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Applications of the Bethe-Salpeter
equation in nuclear physics

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I. INTRODUCTION

Elementary particles are currently grouped into three major classes. Those that do not participate in the strong interaction are referred to as leptons; those that do participate in the strong interaction are known as hadrons (baryons and mesons) and those which mediate interactions are known as gauge bosons. At present it is generally believed that the hadrons are composed of still more fundamental particles known as quarks which are thought to interact through the exchange of one type of gauge boson known as gluons. The only widely accepted theory of strong interactions is Quantum Chromodynamics (QCD). To date no one has solved this theory. Many models which are more easily solved have been inspired by QCD. In one of the simplest forms of these models, known as the naive quark model or more simply as the quark model, hadrons having integer intrinsic angular momentum are considered to be composed of a quark and an antiquark while hadrons having half-integer intrinsic angular momentum are considered to be composed of three quarks.

The research described here originally grew out of an interest in constructing models for baryons which would properly take into account the effects of the special theory of relativity. Because the theory which describes the interaction between quarks, QCD, is a renormalizable quantum field theory, a system of quarks should be correctly described by the relativistic wave equation known as the Bethe-Salpeter equation[1]. The Bethe-Salpeter equation is a four-dimensional equation and is usually used in one of its three-dimensional reductions. Study of the literature revealed no fully satisfactory method for its solution even in these reductions, hence the bulk of the research described here consists of development of satisfactory methods for solution of the Bethe-Salpeter equation within various three-dimensional reductions for quark models of the mesons.

Three essential elements of most quark models for hadrons are, that quarks are spin-half particles, that when quarks are close together they interact by the one gluon exchange potential (OGE) given by perturbative QCD, and the hypothesis of a particular form of confinement. The confinement hypothesis states that the quarks which make up a hadron are confined to remain within a finite distance of each other. The last of these three assumptions, the confinement hypothesis, is a major cause of difficulty in attempting to construct quark models for hadrons using the Bethe-Salpeter equation. To describe confinement using the Bethe-Salpeter equation requires the addition of a term which becomes infinite as the distance between the quarks goes to infinity. Because the kernels describing relativistic interactions of particles are generally nonlocal, the Bethe-Salpeter equation is usually written as a integral equation in momentum space. This implies a singularity in the kernel at zero momentum transfer.

One attempt to treat confinement within the Bethe-Salpeter framework is presented in the work of Mittal and Mitra[2], Mitra and Kulshreshtha[3] and, Singh et al.[4] who succeeded in constructing a Bethe-Salpeter based quark model for both the hadrons composed of quark and antiquark pairs (mesons) and those composed of three quarks (baryons). Furthermore they were able to show how to assure covariance of their results. Their procedure was to assume a harmonic oscillator form for the confinement portion of the kernel. This simplified the Bethe-Salpeter equation from an integral equation to a partial differential equation. The difficulty with this procedure is that both lattice gauge calculations and detailed fits of hadron spectra seem to indicate that the harmonic oscillator is not the correct form for the confinement kernel but rather that a linear confinement kernel is best.

Another approach to constructing a quark model for hadrons is to replace the Bethe-Salpeter equation with another equation. The simplest choice for an alternative equation is a nonrelativistic Schroedinger equation. The next simplest choice is an equation obtained from the Bethe-Salpeter equation by making some nonrelativistic reduction. Perhaps the most comprehensive treatment of this type is the work of Godfrey and Igsur[5] who used a wave equation of a form which could be obtained by making a nonrelativistic reduction of the Bethe-Salpeter equation through a second order expansion in powers of p^2/m . Using such an equation and a potential containing five parameters they were able to obtain a good global fit to the meson spectrum. There are, however, several difficulties with these approaches.

One difficulty with using nonrelativistic reductions is that the reliability of such a reduction depends on the expectation value of p/m for a given eigenstate, where p is the magnitude of the particles momentum and m denotes its rest mass. As reported in Chapter II, I find the expectation values of p/m for various charmonium eigenstates to be greater than $0.47c$. These values are large enough to cast doubt on the reliability of a nonrelativistic reduction based upon an expansion in powers of p/m. A second, and related objection is that nonrelativistic reductions of a OGE potential contain terms proportional to r^{-3} and hence these terms can only be included perturbatively or by using some regularization scheme.

A treatment of the heavy meson spectrum based upon solution of a three-dimensional reduction of the Bethe-Salpeter equation with a realistic interaction was presented in the work of Chris Long[6]. Working in a mixture of coordinate and momentum representations and using a harmonic oscillator basis, he obtained mass eigenvalues for charmonium and b-quarkonium for one of the cases considered in Chapter II. One of the difficulties with this procedure is in generalizing it to treat the baryon. In practice, relativistic three-body calculations are usually done in the momentum representation using a basis of three-particle angular momentum eigenstates for partial wave decomposition of the wave equations. It is not obvious how to generalize this mixture of coordinate and momentum representation methods to deal with such a partial wave decomposition.

When considering relativistic quark models for mesons the work of Crater and Van Alstine[7] is particularly relevant. Using a two particle wave equation based upon Dirac's constraint form of relativistic quantum mechanics they were able to obtain a good fit to the meson spectrum with a one parameter potential. Although this shows the importance of using a wave equation satisfying Lorentz covariance, it is not obvious how the wave equation they used is related to quantum field theory.

II. SOLVING THE BETHE-SALPETER EQUATION WITH SPIN FOR CONFINING POTENTIALS

There is considerable interest in developing a covariant description of the mass spectra and amplitudes of elementary particles based on QCD. Until such time as methods to directly solve QCD are perfected, the Bethe-Salpeter[1] (BS) integral equation with a phenomenological but QCD inspired confining interaction should serve as an instructive model for the properties of the elementary particles. In this connection it is important to have stable and accurate methods to solve the BS equation in momentum space with nonlocal and singular kernels. We have adopted the quarkonium problem to illustrate a method for treating singular potentials in the BS equation with spin. In this chapter, which is based upon a paper submitted for publication in Physical Review D , the problem of obtaining solutions to the Bethe-Salpeter equation for such potentials is solved for the case in which the coupling between positive and negative frequency states is neglected. We begin by showing how confinement potentials for particles with spin may be defined in momentum space in such a manner that their partial wave decomposition is straightforward and yields a numerically tractable framework. A suitable basis for the numerical solution of integral equations containing such potentials is chosen and the appropriate integrals are evaluated accurately and efficiently. We then apply this combination of techniques to evaluate the spectra of charmonium and b-quarkonium within the Blankenbecler-Sugar (BbS) reduction and within the instantaneous approximation (IA) to the BS equation. Results are compared with experiment and with results of related efforts.

In an earlier work[8] which we refer to as I, confinement potentials in momentum space integral equations for spinless particles were treated by introducing a cutoff. In a succeeding work[9] which we refer to as II, we introduced techniques to evaluate analytically the limit as the cutoff goes to zero. In the current work we employ the procedures of II together with standard results from the meson theory of nuclear forces to treat confinement potentials for particles with spin. As a result, many of the techniques of relativistic nuclear physics developed to treat such potentials can be adapted in a straightforward manner. Thus the current work improves on the results of I and II in several ways and retains the advantage that general nonlocal potentials can be treated in addition to these singular potentials.

