all of the market participants are informed.

The "noisy" rational expectations framework has found numerous applications in previous studies. Bray (1981) used the "noisy" rational expectations framework to study the futures price as a sufficient statistic to forecast the spot price mean. The futures price is only sufficient in the case where there is information available on only one side of the spot market (supply or demand).
Brannen and Ulveling (1984) used the "noisy" rational expectations framework to examine the predictive power of the price system. They also examined how the predictive power is affected by the introduction of a futures market. Their results show a positive correlation between the lack of predictive power in a price system and the subsequent introduction of a futures market.

Stiglitz (1985) studied some of the paradigms of the "new classic" economics in the context of "noisy" rational expectations equilibrium. The assumption of "noisy" rational expectations radically alters many of the paradigms, especially those related to the "law of one price".

There is a dearth of research in examining options from an information perspective. Patell and Wolfson (1979) studied the ability of the Black-Scholes option implied volatility to anticipate future financial reporting events. Implied volatility was found to increase just before the announcement and declined quickly thereafter.

Manaster and Rendlemen (1982) examined the information content of the Black-Scholes implied stock price as compared to the actual stock price. The implied stock price was found to contain information not already implicit in the actual stock price.

Gardner (1977) discussed the usefulness of commodity option implied volatility as a useful tool for producers and policymakers. Fackler (1987) discussed a procedure using the option price in deriving the implied probability distribution for the future prices.
Grossman's noisy rational expectations model

Because the results of this chapter are an extension of Grossman's study, a brief review of his paper is in order. Grossman began with a two-period model of a commodity that is consumed in both periods, is produced only in period one, is storable, but cannot be carried over after period two. Using his notation, consider the system:

\begin{align*}
(5.1) \quad & P_2 = D_2(I, W_2), \\
(5.2) \quad & D_2(I, W_2) = h_1 I + h_2 W_2, \\
(5.3) \quad & D_1(P_1) = Q - I, \\
(5.4) \quad & D_1(P_1) = h_3 + h_4 P_1.
\end{align*}

\(D_2(I, W_2)\) is the period two inverse demand curve, which is linear with \(h_1 < 0\) and \(h_2 > 0\). \(D_1(P_1)\) is the period one demand curve which is linear with \(h_3 > 0\) and \(h_4 < 0\). \(Q\) denotes the quantity harvested, \(I\) is the amount of the harvest that is carried over into period two, and \(W_2\) is the realization of a random demand variable \((W_2^*)\). Equations (5.1) through (5.4) are solved for the two equilibrium prices. Note that the period two price \((P_2)\) is random, the distribution of which depends on \(W_2^*\).

In this setting, Grossman considered the problem of deciding on the level of inventory to carry from period one to period two for two groups of traders. He differentiated between these groups on the basis
of whether they are informed or uninformed. Informed traders know a conditional distribution of $W_2^*$. Uninformed traders know only the unconditional distribution of $W_2^*$. Both types of traders are assumed to be risk neutral and have the same storage cost function ($h^2 l^2$).

Grossman showed that, in equilibrium, all relevant information about the period two demand curve will be made available to the uninformed traders, as a group, via the period one spot price. This raised the question of how informed traders are compensated for their efforts at information collection.

To have incentives for the collection of information, it was necessary to allow for another source of uncertainty. This was accomplished by replacing the known harvest ($Q$) with the realization ($W_1$) of a random variable ($W_1^*$). The new set of demand equations is therefore

(5.5) $P_2 = D_2(I, W_2)$,
(5.6) $D_1(P_1) = W_1 - I$.

For crops such as wheat, corn, and soybeans, it is conceivable to have a random harvest variable after the harvest has occurred. This is a result of the time lag between the time that harvest occurs and the time at which the final estimates of the harvest are released (for example, the USDA stocks reports). Thus, even though actual $Q$ is fixed after harvest, to the individual investor it is still random since actual $Q$ has not been made known to him/her through public information.
Grossman showed that informed traders, as a group, will make greater profits than uninformed traders. The intuition behind this result is relatively straightforward. With only one signal, the spot price, uninformed traders are unable to gather information about two sources of uncertainty, W_1^* and W_2^*.

Then, Grossman introduced a futures market into the model. Let X represent a promise made in period one to purchase X units of the commodity in period two at a price P_f, and -X be a promise to deliver X units at the same price. The profit that the traders make in the futures and the storage markets can be defined as

\[(5.7) \quad \pi_i^* = (P_2^* - P_1) \cdot I_i - h_i \cdot I_i^2 + (P_2^* - P_f) \cdot X_i,\]

where i indexes the group of traders (i-a refers to informed traders and i-b refers to uninformed traders). This additional price (P_f), coupled with P_1, again signals all of the available information on the period two demand to the uninformed group of traders. Profit-maximizing informed traders, who are risk neutral, will bid P_f up or down until it conforms with their expectations about P_2. Uninformed traders use P_f to make their storage decisions. Thus, they will store exactly as much as the informed traders. The expected profits from futures trading are zero for all participants because P_f will, on average, be equal to P_2. This result again leads to the unsatisfactory situation where returns to information collection are zero.
One can, conceivably, introduce another source of noise, providing a new incentive to collect information. In turn, this will create the conditions for another market to exist and the sequence will continue indefinitely. At some point, however, the speculative capital will be exceedingly diluted by the proliferation of markets. This will cause the process to break down. Grossman, therefore, stopped the process at this point. However, he did develop an important further step by allowing for risk aversion. This was done to solve an indeterminancy in the equation specifying the size of futures contracts. This indeterminancy occurs because the risk-neutral informed traders demand an infinite quantity of futures contracts whenever the futures price does not conform to their expectations.

This model not only resolves the question of the size of futures positions in equilibrium but also introduces two new sources of noise. These are the period two spot price variance and the investors' risk-aversion coefficients. This noise creates the conditions where informed investors can earn returns to information collection.

In summary, Grossman's model provided a compelling justification for the existence of futures markets on the basis of their information role. Indeed, his analysis has emphasized the information role of all markets, inasmuch as the spot market can disseminate all the relevant information when there is only one source of uncertainty.
The Grossman Model with futures and options markets

Grossman's noisy rational expectations model was extended to include a put option using the model from Chapter Two. Let (3.1) and (3.2) be rewritten as

\[(5.8) \quad \pi_i^* = (P_2^* - P_1) \cdot I_i - \beta g \cdot I_i^2 + (P_2^* - P_f) \cdot X_i + (P_R - \text{Max}(0, K P_2^*)) \cdot R_i, \]

\[(5.9) \quad E[U_i(\pi_i^*)] = E[\pi_i^*|Z_i] - \mu A_i \cdot \text{Var}[\pi_i^*|Z_i], \]

where \(Z_i\) is the information set of the traders of type \(i\). The uninformed traders' information set is restricted to observable market prices; that is, \(Z_b = (P_1, P_f, P_R)\). The informed traders know, in addition, the realizations of a set of random variables (\(\epsilon^*\)). These variables are jointly distributed with the sources of noise (\(W_1^*\) and \(W_2^*\)); that is, \(Z_a = (Z_b, \theta)\).

The definition of information in this study applies only to that available to the traders. No provision is made for traders who interpret the information set differently, whether because of different beliefs or different skill levels in interpretation. However, this can easily be handled in this framework through the introduction of filters into the trader's information set \((Z_i)\). An trader with a relatively precise filter would be superior in interpretative ability. A filter representing beliefs would impose a predetermined bias on the informa-
tation set. The Bayesian prior is one of the most commonly used infor-

Using equations (5.8) and (5.9), the demand equations [(3.6) through (3.8)] can be rewritten as

\[(5.10) \quad I_1^d = \frac{F_n - P_1}{g} \]

\[(5.11) \quad X_1^d = \frac{E[P_2^*|Z_1] - F_n}{A_1V_{111}^*} \cdot \frac{V_{121}^*}{V_{111}^*} R_1^d - I_1^d, \]

\[(5.12) \quad R_1^d = \frac{P_R - \alpha_1^* B_{21}^*}{A_1V_{221}^*} \cdot \frac{V_{121}^*}{V_{221}^*} (I_1^d + X_1^d), \]

where

\(V_{111}^* = \) trader \(i\)'s estimate of the future spot price volatility conditional on \(Z_1\),

\(V_{121}^*, V_{221}^* = \) trader \(i\)'s estimates of the option components of the variance-covariance matrix conditional on \(Z_1\),

\(\alpha_1^* = \) trader \(i\)'s estimate of the probability of option exercise conditional on \(Z_1\),

\(B_{21}^* = \) trader \(i\)'s estimate of the exercise value of the option conditional on \(Z_1\).

The * superscript indicates a conditional random variable and \(\alpha_1^*, B_{21}^*, V_{121}^*, V_{221}^*\) are functions of \(E[P_2^*|Z_1]\) and \(V_{111}^*\).
An equilibrium can now be defined as a set of random variables
\((P_1^*, P_2^*, P_F^*, P_R^*)\) such that

\[
\begin{align*}
(5.13) \quad X^d_a + X^d_b &= 0, \\
(5.14) \quad R^d_a + R^d_b &= 0, \\
(5.15) \quad D_1(P_1^*) &= W_1 - (I_a^d + I_b^d), \\
(5.16) \quad P_2^* &= D_2(I_a^d + I_b^d, \bar{W}_2).
\end{align*}
\]

Equations (5.13) and (5.14) are the futures and option equilibrium equations. They imply that both futures and options are zero-sum markets. Thus, if the traders of informed group \(a\) are long futures, then the traders of uninformed group \(b\) must be short an equivalent amount of futures. This is required for the futures market to clear. When demand is set equal to supply in the spot market for both periods, equations (5.15) and (5.16) are equivalent to equations (5.1) through (5.4).

Thus, investors will simultaneously adjust their portfolio positions to the reception of information related to the mean and variance of the future spot price. This implies a simultaneous adjustment in both the futures price and option premium with no causality running from one market to the other.

To appreciate the information role of option markets, it is useful to compare (5.11) and (5.12) to the Grossman model without option markets. Omitting the option market does not change the inventory demand equation. On the other hand, without option markets, the futures market position would be
(5.17) \[ X_1^d = \frac{E[P_2^*|Z_1] - P_F}{A_1 V_{11}^*} \cdot I_1^d. \]

There will be no equivalent of the option market demand equation (5.12).

To stress the information side of the market, assume that the coefficient of risk aversion is the same for all traders. However, without an option market uninformed traders, as a group, still cannot gather the two relevant pieces of information. These are the informed trader's estimates of the future spot price moments \( E[P_2^*|Z_a] \) and \( V_{11a}^* \). The uninformed traders cannot gather this information from observed market conditions because they have only equation (5.17) to use in their decision making.

