Introduction

The classic Banach-Stone Theorem establishes a form for surjective, complex-linear isometries (distance preserving functions) between function spaces. Mathematician Takeshi Miura from Niigata University questioned the failure of the classification for the complex-linear isometries (distance preserving functions) between function spaces. Mathematician Takeshi Miura from Niigata University questioned the failure occurs when we have only real-linearity instead of complex-linearity.

We seek to understand the failure of the classification for the counter-example that demonstrated the shortcomings of the Banach-Stone Theorem proof, we hope to gain insight into why the theorem does not hold in general and to find a form for real-linear isometries between function spaces.

Objectives

- We seek to determine why the Banach-Stone Theorem fails to classify forms of isometries for general subspaces.
- We seek to understand the failure of the classification for the counterexample.
- We seek to prove a conjecture as to the correct form of such isometries of the unit circle in the complex plane.

Methods

- Understand problem background and definitions needed to understand the Banach-Stone Theorem.
- Begin by examining a proof of the Banach-Stone Theorem and see where the failure occurs when we have only real-linearity instead of complex-linearity.
- Seek to prove the conjecture for forms of real-linear isometries.

Classic Banach-Stone Result

The Banach-Stone Theorem says if $f$ is a surjective, complex-linear isometry between function spaces, then there is a weighting function $\varphi$ and a continuous bijection $\psi$ such that

$$T(f) = \varphi \cdot (f \circ \psi)$$

for all $f$ in the function space.

![Diagram of a surjective, complex-linear isometry](Diagram)

Note: This function is in fact a surjective isometry and is real-linear, but it does not follow the form set by the Banach-Stone Theorem for complex-linear isometries. Also note that the isometry can be written in the following way:

$$T(az + b) = \text{Re}(az + b) - i \text{Im}(a(-z) + b),$$

which matches one of the conjectured forms.

Forms of Isometries Between Function Spaces

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Projects Advisor: Dr. Kristopher Lee

Results

In the Banach-Stone proof we referred to, we were able to relate the functions we sought, $\psi_1$ and $\varphi_1$, to the inverse image of a set of functions, $Q$ such that for $w, \alpha \in \mathbb{T}$,

$$Q(w, \alpha) = \{f \in A : f(w) = \alpha, \|f\|_{\mathbb{T}} = 1\}.$$  

We found that

$$T^{-1}Q(w, 1) = Q(\psi_1(w), \varphi_1(w)).$$

This along with the complex-linearity of $T$ allowed us to access any of the complex-numbers. However, since we sought to create such a generalization for a real-linear function, we needed to use two different such sets, one to allow us to access any real part and one to allow us to access any imaginary part:

$$T^{-1}Q(w, 1) = Q(\psi_1(w), \varphi_1(w)) \quad \text{and} \quad T^{-1}Q(w, i) = Q(\psi_1(w), \varphi_1(w)).$$

Conclusion

As stated by the original Banach-Stone Theorem, complex-linear isometries $T$ between function spaces take this form:

$$T(f) = T(1) \cdot (f \circ \psi).$$

However, real-linear isometries $T$ between function spaces take one of the following forms:

$$T(f)(z) = \text{Re}[\psi_1(z)f(\psi_1(z))] + i \text{Im}[\psi_1(z)f(\psi_1(z))],$$

or

$$T(f)(z) = \text{Re}[\psi_1(z)f(\psi_1(z))] - i \text{Im}[\psi_1(z)f(\psi_1(z))].$$

We found that this was indeed possible to prove as we had hoped. So we now have a “nice” formula to tell us the forms of such isometries.

References