A. The Equations

Working in the center of momentum (CM) frame, we consider two three-dimensional reductions of the BS equation. The first reduction is an instantaneous approximation

(IA), known as Salpeter's equation[1,10]

$$[E - E_a(\vec{q}) - E_b(\vec{q})]\,\Lambda_a^+(\vec{q})\Lambda_b^+(\vec{q})\phi(\vec{q}) = \Lambda_a^+(\vec{q})\Lambda_b^+(\vec{q}) \int d^3q' G(\vec{q}-\vec{q}\,') \left(\Lambda_a^+(\vec{q}\,')\Lambda_b^+(\vec{q}\,') + \Lambda_b^-(\vec{q}\,')\Lambda_b^-(\vec{q}\,')\right) \phi(\vec{q}\,') \quad (1a)$$

and

$$[E + E_a(\vec{q}) + E_b(\vec{q})]\,\Lambda_a^-(\vec{q})\Lambda_b^-(\vec{q})\phi(\vec{q}) = -\Lambda_a^-(\vec{q})\Lambda^-(\vec{q}) \int d^3q' G(\vec{q}-\vec{q}\,') \left(\Lambda_a^+(\vec{q}\,')\Lambda_b^+(\vec{q}\,') + \Lambda_a^-(\vec{q}\,')\Lambda_b^-(\vec{q}\,')\right) \phi(\vec{q}\,') \quad (1b)$$

The $\Lambda^+(\vec{q})(\Lambda^-(\vec{q}))$ are projection operators that project positive (negative) energy free particle states and are defined by

$$\Lambda_a^\pm(q) = \frac{1}{2}\left[1 \pm \frac{\vec{\alpha}_a \cdot \vec{q} + \beta_a m_a}{E_a(\vec{q})}\right] \quad (2a)$$

$$\Lambda_b^\pm(q) = \frac{1}{2}\left[1 \pm \frac{-\vec{\alpha}_b \cdot \vec{q} + \beta_b m_b}{E_b(\vec{q})}\right] \quad (2b)$$

$$E_a(\vec{q}) = +\left(m_a^2 + \vec{q}^{\,2}\right)^{\frac{1}{2}} \quad (2c)$$

$$E_b(\vec{q}) = +\left(m_b^2 + \vec{q}^{\,2}\right)^{\frac{1}{2}} \quad (2d)$$

The kernel $G(\vec{q}-\vec{q}\,')$ represents the interaction between the particles, and denoting $\alpha^0 = \beta$, the matrices α_a^μ and α_b^μ satisfy

$$\alpha_a^\mu \alpha_a^\nu + \alpha_a^\nu \alpha_a^\mu = 2\delta^{\mu\nu}$$

$$\alpha_b^\mu \alpha_b^\nu + \alpha_b^\nu \alpha_b^\mu = 2\delta^{\mu\nu}$$

$$\alpha_a^\mu \alpha_b^\nu - \alpha_b^\nu \alpha_a^\mu = 0$$

where μ, ν=0, 1, 2, 3.

The inclusion of coupling between positive and negative frequency states is particularly important for light mesons for which the energy gap between positive and negative frequency states is relatively small. It is also more important when considering interactions containing pseudo-scalar terms because such terms couple positive and negative frequency states strongly. For these reasons inclusion of this coupling is crucial to the problem of attempting to determine the Lorentz structure of the confinement kernel and when fitting both light and heavy meson spectra together and will be treated later. However the inclusion of the coupling between positive and negative frequency states is not important for the present examination of methods for solution of the BS equation so we neglect them and equation (1a) becomes

$$\begin{aligned} &[E - E_a(\vec{q}) - E_b(\vec{q})]\, \Lambda_a^+(\vec{q})\Lambda_b^+(\vec{q})\phi(\vec{q}) = \\ &\Lambda_a^+(\vec{q})\Lambda_b^+(\vec{q}) \int d^3q' G(\vec{q} - \vec{q}\,')\Lambda_a^+(\vec{q}\,')\Lambda_b^+(\vec{q}\,')\phi(\vec{q}\,') \end{aligned} \tag{3a}$$

and similarly for (1b).

A second three-dimensional reduction of the BS equation also explicitly treated here is the Blankenbecler-Sugar (BbS)[11] reduction which for the case $m_a = m_b = m$ has the form

$$\left[\varepsilon - \frac{\vec{q}^{\,2}}{m}\right] \Lambda_a^+(\vec{q})\Lambda_b^+(\vec{q})\bar{\phi}(\vec{q}) = \Lambda_a^+(\vec{q})\Lambda_b^+(\vec{q}) \int d^3q' \bar{G}(\vec{q}-\vec{q}\,')\Lambda_a^+(\vec{q}\,')\Lambda_b^+(\vec{q}\,')\bar{\phi}(\vec{q}\,') \quad (3b)$$

Note that both the amplitude $\bar{\phi}(\vec{q}\,')$ and the el $\bar{G}(\vec{q}-\vec{q}\,')$ are different than for the IA.

B. The Kernels

In relativistic nuclear physics systems containing two or three nucleons are often described by the use of some three-dimensional reduction of the BS equation, or of a Faddeev equation. The nucleons interact by means of a sum of one boson exchange potentials[12,13,14,15] (OBEP). These potentials are obtained by making the appropriate three-dimensional reductions of four-dimensional OBEP kernels of the form

$$G(q,q',\mu) = \frac{4\pi\gamma_0 \otimes \gamma_0\Gamma_a \otimes \Gamma_b}{(-(q-q')^2 + \mu^2)} \quad (4)$$

where μ is the mass of the exchanged meson and

$$\Gamma_a \otimes \Gamma_b = 1 \otimes 1, \gamma_5 \otimes \gamma^5, \gamma_\mu \otimes \gamma^\mu, \ldots$$

$$q = \frac{q_a - q_b}{2} \qquad q' = \frac{q'_a - q'_b}{2}$$

$$q_a = (q_{oa}, \vec{q}_a) \qquad q_b = (q_{ob}, \vec{q}_b)$$

Here the γ_μ denote the usual Dirac matrices.

For the IA such a three-dimensional reduction gives a potential which we denote by $V^{-1}(\vec{q}, \vec{q}', \mu)$ and which is given by

$$V^{-1}(\vec{q}, \vec{q}', \mu) = \frac{4\pi\gamma_o \otimes \gamma_o \Gamma_a \otimes \Gamma_b}{((\vec{q} - \vec{q}')^2 + \mu^2)} \tag{5}$$

Since $\left(\lim_{\mu \to 0}\right)(-\partial/\partial\mu)^{i+1} \exp(-\mu r)/r = r^i$, we define the momentum-space kernel which corresponds to the coordinate space kernel $V^i(r;\mu) = \gamma_0 \otimes \gamma_0 \Gamma_a \otimes \Gamma_b r^i e^{-\mu r}$ to be[8]

$$V^i(\vec{q}, \vec{q}'; \mu) \equiv \left(-\frac{\partial}{\partial \mu}\right)^{i+1} V^{-1}(\vec{q}, \vec{q}'; \mu) \tag{6a}$$

with a class of singular potentials including confining forms being given by

$$V^i(\vec{q}, \vec{q}'; 0) \equiv \lim_{\mu \to 0} \left(-\frac{\partial}{\partial \mu}\right)^{i+1} V^{-1}(\vec{q}, \vec{q}'; \mu) \tag{6b}$$

The limit $\mu \to 0$ is to be taken after integration over $\vec{q}'$.

Since the OBEP is simply the special case of (6a) with $i = -1$ many of the techniques and results developed for use with the OBEP may now be adapted with little change provided we know how to take the $\mu \to 0$ limit. Techniques to handle this limiting procedure were developed in I and II. It is worth noting that for the case

$i = +2$ (6a) reduces to the form used by Singh, Mathur and Mitra[4] for their covariant harmonic oscillator model for the mesons.