However, uninformed traders, as a group, can learn \( E[P_2^*|Z_a] \) and \( V_{11a}^* \) by using their implicit knowledge of equations (5.11) and (5.12). This requires an existing option market. The uninformed traders can decipher an approximate solution to these equations because they observe \( P_R \) and \( P_F \) along with the trading volume for both markets (which is equal to \( X_a^d \) for futures and \( R_a^d \) for options). Also, \( a_a^*, V_{12a}^*, \) and \( V_{22a}^* \) will be functions of the unconditional probability moments if we assume a normal distribution. These results give two equations in two unknowns \( (E[P_2^*|Z_a] \) and \( V_{11a}^* \)). Therefore, a solution for these unknowns will become implicit in the expectations of the uninformed traders. Under these conditions, the uninformed traders will refuse to take an opposite
position to the informed traders. Thus, no speculative trade will occur.

Conclusions

Option markets have an information role. It is shown that the option market premium contains information on the second moment of the future spot price distribution. This is done by using a simple two-period inventory model with the assumptions of constant absolute risk aversion, identical carrying-cost functions, and rational expectations among all traders.

The results have practical implications for the usage of option pricing models. First, these models are useful for individual investors who possess a subjective estimate of the variance. However, individual traders should not make trades based upon the historic volatility measure as suggested by Rubenstein and Cox (p. 262) unless their subjective forecast of the future spot price variance corresponds with the historic volatility.

Second, option pricing formulae should not be used to determine whether option markets are efficient (as in Galai) unless one can provide a more accurate estimate of price variance than the market. Historic volatility only measures the unconditional variance of prices, whereas informed traders possess a conditional estimate of variance that may be different from the unconditional one.

The option implied volatility can have several uses to any individual with an interest in the particular commodity market. The
Implied volatility can be used by producers in their enterprise selection decisions. For instance, a producer, with a financial position that makes little allowance for risk, can choose the commodity with the lowest implied volatility relative to other commodities for his/her productive enterprise.

The implied volatilities can be used as variance inputs into a mean-variance decision model. One problem with this use, however, is there hasn't been a model designed to derive the implied covariances between different futures prices. Thus, the variance-covariance matrix will be diagonal for an investor using option implied volatilities with mean-variance analysis.

The implied volatilities can be used by producers and agribusiness managers to assess their potential need for future risk management strategies. An example of this would be a selective hedging strategy that is based upon the option implied volatility. The percentage of the cash position hedged could be an increasing function of the option implied volatility.
CHAPTER SIX
VALUE-ADDED OF OPTION MARKETS
IN A PORTFOLIO CONTEXT

Options can add value to a portfolio through an expanded set of possible return structures. These structures can be used to capitalize upon information and to alleviate risk. The goal of this chapter is to examine some questions relating to the value-added of options. This will be accomplished by using a simulation model that measures value-added by the addition of an option into an investor’s portfolio.

In this chapter, two questions will be examined which relate to the value-added of an option. The first question will be concerned with statistical and economic significance of the value-added by an option. The second question will be concerned with the relative importance of information and risk factors that may influence the value-added.

Examining the statistical and economic significance of value-added is important in determining the viability of an option contract. An investor will only hold positions in an option when the marginal value is greater than the marginal cost of adding the option to the portfolio.

It can be shown, by application of the LeChâtelier Principle, that the introduction of an option will never lower the value of an investor’s portfolio. This is conditional upon the investor being unconstrained as to what positions can be taken in the option. The addition of an option to an investor’s portfolio expands the space of possible
return patterns. Thus, it removes previous constraints upon the investor (the LeChâtelier Principle application).

Also, it can be shown, by application of Proposition 3 from Chapter Four, that the option will have zero value-added for an investor who is either uninformed about the moments of the future spot price or agrees with the market's estimates of the moments. Thus, the "classic" hedger enjoys little or no value-added with options.

However, the results of Proposition 3 do not give the implication that all hedgers will have zero value-added, as has been incorrectly maintained in some previous studies. Not all hedgers will follow the mode of the "classic" hedger. In fact, many hedgers will have information about the future spot price moments, leading them to disagree, in many instances, with the market estimates. Since most hedgers are closely involved with the production and the marketing of the commodity, it would be unrealistic to assume that hedgers operate in an information vacuum. It is not so clear that exploiting the information of a hedger is significant in determining the value-added of an option.

Also, the mean-variance utility function assumes that the investor has an equal aversion to upside as well as downside price risk (around a particular price level such as breakeven). However, the average agricultural producer's utility function may be better represented using a mean-semivariance utility function, where upside price variance is viewed favorably and downside price variance is viewed negatively. An argument can then be constructed for options having value to an individual whose spot price moment forecasts coincide with the market's
estimates. This argument is based upon the fact that an investor can achieve downside price protection while not completely giving up any upside price gains through the use of an option hedge.

As discussed in Chapter Two, Hanson concluded that options have little value-added to the hedger. Hanson bases this conclusion upon his analytical model, in which the cash position does not directly enter the option demand equation, and upon simulation results using his model.

A money metric, as discussed in Roe and Antonovitz (1985), was used to arrive at the value-added for a put option. The form of the money metric is

\[ \text{EU}(Y(x^*,0) + V) - \text{EU}(Y(x^{**},y^{**})) \]

where \( x^* \) is the optimal futures position when the portfolio contains only inventory and futures; \( x^{**} \) and \( y^{**} \) are the optimal futures and options positions from a portfolio containing inventory, futures, and options; \( \text{EU}(\cdot) \) is the expected utility function of the investor; and \( V \) is the value-added of the option.

An examination of the relative importance of information and risk factors influencing the value-added of an option is important for two reasons. First, an analysis of these factors will facilitate an examination regarding the correlation of information availability and option value. Second, an analysis of these factors may shed some light upon some of the symptoms that may cause a nonviable option market due to low trading volume.
Description of the model

This chapter contains two simulation experiments. One simulation experiment will examine the statistical and economic significance of value-added when an option is added to the portfolio. A second simulation experiment will use ordinary least squares (OLS) upon the portfolio results to examine the relative importance of some information and risk factors that may influence the value-added of the option.

The construction of the simulation model used in this chapter is based upon a discussion in Robison and Barry (pp. 39-40). When a mean-variance utility function is assumed to represent the decision maker's preferences, the resulting expected utility is equal to the certainty equivalent income from the investor's portfolio. The certainty equivalent income can be expressed by the following relationship:

\[(6.1) \quad y_{CE} = E(y) - \frac{1}{2} R(y) \cdot \sigma_y^2,\]

where

- \(y_{CE}\) = certainty equivalent income,
- \(y\) = actual income,
- \(R(\cdot)\) = Arrow-Pratt absolute risk aversion function,
- \(\sigma_y^2\) = variance of income.
Thus, the expected utility from a mean-variance portfolio is equal to the expected return from the portfolio minus a risk premium that is based upon the form of the investor's utility function. The certainty equivalent income is defined as the amount of certain income that is equivalent, in expected utility, to the risky portfolio. Thus, $y_{CE}$ can be regarded as the monetary value of the portfolio to an individual investor.

Given two portfolios (a and b), the difference between the certainty equivalent income levels ($y_{cea} - y_{ceb}$) should give the incremental value-added of holding portfolio a over portfolio b when preferences are represented by a mean-variance utility function. This approach is used by Marshall (pp. 298-308) to analyze different hedging strategies for a grain elevator.

The simulation model used in this chapter will contain two portfolios. The first portfolio (f) contains inventory and futures. The demand equations can be represented as

\begin{align*}
(6.2) \quad I_f^d &= \frac{P_f - P_l}{\xi} \\
(6.3) \quad X_f^d &= \frac{E[P_2] - P_f}{AV_{11}} - I_f^d.
\end{align*}

The certainty equivalent of this portfolio is
(6.4) \[ y_{CEF} = (E[P_2] - P_1)I_{f}^d - \frac{b_g}{2}(I_{f}^d)^2 + (E[P_2] - P_f)X_{f}^d - \frac{a_A}{2}V_{11}(I_{f}^d + X_{f}^d)^2. \]

The second portfolio (\( o \)) contains inventory, futures, and put options. The portfolio demand equations can be represented as

(6.5) \[ I_{o}^d = \frac{P_f - P_1}{g}, \]

(6.6) \[ X_{o}^d = \frac{E[P_2] - P_f}{AV_{11}} - \frac{V_{12}}{V_{11}}R_{o}^d - I_{o}^d, \]

(6.7) \[ R_{o}^d = \frac{P_R - \alpha B_2}{AV_{22}} - \frac{V_{12}}{V_{22}}(I + X)_{o}^d. \]

The certainty equivalent of this portfolio is

(6.8) \[ y_{CEO} = (E[P_2] - P_1)I_{o}^d - \frac{b_g}{2}(I_{o}^d)^2 + (E[P_2] - P_f)X_{o}^d + (P_R - \alpha B_2)R_{o}^d - \frac{a_A}{2}\text{Var}[P_2|P_2 \leq K]\cdot(I_{o}^d + X_{o}^d + R_{o}^d)^2 + (1 - \alpha)\cdot\text{Var}[P_2|P_2 > K]\cdot(I_{o}^d + X_{o}^d)^2 + \alpha(1 - \alpha)\cdot(B_1\cdot(I_{o}^d + X_{o}^d) + B_2R_{o}^d)^2. \]

The notation used in this chapter is the same as that used in Chapters Three and Four with the exception of the subscripts on the portfolio demand equations. An \( f \) subscript is used for the demand equations in the portfolio containing inventory and futures. The \( o \) sub-
script is used for the demand equations in the portfolio containing inventory, futures, and a put option.

Methods

The two questions examined in this chapter use a numerical simulation model. The simulation model is identical to one used in Chapter Four, with the addition of the certainty equivalent equations (6.4) and (6.8). Also, a money metric representing the value of the option (which is equal to $y_{CEo} - y_{CEF}$) was added.

For each commodity, two experiments were done using the simulation model. The first experiment examined the statistical and economic significance of the value-added for an option. The percentage increase in value-added was calculated for each observation and the mean of the percentages was used to represent the increase in value to an average investor. A t-statistic was used to test the null hypothesis of a mean less than or equal to zero. If the null hypothesis was rejected, it was assumed that the observed mean is statistically significant.

The second experiment examined the relative importance of various information and risk factors in determining the value-added of the option. The absolute difference between the expected spot price mean and the futures price was used as an explanatory variable for measuring the information available to the investor about the spot price mean. The absolute difference between the future spot price variance estimate and the market implied volatility was used as an explanatory variable in measuring the information available to the investor about the spot price.
An OLS regression model, which allowed for interaction between the explanatory variables, was used to rank the various factors in order of importance. The regression model was estimated as follows:

\[
\text{Value-added} = a_0 + a_1 A + a_2 \text{Dev}_1 + a_3 \text{Dev}_2 + a_4 (\text{Dev}_1 \times \text{Dev}_2) + a_5 (A \times \text{Dev}_1) + a_6 (A \times \text{Dev}_2) + u,
\]

where

- \text{Dev}_1 = \text{absolute value of the difference between the expected spot price mean and the futures price},
- \text{Dev}_2 = \text{absolute value of the difference between the forecasted spot price variance and the market implied volatility},
- A = \text{risk aversion coefficient},
- u = \text{error term}.

