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# Uncovering the "will of the people": heterogeneity and polarization within electorates

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**Keywords**
Condorcet cycles, voting paradoxes, plurality, Borda Count, polarization

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# Uncovering the ”Will of the People”: Heterogeneity and Polarization within electorates

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Very preliminary, for comments only

**Abstract**

The present paper examines when, for any given preference profile or set of individual preference orders, is it possible to define a *procedure independent* or *objective* aggregate ranking of the alternatives, such that the aggregate ranking qualifies as the ”will of the people”. It also investigates what message is being conveyed by the profile, when different procedures come up with different aggregate rankings. Specifically, the paper establishes a profile decomposition methodology that allows us to answer these two questions and tests this methodology on ballot data from the Cambridge City Council elections. Our method is easy to implement and admits any number of candidates. The empirical results based on the

*We wish to thank the Cambridge City Council, MA for providing the ballot data for this study and Lesley Waxman in particular for helping us gain access to this data. Any opinions, conclusions or recommendations expressed in the paper are, however ours and do not reflect the views of the Cambridge City Council.

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JEL Numbers: D70, D71, D72

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# 1 Introduction

Aggregation paradoxes that arise when a collective ranking of multiple alternatives is constructed from a set of individual preference orders on them, pose significant challenges for a democracy. Two such problems have drawn the most attention from scholars, beginning with the 18th and 19th century writings of Borda (1781), Condorcet (1785), Dodgson[1] (1884) and Nanson (1882). First, for a fixed set of the individual voters' rankings of the alternatives, different aggregation procedures may yield different aggregate or social rankings of the alternatives with no natural reason to prefer one procedure over another. Second, under any given aggregation procedure, rank reversals in the aggregate ranking may occur when alternatives are dropped or added. We describe these problems, henceforth as *procedure dependency* and *internal inconsistency*, respectively.

As the first ranked alternative in an aggregate ranking is most often the chosen or elected alternative in a democratic process, such problems imply that the (choice) outcome may reflect the somewhat arbitrary choice of the procedure rather than the true "will of the people" and, further, can be manipulated. Two existing theorems in social choice theory illustrate the potentially extreme form of this arbitrariness. One states that *with more than three candidates (alternatives), a profile of individual preferences may be found, such that each candidate or alternative is top ranked, second ranked... last ranked in the aggregate ranking with an appropriate choice of an aggregation method.* (Theorem 9, Saari 2000b). The other says, *a profile may be found such that the voters' sincere plurality ranking of each subset of candidates matches an arbitrarily chosen one.* (Theorem 1, Saari 2000b). Thus, the profile or set of individual preference orders is fixed, implying that the "will of the people" - whatever it is - is constant. However, an attempt to express it by means of an aggregate ranking produces different results under different procedures and for different subsets of the candidates. In fact, a most natural question under the circumstances is, *what is the "will of the people"*.

A broad objective of the present paper is to examine when can a clear answer to the above question be provided. It explores, firstly, if, for any given profile of individual preference orders, it is possible to define a *procedure independent* (all procedures providing the same answer) or *objective* aggregate ranking of the alternatives, such that this ranking may be described as the "will of the people". Secondly, we investigate what message, from a collective perspective, is conveyed by a profile, when different procedures come up with different aggregate rankings. Specifically, the paper establishes an implementable methodology that decomposes any given profile into component profiles with *meaningful collective characteristics* and enables

[1] better known as "Lewis Caroll", the author of *Alice in Wonderland*

us to answer these two questions. Further, we test this methodology on profiles of individual preference orders revealed by the ballot data from the Cambridge City Council, Massachussetts, elections.

By showing that no aggregation method simultaneously satisfies even a minimal set of four desirable properties - *unrestricted domain, unanimity, independence of irrelevant alternatives (IIA) and non-dictatorial* - Arrow's seminal work (1951) demonstrates why no method is naturally better than others. The literature on aggregation methods has developed in multiple directions since then. A line of research focuses on comparative studies of the procedures, specially with respect to aggregation properties that are often deemed essential or desirable, including but not limited to the four laid out by Arrow.[2] A second line of research following Black's (1958) seminal paper, investigates whether some of these properties that are incompatible on the unrestricted domain of all possible rankings of the alternatives are compatible on a (proper) subset of the set of all possible rankings, at least. Amongst the major contributors are May (1952), Dasgupta and Maskins (2008).

A very recent and complementary approach, pioneered by Saari (1999; 2000a; 2000b amongst others) and described as geometric voting theory, explores whether some of these properties are compatible on the the set of all possible rankings but for a *restricted* class of profiles or distribution of voters across all possible rankings. Hodge and Klima (2005) and to some extent, Balinski and Laraki (2010) provide expositions of the linear algebraic framework, techniques and many of Saari's important results. The present paper adopts this approach as the most useful one for addressing our questions.

A highlight of this linear algebraic approach and one of Saari's most important contributions is the idea that a given preference profile (or distribution of voters across all possible rankings) can be expressed as a sum of different types of component profiles such that specific component profiles influence the outcome of specific aggregation procedures but not of others. For example, Saari identifies component profiles that influence aggregation outcomes under any specific procedure for different subsets of candidates and profiles that influence outcomes under pairwise but not under positional aggregation methods.[3] The profile decomposition approach can be adapted to serve different objectives. It can be used as Saari has used it, to gain a deep understanding of the aggregation paradoxes and why they occur. With this goal in mind, he constructs a set of orthogonal component profiles that would act as a basis for the space of all possible profiles, with

[2] Major contributions include the work of Sen and Pattanaik (1969), Young (1974, 1975, 1988), Young and Levenglick (1978), Smith (1973), Fishburn (1984), Moulin (1988), Nurmi (2004) amongst others. Nurmi (1996, 2002), Balinski and Laraki (2010), and Brahms(2002) provide comprehensive and excellent surveys of this vast literature. One way to select an appropriate procedure is to recognize that aggregation methods serve two very different purposes - *determining a collective winner amongst several alternatives* and *providing a collective or social ranking of the alternatives*. Thus some methods are more appropriate for determining winners and some for ranking candidates. Balinski and Laraki (2010) provides a detailed and comprehensive treatment of this debate.

[3] The first type of profiles are described by him as Departure profiles. The second type of profiles are the well known Condorcet profiles.

specific basis components influencing specific procedures but not others. This is a computationally demanding task and requires, amongst other things, the construction and inversion of a matrix of $n!$ dimension, for a $n$ candidate field.

Alternatively, as we do in this paper, the approach can be used to decompose a given profile into only component profiles that admit *distinctive collective identities*. When, for any given profile, different aggregation procedures arrive at different answers, such components convey information about the electorate that is useful. The main methodological contribution of this paper is an easily implementable technique to do this. The advantages of our technique is that it allows any number of candidates without introducing the curse of dimensionality mentioned above and further, avoids references to component profiles that may help explain aggregation paradoxes but otherwise admit little or no collective character.[4] We follow up our methodological contribution with an analysis of the Cambridge City Council, Massachussetts, ballot data for the period 1997-2011, using our techniques.

The paper focuses on three types of component profiles with meaningful collective characters. The first is the long well known *Condorcet profile*, a set of cyclic rankings, each supported by an equal number of voters. Under such a profile, each candidate is supported in each position by an equal number of voters, thus indicating a form of preference *heterogeneity*. The second type of component profile is described by us as *Reverse profiles* and assigns an equal number of voters to each ranking that has a specific candidate in the first and last places. Such profiles, identified early on by Young (1975) amongst others, indicate a form of preference *polarization* within an electorate, as they reflect blocks of voters with diametrically opposite preference rankings. Thus the weights of the Condorcet and Reverse profiles within a given profile measure how fragmented society is, in *two different senses*. Heterogeneity indicates diversity of preferences, polarization indicates clustering into groups of opposites. A useful feature of our approach is that it distinguishes between these two different ways in which society may be divided and provides characterizations and measures of both.[5] A substantive contribution of our paper is that, based on preferences revealed by a section of the US voters, it provides hard evidence of increasing political polarization.

The third type of component profile with a distinctive collective character and identified by Saari, is the *Basic* profile.[6] Under a Basic profile there are more voters who rank a specific candidate in the first place compared to the number of voters who rank each of the other alternatives in the first place and no voter ranks the specific candidate in question in the last place. A most interesting property of a Basic profile is

[4]Saari (2000b) notes for instance, that Departure profiles that cause internal inconsistency have "sufficiently bizarre" forms and hence highly unlikely to ever arise. See Remark 1 below, for more discussion.

[5]A similar distinction is made in the literature on income distribution. See Esteban and Ray (1994) amongst others.

[6]See Saari (1999; 2000a,b) amongst others. Although he is credited with introducing and characterizing Basic profiles, the decomposition method used in these papers cannot be easily used to extract the weights of these profiles - a gap filled by our work.

that on such a profile, under *any* aggregation procedure, the specified candidate is always placed first in the aggregate ranking and all the other candidates are tied for the second place. Thus all procedures agree on Basic profiles. Further, the aggregate ranking thus obtained, satisfy Arrow's IIA property and hence is consistent over any subset of candidates.

As all procedures agree on Basic profiles, an objective or procedure independent aggregate ranking can be obtained for any given profile, *if* the latter can be expressed either as a pure combination of Basic profiles or as a combination of Basic, Condorcet and Reverse profiles, where the weights of the latter are not significant relative to the weights of the Basic profiles. As the weights of the Condorcet and Reverse profiles get larger relative to the Basic profile weights, aggregation outcomes under different procedures become more disparate from each other and move farther away from the aggregation outcome on Basic profiles. Thus the higher the weights of the Condorcet and Reverse components of a profile - that is, the more fragmented society is - the more diluted is the message conveyed by the aggregate ranking based on Basic profiles.

An aggregate ranking based on Basic profiles has the twin advantage of being procedure independent and consistent over subsets, implying that such a ranking cannot be manipulated through choice of procedure or strategic participation in races. Further, both Condorcet and Reverse profiles feature certain symmetries which justify interpreting them as complete ties amongst the candidates so far as elections are concerned, under the *impartiality* or *equal treatment of voters* argument. Thus, impartiality requires that Condorcet and Reverse profile components be ignored and an election be declared based on the aggregate ranking obtained from Basic profiles only. However, a sizable presence of Condorcet and Reverse profiles also indicate that society is very divided. We submit that although an aggregate ranking that is impartial and non-manipulable is the best that we can do under the circumstances, we should not designate it, the "will of the people". Thus, an additional benefit of a profile decomposition technology is that by providing us with the relative weights of all three types of profiles, it also enables us to understand, *when is aggregation meaningful*.