C. Partial Wave Decomposition

A suitable basis for partial wave decomposition of the IA to the BS equation is the LSJ basis. With this basis eigenstates of the BS equation decouple into three[11,12,13] sets: singlet (S = 0, L = J), uncoupled triplet (S = 1, L = J = 1), and coupled triplet (S = 1, L = J ± 1). Then, using the simplified notation $V(\vec{q},\vec{q}\,';\mu) \equiv V^{-1}(\vec{q},\vec{q}\,';\mu)$, the BS amplitude is the solution of

$$\langle JML'S|\psi(q)\rangle = \frac{1}{E-2E(q)} \sum_L \int dq'q'^2 \, \langle JML'S|V(\vec{q},\vec{q}\,';\mu)|JMLS\rangle\langle JMLS|\psi(q')\rangle \tag{7a}$$

where either $L = L' = J$ or $L = J \pm 1$, $L' = J \pm 1$. We next give some standard results for the OBEP case in order to establish notation and then indicate how to adapt the results for our studies. Denoting $i^{(L'-L)} V^{JS}_{L'L}(q,q';\mu) = \langle JML'S|V(\vec{q},\vec{q}\,';\mu)|JMLS\rangle$ and $i^L \psi^J_L(q) = \langle JMLS|\psi(q)\rangle$ equation (7a) becomes

$$\psi^J_{L'}(q) = \frac{1}{E-2E(q)} \sum_L \int dq'q'^2 V^{JS}_{L'L}(q,q';\mu)\psi^J_L(q') \tag{7b}$$

with the sum over $L = J \pm 1$ for the coupled triplet case or over only the single term

$L = J$ for the uncoupled triplet case and the singlet case. The quantities $V^{JS}_{L'L}$ can be written as linear combinations of quantities ${}^{x}V^{J}$; x=0, 1, 12, 34, 55, 66, a, b, with the definitions

Singlet:

$$V^{J0}_{JJ} = {}^{0}V^{J}$$

Uncoupled Triplet:

$$V^{J1}_{JJ} = {}^{1}V^{J}$$

Coupled Triplet:

$$\begin{aligned}
V^{J1}_{J-1,J-1} &= \frac{1}{2J+1}\left[J\ {}^{12}V^{J} + (J+1)\ {}^{34}V^{J} + \sqrt{J(J+1)}\ {}^{a}V^{J}\right] \\
V^{J1}_{J+1,J+1} &= \frac{1}{2J+1}\left[(J+1)\ {}^{12}V^{J} + J\ {}^{34}V^{J} - \sqrt{J(J+1)}\ {}^{a}V^{J}\right] \\
V^{J1}_{J-1,J+1} &= \frac{1}{2J+1}\left[\sqrt{J(J+1)}\ {}^{b}V^{J} - J\ {}^{55}V^{J} + (J+1)\ {}^{66}V^{J}\right] \\
V^{J1}_{J+1,J-1} &= \frac{1}{2J+1}\left[\sqrt{J(J+1)}\ {}^{b}V^{J} + (J+1)\ {}^{55}V^{J} - J\ {}^{66}V^{J}\right]
\end{aligned}$$

$${}^{a}V^{J} \equiv {}^{55}V^{J} + {}^{66}V^{J}$$

$${}^{b}V^{J} \equiv {}^{12}V^{J} - {}^{34}V^{J}$$

The details of the expressions for the ${}^{x}V^{J}$ differ with the Lorentz structure of the kernel and for other reductions of the BS equation. Our procedure for generalizing the OBEP results to treat confinement is the same in each case so we present only the scalar case with equal mass particles and within the IA.

For the scalar case define

$$F_S^{(0)} = -(E'E + m^2)$$

$$F_S^{(1)} = q'q$$

$$F_S^{(2)} = m(E' + E)$$

$$I_J^{(N)} \equiv \frac{Q_J^{(N)}(Z)}{q'q} \qquad N = 0, 1, 2, 3$$

$$Q_J^{(0)}(Z) \equiv Q_J(Z)$$

$$Q_J^{(1)} \equiv ZQ_J(Z) - \delta_{J,0}$$

$$Q_J^{(2)}(Z) \equiv \frac{1}{J+1}(JZQ_J(Z) + Q_{J-1}(Z))$$

$$Q_J^{(3)}(Z) \equiv \sqrt{\frac{1}{J+1}}(ZQ_J(Z) - Q_{J-1}(Z))$$

$$E = \left(q^2 + m^2\right)^{\frac{1}{2}}$$

$$E' = \left(q'^2 + m^2\right)^{\frac{1}{2}}$$

$$C_S = \frac{1}{2\pi EE'}$$

$$Z = \frac{q^2 + q'^2 + \mu^2}{2qq'}$$

where $q(q')$ now denotes the magnitude of $\vec{q}(\vec{q}\,')$ and $Q_J(Z)$ are the Legendre functions of the second type as defined below.

The explicit forms for the ${}^xV^J$'s for this case are then

$$
\begin{aligned}
{}^0V_S^J &= C_S\left(F_S^{(0)}I_J^{(0)}(Z) + F_S^{(1)}I_J^{(1)}(Z)\right)\\
{}^1V_S^J &= C_S\left(F_S^{(0)}I_J^{(0)}(Z) + F_S^{(1)}I_J^{(2)}(Z)\right)\\
{}^{12}V_S^J &= C_S\left(F_S^{(1)}I_J^{(0)}(Z) + F_S^{(0)}I_J^{(1)}(Z)\right)\\
{}^{34}V_s^J &= C_S\left(F_S^{(1)}I_J^{(0)}(Z) + F_S^{(0)}I_J^{(2)}(Z)\right)\\
{}^{55}V_S^J &= C_S F_S^{(2)}I_J^{(3)}(Z)\\
{}^{66}V_S^J &= C_S F_S^{(2)}I_J^{(3)}(Z)
\end{aligned}
$$

The results for the BbS reduction are similar.[15]

D. Treatment of Singularities

The Legendre functions of the second type, $Q_J(Z)$ are defined by

$$
\begin{aligned}
Q_0(Z) &= \tfrac{1}{2}\ln\left(\tfrac{Z+1}{Z-1}\right)\\
Q_{J+1}(Z) &= \left(\tfrac{2J+1}{J+1}\right) Z Q_J(Z) - \tfrac{J}{J+1} Q_{J-1}(Z) + \delta_{J,0}
\end{aligned}
\tag{8}
$$

Because the $Q_J(Z)$ have logarithmic singularities when $q = q'$ and $\mu = 0$, quantities of the form $\left(\lim_{\mu\to 0}\right)(-\partial/\partial\mu)^{i+1}Q_J(Z)$ require care in their interpretation.[13] As an example of our procedure for treating these singularities consider the linear confinement case. If $K(q,q')$ is finite as $q \to 0$ we may generalize the reasoning of II, starting with

$$\left(\lim_{\mu \to 0}\right)\left(-\frac{\partial}{\partial\mu}\right)^2 Q_0(Z) = P\frac{d}{dq}\left[\frac{1}{(q'+q)} - \frac{1}{(q'-q)}\right]$$

and then integrating by parts twice and using the boundary conditions gives

$$\begin{aligned}
&\left(\lim_{\mu\to 0}\right)\left(-\tfrac{\partial}{\partial\mu}\right)^2 \textstyle\int_0^\infty Q_0(z)K(q,q')\psi(q')dq' \\
&= -2/q\,K(0,q)\psi(0) - P \textstyle\int_0^\infty \left[\frac{1}{(q'+q)} - \frac{1}{(q'-q)}\right]\frac{\partial}{\partial q'}K(q,q')\psi(q')dq' \\
&= -2/q\,K(0,q)\psi(0) + \textstyle\int_0^\infty \ln\left(\left|\frac{q'+q}{q'-q}\right|\right)\frac{\partial^2}{\partial q'^2}K(q,q')\psi(q')dq' \\
&= -2/q\,K(0,q)\psi(0) + \left(\lim_{\mu\to 0}\right)\textstyle\int_0^\infty Q_0(Z)\frac{\partial^2}{\partial q'^2}K(q,q')\psi(q')dq'
\end{aligned} \qquad (9)$$

Together with the relationship

$$\left(\lim_{\mu \to 0}\right)\left(-\frac{\partial}{\partial\mu}\right)^2 ZQ_J(Z) = \left(\lim_{\mu \to 0}\right)\left(\frac{1}{qq'}Q_J(Z) + Z\frac{\partial^2}{\partial\mu^2}Q_J(Z)\right)$$

equations (8) and (9) define the quantities

$$\left(\lim_{\mu \to 0}\right)(-\partial/\partial\mu)^2 Q_J(Z).$$

E. Solution of the Equations

As a specific case we consider the IA for equal mass particles interacting only through a linear confinement kernel. In the notation of eq. (7b) we have

$$[E - 2E(q)]\,\psi_{L'}^{JS}(q) = \begin{pmatrix} \lim \\ \mu \to 0 \end{pmatrix} \sum_{L} \int_0^\infty dq' q'^{\,2} \left(-\frac{\partial}{\partial \mu}\right)^2 V_{L'L}^{JS}(q,q';\mu)\psi_L^{JS}(q')$$