The t-statistics on the coefficient values were used to determine the statistical significance of the factors influencing the value-added of the option. The data set used for the simulations was generated using all of the observations from Table 2. For each observation in Table 2, seventy-five data points were numerically produced. Values of the expected spot price mean (E[P_2]) were numerically generated using a normal probability distribution and a Monte Carlo sampling procedure. The sample mean was set equal to the futures price and the sample standard deviation was set equal to three percent of the futures price.
The variance forecast values were also obtained using Monte Carlo and a normal probability distribution. The sample mean was set equal to the market implied volatility and the sample standard deviation was set equal to three percent of the implied volatility.

The risk aversion coefficients were generated using Monte Carlo and a triangular distribution. The distribution used .000001 as the minimum value, .01 as the maximum value, and .0001 as the most likely value. The @Risk computer simulation package was used for the Monte Carlo procedure.

The normal distribution was justified for the forecasted means by assuming a large number of traders in the market and application of the central limit theorem. The mean was assumed equal to the futures price because effective arbitrage will keep the futures price approximately at the average of all traders' forecasts. The standard deviation was assumed equal to three percent of the futures price as an ad hoc rule. One implication of this rule is that the dispersion of market forecasts around the futures price will become larger as the futures price becomes larger. Thus, the forecasts will behave approximately as a lognormal rather than a normal distribution. The same lines of reasoning apply to the choice of sampling method for the variance forecasts.

The triangular distribution was used for generating risk aversion coefficients because it allows for the specification of a minimum and maximum value. A minimum value is required because a negative risk aversion coefficient will cause the second-order conditions of the simulation model to break down. Also, a triangular distribution is easy to
understand in that its moments can be articulated through the following questions:

1. What is the lowest value that this variable can possibly assume?
2. What is the highest value that this variable can possibly assume?
3. What is the most likely value for this variable to assume?

By answering these three questions, a triangular distribution can be constructed.

Results

The percentage increase in value-added is illustrated by commodity with the corresponding t-statistics in the following table:

<table>
<thead>
<tr>
<th>Commodity</th>
<th>Mean Percent Increase</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corn</td>
<td>17.45</td>
<td>16.32***</td>
</tr>
<tr>
<td>Soybeans</td>
<td>10.00</td>
<td>12.43***</td>
</tr>
<tr>
<td>Live Cattle</td>
<td>64.83</td>
<td>1.96**</td>
</tr>
<tr>
<td>Live Hogs</td>
<td>18.54</td>
<td>18.54***</td>
</tr>
</tbody>
</table>

For the t-statistics, two stars indicates significance at the 95 percent confidence level, and three stars indicates significance at the 99 percent confidence level.
Table 5 OLS Estimates of Option Value Equation (standard errors in parentheses)

<table>
<thead>
<tr>
<th>Commodity</th>
<th>Variable</th>
<th>Corn</th>
<th>Soybeans</th>
<th>Live Cattle</th>
<th>Live Hogs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td></td>
<td>56.58</td>
<td>-43.26*</td>
<td>-354.23*</td>
<td>-326.46</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(101.74)</td>
<td>(23.72)</td>
<td>(175.42)</td>
<td>(214.91)</td>
</tr>
<tr>
<td>A</td>
<td></td>
<td>-32,214.72</td>
<td>4,351.74</td>
<td>13,191.22</td>
<td>-53,329.89</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(21,207.49)</td>
<td>(4,698.47)</td>
<td>(35,819.06)</td>
<td>(46,757.05)</td>
</tr>
<tr>
<td>Dev1</td>
<td></td>
<td>3,780.58***</td>
<td>548.24***</td>
<td>616.14***</td>
<td>908.87***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(808.55)</td>
<td>(80.49)</td>
<td>(65.81)</td>
<td>(128.56)</td>
</tr>
<tr>
<td>Dev2</td>
<td></td>
<td>-46.66</td>
<td>.18</td>
<td>-3.24</td>
<td>1.77</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(.35.84)</td>
<td>(.67)</td>
<td>(3.17)</td>
<td>(3.74)</td>
</tr>
<tr>
<td>Dev1*Dev2</td>
<td></td>
<td>-488.71</td>
<td>-4.55*</td>
<td>-2.12*</td>
<td>-9.89***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(278.57)</td>
<td>(2.17)</td>
<td>(1.04)</td>
<td>(2.23)</td>
</tr>
<tr>
<td>A*Dev1</td>
<td></td>
<td>-423,727.80***</td>
<td>-64,494.41***</td>
<td>-80,981.22***</td>
<td>-61,975.99***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(50,973.09)</td>
<td>(6,705.82)</td>
<td>(9,868.40)</td>
<td>(12,548.78)</td>
</tr>
<tr>
<td>A*Dev2</td>
<td></td>
<td>16,445.48**</td>
<td>57.09</td>
<td>1,377.92**</td>
<td>1,671.18**</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(7,330.32)</td>
<td>(130.04)</td>
<td>(567.01)</td>
<td>(810.53)</td>
</tr>
<tr>
<td>R^2</td>
<td></td>
<td>.52</td>
<td>.56</td>
<td>.46</td>
<td>.29</td>
</tr>
<tr>
<td>F</td>
<td></td>
<td>39.13***</td>
<td>45.45***</td>
<td>30.92***</td>
<td>15.01***</td>
</tr>
<tr>
<td>DW</td>
<td></td>
<td>1.95</td>
<td>1.95</td>
<td>2.18</td>
<td>1.98</td>
</tr>
</tbody>
</table>

*Significant at 90% confidence level.
**Significant at 95% confidence level.
***Significant at 99% confidence level.

The t-statistics indicate that the percentage increases in value-added are all significantly different from zero. The percentages can also be regarded, by some individuals, as economically significant since they range from plus 10 percent to plus 65 percent, and if an investor
has an initial value of $5,000 for his/her portfolio containing invento-
y and futures, the addition of a put option could add from $500 to
$3,250 in additional value (using the minimum and maximum percentages).

Table 5 displays the OLS estimates of equation (6.9) by commodity.
Information about the future spot price mean is highly significant in
increasing the value of the option in all of the equations. This
result, when considered with the comparative statics results of Chapter
Four, indicates that a major use of options is to hedge speculative
positions in futures plus inventory.

The interaction between risk aversion and information about the
future spot price mean is highly significant in decreasing the value-
added of the option in all of the equations. Also, interaction between
risk aversion and information about the future spot price variance is
significant to a lesser degree in three of the four equations. These
results, when combined with the lack of significance for risk aversion
in all equations, indicate that risk aversion is only significant when
considered in the context of its interaction with the moments of the
price distribution. Thus, the impact of increases in risk aversion upon
the value-added of the option will be greater when the spot price
distribution moments are more distant from the market estimates.

Information about the future spot price variance is only signifi-
cant when considered in the context of interaction with risk aversion
(three of the four equations) and with information on the future spot
price mean (three of the four equations). This indicates that the use
of options as a speculative medium for the variance is not of signifi-
cant value to the average investor. Only investors with extremely low risk aversion coefficients will find any value in placing the "straddle" position. This is probably because forecasting the future spot price variance is a more risky practice than forecasting the future spot price mean. There are more unpredictable market factors affecting the volatility than there are affecting the mean. Some of these factors include unpredictable events such as floods and other economic disasters. These events are more likely to have a larger impact upon the implied volatility of the option that upon the futures price because of the daily price limits on futures. These limits do not apply to the option premium.

Conclusions

This chapter examined the statistical and economic significance of value-added through the incorporation of options into an investor's portfolio. Also, the factors influencing this value were examined. A numerical simulation model that incorporated equations reflecting the values of two portfolios and their difference in value was used in this analysis. One of the portfolios contained inventory and futures while the other portfolio contained inventory, futures, and a put option.

Some observations can be made from the results of the simulation models. The addition of an option to an investor's portfolio adds statistically significant value in most cases, ranging from 10 to 65 percent.
Information about the future spot price mean appears to have a significant positive effect upon the value-added of an option. This result, when combined with lack of significance for information about the variance, indicates that the option is more valuable to the investor when used as a hedge on the speculative futures and inventory positions rather than as a medium for speculating on the future spot price variance.

Increases in risk aversion have a negative impact upon the value-added of the option only when taken in the context of interaction with information on the future spot price distribution moments. Risk aversion has a greater impact upon the value-added of the option when the underlying volatility of the expected spot price is great. When this volatility is low, risk aversion has a minor impact upon the value-added.
The introduction of a new market, such as options, can have an impact upon the existing markets. The individual investor impact of the introduction of a new market can be determined by analyzing the effect of the additional market upon the individual investor's utility function. The preceding chapter has addressed this particular question for the case of adding put options to a portfolio containing inventory and futures. The social impact of the introduction of a new market can be determined by analyzing the impact of the market in either a partial or general equilibrium setting. Chapter Five examined the social aspect of information provision for put option markets using a noisy rational expectations framework.

The objective of this chapter is to examine, in a partial equilibrium framework, some additional questions related to the social impact of put option markets. Whether these social impacts are positive or negative will depend upon the utility functions of the public and in particular, the interpretation of public utility by policymakers.

This chapter examines several questions related to the social impact of option markets. The impact of put option markets upon the moments of the equilibrium prices and upon the volumes for futures and inventory is one of the questions to be examined. Previous research has shown that the introduction of a futures market will lower the long-run
mean and variance of the spot market price. It is important to determine whether the introduction of an option market will produce similar stabilizing effects or if its introduction counters the stabilizing effect from futures prices. If option introduction has a destabilizing effect upon the spot and/or futures prices, then the usefulness of option markets is questionable from a social point of view (in spite of its information role). However, if the results indicate that option introduction provides further stabilizing properties to spot and futures equilibrium, then this is an impact that will probably be viewed favorably by the public.

Also, if options are shown to change the mean prices for the futures and spot markets, then the impact of option market introduction is of interest to anyone who has a vested interest in the commodity markets. The impact of the price changes upon the incomes of various segments of the economy will depend upon the elasticities of demand and the supply for the commodity. Also, it will depend upon the competitive structures at the various levels of the food marketing system.

The impact of option markets upon the mean levels of volume in inventory and futures markets is of interest particularly to members of the futures exchanges and those involved in the storage of commodities. If option markets enhance the volume of futures markets then the introduction of option markets is of particular benefit to exchange members. If option markets induce individuals to hold larger inventories, then the market storage agents (particularly grain elevators) will benefit through the introduction of option markets.
Also examined in this chapter is the relationship of the deviation in the equilibrium option price from its pooled information equilibrium price and the significance of certain factors that contribute to this deviation. The pooled information equilibrium option price is defined as the unweighted average of all investors' subjective values of the option. As will be shown later, the option market equilibrium price will be a weighted average of the uninformed and informed investors' valuations of the option plus or minus some extraneous terms. Thus, the actual equilibrium option price will usually deviate from the pooled information equilibrium.