The structure of the rest of the paper is as follows. Section 2 presents the existing concepts and tools of geometric voting theory that are useful for us. Sections 3 and 4 introduce Reverse profiles, our main theoretical results and the decomposition technology. The weights of the Condorcet and Reverse components of a profile provide natural measures of heterogeneity and polarization among an electorate that are discussed in Section 5. Finally, in Section 6, we test our method and measures on ballot data from the Cambridge City Council, Massachussetts, elections.

# 2 The algebra of geometric voting theory

## 2.1 Basic and Condorcet profiles

Voters have strict transitive preferences over $n$ candidates indexed $i = 1 \dots n$. Hence there are $n!$ different ways of ranking these candidates. Assume an electorate of a given and fixed size. A profile $p = (p_1 \dots p_{n!}) \in \mathbf{R}_+^{n!}$ is a distribution of voters across these rankings, with $p_j =$ the number of voters with preferences given by the $j$th ranking of the candidates. A profile differential $p' \in \mathbf{R}^{n!}$ is the difference between two different profiles for an electorate of a given size. Thus $p'$ may have negative components and further, the components of $p'$ add up to zero.[7]

An analytically useful type of profile with a collective character is the $K^n \in \mathbf{R}_+^{n!}$ profile, that has one voter for each possible ranking - in other words, a $K^n$ profile characterizes an equitable distribution of voters across all possible rankings. Further, a $K^n$ profile yields a tied outcome across all candidates under any procedure and hence the relative ranking of candidates are not affected by addition or subtraction of a $K^n$ profile to any other given profile, under any procedure. Addition of a $K^n$ profiles to other profiles changes the aggregate tallies but not the differences in tallies between any two candidates.[8]

Assuming that the total number of voters distributed over the $n!$ rankings is $V$, it is useful to view any given profile $p$ as a perturbation from a $\frac{V}{n!}K^n$ profile. In other words, it is convenient to define $p$ as $\frac{V}{n!}K^n + p'$ for some profile differential $p' \in \mathbf{R}^{n!}$. For example, consider the profile $p = (13, 11, 9, 8, 11, 0)$ of 52 voters with three candidates described in Table 2. $p$ can be expressed as $\frac{52}{6}K^3 + p'$ where $p' = (\frac{26}{6}, \frac{14}{6}, \frac{2}{6}, -\frac{4}{6}, \frac{14}{6}, -\frac{52}{6})$. To understand the usefulness of viewing a profile thus (as a sum of a weighted $K^n$ profile and a profile differential), first note that as $\frac{V}{n!}K^n$ has completely tied outcomes for all candidates under any procedure, the profile differential $p'$ yields the same ranking of the candidates as $p$ under any procedure. That is under any procedure, the tallies of $p$ and $p'$ may differ but the ranking outcome is the same. Thus for analytical purposes, a profile $p$ and an appropriate profile differential $p'$ are equivalent. Moreover, viewed thus, a profile $p$ of an electorate of size $V$ is obtained from an equitable distribution of the voters (a $\frac{V}{n!}K^n$ profile) by moving voters away from specific rankings and adding them to other specific rankings. That is, any profile $p$ is a result of "padding" and "thinning" of specific rankings of an equitable distribution of voters. This view is particularly useful for us as the various types of component profiles introduced below are structured "padding" and "thinning" of a weighted $K^n$ profile.

[7]In presenting the necessary concepts from geometric voting theory, we stick to Saari's terminology for the most part and sometimes his notations, in recognition of the pioneering nature of his work. We do however, occasionally deviate and use our own names, if we feel that these are more intuitive or if a subtle distinction in our specific context is necessary.

[8]There are many other types of profile with the feature that under *any* procedure, they produce completely tied outcomes. Saari(2000a,b) terms the set of such profiles as the Universal Kernel.

The preceding discussion shows that a profile differential rather than a profile can be used as the analytical building block of the geometric approach. The advantage of this (as noted by Saari) is that as a profile differential $p'$ is orthogonal to $K^n$, a decomposition of $p'$ does not include neutral $K^n$ effects. An alternative and useful view of a profile differential is that it is a profile with the number of voters normalized to zero.

To understand the structure of a Basic profile first fix a candidate, say $i$. Take a $K^n$ profile and shift a voter from each ranking which has $i$ last ranked and add the voter to a ranking which has $i$ first ranked, taking care not to add more than one such voter to a ranking. The profile $p(i)$ thus obtained equals $K^n + p'(i)$ where the profile differential $p'(i)$ has one voter for each ranking that has $i$ top ranked, (-1) voter for each ranking that has $i$ bottom ranked and 0 voter for each ranking that has $i$ ranked somewhere in the middle. We define the profile differential $p'(i)$ as a *Basic profile* favoring candidate $i$ and denote it by $B_i^n$. Following our previous discussion, $p(i)$ and $B_i^n$ yield the same ranking of the candidates and are interchangeable. Thus the term Basic profile may be used to refer to either. Under a Basic profile, candidate $i$ is some voter's first choice and nobody's last choice. Further, the number of voters who rank candidate $i$ first is greater than the number of voters who rank any other candidate $j$, first. Thus under any pairwise or positional procedure, the aggregate ranking for this profile will have the $i$-th candidate top ranked and everyone else tied in the second place. Voters making up $B_i^n$ collectively like the $i$-th candidate more than any other candidate and are completely indifferent across the others. By a slight stretch of the imagination (and perhaps abuse of terms), a Basic profile may be used to describe the preferences of the voters making up the popular "base" of a candidate.

Different Basic profiles are easy to aggregate. Suppose a given profile $p$ can be expressed as, $p = \frac{V}{n!}K^n + a_1B_1^n + a_2B_2^n + \ldots + a_nB_n^n$, where $a_i$'s are given constants. The pairwise score difference (see the next subsection) between candidates $i$ and $j$ can be shown to be $a_i - a_j$. The difference in positional tallies under any positional procedure can also be shown to be a multiple of $a_i - a_j$, with a common multiplier across all pairs. Thus under any procedure, in an aggregate ranking, the relative rank of two candidates depend only on their relative Basic profile weights and are unaffected by the presence of a third candidate. For a profile that is a sum of Basic profiles, as the relative rank of the $(i, j)$ pair in an aggregate ranking under any procedure depends only on $(a_i - a_j)$, it cannot be altered by an appropriate choice of procedure or strategic participation by other candidates in the race. We therefore claim that under such a profile, the difference between the two candidates or their relative ranking is "true" or objective.

To define a Condorcet profile, first specify and fix a *reference ranking* of the candidates, say $1 > 2 > 3 \ldots > n$, and denote this ranking as (1). A *Condorcet n-tuple* generated by the reference ranking (1), $c_{(1)}^n$, is the set of $n$ rankings of the candidates described by the first column of Table 1. The reverse of this set, $\rho(c_{(1)})^n$, is another set of $n$ rankings of the candidates described by the second column of Table 1. The sets $c^n_{(1)}$ and $\rho(c_{(1)})^n$ are thus two specific sets of cyclic rankings of the candidates, with the feature that each ranking in the set $\rho(c_{(1)})^n$ is a reversal of a ranking in $c^n_{(1)}$. There are many distinct Condorcet $n$-tuples in the set of $n!$ possible rankings of the candidates, each with an associated reversal set and each identified by its first or reference ranking.

Table 1:

| $c^n_{(1)}$ | $\rho(c_{(1)})^n$ |
|---|---|
| $1 > 2 > 3 \ldots > n$ | $n > n-1 > n-2 \ldots > 1$ |
| $2 > 3 > 4 \ldots > 1$ | $n-1 > n-2 > n-3 \ldots > n$ |
| $3 > 4 > 5 \ldots > 2$ | $n-2 > n-3 > n-4 \ldots > n-2$ |
| $\ldots$ | $\ldots$ |
| $n > 1 > 2 \ldots > n-1$ | $1 > n-1 > n-2 \ldots > 2$ |

A *Condorcet profile* $C^n_{(1)}$ associated with the reference ranking $(1)$, is a profile that has one voter for each ranking in $c^n_{(1)}$ and (-1) voter for each ranking in $\rho(c^n_{(1)})$ and zero voter for each remaining ranking in the profile. $C^n_{(1)}$ has the same structure and tallies as $aK^n$ profile with a voter moved away from each of the $\rho(c^n_{(1)})$ rankings and added to each of the $c^n_{(1)}$ rankings. Under a $C^n_{(1)}$ profile, *each candidate is placed in each position by exactly the same number of voters*. Thus under any positional method, such a profile produces a complete tie - a feature that is common to both the $K^n$ profile and Condorcet profiles. A $K^n$ profile however also produces a pairwise tie for every candidate pair as a ranking and its reversal are supported by the same number of voters. This is not true of Condorcet profiles as voters have been shifted from the $\rho(c^n_{(1)})$ rankings and added to the $c^n_{(1)}$ rankings.

As a Condorcet profile is uniquely defined by its first reference ranking, a $n$ candidate field has $\frac{(n-1)!}{2}$ distinct Condorcet profiles - a number which gets large very quickly, as $n$ increases. By contrast, an $n$-candidate field has $n$ Basic related by $\sum_i B^n_i = 0$ - implying only $n-1$ of these are distinct.

## 2.2 Pairwise scores

Pairwise aggregation methods are based on pairwise comparisons of candidates. Condorcet's *successive reversal* and the *maximal agreement* procedures, Kemeny's method, Copeland's method are commonly used examples. Given a preference-profile $p$, we begin by counting the number of voters who rank $i$ over $j$ and call this $i$'s pairwise tally against $j$. The *normalized difference in the pairwise scores* of candidates $i$ and $j$

is then given as

$$a_{ij} = \frac{(i\text{'s tally against } j - j\text{'s tally against } i)}{\text{total number of voters}}$$

Thus for each pair of candidates, $-1 \leq a_{ij} \leq 1$, with a value of 0 indicating a pairwise tie between the two candidates, and values of -1 and + 1 respectively indicating unanimous loss or win by candidate $i$ against candidate $j$. Further note that $a_{ij} = -a_{ji}$. A vector $a = \{a_{ij}\}_{i,j=1...N,i<j}$ of normalized pairwise score differentials defines a point in the cube $BS(n) \subset \mathbf{R}^{n_{C_2}}$, defined by the $n_{C_2}$ intervals $[-1,1]$ (BS stands for Binary Score).