To solve these equations the functions $\psi_L^{JS}(q)$ were assumed to have the convenient form $\psi_L^{JS}(q) = q^J \chi_L^{JS}(q)$. The functions $\chi_L^{JS}(q)$ were approximated as linear combinations of cubic B-splines with two continuous derivatives

$$\chi_L^{JS}(q) \approx \sum_{\mu=1}^{N} \beta_\mu^{JSL} B_\mu(q)$$

and the singular quantities $\binom{\lim}{\mu\to 0}(-\partial/\partial\mu)^2\, V_{L'L}^{JS}(q,q';\mu)$ were converted to integro-differential operators in the manner shown above. Then the coefficients β_μ^{JSL} were determined by a Galerkin[17] method chosen to yield symmetric matrices. The functions $B_\mu(q)$ are then defined in terms of N+4 (distinct) knots $\{\tau_j\}$ by a recursion relation.[18] For $j > 4$ these knots were chosen to be the images of the zeros $\{x_j\}$ of a Chebyshev polynomial

$$x_j = -\cos\left(\frac{(2j-1)}{2N\pi}\right)$$

under a mapping

$$\tau_{j+4} = \bar{q}\sqrt{\frac{1+x_j}{1-x_j}} + \delta.$$

For $j \leq 4, \tau_1 = 0$ and the remaining knots were chosen symmetrically so that

$$\tau_{4-j} = -\tau_{j+4}, \qquad j = 1,2,3$$

Satisfactory choices for $\bar{q}$ and δ, giving numerically stable results in all partial waves, were $\underline{a}rq = 0.5$ GeV and $\delta = 0.025$ GeV.

F. Evaluation of Integrals

In order to evaluate needed integrals of the form

$$\left(\lim_{\mu \to 0}\right)\left(-\frac{\partial}{\partial\mu}\right)^{i+1} \int_0^\infty dq' \frac{Q_J^{(N)}(Z)}{(2qq')} F^{(M)}(q,q') B_\mu(q') \tag{10}$$

the integrals

$$\left(\lim_{\mu \to 0}\right)\left(-\frac{\partial}{\partial\mu}\right)^{i+1} \int_{\tau_j}^{\tau_{j+1}} dq' \frac{Q_J^{(N)}(Z)}{(2qq')} q'^v F^{(M)}(q,q') \tag{11}$$

$$v = 0, 1, 2, \ldots$$

were evaluated with methods described below and the results substituted into the recursion relation defining the splines $B_\mu(q')$. To evaluate (11) each $F^{(M)}(q,q')$ was approximated on the interval $[\tau_j, \tau_{j+1}]$ by a cubic hermite spline $\tilde{F}_j^{(M)}(q,q)$

$$\tilde{F}_j^{(M)}(q,q') = \sum_{n=0}^{3} C_{nj} q'^n$$

satisfying both

$$\tilde{F}_j^{(M)}(q,q') = F^{(M)}(q,q')$$

and

$$\frac{\partial}{\partial q'} \tilde{F}_j^{(M)}(q,q') = \frac{\partial}{\partial q'} F^{(M)}(q,q')$$

at the end points $q' = \tau_j, \tau_{j+1}$. This yields the approximation

$$\begin{pmatrix} \lim \\ \mu \to 0 \end{pmatrix} \left(-\frac{\partial}{\partial \mu}\right)^{i+1} \int_{\tau_j}^{\tau_{j+1}} dq' \frac{Q_J^{(N)}(Z)}{(2qq')} q'^v F^{(M)}(q,q') \approx$$

$$\begin{pmatrix} \lim \\ \mu \to 0 \end{pmatrix} \left(-\frac{\partial}{\partial \mu}\right)^{i+1} \sum_{n=0}^{3} C_{nj} \int_{\tau_j}^{\tau_{j+1}} dq' \frac{Q_J^{(N)}(Z)}{(2qq')} q'^{n+v}$$

These integrals were evaluated analytically and the second (nonsingular) integration necessary for the Galerkin method was done numerically. An advantage of this procedure is that the quantities

$$\int_{\tau_j}^{\tau_{j+1}} dq' Q_0(Z) q'^m \quad m = 0, 1, 2, \ldots$$

and

$$\int_{\tau_j}^{\tau_{j+1}} dq' \, q'^m \qquad m = 0, 1, 2, \ldots$$

need to be evaluated only once and the evaluation of integrals of the form (10) becomes a matter of forming linear combinations of them.

Being able to evaluate the first integrals analytically is a major advantage of this method, however certain limitations need to be kept in mind. First, because this method deals with polynomials of moderately high degree in the momenta, it is desirable to choose units so that the values of the momenta characteristic of the problem are of order unity, which in this case means making $\bar{q}$ of order unity. Second, it was found desirable to use a method of solution for the equations which utilized symmetric matrices. This was the case because the method worked best for the upper triangular portion of the matrices and the lower triangular portion of the symmetric matrices can be obtained from the upper portion by reflection. Third, as is usually true when dealing with numerical solution of integral equations, it was found desirable to use double precision when working on a machine with a 32 bit word length.

G. Results and Discussion

Using a basis of 31 splines for each value of L we solved for eigenenergies and eigenfunctions of the IA and BbS reductions of the BS equation with Coulomb plus confinement kernels

$$\frac{-4\pi\, a\gamma_o\gamma_\mu \otimes \gamma_o\gamma_\mu}{(-(q-q')^2)} + 4\pi b\begin{pmatrix} \lim \\ \mu \to 0 \end{pmatrix}\left(-\frac{\partial}{\partial\mu}\right)^2 \frac{\gamma_o \otimes \gamma_o \Gamma \otimes \Gamma}{(-(q-q')^2 + \mu^2)} \tag{12}$$

The results are presented in Table 1. The calculation labeled IA(I) was done with

$\Gamma\otimes\Gamma = 1\otimes 1+\gamma_5\otimes\gamma^5$ and the remainder labeled with IA(II) and BbS with $\Gamma\otimes\Gamma = 1\otimes 1$. In columns eight and nine results for a related calculation[6], using a coordinate space kernel corresponding to (12) with $\Gamma\otimes\Gamma = 1\otimes 1$ are presented for comparison. All calculations are done with a parameter set given in Ref. 6. That is, we use a charmed quark mass of 1.25 GeV, a bottom quark mass of 4.58 GeV, a=0.25 and b=0.29 GeV^2.

As the purpose of this chapter is the treatment of general methods for solution of the BS equation with spin and singular kernels and not with the separate question of fitting the meson spectra, no attempt to optimize or adjust these parameters has been made. The calculations of Ref. 6 are most closely related to IA(II), but include coupling between positive and negative frequency states. The results of Ref. 6 are those obtained with a mixture of coordinate space and momentum space methods, using harmonic oscillator eigenfunctions as a basis. There is a general consistency among the three calculations performed using the IA, IA(I), IA(II) and also with the results of Ref. 6. With the exception of the 0^- states and 2^+ states the differences between the calculations are of the order of magnitude of the effect of our omission of coupling between negative and positive frequency states (5–10MeV quoted in Ref. 6). In the b-quarkonium spectrum we found, in addition to states corresponding to the observed 1^- states, which were primarily S states, two unobserved 1^- states having eigenenergies near 10100MeV and near 10450MeV in our calculations. These states were almost pure D states and their existence represents a prediction of these calculations.