An examination of significance for factors that contribute to the deviation of the market option price from its pooled information price is important for two reasons. Market participants will have a better idea of how to interpret the equilibrium option price and its corresponding implied volatility. Also, these determining factors are of interest to policymakers because they may illuminate the areas of information difference that are most likely to maintain a viable, but informationally inefficient (in the Grossman-Stiglitz sense) equilibrium.

Several studies have looked at the impact of the futures market upon the spot market equilibrium. Stein (1961) used a mean-variance partial equilibrium model containing spot and futures markets. He concluded that the spot and futures prices are simultaneously determined. Stein also showed that it is possible to determine whether changes in the futures price are due to (a) changes in the excess supply
of current production or (b) changes in price expectations. This can be done by observing the correlation of changes between the basis, spot price, and excess supply.

McKinnon (1967) argued that government intervention in stabilizing farm income would be more effective if the government concentrated on making long-term futures markets more viable instead of attempting to stabilize the spot price.

Danthine (1978) used a rational expectations equilibrium model to show that the futures price is a sufficient statistic (a complete summary of all information in the market) and that its informative role has a stabilizing influence on the spot price. Also, it was shown that hedgers compensate speculators for risk sharing through the futures price. This causes the futures price to be a biased estimate of the future spot price. This bias, in itself, can generate speculative trading in commodity futures.

Turnovsky (1983) used a partial equilibrium model of futures trading to show how the introduction of futures markets reduced the variance of the spot price and reduced its long-run mean.

As mentioned in Chapter One, Stein (1986) provides an excellent summary of previous research in partial equilibrium models using futures and spot markets.

Description of the models

The models used in this chapter consist of a spot and futures market without an option market, and a spot and futures market with a
The demand equations for the model with spot and futures only are as follows:

(7.1) \[ I_i^d = \frac{P_F - P_1}{g} \]

(7.2) \[ X_1^d = \frac{E[P_2^*|Z_1] - P_F}{A_1^\dagger} - I_1^d \]

where \( i = a \) for the informed traders and \( i = b \) for the uninformed traders as in Chapter Five. Equation (7.1) is the inventory demand and equation (7.2) is the futures demand.

The market equilibrium equations for this model are

(7.3) \[ X_a^d + X_b^d = 0, \]

(7.4) \[ h_1 - h_2^*P_1 = Q - (I_a^d + I_b^d), \]

where

- \( h_1 \) = intercept of period one spot demand (\( > 0 \)),
- \( h_2 \) = slope of period one spot demand (\( > 0 \)),
- \( Q \) = total harvest in period one.
Equation (7.3) is the zero-sum market clearing equation for futures markets and equation (7.4) represents the period one spot market equilibrium.

In this model, Q is fixed since the harvest has already been realized and will be reflected in the spot market prices. However, to the individual trader, the harvest will still be random because of the information lag in receiving the final harvest estimate (as was mentioned in Chapter Five).

Solving for equations (7.3) and (7.4) gives the following equilibrium price equations for spot and futures:

\[
\begin{align*}
(7.5) \quad P_1^e &= \frac{(h_1 + I_a + I_b - Q)}{h_2}, \\
(7.6) \quad P_f^e &= \frac{G_b}{G_T} E[P_2|Z_a] + \frac{G_a}{G_T} E[P_2|Z_b] - \frac{G_a G_b}{G_T} I_T^d,
\end{align*}
\]

where \( G_i \) equals \( A_i V_{11} \) and subscript \( i \) equal to \( T \) represents the sum of the variable for both groups of traders (for example, \( G_T = G_a + G_b \)).

Equation (7.6) can be rewritten as follows:

\[
(7.7) \quad P_f^e = \omega_a E[P_2|Z_a] + \omega_b E[P_2|Z_b] - \phi I_T^d.
\]

Note that \( \omega_a + \omega_b = 1 \) and \( \phi > 0 \). Thus the equilibrium futures price is equal to a weighted average of the forecasted future spot price means for each type of trader, and a component representing the net hedging
influence. The last component is often referred to as normal backwardation if \( I_T^d \) is greater than zero (net short hedge interest in market). This equation corresponds to equation (2.45) in Stein (1986, p.53).

The model containing inventory, futures, and options will include equations (7.1), (7.3), and (7.4). Equation (7.2) is replaced by the following equation:

\[
X_1^d = \frac{E[P_2|Z_1] - P_F}{A_1V_{111}^*} - \frac{V_{121}^*}{V_{111}^*} R_1^d - I_1^d,
\]

and the following equations are also added to the model:

\[
R_1^d = \frac{P_R - \alpha_1^* B_{21}^*}{A_1 V_{221}^*} - \frac{V_{121}^*}{V_{221}^*} (I_1 + X_1)^d,
\]

\[
R_a^d + R_b^d = 0.
\]

Equation (7.8) represents futures demand with options, equation (7.9) represents option demand, and equation (7.10) represents market clearing equilibrium for options. Note that the model containing options is identical to the model in Chapter Five.

Solving the model yields the following price equilibrium equations for futures and options (the spot price equation is the same as in the model without options):
Equations (7.11) and (7.12) can be rewritten as

\begin{align*}
(7.13) \quad P_f^e &= \gamma_a E[P_2|Z_a] + \gamma_b E[P_2|Z_b] + \lambda_a a_2 a + \lambda_b a_2 b - \psi T^d, \\
(7.14) \quad P_R^e &= \xi_a E[P_2|Z_a] + \xi_b E[P_2|Z_b] + \tau_a a_2 a + \tau_b a_2 b + \Omega T^d.
\end{align*}

Note that \( \gamma_a + \gamma_b = 1, \lambda_a + \lambda_b = 0, \xi_a + \xi_b = 1, \) and \( \tau_a + \tau_b = 0. \)

Equation (7.13) implies that the equilibrium futures price is equal to the weighted average of the means of the future spot price and of terms representing the impact of the values of the option and a net hedging
interest term. If $a_a B_{2a} = a_b B_{2b}$, then the option values will have no direct impact on the futures price. Equation (7.14) implies that the equilibrium option price is equal to the weighted average of the trader's subjective option values plus terms representing the impact of the trader's expectations of the future spot price mean, and a net hedging interest term. If $E[P_2|Z_a] = E[P_2|Z_b]$, then the expected future spot prices will have no direct impact on the option price.

Methods

The questions relating to equilibrium price determination in futures and options were examined within the context of a numerical simulation model. This model incorporates equations (7.1) through (7.3), (7.7) through (7.9), and (7.13) through (7.14) to arrive at the market equilibrium prices and volumes.

The following procedure was used (for each commodity) in examining the impact of option market introduction upon the volume and prices for inventory and futures:

1. The simulation model and data set were used to generate equilibrium volumes and prices for both inventory and futures. The prices and volumes were generated at each sample point for both the portfolio not containing a put option and the portfolio containing a put option. The percentage change, from the ex ante portfolio (inventory and futures markets) to the ex post portfolio
(inventory, futures, and options), in the prices and volumes was calculated.

2. The mean and variance of the percentage changes in prices and volume were calculated for the entire simulation run.

3. The null hypothesis that option introduction has no impact upon the variability of the prices for inventory and futures was tested using the F-test upon the variances derived from each portfolio. Rejection of the null hypothesis would support the notion that option introduction has an impact upon variability of the prices. Note that variability is used in the context of variability among different investor's expectations and not variability over time. However, it is reasonable to associate a high variability among different expectations with a high variability over time since expectations generally change over time.

4. The null hypothesis that option introduction has no impact upon the mean level of the volumes and prices for inventory and futures was tested using a two-tailed T-test upon the means of the percentage changes.

Factors influencing the bias of the option price were examined in the context of an OLS regression model. The form of the regression model is as follows:

\[
(7.15) \quad \text{Bias} = a_0 + a_1 \text{Diff1} + a_2 \text{Diff2} + a_3 \text{Diff3} + a_4 \text{Net}_\text{hedge} + u,
\]
where

\[ \text{Bias} = \text{absolute value of the difference between } P_R \text{ and the pooled information price } \frac{(aB_{2a} + aB_{2b})}{2}. \]

\[ \text{Diff1} = \text{absolute value of the difference between } A_a \text{ and } A_b, \]

\[ \text{Diff2} = \text{absolute value of the difference between } E[F_2|Z_a] \text{ and } E[F_2|Z_b]. \]

\[ \text{Diff3} = \text{absolute value of the difference between } V_{11a} \text{ and } V_{11b}, \]

\[ \text{Net_hedge} = \text{net hedging influence in the market } (-\Omega_{T^{d}}), \]

\[ u = \text{error term}. \]

The independent variables were chosen because of the structure and implications of the theoretical model.

The significance of each factor in influencing the bias of the market option price was determined by testing the null hypothesis that the relevant regression coefficient was equal to zero. This was tested using a two-tailed \( t \)-test on the regression coefficient.

The data set used in the numerical simulation analysis of this chapter was derived using the following assumptions:

1. The uninformed investor's information set with regards to the future spot price mean and variance is limited to the present futures price for the mean and the option implied volatility for the variance. Thus, for all cases the uninformed trader uses the futures price and the implied volatility for the values of \( E[F_2|Z_b] \) and \( V_{11b} \).
2. The informed investor is assumed to have some information that may lead to a deviation from the futures price and/or the implied volatility in estimating the moments of the future spot price distribution. In particular, informed trader's estimates of the moments will come from a normal distribution with the market estimates as the means and three percent of the market estimates as the standard deviation.

3. Both groups of investors will have risk aversion coefficients that come from a triangular distribution with a minimum of .000001, a maximum of .01, and .0001 as the most likely value.

The rationale behind the first assumption comes from the definition of the information set available to the uninformed investor in Chapter Five ($Z_b = (P_1, P_f, P_R)$). The rationale behind the second and third assumptions is the same as used for the data set used in Chapter Six and is based upon selecting values that nearly represent actual distributions for the moment forecasts and the risk aversion coefficients.

All of the observations listed in Table 2 were used in generating the data set. As in Chapter Six, the data set was divided by commodity and there were 75 sample points generated for each observation in Table 2.

For all of the simulation experiments, the equilibrium prices were determined using the Gauss-Seidel algorithm that is incorporated into RATS. The storage cost coefficient ($g$) was set to a value of .5, the period one harvest ($Q$) was set to 200, the period one spot price demand
Table 6 Percentage Impacts of Option Introduction Upon Spot and Futures Price Variance (F-statistics in parentheses)

<table>
<thead>
<tr>
<th>Commodity</th>
<th>Spot Price</th>
<th></th>
<th>Futures Price</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Corn</td>
<td>+6.9%</td>
<td>(1.07)</td>
<td>+6.9%</td>
<td>(1.07)</td>
</tr>
<tr>
<td>Soybeans</td>
<td>+3.5%</td>
<td>(1.03)</td>
<td>+3.5%</td>
<td>(1.03)</td>
</tr>
<tr>
<td>Live Cattle</td>
<td>-2.5%</td>
<td>(1.03)</td>
<td>-2.5%</td>
<td>(1.03)</td>
</tr>
<tr>
<td>Live Hogs</td>
<td>+0.6%</td>
<td>(1.01)</td>
<td>+0.6%</td>
<td>(1.01)</td>
</tr>
</tbody>
</table>

equation intercept \( (h_1) \) was set to 250, and the period one spot demand equation slope \( (h_2) \) was set to 25. The values were chosen because they allowed the Gauss-Seidel algorithm to converge to a solution. Numerical variation of these parameters will have an impact upon the output of the model through changes in the inventory position. However, this is of minor consequence to the percentage and regression results of this chapter since the futures and options market results will vary proportionally to the inventory market results (as was observed in Chapter Four).