Pairwise score differences are not defined for profile differentials as the number of voters is normalized to zero. However as the pairwise tally differences are defined for profile differentials, it is possible to characterize the *directional* vectors for the pairwise score differences. A Basic profile $B_i^n$ generates a directional vector of pairwise score differences, $T_i^n \in BS(n)$ that has the following structure: $a_{ij} = 1$ for all $j \neq i$ and $a_{jk} = 0$ for all $j,k \neq i$. That is, under a $B_i^n$ profile, $i$ defeats all $j$'s unanimously in pairwise contests and the other pairwise contests not involving $i$ are all ties. To see why, note that under a $B_i^n$ profile, $(n-1)!$ voters rank $i$ over $j$ and $-(n-1)!$ voters rank $j$ over $i$. The latter are accounted for by the voters who rank $i$ last and so rank $j$ above $i$. Rankings in which $i$ is not first or last placed are supported by 0 number of voters each.

The collection $\{T_i^n\}_{i=1}^n$ of directional vectors generated by the $n$ Basic profiles, in $BS(n)$, can be shown to span a $(n-1)$ dimensional subspace $\mathbf{T} \subset BS(n)$ called the *Transitivity plane* by Saari. Basic profiles and the Transitivity plane have a very useful *additive transitivity* property: For any subset of $k$ out of $n$ candidates and any permutation of the indices, $\sum_{j=1}^{k-1} a_{jj+1} = a_{1k}$. Thus, a pairwise score between the $(i,j)$ pair can be expressed as the sum of pairwise scores of other pairs chosen in an order. This property ensures that procedural dependency and internal inconsistency never happen with Basic profiles.

Directional vectors of pairwise score differences generated by the Condorcet profiles are more difficult to characterize (especially for large $n$) than the ones generated by the Basic profiles, as such a characterization depends on the specific reference ranking for each distinct Condorcet profile. However the following algorithm is useful to understand how such a vector may be generated, once a reference ranking is fixed. Note, that the pairwise tallies of $(i,j)$ for a Condorcet profile are given by $(n-2s):(2s-n)$ if $i$ is ranked $s$ candidates above $j$ in the reference ranking, where either $n-2s$ or $2s-n$ is negative unless $n=2s$. This implies that the normalized pairwise score differentials of any pair, $a_{ij}$, in a Condorcet profile corresponding to the specific reference ranking, is 1 if $n-2s>0$, implying $i$ wins unanimously over $j$: it is -1 if $n-2s<0$, implying $j$ wins unanimously over $i$; or it is 0 if $n-2s=0$, implying both are tied. The subspace spanned by the set of vectors of pairwise score differentials generated by the distinct Condorcet profiles is described as the *Condorcet subspace* and denoted $\mathbf{C} \subset BS(n)$.[9]

A Condorcet profile generates an intransitive pairwise ranking amongst the alternatives, although individuals making up the profile are fully rational and have strict transitive rankings. When a component of a larger profile, they cause pairwise aggregation methods to disagree on their aggregate ranking, depending on their weights relative to the Basic profiles. Significant weights result in intransitive aggregate rankings. This results from a feature of pairwise scores in general, namely such scores cannot distinguish between a Condorcet profile and a profile of irrational voters with intransitive rankings.[10]

A very useful result by Saari (2000a, Proposition 5) shows that pairwise score differences, $a_{ij}$, are determined only by the Basic and Condorcet components of a preference profile. Other types of profiles contribute nothing towards these values. Denote $a \in BS(n) = \{a_{ij}\}$. The result says $a = a^T + a^C$, where $a^T \in \mathbf{T}$ and $a^C \in \mathbf{C}$. $a^T$ and $a^C$ are described respectively as the Transitive and Condorcet component vectors of the pairwise score vector $a$ by Saari - a terminology we retain. The component vector $a^T$ is determined by the weights of the Basic profiles. Specifically, the component $a^T_{ij} = a_i - a_j$. The component vector $a^C$ is contributed by the set of Condorcet profiles. Unlike $a^T_{ij}$ however, the component $a^C_{ij}$ does not have a neat expression.

The Appendix I provides 3-candidate illustrations of all the major ideas and results presented in this sub-section.

Under Condorcet profiles each candidate is supported in each position by an equal number of voters. Preference heterogeneity has multiple connotations. By pointing towards an even distribution of voters across candidates for a specific place in a ranking, a Condorcet profile reflects an extreme form of heterogeneity.This is however not the only type of profile that reflects heterogeneity. A $K^n$ profile does the same and yields the same positional tallies as a Condorcet profile. A Condorcet profile however affects pairwise scores, whereas a $K^n$ profile does not.

## 3 Reverse profiles, polarized preferences and plurality

*Positioinal* or *sum-scoring* methods assign fixed points to a candidate depending upon his/her position in an individual's ranking.[11] Assume an electorate of size 1 and denote the space of normalized profiles as

[9]The $\frac{(n-1)!}{2}$ distinct Condorcet profiles span a subspace of dimension $n_{C_2} - (n-1) = (n-1)_{C_2}$ in $BS(n)$, a higher dimensional subspace compared to the Transitivity plane for $n > 4$.

[10]As pointed out by Saari, pairwise methods produce consistent rankings over subsets of candidates but at the cost of "weakening or ignoring the rationality of voters".

[11]Plurality and Borda Counts are commonly used examples.

$P(n!) = \{p = (p_1 \dots p_{n!}) | \sum p_j = 1, p_j \geq 0\}$. The plurality method involves a voter awarding 1 point to his/her first ranked candidate and 0 to all other candidates placed in other positions in his/her ranking. Plurality tallies are the sum of all the points awarded by all the voters. A *normalized plurality score* is a mapping, $PS(p) = (ps_1 \dots ps_n) : P(n!) \longrightarrow S(n)$, where $ps_i$ is the proportion of the electorate who has $i$ first ranked and $S(n) = \{ps_i\} | \sum_{i=1}^{n} ps_i = 1$ is the unit simplex in $R^n_+$.

The Borda Count (BC) assigns $n-1$ points to the first ranked candidate of each voter, $n-2$ points to the second ranked candidate and so on. BC ranking can also be shown to be equivalent to ranking the candidates according to the sum of their normalized pairwise scores against other candidates - that is, ranking the candidates by assigning the $i$th candidate a score of $\sum_{j \neq i} a_{ij}$. Thus a Borda Count can be shown to be a pairwise as well as a positional or sum scoring aggregation method.

The main task of this section is to introduce profiles that (1) produce pairwise ties (2) are orthogonal to Basic profiles and (3) influence the plurality tallies of specific candidates but not of others. We begin with a definition.

**Definition 1** *Fix an integer k, such that* $2 \leq k \leq \frac{n+1}{2}$. *For* $k < \frac{n+1}{2}$, *a generic (first place) Reverse profile,* $R^n_i$, *has 1 voter for each ranking in which the i-th candidate is first and last ranked, (-1) voter for each ranking in which he/she is k-th ranked and to the reversal of this ranking and 0 voters for all other rankings. If* $k = \frac{n+1}{2}$, *the profile has 1 voter for each ranking in which the candidate is first and last ranked, (-2) voters for each ranking in which the candidate is k-th ranked and 0 voters for all other rankings.*

To understand the structure of the Reverse profile $R^n_i$, assume $n > 3$ and $k = 2$. $R^n_i$ has the same structure and tallies as a $K^n$ profile with a voter moved from each ranking in which $i$ is either second or $(n-1)$th ranked and added to a ranking in which $i$ is first or last ranked. In words, such a profile is obtained by padding the rankings in which $i$ is placed at the two extremes and thinning the rankings in which $i$ is placed somewhere in the middle. The exact choice of $k$ is not material to the concept of a Reverse profile, as we explain further.

**Remark 1**: Generic (first place) Reverse profiles are similar but not identical in construction to the Symmetric profiles of Saari (2000b) - hence, our use of a different name. Saari's construction is motivated by the fact that positional tallies needed to be expressed as deviations from the Borda Count (2000b, Proposition 1) as the weights of the Basic profiles are not directly available. In the absence of direct information about the weights of the Basic profiles, the Borda Count can be used as a surrogate, because Borda scores are influenced only by Basic profiles. In addition to Symmetric profiles, Saari introduces many other types of profiles in the paper, to continue with the task of decomposition in the absence of direct information about the weights of Basic profiles. A problem with some of these profiles however is that their structures do not readily admit meaningful collective interpretations as Condorcet and Reverse profiles do, although they are very useful for explaining exactly how aggregation paradoxes happen. Our approach provides direct information about Basic profiles and is able to bypass such problems.

**Remark 2**: Adding an appropriate $K^n$ profile shows that under a generic (first place) Reverse profile, each ranking and its reverse are supported by the same number of voters - hence the name"Reverse" profile. Impartiality or equal treatment of voters justify cancelling a ranking against its reversal because of this and hence interpreting these profiles as complete ties between the candidates, as far as elections are concerned. Note however that socio-politically such a profile indicates polarized preferences, as an equal number of voters place a specific candidate in the first and last positions. Hence, as in the case of Condorcet profiles, depending on the weight of these Reverse profiles, excluding them implies loss of important information about voters' preferences.

**Remark 3**: The main purpose behind defining the Reverse profiles with reference to a fixed $k$, is to ensure that $R_i^n$ is a profile differential. The specific choice of $k$ does not matter for our analysis, as the proofs below show and hence the description "generic" Reverse profiles. Further, as our specific focus is plurality, these Reverse profiles are defined with a positive number of voters for every ranking with $i$ in the first place (hence the description "first place" Reverse profiles). However, the definition can be generalized for the $m$-th place to study other sum scoring or positional methods, such as assigning 1 point each to the first two candidates in any individual ranking.