Comparing the BbS calculation with the IA calculation using the same kernel, IA(II), we note that there is good agreement for the eigenenergies of b-quarkonium. However, in the charmonium case where the quark mass is lighter, the agreement between IA(II) and BbS rapidly deteriorates with increasing charmonium mass. For this choice of parameters the IA(I) and IA(II) provide a better phenomenological description of the quarkonia spectra than does the BbS. Both IA(I) and IA(II) also provide descriptions comparable to that provided in Ref. 6. The rms deviations between calculations and experiment for charmonium are 35MeV, 40MeV and 43MeV for IA(I), IA(II) and Ref. 6 respectively. These rms deviations are of the order of 1.0 percent of the total mass. The rms deviations for B-quarkonium are 48MeV, 49MeV and 53MeV for IA(I), IA(II) and Ref. 6 respectively. These rms deviations are of the order of 0.5 percent of the total mass. The differences between the IA results and the BbS results appeared to be due to the difference between the IA and BbS confinement kernels. This was determined by substituting the BbS kernel into the IA calculations and observing that the modified IA results were considerably closer to the BbS results.

In Figure 1 we give the radial dependence of the IA(II) eigenfunction for the first 0^- state of charmonium and in Figure 2 that of the second 0^- state. Likewise, in Figure 3 we give both the S and D components of the first 1^- state of charmonium and in Figure 4 those of the third 1^- state. Figures 3 and 4 demonstrate that L is not a good quantum number for this problem. It is also interesting to find a calculated 1^- state, which is primarily a D state, such as shown in Figure 4, in among other 1^- states which are primarily S states. These eigenfunctions ϕ are normalized such that $\int_0^\infty \phi^2(q)\ q^2 dq = 1$. It is also interesting to note that, at small values of the momentum, the eigenfunctions look much like harmonic oscillator eigenfunctions as might be expected for a confinement kernel. At larger values of the momentum, the eigenfunctions fall off slowly with the long low tail characteristic of momentum space Coulomb eigenfunctions. Noting that the x-axis for these figures is labeled in units of GeV/c and that we have taken the charmed quark mass to be 1.25 GeV, the relativistic nature of the problem is apparent. The relativistic nature of the problem is also apparent when comparing the eigenenergies in column IA(I) of Table (1), for scalar plus pseudoscalar confinement, with these in columns IA(II), and BbS for the corresponding IA and BbS calculation with scalar confinement. In the nonrelativistic limit these would all be equal, as is much the case with b-quarkonium. In the case of charmonium, the differences are already significant.

Another argument for the relativistic nature of charmonium is given in Table 2. There we give expectation values for a nonrelativistic velocity operator $v_{\rm NR} = q/m$ and a corresponding relativistic expression $v_{\rm REL} = q/E(q)$. From the fact that the values presented are all larger than 0.48c we conclude that the motion of the constituents in this model is substantially relativistic We therefore expect that the calculation of additional observables could readily yield results at variance with corresponding results from nonrelativistic models.