Results

Table 6 shows the percentage impact of option market introduction upon the price variance for spot (inventory) and futures. The introduction of options into the portfolio model appears to have little in-
Table 7 Percentage Impacts of Option Introduction Upon Spot and Futures Price Means (t-statistics in parentheses)

<table>
<thead>
<tr>
<th>Commodity</th>
<th>Spot Price</th>
<th>Futures Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corn</td>
<td>0.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td></td>
<td>(1.28)</td>
<td>(1.36)</td>
</tr>
<tr>
<td>Soybeans</td>
<td>+0.3%***</td>
<td>+0.6%***</td>
</tr>
<tr>
<td></td>
<td>(9.55)</td>
<td>(9.52)</td>
</tr>
<tr>
<td>Live Cattle</td>
<td>+0.1%***</td>
<td>+0.4%***</td>
</tr>
<tr>
<td></td>
<td>(3.65)</td>
<td>(3.68)</td>
</tr>
<tr>
<td>Live Hogs</td>
<td>0.0%*</td>
<td>0.0%*</td>
</tr>
<tr>
<td></td>
<td>(1.86)</td>
<td>(1.90)</td>
</tr>
</tbody>
</table>

*Significant at 90% confidence level.
***Significant at 99% confidence level.

fluence upon the variance of the spot and futures prices. This may indicate that information on the future spot price mean (through futures prices) is more likely to reduce spot price volatility than information on the variance (through option prices).

Table 7 shows the percentage impact of option market introduction upon the mean of prices for spot and futures. With the exception of corn and live hogs, the introduction of option markets increases the mean spot and futures prices by a statistically significant amount. However, the economic significance of the percentage changes in Table 7 (ranging from 0.1 to 0.6 percent) is questionable. For example, if live cattle cash prices have averaged around the $74.00 per hundredweight, then a 0.1 percent increase in price will result in an extra 7.4 cents per hundredweight or a total increase of 88.8 cents in value for a 1,200
Table 8 Percentage Impacts of Option Introduction Upon Spot and Futures Volume Mean (t-statistics in parentheses)

<table>
<thead>
<tr>
<th>Commodity</th>
<th>Spot Volume</th>
<th>Futures Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corn</td>
<td>0.0%</td>
<td>+69.2%***</td>
</tr>
<tr>
<td></td>
<td>(1.13)</td>
<td>(3.06)</td>
</tr>
<tr>
<td>Soybeans</td>
<td>+0.2%***</td>
<td>+246.5%</td>
</tr>
<tr>
<td></td>
<td>(9.42)</td>
<td>(1.63)</td>
</tr>
<tr>
<td>Live Cattle</td>
<td>+0.1%***</td>
<td>+100.5%**</td>
</tr>
<tr>
<td></td>
<td>(3.74)</td>
<td>(2.15)</td>
</tr>
<tr>
<td>Live Hogs</td>
<td>+0.1%</td>
<td>+40.9%**</td>
</tr>
<tr>
<td></td>
<td>(1.36)</td>
<td>(2.50)</td>
</tr>
</tbody>
</table>

**Significant at 95% confidence level.
***Significant at 99% confidence level.

Table 8 shows the percentage impact of option market introduction upon the mean of volume for spot and futures. With the exception of corn and live hogs, the introduction of option markets increases the mean volume in the spot market by a statistically significant amount. With the exception of soybeans, the introduction of option markets increases the mean volume in the futures market by a statistically significant amount. As with the results from Table 7, the economic significance of a 0.1 to 0.2 percent increase in spot market volume is questionable. However, the results for the futures volume appear to be economically significant. The rationale behind the increase in futures volume is that the introduction of put options facilitates the assump-
tion of larger speculative positions in the futures market because of the ability to conditionally hedge part of the speculative position.

Table 9 shows the OLS estimates of the equation illustrating the impact of various factors upon the bias of the option price from the pooled information value. There were no independent variables with more than two statistically significant coefficient values from the four regressions. However, there was at least one significant coefficient value among the four regressions for each independent variable. These results, when combined with the low R-squared statistics for the live cattle and live hogs equations indicate that there exists no consistent relationship across commodities between the bias and the explanatory variables. A check of alternate functional forms using Box-Tidwell regressions showed little improvement in the regression results.

Conclusions

In this chapter, the objective was to examine the implications of introducing an option market into a two-trader partial equilibrium model where futures and spot markets already exist. The equilibrium futures and spot price equations were derived both for the case of no option markets for the particular commodity and for the case of an existing option market. Also, the equilibrium option market price equation was derived for the case where option markets existed.

For the case where option markets do not exist, the futures price equation is equal to a weighted average of the individual trader's expected spot price means plus or minus a component representing the net
Table 9 OLS Estimates of Option Market Price Bias Equations (standard errors in parentheses)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Corn</th>
<th>Soybeans</th>
<th>Live Cattle</th>
<th>Live Hogs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>-2.12</td>
<td>-0.02*</td>
<td>-8.31***</td>
<td>-0.18</td>
</tr>
<tr>
<td></td>
<td>(2.72)</td>
<td>(0.01)</td>
<td>(1.66)</td>
<td>(0.49)</td>
</tr>
<tr>
<td>Diff1</td>
<td>0.54</td>
<td>4.99***</td>
<td>-10.73</td>
<td>-17.52</td>
</tr>
<tr>
<td></td>
<td>(0.80)</td>
<td>(0.46)</td>
<td>(22.50)</td>
<td>(10.86)</td>
</tr>
<tr>
<td>Diff2</td>
<td>-0.11***</td>
<td>0.10***</td>
<td>-0.00</td>
<td>-0.01</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.01)</td>
<td>(-0.04)</td>
<td>(0.03)</td>
</tr>
<tr>
<td>Diff3</td>
<td>83.23</td>
<td>0.05</td>
<td>0.72***</td>
<td>0.19***</td>
</tr>
<tr>
<td></td>
<td>(92.36)</td>
<td>(0.07)</td>
<td>(0.17)</td>
<td>(0.09)</td>
</tr>
<tr>
<td>Net_hedge</td>
<td>-0.05***</td>
<td>0.00</td>
<td>0.04***</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
<td>(0.00)</td>
<td>(0.01)</td>
<td>(0.01)</td>
</tr>
<tr>
<td>R²</td>
<td>0.86</td>
<td>0.62</td>
<td>0.21</td>
<td>0.04</td>
</tr>
<tr>
<td>F</td>
<td>23.98***</td>
<td>87.99***</td>
<td>14.38***</td>
<td>2.18*</td>
</tr>
<tr>
<td>DW</td>
<td>1.88</td>
<td>1.98</td>
<td>1.75</td>
<td>1.97</td>
</tr>
</tbody>
</table>

*Significant at 90% confidence level.
***Significant at 99% confidence level.

hedging influence in the market. For the case where option markets do exist, the equilibrium futures price is equal to a weighted average of the individual trader's expected spot price means plus or minus terms representing each trader's subjective valuation of the option price and also including a net hedging influence component. The equilibrium option price is equal to a weighted average of the individual trader's subjective values of the option plus or minus terms representing the
trader's expected spot price means and also including a net hedging influence component.

Two examples related to the implications of introducing option markets into the two-trader model were examined using simulation experiments. The first simulation experiment examined the impact of option market introduction upon the mean and variance of price and the mean of trading volume for futures and inventory. The introduction of option markets substantially increased the mean volume in the futures market, but had little or no significant influence upon any other variables.

The second experiment examined the importance of various factors influencing the divergence of the option price from the pooled information option price. The results indicate that no consistent linear relationship exists between the factors studied and the bias in the equilibrium option price.
One of the major objectives of this study was to extend the mean-variance portfolio model to realistically incorporate option markets. Previous attempts in the literature to extend the mean-variance model to include options have been only partially successful because they have been unable to realistically handle the truncation of returns that is associated with options. This study uses a statistical theorem that relies upon the definition of a conditional moment to arrive at the actual mean vector and variance-covariance matrix of returns for the portfolio containing options.

Another objective of this study was to use the portfolio model to examine several questions related to the information role of option markets. In Chapter Four, the model was used to examine how information about the future spot price moments is used by the risk averse investor to arrive at the optimal portfolio positions. The analytic form of the model indicated that the optimal positions are determined by the interaction of the ratio of expected marginal speculative returns with components related to the hedging effectiveness of futures and options.

Numerical simulations of the model indicated that for changes in the forecasted spot price mean, the net futures plus inventory position is the primary speculative medium. Options are used to partially hedge the speculative position. For changes in the future spot price vari-
ance, the option is used as the primary speculative medium through the use of "straddle" positions in the portfolio.

Chapter Five extended Grossman's noisy rational expectations model to provide an information rationale for the existence of option markets. It was shown that information differentials with respect to the future spot price variance were sufficient to enable speculative trading in a viable option market. This option market is informationally inefficient in the Grossman-Stiglitz sense.

Chapter Six examined the value-added of option markets. Options can add extra value to an investor's portfolio due to an expanded set of possible returns. The expanded set of possible instruments allows the individual investor to capitalize more fully upon the information available.

A money metric was used to measure the value-added of the option. Simulation results of the money metric gave mean percentage increases in value that were statistically significant and appeared to be of economic significance. Also, the simulation results indicated that the option is more valuable when there is information available regarding the future spot price mean.

In Chapter Seven, the impact of option market introduction upon the existing markets (spot and futures) was examined using a two-trader simulation model. Also, the divergence of the equilibrium option price from a price representing the pooled values for all investors was examined.
The simulation results showed that the introduction of an option market has little or no effect upon the variance of the spot and futures prices, the mean level of futures and spot prices, and the mean level of trading volume in spot markets. However, the introduction of options was found to significantly increase the mean volume in the futures markets. The equilibrium option price was shown to usually deviate from the pooled information price and an investigation of the possible factors influencing the size of the bias showed no consistent linear relationship.

The model used in this study is quite rich and can be used in numerous other applications. For instance, any problem involving truncated variables can be incorporated into a mean-variance portfolio using the model contained in this study. Some examples of these problems include government price programs, insurance contracts, and situations involving contingency variables.

One of the major limitations upon the results of this study is the choice of assumptions upon which the model is based. For instance, the model can also be extended to include other decision functions besides the mean-variance objective function. Examples of these would include mean-semivariance models, mean-target deviation (MOTAD) models, and models involving higher moments of the probability distribution of returns.

In this model, puts and calls could not be combined with futures due to asset redundancy problems. However, in the real world, all three markets coexist successfully for several commodities. Extensions of
this model to include transactions costs and/or basis risk might provide clues as to why this observed phenomenon occurs.