For any given $k$, there are $n$ (first place) Reverse profiles for a $n$-candidate field, related by $\sum_{i=1}^{n} R_i^n = 0$. The following proposition lays out the properties of these profiles, for a given $k$.

**Proposition 1** *For a given k,*

1. *The set of* $\{R_i^n\}_{i=1...n}$ *profiles are not pairwise orthogonal to each other and span a* $(n-1)$ *dimensional subspace of the profile space.*

2. *The set of* $\{R_i^n\}_{i=1...n}$ *profiles are pairwise orthogonal to the set of* $\{B_i^n\}_{i=1...n}$ *profiles.*

3. *The plurality tallies of* $B_i^n$ *and* $R_i^n$ *profiles are identical, with candidate I receiving* $(n-1)!$ *points and every other candidate receiving* $-(n-2)!$ *points each. The pairwise scores for each candidate pair under a* $R_i^n$ *profile is a complete tie.*

**Proof**: See Appendix II. The Appendix also provides 3 and 4 candidate illustrations whenever useful.

As the proofs make clear that the specific choice of $k$ does not matter for these properties, we assume without loss of generality that $k = 2$ for the rest of the paper. We also refer to these profiles simply as Reverse profiles. The following is a main result of this paper.

**Theorem 1** *Differences in plurality tallies for any two candidates are fully explained by Basic and generic first place Reverse profiles.*

**Proof**: Differences in tallies under any specific procedure are not affected by neutral profiles such as $K^n$ which influences these tallies uniformly for all candidates. Hence these differentials are explained by Basic profiles and profiles orthogonal to Basic profiles that affect plurality tallies. Condorcet profiles do not affect differences in any positional and specifically plurality tallies, as each alternative is supported in each position by the same number of voters, implying these tallies are the same for all candidates. Profiles orthogonal to Basic profiles that affect plurality tallies for a specific candidate must have an equal number of voters for each ranking with the candidate in the first and last places. Note that the structure of the rest of the profile - for the rankings which have the specific candidate in other positions - does not matter. Such profiles are therefore fully characterized by generic Reverse profiles. $\Delta$.

A main implication of Proposition 1 is that although the Reverse profiles are orthogonal to the Basic profiles, the plurality tallies lie in an identical direction. Thus to extract the component attributable to Reverse profiles only, the weights of the Basic profiles need to be isolated first.

# 4 Isolating component profiles

## 4.1 Isolating Basic and Condorcet profiles

The previous two sections show that Condorcet and Reverse profiles cause aggregate rankings to differ from the objective ranking induced by Basic profiles under standard procedures, and to differ from each other. Further, although these profiles can be interpreted as ties across the candidates as far as elections are concerned, they convey important information about voter's collective preferences that cannot be ignored. In this section, we focus on the task of extracting the weights of the Basic, Condorcet and Reverse profiles.

We assume that a given profile $p$ is obtained from a $\frac{V}{n!}K^n$ profile by padding it with $(n-1)$ Basic, $\frac{1}{2}(n-1)!$ Condorcet and $(n-1)$ distinct Reverse profiles. In other words $p$ is a linear combination of all these component profiles. The decomposition techniques discussed in this section requires pairwise and plurality tallies. We therefore assume that such scores are available. Further, denote by $a_i$, the coefficient of the Basic profile $B_i^n$, in this combination.

The first step is to isolate the weights $a_i$'s. An obvious way is to use Saari's Proposition 5 (2000a) to extract the component vector $a^T$ from the pairwise scores. The curse of dimensionality however makes a direct application of this proposition difficult for any arbitrary $n$, because the first step requires characterizing all the directional vectors of the pairwise score differentials generated by the $\frac{(n-1)!}{2}$ distinct Condorcet profiles. We therefore use a result and an algorithm presented in Chandra and Roy (2012/2013) which allows us to extract the component $a^T$ rather easily from the pairwise scores.

Let $a_{ij}^{(0)}$ denote the given initial pairwise score differences in an election between $i$ and $j$. This is the data. The Chandra and Roy (2013) method consists of revising the initial given scores according to the formula

$$a_{ij}^{(1)} = a_{ij}^{(0)} + CF. \sum_{k \neq i,j} (a_{ik}^{(0)} + a_{kj}^{(0)}), \forall i, j \tag{1}$$

where $CF$, a *confidence factor*, is a number chosen from the interval $[0, 1/2]$.

To understand the formula, note that pairwise scores generate an intransitive ranking over a Condorcet profile - although the individual voters making up the profile have strict transitive rankings - because such scores do not use the full information provided by the complete set of *multilateral* rankings making up the profile. Instead they use *partial* information about the profile in the form of *selective binary* components of these rankings. In Saari's words, pairwise scores do not recognize or use the transitivity property of the individual rankings, for the notion of transitivity is irrelevant over two candidates. The aggregate ranking $A > B$, $B > C$ and $C > A$ may as well be generated by a Condorcet profile of voters with strictly transitive preferences as by a profile of irrational voters with intransitive preferences across the candidates.

The Chandra-Roy (2013) algorithm essentially restores some of the lost information contained in the original multilateral rankings. It revises the pairwise score for a candidate pair $(i, j)$ by placing a positive weight on pairwise scores of $i$ against other candidates (the $a_{ik}$ 's) and pairwise scores of $j$ against other candidates (the $a_{kj}$). In other words, the data difference between $i$ and $j$ is *re-assessed* using all possible indirect evidence concerning $i$ and $j$ against other candidates. Thus, if $i$ won massively against $j$ but lost against $k$, whereas $j$ won massively against $k$, our method will reduce the margin by which $i$ won against $j$. These revisions are meant to recapture the spirit of the original multilateral rankings which provided information about how each candidate stood within the entire group of candidates in voters' preferences (rather than how each candidate stood relative to a specific another).

The main result of Chandra-Roy (2013) shows that setting the confidence factor, $CF = 1/2$ removes the Condorcet component vector $a^C$ from the pairwise scores. The result is reported here without proof. The interested reader is referred to the earlier paper.

**Theorem 2** *(Chandra and Roy (2013)) For CF $= 1/2$, the vector of revised scores $a_{ij}^{(1)}$ lies in the Transitivity plane. Specifically, for each $(i,j)$ pair, $a_{ij}^{(1)} = (1+\frac{1}{2}(n-2))a_{ij}^{T} = (1+\frac{1}{2}(n-2))(a_i - a_j)$.*

The Chandra-Roy algorithm enables us to extract the differences $a_i - a_j$ for all $(i,j)$ pairs from the given pairwise scores. Specifically, $(a_i - a_j) = a_{ij}^{(1)} \frac{1}{(1+\frac{1}{2}(n-2))}$. To extract the coefficients $a_i$ themselves, a normalization is needed - an issue that was not important in the earlier paper but is important now.

Any one of the coefficients $a_i$, $i = 1 \ldots n$ may be set to zero to obtain the remaining $(n-1)$ coefficients, in principle. From the point of view of interpretation, however, it is useful to choose the normalizing coefficient (the zero coefficient) in such a way that the weights of the remaining $(n-1)$ independent Basic profiles are non-negative. Hence we use the following algorithm to select the normalizing coefficient.

As the differences $(a_m - a_n)$ are ordered, choose the $(m,n)$ pair for which this difference is maximized. Suppose $\max_{(m,n)}(a_m - a_n) = (a_i - a_j)$. Note that $(a_i - a_j) \geq 0$ and therefore $(a_j - a_i) = \min_{(m,n)}(a_m - a_n) \leq 0$. Set $a_j = 0$ to be the normalizing coefficient.

Note that $a_j = 0$ implies $a_i \geq 0$. Note that $\forall m \neq j$, $(a_m - a_j) = (a_m - a_i) + (a_i - a_j)$. Since $(a_j - a_i) \leq (a_m - a_i) \leq (a_i - a_j)$, $(a_m - a_j) \geq 0$, implying $a_m \geq a_j = 0$. Thus all other coefficients are positive.

The Chandra-Roy algorithm provides us with the Condorcet components $a_{ij}^{C} = a_{ij} - a_{ij}^{T} = a_{ij} - (a_i - a_j)$, of the pairwise scores but *not* the weights of the distinct $\frac{1}{2}(n-1)!$ Condorcet profiles themselves. Our main objective however does not require us to isolate the weights of the individual Condorcet profiles, as additional collective insight is not gained by distinguishing between the different Condorcet profiles themselves. Collectively speaking, they all represent heterogenity.

**Remark 4**: It is important to clarify what the weights of component profiles (Basic, Condorcet and Reverse) exactly mean. In particular, such weights may in general take on any real value and hence *cannot* be interpreted as a "share or proportion" of the electorate. Using our view of a given profile as a padded and thinned $K^n$ profile (voters moved from specific rankings to other rankings), a weight $a_i$ of the $B_i^n$ profile measures the "thickness" of the padding and thinning performed relative to $\frac{V}{n!}$, the weight of the $K^n$ profile, to obtain the given profile. For example, if a profile can be expressed as $p = \frac{V}{n!}K^n + a_i B_i^n$, where $a_i$ is also found to be equal to $\frac{V}{n!}$, then we conclude that the entire profile has the same structure as a $B_i^n$ profile. The relative values of the $a_i$ coefficients therefore provide direct measure of the relative importance or strength of these profiles. In order to facilitate this interpretation, it is important that the coefficients $a_i$s (and coefficients of some other component profiles as well) are non-negative. The normalization above achieves this goal.

### 4.2 Isolating Reverse profiles

Assume that plurality tallies of all candidates and pairwise scores of all candidate pairs are available. The following is a second main result of the paper.

**Theorem 3** *A unique decomposition of the plurality tally differences into two components, one determined by Basic profiles and the other by Reverse profiles, can be implemented.*

**Proof**: From Theorem 1, plurality tallies are determined by Basic, Reverse and $K^n$ profiles. Assume that our given profile is a linear combination of $n$ Basic profiles, $n$ Reverse profiles and a $\frac{V}{n!}K^n$ profile. That is $p = \sum_{i=1}^n a_i B_i^n + \sum_{i=1}^n r_i R_i^n + \frac{V}{n!}K^n$ where the unknown coefficients or weights of the Basic and Reverse profiles are to be obtained from the given data. Moreover, only $(n-1)$ of the Basic profiles and $(n-1)$ of the Reverse profiles are independent.