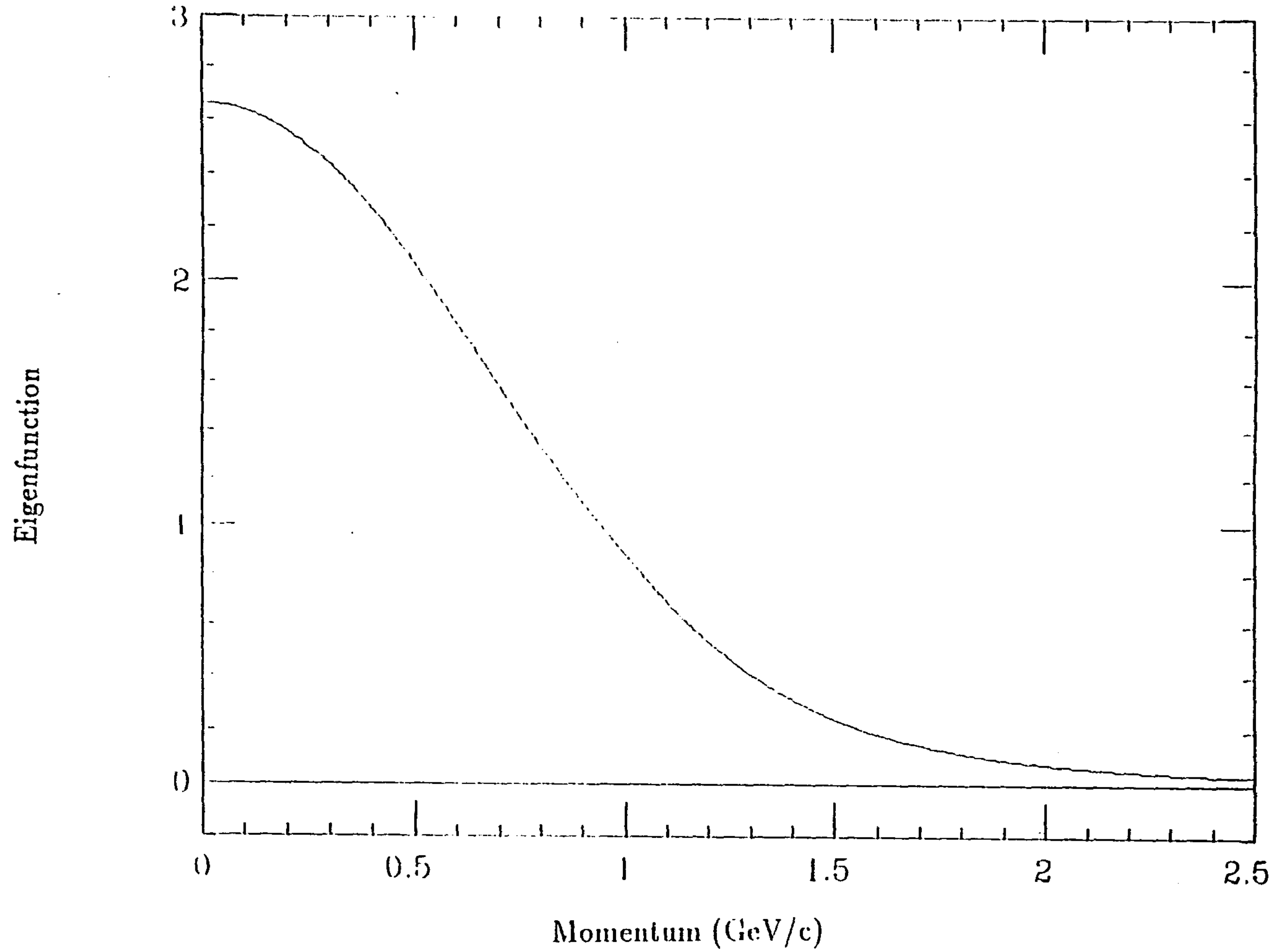

Figure 1. Charmonium 0^- First Eigenstate

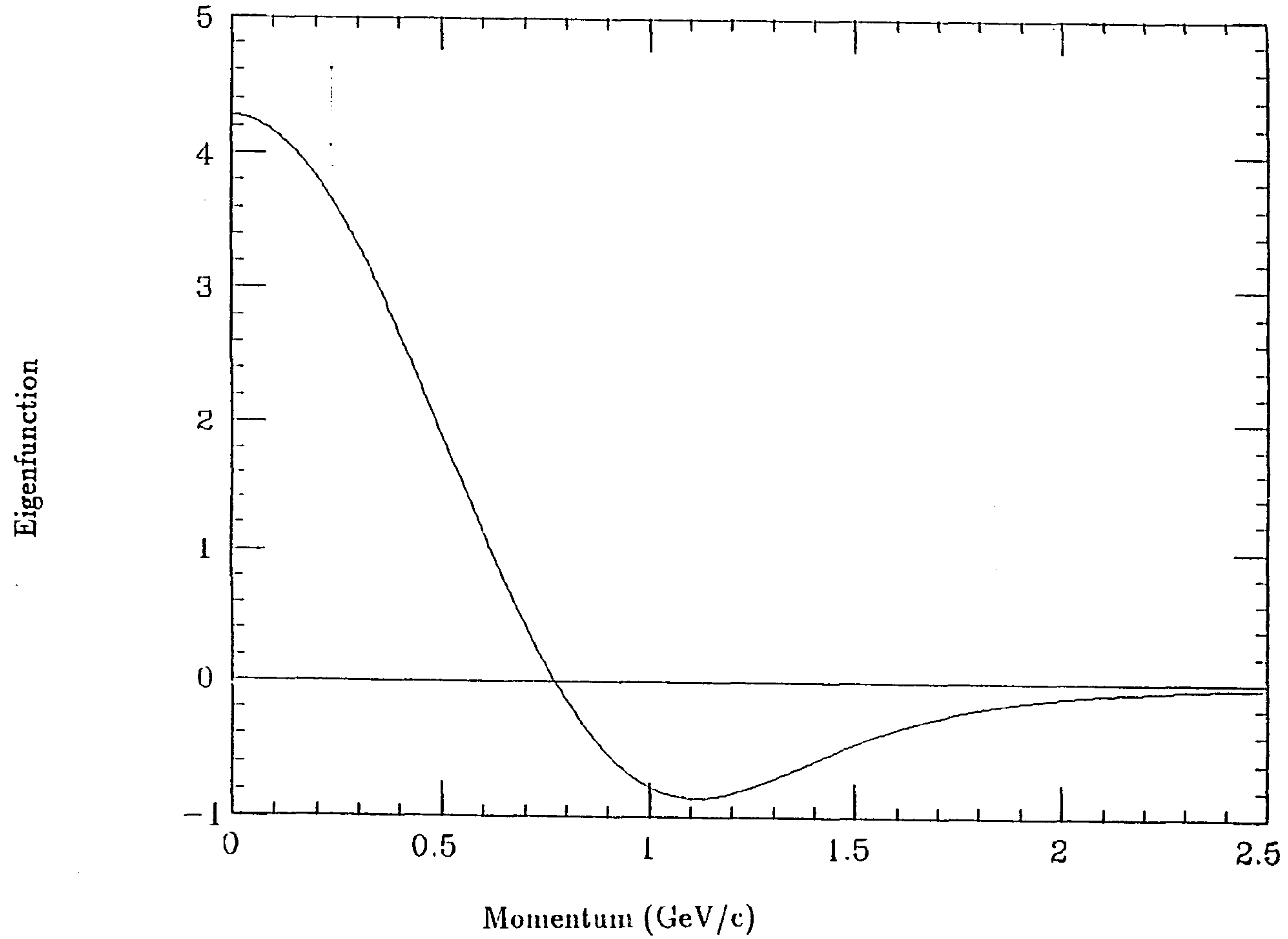

Figure 2. Charmonium 0^- Second Eigenstate

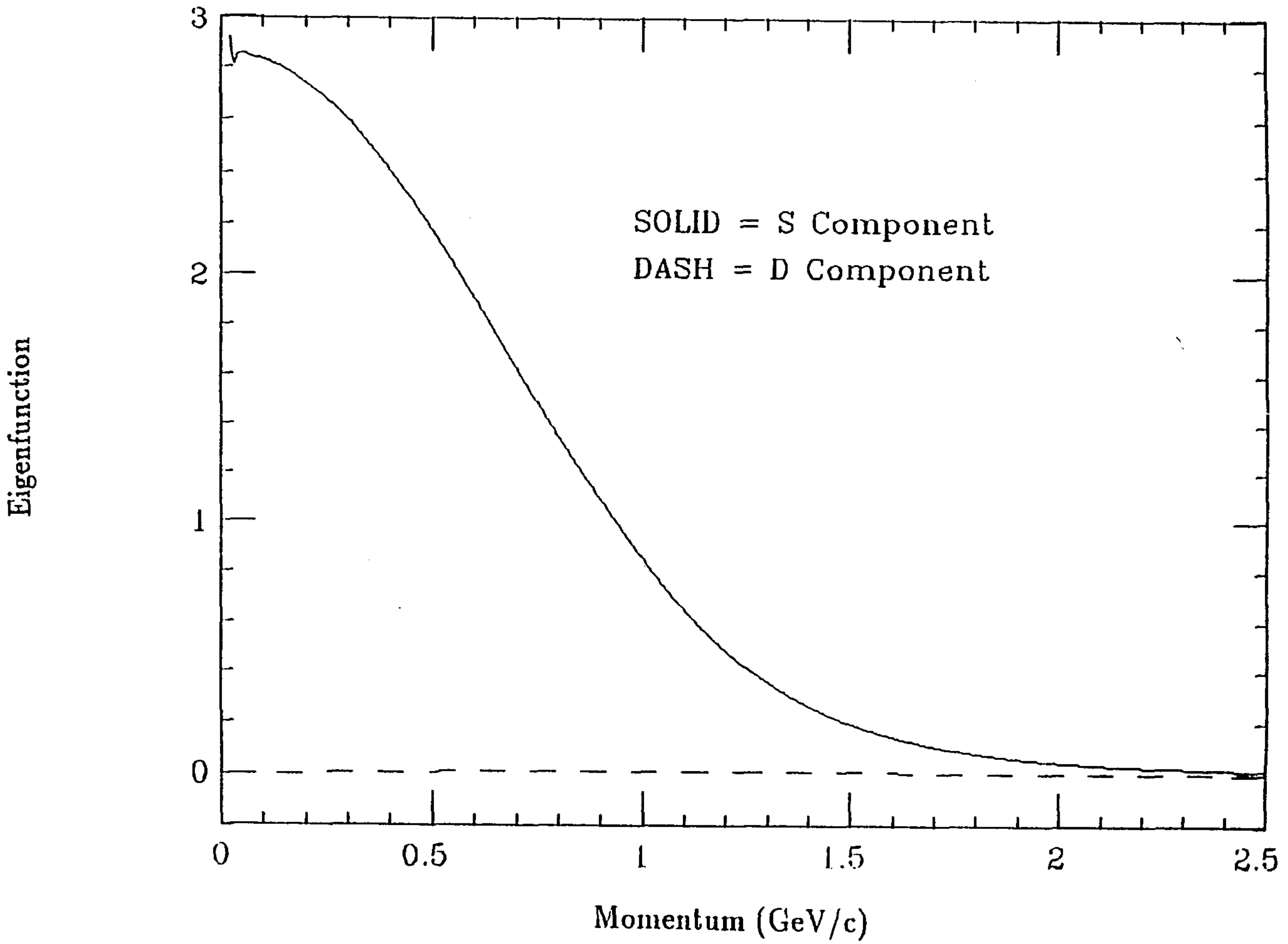

Figure 3. Charmonium 1^- First Eigenstate

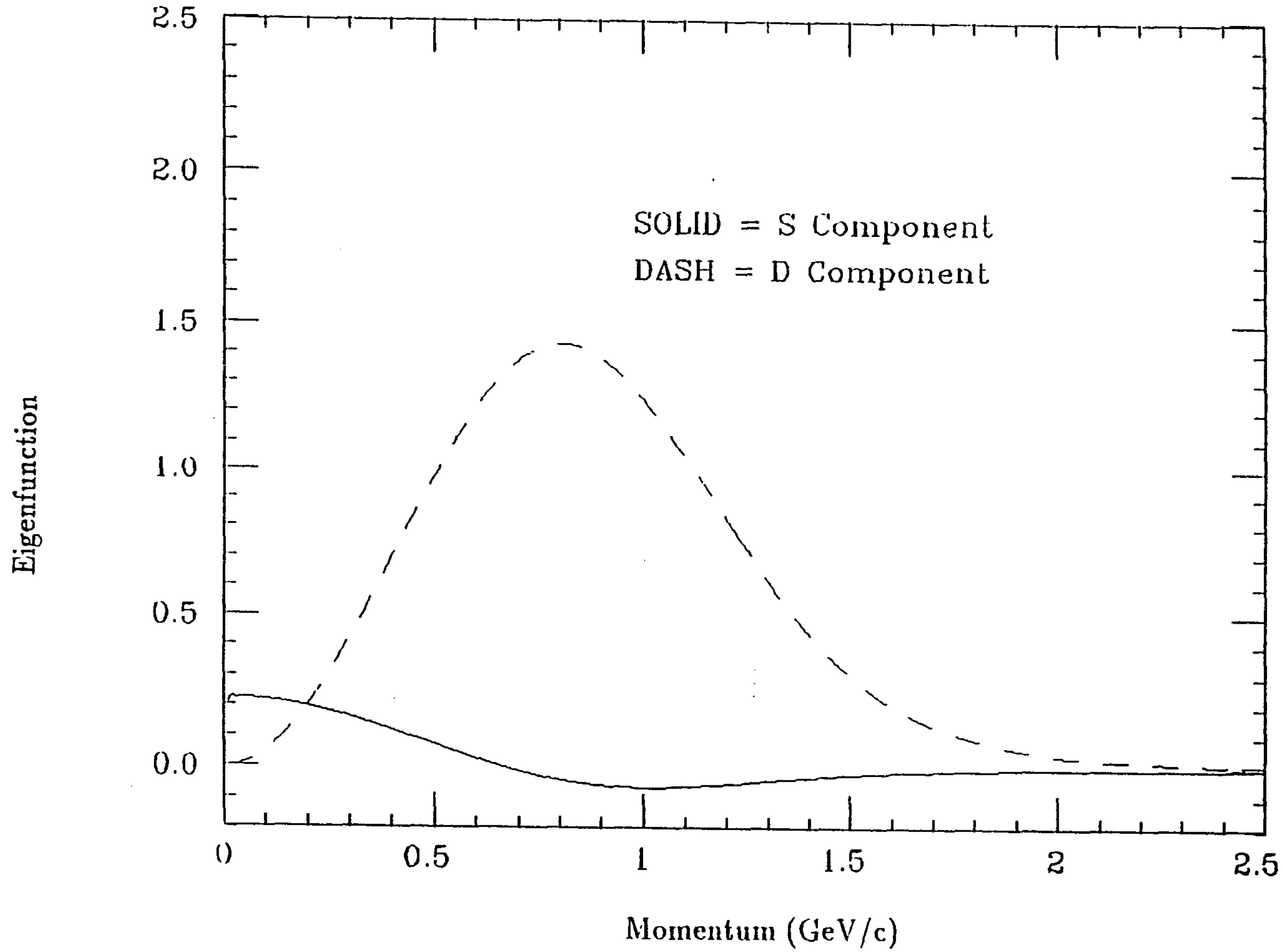

Figure 4. Charmonium 1^{--} Third Eigenstate

Table 1. Quarkonium Masses in MeV with kernels as given in text

			IA(I)		IA(II)		BbS		Ref. 6	
Meson	J^P	M_{EX}	M_{TH}	M_{TH} $-M_{\mathrm{EX}}$	M_{TH}	M_{TH} $-M_{\mathrm{EX}}$	M_{TH}	M_{TH} $-M_{\mathrm{EX}}$	M_{TH}	M_{TH} $-M_{\mathrm{EX}}$
η_c	0^-	2980	3003	+23	3049	+69	3064	+84	2966	−14
J/ψ	1^-	3097	3117	+20	3105	+8	3145	+48	3095	−2
χ_0	0^+	3415	3468	+53	3437	+22	3555	+140	3434	+19
χ_1	1^+	3510	3460	−50	3462	−48	3591	+81	3475	−35
χ_2	2^+	3556	3536	−20	3528	−28	3673	+117	3447	−109
η_c	0^-	3594	3628	+34	3651	+57	3782	+188	3622	+28
ψ	1^-	3685	3694	+9	3691	+6	3842	+157	3682	−3
ψ	1^-	3770	3749	−21	3741	−29	3953	+183	3735	−35
ψ	1^-	4030	4093	+63	4094	+64	4371	+341	4085	+55
ψ	1^-	4160	4130	−30	4127	−33	4449	+289	4119	−41
ψ	1^-	4415	4411	−4	4414	−1	4820	+405	4405	−10
Υ	1^-	9460	9480	+20	9480	+20	9476	+16	9471	+11
$1\,^3P$	0^+	9873	9826	−47	9825	−48	9836	−37	9822	−51
$1\,^3P$	1^+	9895	9839	−56	9842	−53	9854	−41	9837	−58
$1\,^3P$	2^+	9915	9909	−6	9907	−8	9926	+11	9843	−72
Υ	1^-	10023	10001	−22	10004	−19	10016	−7	9997	−26
	1^-		10097		10099		10126			
$2\,^3P$	0^+	10233	10227	−6	10229	−4	10255	+22	10225	−8
$2\,^3P$	1^+	10254	10238	−16	10244	−10	10273	+19	10237	−17
$2\,^3P$	2^+	10271	10300	+29	10299	+28	10328	+57	10244	−27
Υ	1^-	10356	10379	+23	10384	+28	10416	+60	10376	+20
	1^-		10443		10448		10492			
Υ	1^-	10573	10694	+121	10701	+128	10756	+183	10693	+120

Table 2. Expectation Value of Velocity Operators for Selected Eigenstates of Charmonium

J^P	M_{EX}	V_{NR}	V_{REL}
0^-	2980	.6278	.5005
0^-	3594	.7328	.5305
1^-	3097	.5829	.4898
1^-	3685	.7174	.5306
1^-	3770	.7903	.6054
1^-	4030	.8394	.5846
1^-	4160	.8813	.6261
1^-	4415	.9372	.6229

III. THE LORENTZ STRUCTURE OF THE CONFINEMENT KERNEL

In this chapter, which is based upon work being prepared for submission for publication in Physical Review D, we take the next steps in our program for developing a relativistic QCD inspired model for the hadrons. In the previous chapter we developed general methods for solution of the BS equation with spin for a OGE plus linear confinement kernel having arbitrary Lorentz structure. In this chapter we attempt to determine the Lorentz structure of the confinement kernel and to generalize our earlier treatment so as to be applicable eventually to treatment of the light mesons.