Another of the major limitations upon the results of this study is the reliance upon analytical and simulation results. Many of the results of this model can be empirically tested using actual price and volume data. Also, the merits of the option valuation formula in this dissertation ($\alpha B_2$) as compared to other option valuation formulas (Black's formula) can be empirically tested using actual price data.
REFERENCES


I wish to thank the following individuals for their contributions and assistance during the preparation of this study: Dr. Dermot Hayes, Dr. Giancarlo Moschini, Dr. Marvin Hayenga, Dr. George Ladd, Dr. Ronald Deiter, Dr. Gene Futrell, and Dr. Frances Antonovitz of the Iowa State University Economics Department; Dr. William Meeker of the ISU Statistics Department; and Roxanne Clemens of the ISU clerical staff.

This study is dedicated to my late father, Gary E. Bullock, who instilled in me the will to achieve and the desire to learn. This study is also dedicated to my mother, Mildred Green; my stepfather, Merle Green; my wife, Lori Bullock; and the rest of my family for their love and support during my five years at ISU.

David W. Bullock
APPENDIX A
DERIVATION OF THE MEAN AND VARIANCE OF PROFITS

Introducing a put option into the individual investor's portfolio causes his/her profit function to become nonsmooth at the point where \( P_2 \) is equal to \( K \). At points where \( P_2 \leq K \), the profit function is

\[
\pi = (P_2 - P_1) \cdot I - \beta g \cdot I_1^2 + (P_2 - P_f) \cdot X + (P_R + K - P_2) \cdot R,
\]

and at points where \( P_2 > K \), the profit function is

\[
\pi = (P_2 - P_1) \cdot I - \beta g \cdot I_1^2 + (P_2 - P_f) \cdot X + P_R R.
\]

Note that this function is continuous but nonsmooth at \( P_2 = K \).

To derive the mean and variance of \( \pi \), the following theorem is used. This theorem is a generalization of results discussed in Taylor and Karlin (1984, pp. 34-36) and in Beaumont (1986, pp. 207-209).

**Theorem**

If \( g(P) = g_1(P) \) for all \( P \leq K \), \( g(P) = g_2(P) \) for all \( P > K \), and \( P \) is a random variable with cumulative distribution function \( F(P) \), then

\[
(A) \quad E[g(P)] = \alpha E[g_1(P) \mid P \leq K] + (1 - \alpha) E[g_2(P) \mid P > K],
\]
(B) \[ \text{Var}[g(P)] = \alpha \text{Var}[g_1(P) | P \leq K] + (1-\alpha) \text{Var}[g_2(P) | P > K] \]
\[ + \alpha(1-\alpha) \left( E[g_2(P) | P \leq K] - E[g_1(P) | P > K] \right)^2, \]

where

\[ \alpha = F(K) = \text{Prob}(P \leq K). \]

**Proof of (A)**

Let

\[ (A.1) \quad E[g(P)] = \int_{-\infty}^{K} g_1(P) dF(P) + \int_{K}^{\infty} g_2(P) dF(P). \]

From the definition of conditional expectations of \( g_1(P) \) and \( g_2(P) \), the following can be written:

\[ (A.2) \quad \int_{-\infty}^{K} g_1(P) dF(P) = \int_{-\infty}^{K} dF(P) \cdot E[g_1(P) | P \leq K], \]

\[ (A.3) \quad \int_{K}^{\infty} g_2(P) dF(P) = \int_{K}^{\infty} dF(P) \cdot E[g_2(P) | P > K]. \]

Substituting equations (A.2) and (A.3) into (A.1) and noting that

\[ \int_{-\infty}^{K} dF(P) = F(K) = \alpha \quad \text{and} \quad \int_{K}^{\infty} dF(P) = [1-F(K)] = (1-\alpha) \]

gives the result illustrated in A. Q.E.D.
Proof of (B)

Let

\((A.4)\) \[ g(P) = 1(P\leq K)g_1(P) + 1(P > K)g_2(P), \]

where \(1(y)\) is the indicator function such that \(1(y) = 1\) if condition \(y\) is met and \(1(y) = 0\) otherwise. From the definition of variance, it can be written

\[(A.5)\] \[ \text{Var}[g(P)] = E[g^2(P)] - E^2[g(P)], \]

where, in this case,

\[(A.6)\] \[ E[g^2(P)] = E(1(P\leq K)g_1^2(P) + 1(P > K)g_2^2(P)) \]
\[+ 2\cdot1(P\leq K)\cdot1(P > K)g_1(P)g_2(P)) \]
\[= \alpha E[g_1^2(P)|P\leq K] + (1-\alpha\cdot E[g_2^2(P)|P > K], \]

\[(A.7)\] \[ E^2[g(P)] = \alpha^2 E^2[g_1(P)|P\leq K] + (1-\alpha)^2 E^2[g_2(P)|P > K] \]
\[+ 2\alpha\cdot(1-\alpha)\cdot E[g_1(P)|P\leq K]\cdot E[g_2(P)|P > K]. \]

Substituting equations (A.6) and (A.7) into equation (A.5), noting that \(E[g_1^2(P)|y] = \text{Var}[g_1(P)|y] + E^2[g_1(P)|y]\) gives
(A.8) \[ \text{Var}[g(P)] = a\text{Var}[g_1(P)|P<\text{K}] + (1-a)\text{Var}[g_2(P)|P>\text{K}] \]
+ \[a(1-a)(E^2[g_1(P)|P<\text{K}] + E^2[g_2(P)|P>\text{K}]) \]
- \[2(a-1-a)E[g_1(P)|P<\text{K}]E[g_2(P)|P>\text{K}], \]

which simplifies to the result in (B) since

\[ E^2[g_1(P)|P<\text{K}] + E^2[g_2(P)|P>\text{K}] = \]
\[(E[g_1(P)|P<\text{K}] - E[g_2(P)|P>\text{K})^2 \]
+ \[2E[g_1(P)|P<\text{K}]E[g_2(P)|P>\text{K}]. \] Q.E.D.
APPENDIX B

DERIVATION OF THE VARIANCE-COVARIANCE MATRIX OF RETURNS AND THE DEMAND EQUATIONS FOR THE MEAN-VARIANCE PORTFOLIO MODEL

The variance of profits from the mean-variance portfolio model can be represented as [from equation (3.4)]

\[(B.1) \quad \text{Var}[\pi] = a \text{Var}[P_2 | P_2 \leq K] \cdot (I+X+R)^2 + (1-a) \text{Var}[P_2 | P_2 > K] \cdot (I+X)^2 + a(1-a) \cdot b_1 \cdot (I+X+R)^2.\]

Multiplication of the squared terms in equation (B.1) gives the following representation of the variance of profits:

\[(B.2) \quad \text{Var}[\pi] = (a \text{Var}[P_2 | P_2 \leq K] + (1-a) \text{Var}[P_2 | P_2 > K] + a(1-a) \cdot b_1^2) \cdot (I^2+X^2+2 \cdot I \cdot X) + a \text{Var}[P_2 | P_2 \leq K] + a(1-a) \cdot b_1 \cdot b_2 \cdot (2 \cdot I \cdot R+2 \cdot X \cdot R),\]

or by using substitutions:

\[(B.3) \quad \text{Var}[\pi] = V_{11} \cdot (I^2+X^2+2 \cdot I \cdot X) + V_{22} R^2 + V_{12} \cdot (2 \cdot I \cdot R+2 \cdot X \cdot R),\]

or by using matrices and vectors:
The vector of partial derivatives of \( \text{Var}[\pi] \) with respect to \( I, X, \) and \( R \) multiplied by \( \mathbf{HA} \) is

\[
\begin{bmatrix}
\frac{\partial \text{Var}[\pi]}{\partial I} \\
\frac{\partial \text{Var}[\pi]}{\partial X} \\
\frac{\partial \text{Var}[\pi]}{\partial R}
\end{bmatrix}
= \mathbf{A} \cdot
\begin{bmatrix}
v_{11} & v_{11} & v_{12} \\
v_{11} & v_{11} & v_{12} \\
v_{12} & v_{12} & v_{22}
\end{bmatrix}
\begin{bmatrix}
I \\
X \\
R
\end{bmatrix},
\]

or by using vector notation:

\[
\mathbf{hA} \cdot \nabla \text{Var}[\pi] = \mathbf{A} \cdot \nabla \cdot \mathbf{d},
\]

where

\( \nabla = \text{gradient operator}, \)

\( \nabla = \text{variance-covariance matrix of returns}, \)

\( \mathbf{d} = \text{vector of portfolio demands}. \)
The first-order conditions [equation (3.5)] can be rewritten as

\[ \frac{\partial E[U(\pi)]}{\partial I} = E[P_2] - P_1 - (g+A\cdot V_{11}) \cdot I^d - A\cdot V_{11} \cdot X^d - A\cdot V_{12} \cdot R^d = 0, \]

\[ \frac{\partial E[U(\pi)]}{\partial X} = E[P_2] - P_\ell - A\cdot V_{11} \cdot I^d - A\cdot V_{11} \cdot X^d - A\cdot V_{12} \cdot R^d = 0, \]

\[ \frac{\partial E[U(\pi)]}{\partial R} = P_R - \alpha B_2 - A\cdot V_{12} \cdot I^d - A\cdot V_{12} \cdot X^d - A\cdot V_{22} \cdot R^d = 0. \]

Solving equation (B.7) for \( I^d \) gives the following equation:

\[ I^d = \frac{E[P_2] - P_1}{g+A\cdot V_{11}} - \frac{A\cdot V_{11}}{g+A\cdot V_{11}} \cdot X^d - \frac{A\cdot V_{12}}{g+A\cdot V_{11}} \cdot R^d. \]

Solving equation (B.8) for \( X^d \) gives the following equation:

\[ X^d = \frac{E[P_2] - P_\ell}{A\cdot V_{11}} - \frac{V_{12}}{V_{11}} \cdot R^d - I^d. \]

Substitution of equation (B.11) into (B.10) gives the following equation:

\[ I^d = \frac{P_\ell - P_1}{g}. \]
Solving equation (B.9) for \( R^d \) gives the following equation:

\[
(B.13) \quad R^d = \frac{P_R - aB_2}{A \cdot V_{22}} - \frac{V_{12}}{V_{22}} (I+X)^d.
\]

Note that equations (B.11), (B.12), and (B.13) are the demand equations that are illustrated in Chapter Three.
Graphical analysis, illustrated in Figure 3, can be used to derive the optimal portfolio positions. The points where the portfolio demand curves ($D_1$ and $D_2$) intersect the $R^d$ axis equal the speculative com-

Figure C1 Graphical representation of cases where positions are taken in Quadrant I.
ponents of each demand curve ($\sigma_1$ and $\sigma_2$). The points where the portfolio demand curves intersect the $(I+X)^d$ axis equal the ratio of the speculative component over the hedging component for each demand curve ($\sigma_1/\beta_1$ and $\sigma_2/\beta_2$).