From the pairwise scores for each candidate pair for this profile, the differences in the weights of the component Basic profiles, $(a_i - a_j)$, can be extracted using Theorem 2. From Proposition 1, the tallies of $B_i^n$ and $R_i^n$ are in identical direction, for all $i$. Define $\mathbf{t_i}$ as a vector with $(n-1)!$ as its $i$-th component and $-(n-2)!$ as all the other components. Denote $\mathbf{1} = (1,1,\dots 1)$, a $n$-component vector. Denote the vector of plurality tallies of the candidates by $\tau = (\tau_1 \dots \tau_n)$. For a profile $p = \sum_{i=1}^n a_i B_i^n + \sum_{i=1}^n r_i R_i^n + \frac{V}{n!}K^n$, the plurality tallies can be shown to be

$$\tau = \sum_{i=1}^{n} (a_i + r_i)\mathbf{t_i} + \frac{V}{n}\mathbf{1}$$

Denote by $\alpha = \sum_{i=1}^n (a_i + r_i)(n-2)! - \frac{V}{n}$. A slight manipulation yields,

$$\tau = n(n-2)!\omega - \alpha\mathbf{1}$$

where the $n$-dimensional vector $\omega = \{a_i + r_i\}_{i=1}^n$. The difference in the plurality tallies of the $i$-th and the $j$-th candidate is therefore given by

$$\tau_i - \tau_j = n(n-2)!((a_i - a_j) + (r_i - r_j) \tag{2}$$

The left hand side $(\tau_i - \tau_j)$ is obtained from the data. The difference $(a_i - a_j)$ is obtained from the pairwise scores and Theorem 2. Therefore the difference $(r_i - r_j)$ can be calculated. With an appropriate normalization under which a specific $r_k$ is set to zero, the remaining $r_i$s can be obtained. Hence the claim is true. $\Delta$

In general, any one of the $r_i$ coefficients can be normalized to zero to obtain the other coefficients. Our interpretation of the weights of the component profiles is helped if such weights are non-negative numbers. Hence for the empirical results in Section 6, we adopt the same normalization technique as we did to obtain the $a_i$ coefficients. In general the candidate whose Basic profile weight has been normalized to be zero may not turn out to be the same candidate whose Reverse profile weight has been normalized to be zero.

### 4.3 A 3-candidate illustration

The following preference-profile comes from the election of a president of the *Social Choice and Welfare Society* and also used as an example by Balinski and Laraki (2010).

Table 2:

| Rankings | No. of voters | Rankings | No.of voters |
|---|---|---|---|
| 1. $A > B > C$ | 13 | 4. $C > B > A$ | 8 |
| 2. $A > C > B$ | 11 | 5. $C > A > B$ | 11 |
| 3. $B > C > A$ | 9 | 6. $B > A > C$ | 0 |

It is easy to check that $C$ is the Condorcet winner, the majority rule ranking is $C > A > B$ and the Borda ranking is $A > C > B$. The pairwise scores are $a_{12} = 9/26$, $a_{13} = -1/13$ and $a_{23} = -2/13$. The plurality tallies are $A = 24, B = 9$ and $C = 19$ inducing the plurality ranking $A > C > B$.

Applying the Chandra-Roy algorithm, the revised pairwise scores are $a_{12}^{(1)} = \frac{10}{26}$, $a_{13}^{(1)} = \frac{1}{52}$, and $a_{23}^{(1)} = -\frac{19}{52}$. The weights of the Basic profiles are obtained from the differences, $\hat{a}_{12} = a_1 - a_2 = \frac{20}{78}$, $\hat{a}_{13} = a_1 - a_3 = \frac{1}{78}$, $\hat{a}_{23} = a_2 - a_3 = -\frac{19}{78}$. As $\max(a_i - a_j) = a_1 - a_2$, we normalize $a_2 = 0$, implying $a_1 = \frac{20}{78}$ and $a_3 = \frac{19}{78}$. The Basic profile favoring $A$ has a slightly greater weight than the one favoring $C$. This accounts for the Borda ranking. The Condorcet components obtained from the pairwise scores and the revised pairwise scores are, $a_{12}^c = a_{12} - a_{12}^{(1)} = 9/26 - 10/26 = -1/26$, $a_{13}^c = a_{13} - a_{13}^{(1)} = -1/13 - 1/52 = -5/52$ and $a_{23}^c = a_{23} - a_{23}^{(1)} = -2/13 + 19/52 = 11/52$. These account for the difference between the Borda ranking and the majority rule ranking, specifically the switch between $A$ and $C$.

To obtain the coefficients of the reverse profiles, note that $r_1 - r_2 = 370/78$, $r_1 - r_3 = 129/78$ and $r_2 - r_3 = -241/78$ using the formula of Theorem 3. As the maximum difference is $r_1 - r_2$, we set $r_2 = 0$ and obtain $r_1 = 370/78$ and $r_3 = 241/78$.

# 5 Heterogeneity and Polarization

## 5.1 Condorcet profiles and heterogeneity

The components $\{a^C_{ij}\}$ and $\{a_i - a_j\}$ provide natural measures of heterogeneity and of how sensitive pairwise procedures in general could be to Condorcet profiles.

A pairwise ranking $i > j$ is defined as a *strong reversal* of the pairwise ranking $i < j$ and vice versa, for all $(i, j)$ pairs. The pairwise ranking $i > j$ or $i < j$ is defined as a *weak reversal* of the pairwise ranking $i \sim j$ and vice versa, for all $(i, j)$ pairs. the following proposition provides a way of assessing how different an aggregate ranking based on a pairwise method can be, for a given profile $p$, from an aggregate ranking based on the profile's Basic components only. Recall that all positional methods produce a complete tie for Condorcet profiles. Thus, in the absence of Reverse profiles, the proposition also provides a way of assessing how far rankings under positional methods may differ from rankings under pairwise methods, as such differences under these circumstances are caused by Condorcet profiles only.

**Proposition 2** *Given any profile p, the pairwise majority ranking of the pair $(i, j)$ is a strong reversal of the pairwise majority ranking obtained from the Basic profiles, if $a_i \neq a_j$ and $\frac{a^C_{ij}}{a_i - a_j} < -1$. The pairwise majority ranking of the pair $(i, j)$ is a weak reversal of the pairwise majority ranking obtained from the Basic profiles if one of the following holds: (1) $a^C_{ij} \neq 0$ and $a_i = a_j$ or (2) $a_i \neq a_j$ and $\frac{a^C_{ij}}{a_i - a_j} = -1$.*

**Proof:** Pairwise majority ranking of the pair $(i, j)$ is determined by the the sign of $a_{ij}$. That is $i > j$ if $a_{ij} > 0$, $i < j$ if $a_{ij} < 0$ and $i \sim j$ if $a_{ij} = 0$. The pairwise majority ranking of the pair $(i, j)$ obtained from the Basic profiles only is determined by the sign of $a_i - a_j$. By the previous results, $a_{ij} = a^C_{ij} + (a_i - a_j)$. Thus $a_{ij}$ and $a_i - a_j$ have strictly opposite signs if $a_i - a_j \neq 0$ and $\frac{a^C_{ij}}{a_i - a_j} < -1$. When $a_i = a_j$, the pairwise ranking obtained from Basic profiles is $i \sim j$. The pairwise ranking of the pair $(i, j)$ is determined by the sign of the Condorcet component $a^C_{ij}$ and is a weak reversal of the ranking obtained from the Basic profiles. When $a_i \neq a_j$ and $\frac{a^C_{ij}}{a_i - a_j} = -1$, $a_{ij} = 0$. Thus the pairwise majority ranking of the pair $(i, j)$ is $i \sim j$ but the ranking from the Basic Profiles is a strict inequality. Hence one is a weak reversal of the other. $\Delta$.

Socio-politically, Condorcet profiles represent a form of preference heterogeneity similar to a $K^n$ profile. It is therefore important to have a measure of this heterogeneity. The Condorcet components $a^C_{ij}$ may be used to construct several such natural measures. One is the overall contribution of the Condorcet components to pairwise scores, that is the ratio $\eta_1 = (\frac{(a^C)^T(a^C)}{a^T a})^{1/2}$. Note that Condorcet components in pairwise scores are analogous to residual error terms in linear regression exercises. The ratio $\frac{(a^C)^T(a^C)}{a^T a}$ therefore has a similar interpretation to the $R^2$ statistic in linear regressions. An alternative measure of heterogeneity is the ratio $\eta_2 = (\frac{1}{n_{C_2}} \sum_{i<j} \frac{(a_{ij}^C)^2}{(a_{ij})^2})^{1/2}$. The term $\frac{(a_{ij}^C)^2}{(a_{ij})^2}$ denotes how important the contribution of the Condorcet component is in the pairwise score for the $(i,j)$ pair. The ratio $\eta_2$ thus measures the average contribution of all Condorcet profiles in determining pairwise scores.

Another useful measure is the proportion of pairwise rankings that are strong or weak reversals of the pairwise rankings based on Basic profiles, as such reversals are caused only by Condorcet profiles. A high proportion indicates a sizable presence of such profiles. To this end, define the sets $A^1 = \{(i,k) | i < k, \frac{a_{ik}^C}{a_i - a_k} > -1\}$, $A^2 = \{(i,k) | i < k, \frac{a_{ik}^C}{a_i - a_k} < -1\}$, $A^3 = \{(i,k) | i < k, (a_i - a_k = 0)$ and $a_{ik}^C \neq 0\}$ and $A^4 = \{(i,k) | i < k, \frac{a_{ij}^C}{a_i - a_j} = -1\}$, respectively. Thus, $A^1$ is the set of all pairs $(i,k)$, such that their pairwise rankings and rankings according to their Basic profiles do not differ. $A^2$ is the set of all pairs such that their pairwise rankings and rankings according to Basic profiles strongly differ. $A^3$ and $A^4$ are the sets of all pairs such that their pairwise rankings and rankings according to the Basic profiles weakly differ. Further, denote by Card$(S)$, the cardinality of the set $S$. Then the total number of pairwise rankings that are strong or weak reversals of the pairwise rankings based on Basic profiles is $\text{Card}(A^2) + \text{Card}(A^3) + \text{Card}(A^4)$. As the total number of pairwise orderings is $n_{C_2}$, the measure is provided by the ratio, $\Gamma = \frac{\text{Card}(A^2)+\text{Card}(A^3)+\text{Card}(A^4)}{n_{C_2}}$. $\Gamma$ also provides a measure of how sensitive pairwise procedures in general could be to Condorcet profile components, under a specific situation. A high value for $\Gamma$ indicates in general that different pairwise procedures will yield aggregate rankings that may markedly differ from each other and thus all outcomes should be regarded as procedure sensitive.