In developing a relativistic description of the mass spectra and amplitudes of elementary particles based upon the BS equation with a phenomenological confining interaction, it is important to generalize previous methods for solving the BS equation in momentum space to include the coupling of positive and negative frequency states. The reason for this is, as noted earlier, for the lighter mesons the effects of the mixing of positive and negative frequency components becomes very large. Hence, for an attempt to describe both the light mesons and the heavy mesons with the same equation, inclusion of such coupling is crucial.

A. The Equations and Their Solution

As before we work in the center of momentum frame and use a three-dimensional reductions of the BS equation, the instantaneous approximation, known as Salpeter's equation[1,10]. At first sight the inclusion of the coupling between positive and negative frequency states would seem to offer no difficulty. The methods of the previous chapter can be used to convert the IA with coupling between positive and negative frequency states into a matrix equation. When this is done Salpeter's equation may be written as a matrix equation of the form

$$\begin{pmatrix} R & T \\ -T & -R \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = E \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{13}$$

The difficulty arises from the fact that the matrix

$$\begin{pmatrix} R & T \\ -T & -R \end{pmatrix}$$

is nonhermitian and has trace equal to zero. These facts make the generalized eigenvalue problem of equation (13) numerically unstable. A satisfactory solution has been found by starting with the generalized eigenvalue problem for the square of the energy. Writing equation (13) as

$$Ax = EBx$$

the equation for the square of the energy is

$$E^2 Bx = AB^{-1}Ax.$$

For light mesons this equation is still somewhat unstable numerically. However the equation

$$E'^2 Bx = (AB^{-1}A + cB)x$$

with

$$E^2 = E'^2 - c$$

has been found to be quite stable for c greater than or equal to 10 Gev. The roots of these equations for E^2 come in pairs which are all positive for the heavy mesons. These represent pairs of eigenvalues (positive and negative) of E, which correspond to mesons and antimesons. For light mesons, some pairs of roots of E^2 are negative, corresponding to pairs of purely imaginary eigenvalues for the energy. Following an earlier work[6], we choose the basis so that these imaginary roots are far from the real roots of interest in the complex E plane and then discard the imaginary roots. Presumably, these spurious solutions would be absent in a more complete treatment of the meson spectra.