Figure C1 illustrates three examples where the position is taken in Quadrant I (investor will hold a net long position in futures plus inventory along with a short position in the put option). Note that

Figure C2 Graphical representation of cases where positions are taken in Quadrant II.
these examples correspond to cases where $\sigma_1 > \sigma_2$ and $\sigma_1/\beta_1 > \sigma_2/\beta_2$ which implies that $\theta > \beta_1$.

Figure C2 illustrates three examples where the position is taken in Quadrant II (investor will hold a net short position in futures plus inventory along with a long position in the put option). Note that these examples correspond to cases where $\sigma_1 < \sigma_2$ and $\sigma_1/\beta_1 < \sigma_2/\beta_2$ which implies that $\theta < \beta_2$.

Figure C3 Graphical representation of cases where positions are taken in Quadrant III.
Figure C3 illustrates an example where the position is taken in Quadrant III (investor will hold a net short position in futures plus inventory along with a short position in the put option). Note that this example correspond to cases where $\sigma_1 < \sigma_2$ and $\sigma_1/\beta_1 > \sigma_2/\beta_2$ which implies that $\theta < \beta_2$ and $\theta > \beta_1$. Note that the situation illustrated here is not possible in the model since Proposition 2 implies that $\beta_1 > \beta_2$.

Figure C4 Graphical representation of cases where positions are taken in Quadrant IV.
Figure C4 illustrates an example where the position is taken in Quadrant IV (investor will hold a net long position in futures plus inventory along with a long position in the put option). Note that this example corresponds to cases where $\sigma_1 > \sigma_2$ and $\sigma_1/\beta_1 < \sigma_2/\beta_2$ which implies that $\beta_2 < \theta < \beta_1$.

Figure C5 illustrates two examples where the position is taken along the $R^d$ axis (investor will hold an even position in futures plus

![Diagram showing Case A and Case B](image)

**Figure C5** Graphical representation of cases where positions are taken along the $R^d$ axis.
inventory along with a long or short position in the put option). Note that these examples correspond to cases where \( \sigma_1 = \sigma_2 \) which implies that \( \theta = \beta_2 \). Also note that the case of an outright short position in the option (Case B) is ruled out since \( \theta_1 / \theta_1 > \theta_2 / \theta_2 \) which implies that \( \theta > \beta_1 \), which cannot occur if \( \theta = \beta_2 \).

Figure C6 illustrates two examples where the position is taken along the \((I+X)^d\) axis (investor will hold a net long or short position.

![Diagram](image)

**Figure C6** Graphical representation of cases where positions are taken along the \((I+X)^d\) axis.
in futures plus inventory along with an even position in the put option). Note that these examples correspond to cases where \( \sigma_1/\beta_1 = \sigma_2/\beta_2 \) which implies that \( \Theta = \beta_1 \). Also note that the case of an outright short position in the option (Case A) is ruled out since \( \sigma_1 < \sigma_2 \) which implies that \( \Theta < \beta_2 \), which cannot occur if \( \Theta = \beta_1 \).
APPENDIX D

PARTIAL DERIVATIVES OF THE MEAN-VARIANCE PORTFOLIO MODEL
WITH RESPECT TO THE EXPECTED PRICE MOMENTS AND
RISK AVERSION

This appendix contains the mathematical derivation of the partial
derivatives from Chapter Four. Whenever possible, the sign of the
derivative is obtained analytically; however, numerical procedures were
used when the sign of the analytic derivative was indeterminant.

D1. By assuming that the future spot price is normally distributed, the
following partial derivatives can be signed:

\[ \frac{\partial \alpha}{\partial E[P_2]} < 0, \quad \frac{\partial \alpha}{\partial \text{Var}[P_2]} \geq (>) 0 \quad \text{for } E[P_2] \geq (>) K, \]

\[ \frac{\partial E[P_2|P_2 \leq K]}{\partial E[P_2]} > 0, \quad \frac{\partial E[P_2|P_2 \leq K]}{\partial \text{Var}[P_2]} < 0, \quad \frac{\partial E[P_2|P_2 > K]}{\partial E[P_2]} > 0, \]

\[ \frac{\partial E[P_2|P_2 > K]}{\partial \text{Var}[P_2]} > 0, \quad \frac{\partial \text{Var}[P_2|P_2 \leq K]}{\partial E[P_2]} < 0, \quad \frac{\partial \text{Var}[P_2|P_2 > K]}{\partial \text{Var}[P_2]} > 0, \]

\[ \frac{\partial \text{Var}[P_2|P_2 > K]}{\partial E[P_2]} > 0, \quad \frac{\partial \text{Var}[P_2|P_2 > K]}{\partial \text{Var}[P_2]} > 0. \]

These partial derivatives are a direct result of the functional forms of
the cumulative normal distribution and the conditional moments of the
normal distribution. Figures D1 and D2 illustrate the impacts of increases in $E[P_2]$ and $\text{Var}[P_2]$ upon $\alpha$ and $(1-\alpha)$.

As $E[P_2]$ increases, the area under the distribution to the left of $K$ ($\alpha$) becomes smaller. As $E[P_2]$ approaches infinity, $\alpha$ will approach zero and $1-\alpha$ will approach one. The central point of mass under $\alpha$ ($E[P_2|P_2 \leq K]$) will collapse to $K$ as $E[P_2]$ increases and thus will increase. Also, the central point of mass under $1-\alpha$ ($E[P_2|P_2 > K]$) will

Figure D1 Graphical illustration of the impact of increases in $E[P_2]$ upon $\alpha$ and $(1-\alpha)$. 
increase as $E[P_2]$ increases. Also, the dispersion of mass under $\alpha$ will decrease as $E[P_2]$ increases and thus $\text{Var}[P_2 | P_2 \leq K]$ will decrease. Also, the dispersion of mass under $1-\alpha$ ($\text{Var}[P_2 | P_2 > K]$) will increase as $E[P_2]$ increases.

As $\text{Var}[P_2]$ increases, the area under $\alpha$ will become larger and the area under $1-\alpha$ will become smaller for $K < E[P_2]$. The opposite will hold true for $K > E[P_2]$ and no change will occur if $K = E[P_2]$. Also,

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure_D2.png}
\caption{Graphical illustration of the impact of increases in $\text{Var}[P_2]$ upon $\alpha$ and $(1-\alpha)$.}
\end{figure}
that the center of mass under $\alpha$ will be pushed to the left and thus
decrease, and the center of mass under $1-\alpha$ will be pushed to the right
and thus increase regardless of where $K$ is positioned. The dispersion
of mass under $\alpha$ and $1-\alpha$ will both increase.

\[ D2. \quad \frac{\partial \alpha(1-\alpha)}{\partial E[P_2]} \geq (\leq) 0 \text{ for } E[P_2] \leq (>) K, \quad \frac{\partial \alpha(1-\alpha)}{\partial \text{Var}[P_2]} \geq 0. \]

Note that $\partial \alpha(1-\alpha)/\partial E[P_2] = (1-2\alpha) \cdot (\partial \alpha/\partial E[P_2])$ and $\partial \alpha(1-\alpha)/\partial \text{Var}[P_2] = (1-2\alpha) \cdot (\partial \alpha/\partial \text{Var}[P_2])$. When $E[P_2] = K$ then $\alpha = .5$ and $1-2\alpha = 0$. Also,
when $E[P_2] > (\leq) K$ then $\alpha < (>) .5$ and $1-2\alpha > (\leq) 0$. From D1, $\partial \alpha/\partial E[P_2] \leq (\geq) 0$ and thus $(1-2\alpha) \cdot (\partial \alpha/\partial E[P_2]) \geq (\leq) 0$ for $E[P_2] \leq (>) K$. Also from
D1, $\partial \alpha/\partial \text{Var}[P_2] \geq (\leq) 0$ for $E[P_2] \geq (\leq) K$ and thus $(1-2\alpha) \cdot (\partial \alpha/\partial \text{Var}[P_2]) \leq (\geq) 0$.

\[ D3. \quad \frac{\partial B_2}{\partial E[P_2]} = - \frac{\partial E[P_2|P_2 \leq K]}{\partial E[P_2]} \leq 0, \quad \frac{\partial B_3}{\partial E[P_2]} = \frac{\partial E[P_2|P_2 > K]}{\partial E[P_2]} > 0. \]

\[ D4. \quad \frac{\partial B_2}{\partial \text{Var}[P_2]} = - \frac{\partial E[P_2|P_2 \leq K]}{\partial \text{Var}[P_2]} > 0, \quad \frac{\partial B_3}{\partial \text{Var}[P_2]} = \frac{\partial E[P_2|P_2 > K]}{\partial \text{Var}[P_2]} > 0. \]
Table D1 Values of $B_1$ Relative to $E[P_2]$ and $\text{Var}[P_2]$

<table>
<thead>
<tr>
<th>Values of $E[P_2]-K$</th>
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<td>2.43</td>
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<td>2.43</td>
</tr>
<tr>
<td>$+3$</td>
<td>3.29</td>
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$$
\frac{\delta B_1}{\delta E[P_2]} = \frac{\delta E[P_2|P_2>K]}{\delta E[P_2]} - \frac{\delta E[P_2|P_2\leq K]}{\delta E[P_2]} = \text{(positive)} - \text{(positive)} - \text{(indeterminate)},
$$

$$
\frac{\delta B_1}{\delta \text{Var}[P_2]} = \frac{\delta E[P_2|P_2>K]}{\delta \text{Var}[P_2]} - \frac{\delta E[P_2|P_2\leq K]}{\delta \text{Var}[P_2]} = \text{(positive)} - \text{(negative)} - \text{(positive)}.
$$

Table D1 illustrates the results of a numerical analysis of $B_1$ with respect to changes in $E[P_2]$ and $\text{Var}[P_2]$. From these results it appears that $\frac{\delta B_1}{\delta E[P_2]} \leq (>) 0$ for $E[P_2] \leq (>)$ K.
Table D2 illustrates the results of a numerical analysis of \( V_{23} \) with regards to changes in \( E[P_2] \) and \( \text{Var}[P_2] \). These results indicate that \( \frac{\partial V_{23}}{\partial E[P_2]} \geq (\leq 0 \text{ for } E[P_2] \leq (>) K) \). This result indicates that \( V_{23} \) reaches a maximum when \( E[P_2] = K \). \( V_{23} \) will approach zero as \( E[P_2] \) moves away from \( K \) since either \( \alpha \) or \( 1-\alpha \) will approach zero (from D1).
Table D2  Values of $V_{23}$ Relative to $E[P_2]$ and $Var[P_2]$

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</table>

which is < 0 for $E[P_2] \geq K$ and is indeterminate otherwise.

\[
\frac{\partial V_{22}}{\partial Var[P_2]} = \frac{\partial \alpha}{\partial Var[P_2]} \cdot Var[P_2|P_2\leq K] + \alpha \cdot \frac{\partial Var[P_2|P_2\leq K]}{\partial E[P_2]} + \frac{\partial \alpha(1-\alpha)}{\partial Var[P_2]} \cdot B_2^2 + 2\alpha(1-\alpha) \cdot B_2 \cdot \frac{\partial B_2}{\partial Var[P_2]}
\]

which is > 0 for $E[P_2] \geq K$ and is indeterminate otherwise.