## 5.2 Reverse profiles, plurality and polarization

A $R_j^n$ profile has an equal number of voters placing candidate $j$ in the first and last places. Alternatively, such a profile has an equal number of voters with a given preference order and its reverse. Thus their weights can be useful in multiple ways. A $R_j^n$ profile with a significant weight may be interpreted as, candidate $j$ is a polarizing figure - he/she is loved or hated by an equal number of voters. From a social perspective, these weights may be used to measure how polarized the electorate is - that is how divided the electorate is into groups with opposite preferences.

From the previous section,

$$(\tau_i - \tau_j) = n(n-2)!((a_i - a_j) + (r_i - r_j))$$

As $r_i \geq 0$ for all $i = 1 \dots n$ with our normalization, we may say that candidate $i$ is more polarizing than candidate $j$ if $r_i - r_j > 0$ - that is, more number of voters place candidate $i$ in first and last places than they do candidate $j$. The larger this difference, the larger the contribution of the $R_i^n$ profile towards any electoral win of $i$ over $j$, under the plurality procedure.

An interesting situation arises when the difference $r_i - r_j$ is sizable enough to overturn the objective difference between the two candidates, according to their Basic profiles. From the expression, it follows that $(\tau_i - \tau_j)$ and $(a_i - a_j)$ have the same sign if $\frac{r_i - r_i}{a_i - a_j} > -1$ and opposite signs if $\frac{r_i - r_i}{a_i - a_j} < -1$, when $(a_i - a_j) \neq 0$. Thus, when $(a_i - a_j) \neq 0$ and $\frac{r_i - r_i}{a_i - a_j} < -1$, the relative plurality ranking of the $(i, j)$ pair is a strong reversal of the relative ranking according to their Basic profiles and the candidate that is higher ranked according to plurality is significantly more polarizing than the other. When $(a_i - a_j) = 0$ but $(r_i - r_j)$ is strictly positive or negative, the relative plurality ranking of the $(i, j)$ pair is a weak reversal of the relative ranking according to their Basic profiles and the higher plurality ranked candidate is more polarizing than the other. Thus the ratios $\frac{r_i - r_i}{a_i - a_j}$ or the quantities $(r_i - r_j)$ provide information about how polarizing specific candidates are relative to others. We denote by $\Psi$ the proportion of candidate pairs whose relative ranking under plurality is a strong or weak reversal of the relative ranking based on Basic profiles. In the absence of Condorcet profiles, the ratio $\Psi$ measures how far off is the aggregate plurality ranking of the candidates from an aggregate ranking based on Basic profiles and other pairwise procedures.

We also propose and apply several other direct measures of overall polarization among the electorate. First is the average $r_i$ coefficient over all candidates (denoted $\bar{r}$ in the tables) and its associated standard deviation. More meaningful than the average $r_i$ itself, in some sense, is the ratio of the average $r_i$ to the average Basic profile coefficient (denoted $\bar{a}$ in the tables). The ratio $\frac{\bar{r}}{\bar{a}}$ provides an aggregate measure of how strong the Reverse profiles are relative to Basic profiles and thus to what extent polarized preferences play a part in determining the plurality outcome. For the Cambridge City Council elections that we specifically study with our tools and techniques in the next section, we use two sets of the $\bar{r}$ and $\frac{\bar{r}}{\bar{a}}$ measures - one measured over all the candidates and the other over the set of the nine winning candidates.

# 6 Results from the Cambridge City Council Elections

In this section, we test our method and measures on ballot data from the Cambridge (Massachussetts) City Council elections over the period 1997-2011. Elections are held every two years providing us with eight years of data.

The data set has certain limitations. The traditional model of social choice assumes voters to have strict preferences over all candidates. The Cambridge City electoral laws do not require voters to rank all candidates. Voters must rank at least one of them for the first place and are free to rank as many of the others as they like. On an average there are 18 or19 *official* candidates, indexed $C01, C02\ldots$, on the ballot for every election, out of which 9 city council members are elected. Most voters rank about only 4 or 5 candidates. Thus the major limitation of the data set is that many of the official candidates are *not* ranked by many of the voters. A second but minor limitation of the data set is that under the electoral laws, voters also have the right to vote for unofficial candidates (not on the ballot) by writing their names on the ballot. These candidates, described as write-in candidates, appear in most cases to be people well known within the very small group of voters who ranked them but not widely known outside. There was however one exception that happened in the year 2009. In the year 2009, a candidate who had successfully ran in some of the previous year elections and thus may be deemed as widely known, was suddenly not included in the official list of candidates but was ranked in the first position by a significant number of voters as a write-in candidate. For our analysis for the year 2009, we treated this candidate as an official rather than as a write-in candidate. Finally, for many of the elections prior to 2005, we found that a significant proportion of the ballots had multiple candidates ranked in the same position. By the electoral laws, these ballots should be considered invalid but the total numbers of invalid ballots officially reported for these years are considerably smaller than what they should have been if all such ballots were considered invalid. This problem is not significant beginning with 2005 and the voters seem to have become better informed about the official procedures. Thus for the years 1997-2003, on an average about 8-9% of the total ballots had ties. For the years 2005-2011, this percentage is about 1-2%.

In keeping with our model, we excluded all ballots where multiple candidates were placed in the same position and considered only the ballots with strict rankings. Besides the official candidates, there were typically 7-9 different write-in candidates every year. Instances of the same write-in candidate being ranked by more than ten voters in any year were very rare, with the exception of 2009 discussed above. Instances of a write-in candidate, rather than an official candidate, being ranked first were also very few (with the exception of 2009 elections). We excluded the ballots where a write-in candidate was ranked first, for reasons explained below, but retained the ballots where a write-in candidate was placed in between two official candidates, to use the available information about the pairwise rankings of the official candidates. Thus for the years 1997-2003, about 9% of the ballots and for the years 2005-2011, about 2% of the ballots were discarded. The high percentage of discards for the years 1997-2003 is responsible for some minor difference in the plurality ranking of the candidates reported in our tables (which were directly calculated from the ballots) with the official plurality ranking of them after the first count. These differences are noted in the tables, wherever they exist.

Excluding the write-in candidates helps us to increase the numerical accuracy of the estimates of the

Reverse profile coefficients, for reporting purposes, by reducing the value of $n(n-2)!$ in equation 2. This does not affect the qualitative results, so far as the Reverse profiles are concerned. Note that so far as isolating the Condorcet components are concerned, Theorem 2 requires using the decomposition method on pairwise scores for the *full* set of candidates - official and write-ins. Given that so few voters ranked the write-in candidates anywhere on the ballots, once again we do not think that excluding the write-ins made a difference. A more serious problem however, is that so few voters ranked all the official candidates. As we need voters to order all official candidate pairs to apply Theorem 2, we made the following assumptions about voters' preferences regarding the official candidates that they have not ranked. Firstly, we assumed that if a voter has not ranked a candidate $A$, then the voter strictly prefers all the candidates that he or she has ranked to candidate $A$ - in other words, unranked candidates are ranked *below* the ranked candidates. Secondly, if a voter has not ranked two candidates $A$ and $B$, we assumed that the voter prefers $A$ to $B$ with probability half and $B$ to $A$ with probability half. Thus all voters who did not rank a specific pair $(A,B)$ were equally distributed between $A$ and $B$. We consider these assumptions to be the most reasonable under the circumstances although they have the potential to affect the empirical results. We nevertheless think that these results are useful as a first attempt to apply the methods and measures discussed in this paper.

The nine members of the City Council are elected under a proportional representation (PR) method over several counts of the ballots. Under this method a candidate is elected if he/she wins a certain proportion of the votes, called a quota. The quota is determined by dividing the total number of valid ballots by ten (the number of candidates to be elected plus one) and adding one to the result. The first count involves determining the plurality tallies of all the candidates. All candidates who reach the quota after the first count are declared elected. Any votes they receive beyond the quota are denoted surplus votes. Surplus votes are transferred to the second choice candidates on the surplus ballots. A formula determines which ballots are selected as surplus ballots. After surplus votes are transferred, candidates who have fewer than fifty tallies are eliminated and their votes are transferred to the next in preference. A new ranking is established of the continuing candidates, after this. The candidate with the lowest number of tallies after the two transfers is declared defeated and his/her ballots are transferred to the next continuing candidate marked on each ballot. One a candidate reaches the quota, no more ballots are transferred to him/her. The process continues till all nine members are elected.

The present paper does not attempt to analyze or critique this specific voting procedure used by the City Council. Instead our specific objective is to uncover the size of the Basic, Condorcet and Reverse profiles, for which we directly calculate the necessary pairwise scores and plurality tallies from the ballot data. Our results however is an indirect comment on the specific PR method as well. Condorcet profiles influence the rankings of the continuing candidates when candidates who have already fulfilled the quota or the bottom ranked candidates are dropped and their votes transferred to the next ranked candidate on the ballots. Reverse profiles influence the plurality tallies of the first count, thereby influencing the rest of the PR process and the final outcome. Thus, significant coefficients of these types of profiles would suggest that the PR method itself may have played a significant role in determining the final outcome of these elections. To this extent, comparing the final outcomes with our first count results are useful.

Tables 7-15 presents the numerical results of our analysis. Table 7 presents the values obtained for the various aggregative measures of heterogeneity and polarization discussed in Section 5, for all the years. Tables 8-15 provide the Basic and Reverse profile coefficients for all the candidates along with their aggregate rankings based on plurality, the Borda Count and the weights of their Basic profiles, for specific years. The main findings are summarized below.