B. Results and Discussion

Using a basis of 31 splines for each value of L we solved for eigenenergies of the IA reduction of the BS equation with Coulomb plus confinement kernels

$$\frac{-4\pi\, a\gamma_o\gamma_\mu \otimes \gamma_o\gamma_\mu}{(-(q-q')^2)} + 4\pi b \begin{pmatrix} \lim \\ \mu \to 0 \end{pmatrix} \left(-\frac{\partial}{\partial\mu}\right)^2 \frac{\gamma_o \otimes \gamma_o\, \Gamma \otimes \Gamma}{(-(q-q')^2 + \mu^2)} \qquad (14)$$

The results are presented in Table 3. The calculation labeled IA(I) was done with $\Gamma \otimes \Gamma = 1 \otimes 1 - \gamma_5 \otimes \gamma^5$, that labeled IA(II) with $\Gamma \otimes \Gamma = 1 \otimes 1 + \gamma_5 \otimes \gamma^5$, IA(III) with $\Gamma \otimes \Gamma = 1 \otimes 1$, and IA(IV) was done with $\Gamma \otimes \Gamma = 1 \otimes 1 + \gamma_\mu \otimes \gamma^\mu$. Thus IA(I) corresponds to scalar minus pseudoscalar and IA(II) to scalar plus pseudoscalar confinement. The case IA(III) is pure scalar and IA(IV) is composed of equal parts scalar and vector. The values of the parameters a,b and the charmed and bottom quark masses were determined for each case separately by a least squares fit to the first six 1^- states of charmonium, the first three observed 1^- states of b-quarkonium and the first two 0^- states of charmonium. Our procedure is to use these states to determine a set of parameters for each choice of Lorentz structure of the confinement kernel and then to observe how well each of the corresponding spectra agree with experiment. As before, in the b-quarkonium spectrum we found, in addition to states corresponding to the observed 1^- states, two unobserved 1^- states having eigenenergies near 10100MeV and near 10450MeV in our calculations. We again give their eigenenergies as a prediction of these calculations. In Table 4. we give the parameters for these fits, and the root mean square deviations between theory and experiment.

The fact that these fits are not quite as good as the description presented in the previous chapter is a reflection of the importance of the coupling to the negative frequency states. Further fitting strategies need to be investigated as well. For example one may include more states in the fit but a new issue then arises: the fit could be distorted if a state in the theoretical spectrum of low lying states has not yet been observed experimentally. This would lead to to fitting the wrong eigenstate to a given observed state.

C. Conclusions

Inspecting Table 3. and Table 4. we conclude that IA(III), the pure scalar case, gives the best fit. Testing in this way, we conclude that if confinement is to be described as in equation (14), the Lorentz structure of the confinement kernel is largely or completely scalar. Put briefly, addition of a substantial vector or pseudoscalar portion to the confinement kernel shifts the relative spacings of the eigenstates so as to reduce the quality of the fit to the heavy meson spectra. We determined that the origin of this phenomenon is the coupling of positive and negative frequency states by doing a similar fit without including the coupling. That the origin is this mixing, is illustrated by comparison of IA(I) in Table 1. with IA(II) Table 3. We note that in the calculations of Chapter II, in which this coupling was ignored, addition of a pseudoscalar term to the confinement kernel produced an improved fit. Because this pronounced deterioration in the quality of the fit with addition of a substantial vector or pseudoscalar portion to the confinement kernel is due to coupling of negative and positive frequency states it may not be evident in the lowest order nonrelativistic reduction of the Bethe-Salpeter equation.

D. Outlook

For Coulomb and confinement potentials the combination of procedures outlined here seems to satisfy the criteria of accuracy and stability in a small spline basis. Because one integration was done analytically the amount of computer time was reasonable. The final version of the program took between five and twenty-five minutes of CPU time on a VAX8600 for each partial wave. Because the analytical evaluation of the singular first integration was reduced to forming linear combinations of certain fundamental integrals, the procedures described here are highly suited for vectorization.

Given a stable and accurate procedure for solution of the BS equation and knowledge of the Lorentz structure of the confinement kernel there are several interesting issues to be investigated. First, how well can the entire meson spectrum be described with a flavor independent kernel? Second, do the transition rates and light meson distribution amplitudes obtained with these methods agree with experiment, lattice gauge calculations, and sum rule results? We are proceeding to investigate these questions.

Also underway is an attempt to adapt these methods in order to search for resonances in the positronium continuum postulated as an explanation of some anomalous results in heavy ion experiments. The continuum problem with a one photon exchange kernel is much more difficult than the corresponding bound state problem because in addition to having singularities in the kernel of the BS equation one has singularities in the solution to the BS equation.

Table 3. Quarkonium Masses in MeV. with kernels as given in text

			IA(I)		IA(II)		IA(III)		IA(IV)	
Meson	J^P	M_{EX}	M_{TH}	M_{TH} $-M_{EX}$	M_{TH}	M_{TH} $-M_{EX}$	M_{TH}	M_{TH} $-M_{EX}$	M_{TH}	M_{TH} $-M_{EX}$
η_c	0^-	2980	3003	+23	3187	+107	2969	−11	3070	+90
J/ψ	1^-	3097	3154	+57	3147	+50	3103	+6	3060	−37
χ_0	0^+	3415	3574	+159	3890	+475	3421	+6	3250	−165
χ_1	1^+	3510	3398	−112	3290	−213	3471	−39	3555	+45
χ_2	2^+	3556	3177	−379	3271	−285	3535	+21	3597	+41
η_c	0^-	3594	3865	+269	3490	−104	3631	+37	3347	−247
ψ	1^-	3685	3468	−217	3551	−134	3689	+4	3631	−54
ψ	1^-	3770	3756	−14	3714	−56	3744	−26	3749	−21
ψ	1^-	4030	3999	−31	4016	−14	4080	+50	4078	+48
ψ	1^-	4160	4106	−54	4049	−111	4114	−46	4178	+18
ψ	1^-	4415	4134	−281	4128	−287	4377	−38	4459	+44
Υ	1^-	9460	9371	−89	9482	+22	9457	−3	9501	+41
$1\,^3P$	0^+	9873	9789	−84	9806	−67	9815	−58	10039	+166
$1\,^3P$	1^+	9895	9832	−63	9823	−72	9834	−61	9920	+25
$1\,^3P$	2^+	9915	9919	+4	9879	−36	9907	−8	10008	+93
Υ	1^-	10023	9981	−42	9993	−30	9995	−28	10037	+14
	1^-		10102		10077		10098		10178	
$2\,^3P$	0^+	10233	10208	−25	10214	−19	10245	+12	10470	+237
$2\,^3P$	1^+	10254	10259	+5	10223	−31	10243	−11	10294	+40
$2\,^3P$	2^+	10271	10235	+29	10144	−127	10197	−74	10297	+16
Υ	1^-	10356	10385	+29	10368	+12	10381	+25	10398	+42
	1^-		10470		104421		10450		10498	
Υ	1^-	10573	10709	+136	10679	+109	10702	+129	10696	+123

Table 4. The Parameters Obtained By Fitting the Meson Spectrum With Kernels Having the Lorentz Structures Described in the Text

	M_C	M_B	a	b	DEV_{rms}
IA(I)	1.376	4.590	0.3581	0.3321	143
IA(II)	1.444	4.504	0.1604	0.3388	228
IA(III)	1.256	4.580	0.2666	0.2965	43
IA(IV)	1.287	4.718	0.3825	0.1134	96

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