Table D3 illustrates the results of a numerical analysis of $V_{22}$ with respect to $E[P_2]$ and $Var[P_2]$. These results indicate that $\partial V_{22}/\partial E[P_2] < 0$ and $\partial V_{22}/\partial Var[P_2] > 0$. $V_{22}$ will approach zero as $E[P_2]$ approaches infinity since $\alpha$ will approach zero (from D1).
Table D3  Values of $V_{22}$ Relative to $E[P_2]$ and $Var[P_2]$

<table>
<thead>
<tr>
<th>Values of $E[P_2]-K$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>0.9975</td>
<td>4.2627</td>
<td>7.3311</td>
<td>12.5128</td>
</tr>
<tr>
<td>-2</td>
<td>0.9602</td>
<td>3.5670</td>
<td>6.0971</td>
<td>10.6127</td>
</tr>
<tr>
<td>-1</td>
<td>0.7511</td>
<td>2.6532</td>
<td>4.7375</td>
<td>8.6758</td>
</tr>
<tr>
<td>0</td>
<td>0.3408</td>
<td>1.7042</td>
<td>3.4085</td>
<td>6.8169</td>
</tr>
<tr>
<td>+1</td>
<td>0.0068</td>
<td>0.9268</td>
<td>2.2558</td>
<td>5.1371</td>
</tr>
<tr>
<td>+2</td>
<td>0.0005</td>
<td>0.4225</td>
<td>1.3680</td>
<td>3.7071</td>
</tr>
<tr>
<td>+3</td>
<td>0.0000</td>
<td>0.1613</td>
<td>0.7590</td>
<td>2.5595</td>
</tr>
</tbody>
</table>

$$D8. \quad \frac{\partial V_{33}}{\partial E[P_2]} = \frac{\partial (1-\alpha)}{\partial E[P_2]} \cdot Var[P_2|P_2>K] + (1-\alpha) \cdot \frac{\partial Var[P_2|P_2>K]}{\partial E[P_2]} + \frac{\partial (1-\alpha)}{\partial E[P_2]} \cdot B_3^2 + 2\alpha (1-\alpha) \cdot B_3 \cdot \frac{\partial B_3}{\partial E[P_2]}$$

which is > 0 for $E[P_2] \leq K$ and is indeterminate otherwise.

$$D8. \quad \frac{\partial V_{33}}{\partial Var[P_2]} = \frac{\partial (1-\alpha)}{\partial Var[P_2]} \cdot Var[P_2|P_2>K] + (1-\alpha) \cdot \frac{\partial Var[P_2|P_2>K]}{\partial E[P_2]} + \frac{\partial (1-\alpha)}{\partial Var[P_2]} \cdot B_3^2 + 2\alpha (1-\alpha) \cdot B_3 \cdot \frac{\partial B_3}{\partial Var[P_2]}$$

which is > 0 for $E[P_2] \leq K$ and is indeterminate otherwise.
Table D4 Values of \( V_{33} \) Relative to \( E[P_2] \) and \( Var[P_2] \)

<table>
<thead>
<tr>
<th>Values of ( E[P_2] - K )</th>
<th>( 1 )</th>
<th>( 5 )</th>
<th>( 10 )</th>
<th>( 20 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>0.0000</td>
<td>0.1613</td>
<td>0.7590</td>
<td>2.5595</td>
</tr>
<tr>
<td>-2</td>
<td>0.0005</td>
<td>0.4225</td>
<td>1.3680</td>
<td>3.7071</td>
</tr>
<tr>
<td>-1</td>
<td>0.0068</td>
<td>0.9268</td>
<td>2.2558</td>
<td>5.1371</td>
</tr>
<tr>
<td>0</td>
<td>0.3408</td>
<td>1.7042</td>
<td>3.4085</td>
<td>6.8169</td>
</tr>
<tr>
<td>+1</td>
<td>0.7511</td>
<td>2.6532</td>
<td>4.7375</td>
<td>8.6758</td>
</tr>
<tr>
<td>+2</td>
<td>0.9602</td>
<td>3.5670</td>
<td>6.0971</td>
<td>10.6127</td>
</tr>
<tr>
<td>+3</td>
<td>0.9975</td>
<td>4.2627</td>
<td>7.3311</td>
<td>12.5128</td>
</tr>
</tbody>
</table>

Table D4 illustrates the results of a numerical analysis of \( V_{33} \) with respect to \( E[P_2] \) and \( Var[P_2] \). These results indicate that \( \partial V_{33}/\partial E[P_2] > 0 \) and \( \partial V_{33}/\partial Var[P_2] > 0 \). \( V_{33} \) will approach zero as \( E[P_2] \) approaches negative infinity since \( 1-\alpha \) approaches zero (from D1).

D9. \[
\frac{\partial V_{12}}{\partial E[P_2]} = \frac{\partial V_{22}}{\partial E[P_2]} + \frac{\partial V_{23}}{\partial E[P_2]} = \text{(negative) + (conditional)}
\]

which is \( \leq 0 \) if \( E[P_2] \geq K \) and indeterminate otherwise.

\[
\frac{\partial V_{12}}{\partial Var[P_2]} = \frac{\partial V_{22}}{\partial Var[P_2]} + \frac{\partial V_{23}}{\partial Var[P_2]} = \text{(positive) + (positive)}
\]

\( = \text{(positive)}. \)
Table D5 illustrates the results of a numerical analysis of \( V_{12} \) with respect to changes in \( E[P_2] \) and \( \text{Var}[P_2] \). The results indicate that \( \frac{\partial V_{12}}{\partial E[P_2]} \leq 0. \)

\[
\begin{align*}
\frac{\partial V_{13}}{\partial E[P_2]} & = \frac{\partial V_{33}}{\partial E[P_2]} + \frac{\partial V_{23}}{\partial E[P_2]} = \text{(positive)} + \text{(conditional)} \\
\text{which is} \geq 0 \text{ if } E[P_2] \leq K \text{ and indeterminate otherwise.}
\end{align*}
\]

Table D6 illustrates the results of a numerical analysis of \( V_{13} \) with respect to changes in \( E[P_2] \) and \( \text{Var}[P_2] \). The results indicate that \( \frac{\partial V_{13}}{\partial E[P_2]} \geq 0. \)
Table D6. Values of $V^g$ Relative to $E[P_2]$ and $\text{Var}[P_2]$

<table>
<thead>
<tr>
<th>Values of $E[P_2] - K$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>0.0001</td>
<td>0.4493</td>
<td>1.7139</td>
<td>5.0233</td>
</tr>
<tr>
<td>-2</td>
<td>0.0023</td>
<td>0.9277</td>
<td>2.6354</td>
<td>6.5472</td>
</tr>
<tr>
<td>-1</td>
<td>0.1587</td>
<td>1.6368</td>
<td>3.7591</td>
<td>8.2306</td>
</tr>
<tr>
<td>0</td>
<td>0.5000</td>
<td>2.5000</td>
<td>5.0000</td>
<td>10.0000</td>
</tr>
<tr>
<td>+1</td>
<td>0.8413</td>
<td>3.3632</td>
<td>6.2409</td>
<td>11.7694</td>
</tr>
<tr>
<td>+2</td>
<td>0.9772</td>
<td>4.0723</td>
<td>7.3646</td>
<td>13.4528</td>
</tr>
<tr>
<td>+3</td>
<td>0.9987</td>
<td>4.5507</td>
<td>8.2861</td>
<td>14.9767</td>
</tr>
</tbody>
</table>

\[ \frac{\partial \sigma_1}{\partial A} = \frac{-(E[P_2] - P_f) \cdot V_{12}}{[A \cdot V_{12}]^2} \] which is \( (>0) \) for \( E[P_2] \geq K \).

\[ \frac{\partial \sigma_1}{\partial E[P_2]} = \frac{A \cdot V_{12} - (E[P_2] - P_f) \cdot A \cdot (\partial V_{12}/\partial E[P_2])}{[A \cdot V_{12}]^2} \] which is \( (>0) \) for \( E[P_2] \geq P_f \) and indeterminate otherwise.

\[ \frac{\partial \sigma_1}{\partial \text{Var}[P_2]} = \frac{-(E[P_2] - P_f) \cdot A \cdot (\partial V_{12}/\partial \text{Var}[P_2])}{[A \cdot V_{12}]^2} \] which is \( (>0) \) for \( E[P_2] \geq P_f \).

\[ \frac{\partial \sigma_2}{\partial A} = \frac{-(P_R - \alpha B_2) \cdot V_{22}}{[A \cdot V_{22}]^2} \] which is \( (>0) \) for \( P_R \geq \alpha B_2 \).
which is \( > 0 \) for \( P_R \geq \alpha B_2 \) and indeterminate otherwise.

\[
\frac{\partial \sigma_2}{\partial \text{Var}[P_2]} = -\frac{A \cdot V_{22} \cdot (\partial \alpha B_2/\partial \text{Var}[P_2]) - (P_R - \alpha B_2) \cdot A \cdot (\partial V_{22}/\partial \text{Var}[P_2])}{[A \cdot V_{22}]^2}
\]

which is \( < 0 \) for \( P_R \geq \alpha B_2 \) and indeterminate otherwise.

\[
\frac{\partial \beta_1}{\partial E[P_2]} = -\frac{V_{11} \cdot (\partial V_{12}/\partial E[P_2])}{V_{12}^2}
\]

which is \( > 0 \).

\[
\frac{\partial \beta_1}{\partial \text{Var}[P_2]} = \frac{V_{12} \cdot (\partial V_{13}/\partial \text{Var}[P_2]) - V_{13} \cdot (\partial V_{12}/\partial \text{Var}[P_2])}{V_{12}^2}
\]

which is \( \geq (<) 0 \) for
\[V_{12} \cdot (\partial V_{13}/\partial \text{Var}[P_2]) \geq (<) V_{13} \cdot (\partial V_{12}/\partial \text{Var}[P_2]).\]

\[
\frac{\partial \beta_2}{\partial E[P_2]} = \frac{V_{22} \cdot (\partial V_{23}/\partial E[P_2]) - V_{23} \cdot (\partial V_{22}/\partial E[P_2])}{V_{22}^2}
\]

which is \( < 0 \) for \( E[P_2] \geq K \) and indeterminate otherwise.
\[
\frac{\partial \beta_2}{\partial \text{Var}[P_2]} \quad \frac{V_{22} \cdot (\partial v_{23} / \partial \text{Var}[P_2]) - V_{23} \cdot (\partial v_{22} / \partial \text{Var}[P_2])}{V_{22}^2}
\]

which is \(\geq (\leq) 0\) for

\[V_{22} \cdot (\partial v_{23} / \partial \text{Var}[P_2]) \geq (\leq) V_{23} \cdot (\partial v_{22} / \partial \text{Var}[P_2]).\]