### 6.1 Main results for the period 1997-2011

The values of $\bar{r}$ and $\frac{\bar{r}}{\bar{a}}$ over all the candidates and over the set of winners (denoted $\bar{r_w}$ and $\frac{\bar{r_w}}{\bar{a_w}}$ respectively) show a clear upward trend in polarization among the Cambridge voters since 2001. In particular, after 2005, polarized preferences seem to have played a significantly bigger role in determining the set of winners, compared to before 2005, as evidenced by the values of the ratio $\frac{\bar{r_w}}{\bar{a_w}}$. The proportion of relative ranking reversals under plurality, $\Psi$ does not show a trend over the years but is significant at an average of 36

Our results indicate no significant heterogeneity in the Condorcet sense amongst the Cambridge electorate. Neither $\eta_1$, $\eta_2$, nor $\Gamma$ demonstrate any pattern or trend. It should be noted however that on this issue, the data-set has a limitation discussed earlier that may have affected the results or rather the lack of any.

### 6.2 Specifc results for each election

*1997*

About 9% of the ballots cast this year were discarded for the reasons discussed in the first paragraph of this section. The related differences in our plurality tallies and the official tallies after the first count are pointed out in the table. Our most noticeable findings are the following. Candidate $C12$, Borda and $a_i$ ranked 2nd, was edged out by Candidates $C06$, $C13$, $C14$ and $C15$, all of whom were Borda and $a_i$ ranked below $C12$, in the first round of counting, because the plurality tallies of the latter were boosted by stronger Reverse profiles, as evidenced by the higher $r_i$ values relative to $r_{12}$. $C12$ got elected in the second round after a transfer of surplus votes from the first round. $C14$, Borda and $a_i$ ranked 6th, edged out $C01$, Borda and $a_i$ ranked 5th, in the first round of counting because of a stronger Reverse component profile. $C01$

eventually got elected in the third round of counting. $C06$, Borda and $a_i$ ranked 7th, is plurality ranked above $C04$, Borda and $a_i$ ranked 1st, because $r_6 > r_4$.

*1999*

About 9.3% of the ballots cast this year were discarded. Our findings include: Borda and $a_i$ 6th ranked Candidate $C08$ ($r_8 = 0.068$ approx), edged out Candidate $C18$ who was Borda and $a_i$ 1st ranked ($r_{18} = 0$) in the first count. $C18$ eventually was elected in the 13th round. $C05$ who was Borda and $a_i$ ranked 4th and $C19$ who was Borda and $a_i$ ranked 5th similarly trailed behind $C08$ in the first count and eventually got elected in the 14th and 13th rounds respectively. $C20$ who was Borda and $a_i$ 2nd ranked did *not* get elected.

*2001*

About 6.3% of the ballots cast this year were discarded. We find that $C12$ who was Borda and $a_i$ ranked 7th was elected in the first count, whilst $C16$ who was Borda and $a_i$ ranked 2nd was elected in the 13th count. $C17$, $C03$ and $C18$ who were Borda and $a_i$ ranked 4th, 5th and 6th respectively (that is ranked before $C12$) were elected in the 7th, 9th and 14th counts. In contrast to what happened in 1999, however, *al*l of the first nine Borda and $a_i$ ranked candidates were eventually elected to the Council. (Explain further)

*2003*

About 8.8% of the ballots cast this year were discarded. As it happened in 2001, there is a remarkable consistency between the first nine Borda and $a_i$ rankings and the set of candidates who eventually got elected. A noticeable fact is that candidate $C16$ who is Borda and $a_i$ ranked 3rd got elected in the 13th round, after candidates $C02$, $C04$, $C13$ and $C20$, all of them Borda and $a_i$ ranked lower than $C16$, got elected in earlier rounds. $C16$ has a lower $r_i$ coefficient compared to all of them. Candidate $C06$ was plurality, Borda and $a_i$ first ranked and also the candidate with the lowest $r_i$ coefficient.

*2005*

About 3.4% of the ballots cast this year were discarded. Amongst the most noticeable findings are: $C05$ who was Borda and $a_i$ ranked 6th got elected in the first round, whereas, $C16$ who was Borda and $a_i$ ranked 1st got elected in the 11th round. $C16$ got elected in later rounds than $C17$, $C03$, $C04$, $C18$ and $C13$, all of whom were Borda and $a_i$ ranked lower than him/her. Note that $r_{16} = 0.$ whereas the $r_i$ of all these candidates are higher and significantly so in case of $C05$. Very remarkably, $C12$ who was Borda and $a_i$ ranked 7th did not get elected, whereas, $C10$ who was Borda and $a_i$ ranked 11th got elected. $C10$ has a significantly higher

$r_i$ coefficient compared to $C12$. Even more interestingly, $C12$ ran and was elected in 1999, 2001 and 2003. In all these three years he/she showed remarkable consistency in the Borda/$a_i$ rankings relative to the other candidates, being always placed 7th or 8th. $C10$ ran but lost in 2003 and interestingly enough also had a significantly high $r_i$ coefficient in 2003.

*2007*

About 1.03% of the ballots cast this year were discarded. Some interesting findings are: $C13$ who was Borda and $a_i$ ranked 1st got elected in the 9th round whereas Candidates $C01$, $C15$ $C06$ and $C11$, all of whom were Borda and $a_i$ ranked lower but had higher $r_i$ coefficients (significantly so, for $C11$, $C06$ and $C15$), got elected in earlier rounds. Also notable was that Candidate $C13$ who ran in 2001, 2003, 2005 and 2007, had low $r_i$ coefficients in 2001 and 2003 and had $r_i = 0$ in 2005 and 2007. $C16$ who was Borda and $a_i$ ranked 9th was defeated but $C11$ who was Borda and $a_i$ ranked 10th but had a higher $r_i$ coefficient, was elected. Two candidates $C03$ and $C14$, who have been elected in 1997, 1999, 2001, 2003 and 2005 were defeated in 2007. Further both had low, sometimes 0, $r_i$ coefficients in all the years they were elected. In 2007 when they were defeated, their $r_i$ coefficients were significantly higher compared to the earlier years, specially so for $C03$. Thus both candidates became significantly more polarizing figures in 2007 compared to what they were earlier. The year 2007 also marks the beginning of a period during which a significant number of candidates appear with high $r_i$ coefficients, some of them amongst the winners.

*2009*

Only 0.8% of the ballots cast this year were discarded. Write-in candidate, $WI01$ is designated the 21st official candidate in our table. Amongst the findings are: A significant number of candidates have high $r_i$ coefficients, some of them amongst the winners...Candidate $C19$ who was Borda and $a_i$ ranked 4th and Candidate $C18$ who was Borda and $a_i$ ranked 9th were not elected. Candidates $WI01$ who was Borda and $a_i$ ranked 21st and Candidate $C02$ who was Borda and $a_i$ ranked 10th were elected. Candidate $WI01$ had won as an "official" candidate in 1999, 2001, 2003, 2005 and 2007, had $r_i$ coefficients generally less than 0.1 during these years. In 2009 his/her $r_i$ was the highest at 0.5 approx. $WI01$ also had the lowest $a_i$ coefficient at 0. Four out of the nine elected candidates had $r_i$ coefficients that were significantly higher compared to pre-2007 norms.

*2011*

About 1.3% of the ballots cast this year were discarded. A significantly large number of candidates had significantly higher $r_i$ coefficients compared to pre-2007 norms. Only three out of eighteen candidates had $r_i$ coefficients less than 0.1 and four candidates had $r_i$ coefficients higher than 0.4. Six out of the nine winners had significant $r_i$ coefficients. Other interesting findings are, Candidate $C12$ who was Borda and $a_i$ ranked 4th and Candidate $C17$ who was Borda and $a_i$ ranked 9th were not elected. Instead Candidate $C04$ who was Borda and $a_i$ ranked 11th and Candidate $C16$ who was Borda and $a_i$ ranked 10th were elected. Candidate $C13$ who was Borda and $a_i$ ranked 2nd was elected in the 13th round whereas $C15$ and $CC05$ who were Borda and $a_i$ ranked lower but had higher $r_i$ coefficients were elected in earlier rounds.

# 7 Appendix I

This section provides 3-candidate illustrations of the concepts and results presented in Section 2 of the paper.

## 7.1 Basic, Condorcet profiles, pairwise scores with 3 candidates

In a 3-candidate election, there are $3! = 6$ possible rankings of the candidates $A$, $B$ and $C$. The rankings are numbered as follows:

Table 3:

| | |
|---|---|
| 1. $A > B > C$ | 4. $C > B > A$ |
| 2. $A > C > B$ | 5. $B > C > A$ |
| 3. $C > A > B$ | 6. $B > A > C$ |

Assume $A$ is candidate 1, $B$ is candidate 2 and $C$ is candidate 3. Using the above numbering scheme, the Basic profiles for the 3-candidate field are given by, $B_1^3 = (1,1,0,-1,-1,0)$, $B_2^3 = (0,-1,-1,0,1,1)$, and $B_3^3 = (-1,0,1,1,0,-1)$ where the components of each vector represent the number of voters favoring the specific ranking. Note that $B_1^3 + B_2^3 + B_3^3 = 0$.

To use the terminology of Section 2, the vectors $B_1^3$, $B_2^3$ and $B_3^3$ are strictly speaking profile differentials because they have negative components. However each $B_i^3$ has the same election outcomes as $B_i^3 + K^3$ where $K^3 = (1,1,1,1,1,1)$. As $K^3$ has one voter favoring each ranking, it does not influence any election outcome. Note that $B_1^3 + K^3 = (2,2,1,0,0,1)$. Under the profile $B_1^3 + K^3 = (2,2,1,0,0,1)$, $A$ unanimously wins both pairwise elections against $B$ or $C$. $B$ and $C$ are tied when pitted against each other. Intuitively speaking, the profile $B_1^3$ or $B_1^3 + K^3 = (2,2,1,0,0,1)$ may be thought of as representing the "base" of candidate $A$. Everyone in this group of voters likes $A$ best and is indifferent between the others.

There is a unique reference ranking and only one distinct Condorcet profile for a 3-candidate field. The Condorcet 3-tuple $c^3_{(1)}$ and its reversal set $\rho(c_{(1)})^3$ are the set of rankings in the table below.

Table 4:

| $c^3_{(1)}$ | $\rho(c_{(1)})^3$ |
|---|---|
| $A > B > C$ | $C > B > A$ |
| $B > C > A$ | $A > C > B$ |
| $C > A > B$ | $B > A > C$ |

Using the previous numbering scheme, the Condorcet profile for a 3-candidate field is described by the vector, $C^3 = (1, -1, 1, -1, 1, -1)$

As the number of candidates increases, the dimension of the voters' profile and the number of distinct Condorcet profiles gets large very quickly. In a 4-candidate field the number of all possible rankings of the candidates is 24, implying that the Basic and the Condorcet profiles are 24-dimensional vectors. There are 4 Basic profiles (three of which are independent) and 3 distinct Condorcet profiles. In a 6-candidate field there are 6 Basic profiles and 60 distinct Condorcet profiles, each being a 6!-dimensional vector.

There are three possible pairwise scores in a 3-candidate field and hence $BS(3)$ is a 3-dimensional cube with each side given by the interval $[-1, 1]$. Denoting candidate $A$ as 1, $B$ as 2 and $C$ as 3, a vector in $BS(3)$ is represented as $a = (a_{12}, a_{13}, a_{23})$. For comparison, $BS(4)$ is a 6-dimensional cube.

In a 3-candidate field, the vectors of normalized pairwise score differentials generated by the 3 Basic profiles are $T^3_1 = (1, 1, 0)$, $T^3_2 = (-1, 0, 1)$, and $T^3_3 = (0, -1, -1)$. It is easy to check that these three vectors are linearly dependent and hence span a 2-dimensional subspace of $BS(3)$. The 2-dimensional subspace spanned by the three vectors form the Transitivity plane for the 3-candidate field.

In the 3-candidate field, the unique Condorcet profile generates the directional vector $q = (a_{12}, a_{13}, a_{23}) = (1, -1, 1)$ in $BS(3)$, implying the well known Condorcet triplet that $A$ unanimously beats $B$, $B$ unanimously beats $C$ and $C$ unanimously beats $A$. For comparison, in a 4-candidate field there are three distinct Condorcet profiles and hence three such 6-dimensional directional vectors. In a 5-candidate field there are twelve 10-dimensional directional vectors. As the number of distinct Condorcet profiles increase very rapidly with the candidates, characterizing the directional vectors for these profile becomes a long and involved process. This is one reason why direct profile decomposition is difficult to implement if $n > 4$.

### 7.2 Basic and Condorcet profile decomposition with 3 candidates

Suppose that an electorate can be described by the following combination of Basic and Condorcet profiles

$$p = aB_1^3 + bB_2^3 + cB_3^3 + dC^3$$

where $a$, $b$, $c$ and $d$ are any constants and the vectors $B_1^3$, $B_2^3$, $B_3^3$, and $C^3$ are as defined above.

It is easy to check that the number of voters favoring each possible ranking within the profile are as given in Table 4.

Table 5:

| ranking | no.of voters | ranking | no. of voters |
|---|---|---|---|
| $A > B > C$ | $(a-c+d)$ | $C > B > A$ | $(-a+c-d)$ |
| $A > C > B$ | $(a-b-d)$ | $B > C > A$ | $(-a+b+d)$ |
| $C > A > B$ | $(-b+c+d)$ | $B > A > C$ | $(b-c-d)$ |

The pairwise election tallies are calculated to be $(A:B) = ((2a-2b+d):(2b-2a-d))$, $(A:C) = ((2a-2c-d):(2c-2a+d))$, and $(B:C) = ((2b-2c+d):(2c-2b-d))$. Note that the pairwise tallies depend on the relative weights of the two relevant Basic profiles and the weight of the Condorcet profile. Further each pairwise score difference is a direct sum of a Transitive component and a Condorcet component. For example, the pairwise score difference for the $(A,B)$ pair is $(4a-4b+2d)$ ($A$'s tally minus $B$'s tally). The Transitive component is $4a-4b$ and the Condorcet component is $2d$. The pairwise score difference for the $(A,C)$ pair is $(4a-4c-2d)$ ($A$'s tally minus $C$'s tally). The pairwise score difference for the $(B,C)$ pair is $(4b-4c+2d)$ ($B$'s tally minus $C$'s tally). The Transitive components satisfy the additive transitive property because ($A$'s tally minus $B$'s tally = $4a-4b$) plus ($B$'s tally minus $C$'s tally = $4b-4c$) equals ($A$'s tally minus $C$'s tally = $4a-4c$). Note that the Condorcet components don't satisfy this property.

When all three candidates are running the race (a complete field election), the Borda scores are found to be $A:(4a-2c-2b)$, $B:(4b-2a-2c)$ and $C:(4c-2a-2b)$. Note that under this scenario, the Borda scores are unaffected by the weight $d$ of the Condorcet profile as the Condorcet components cancel out when the pairwise tallies are added.

If however $A$ drops out of the race, the Borda scores for $B$ and $C$ become $2b-2c+d$ and $2c-2b-d$ respectively (the same as the pairwise scores) and now depend on the weight of the Condorcet profile.

# 8 Appendix II

Using the reference ranking of Table ( 3), $R_1^3 = (1,1,-2,1,1,-2)$, $R_2^3 = (-2,1,1,-2,1,1)$ and $R_3^3 = (1,-2,1,1,-2,1)$. The following table provides a reference ranking for a 4-candidate field and the associated $R_1^4$ and $R_2^4$ profiles.

Table 6:

| Reference ranking | $R_1^4$ | $R_2^4$ | Reference ranking | $R_1^4$ | $R_2^4$ |
|---|---|---|---|---|---|
| 1. $A > B > C > D$ | (1) | (-1) | 13. $D > C > B > A$ | (1) | (-1) |
| 2. $A > B > D > C$ | (1) | (-1) | 14. $C > D > B > A$ | (1) | (-1) |
| 3. $A > C > B > D$ | (1) | (-1) | 15. $D > B > C > A$ | (1) | (-1) |
| 4. $A > C > D > B$ | (1) | (1) | 16. $B > D > C > A$ | (1) | (1) |
| 5. $A > D > C > B$ | (1) | (1) | 17. $B > C > D > A$ | (1) | (1) |
| 6. $A > D > B > C$ | (1) | (-1) | 18. $C > B > D > A$ | (1) | (-1) |
| 7. $B > A > C > D$ | (-1) | (1) | 19. $D > C > A > B$ | (-1) | (1) |
| 8. $B > A > D > C$ | (-1) | (1) | 20. $C > D > A > B$ | (-1) | (1) |
| 9. $C > A > B > D$ | (-1) | (-1) | 21. $D > B > A > C$ | (-1) | (-1) |
| 10. $C > A > D > B$ | (-1) | (1) | 22. $B > D > A > C$ | (-1) | (1) |
| 11. $D > A > B > C$ | (-1) | (-1) | 23. $C > B > A > D$ | (-1) | (-1) |
| 12. $D > A > C > B$ | (-1) | (1) | 24. $B > C > A > D$ | (-1) | (1) |

**Proof of Proposition 1**

*Part 1*: Assume $k = 2$ to start with. Also, without loss of generality, consider the pair $(R_1^n, R_2^n)$. $R_1^n$ has non-zero voters for $A$ in the 1-st, 2-nd, $(n-1)$-th and $n$-th places. $R_2^n$ has non-zero voters for $B$ in the 1-st, 2-nd, $(n-1)$-th and $n$-th places. The inner product of $(R_1^n)^T$ and $R_2^n$ have non-zero components for all rankings in which (1) $A$ is in the 1-st place and $B$ is in the 2-nd, $(n-1)$-th or $n$-th place (2) $A$ is in the 2-nd place and $B$ is in the 1-st, $(n-1)$-th or $n$-th place (3) $A$ is in the $(n-1)$-th place and $B$ is in the 1-st, 2-nd or $n$-th place and (4) $A$ is in the $n$-th place and $B$ is in the 1-st, 2-nd or $(n-1)$-th place. In each of these cases (a total of twelve cases), $A$ and $B$ can be placed in their positions in $(n-2)!$ ways. The relevant components of $R_1^n$ and $R_2^n$ belong to the set $\{1,-1\}$. The non-zero components of the inner product equal

$$-(n-2)! - (n-2)! + (n-2)! - (n-2)! + (n-2)! - (n-2)! - (n-2)! + (n-2)!$$
$$-(n-2)! + (n-2)! - (n-2)! - (n-2)! = -4(n-2)!$$

Hence $R_1^n$ and $R_2^n$ are not orthogonal. By way of illustration, for $n = 3$and4, $(R_1^3)^T R_2^3 = -6$ and $(R_1^4)^T R_2^4 =$

$-8$. The argument extends to all pairs of $R_i^n$ profiles for $k=2$.

Next note that all the previous steps of the proof apply directly without any changes to any $k < \frac{n+1}{2}$. Now suppose we choose $k = \frac{n+1}{2}$ which can only happen if $n$ is odd. Note that the candidate $I$ can be in the $k$-th place in $(n-1)!$ rankings and that half of these rankings are reversals of the other half. Each such ranking has (-2) voters by construction. The inner product of $(R_1^n)^T$ and $R_2^n$ have non-zero components for all rankings in which (1) $A$ is in the 1-st place and $B$ is in the $\frac{n+1}{2}$-th or $n$-th place (2) $A$ is in the $\frac{n+1}{2}$-th place and $B$ is in the 1-st, or $n$-th place (3) $A$ is in the $n$-th place and $B$ is in the 1-st, $\frac{n+1}{2}$-th place. The non-zero components of the inner product equal

$$-2(n-2)! + (n-2)! - 2(n-2)! - 2(n-2)! + (n-2)! - 2(n-2)! \quad = \quad -6(n-2)!$$

which is not 0. Hence the non-orthogonality claim is true for any $k$ and for all pairs of generic Reverse profiles.

Consider the sum $\sum_{i=1}^{n} R_i^n$ for $k=2$. Only four out of these $n$ profiles at a time contribute non-zero voters for each ranking. Two of the profiles contribute (1) voter each for the first and last places. The other two profiles contribute (-1) each for the 2-nd and $(n-1)$-th places. Hence the sum is 0. Using similar argument, it is clear that the sum of any $(n-1)$ profiles out of the $n$ profiles is not 0. Hence the set spans a $(n-1)$ dimensional subspace, for $k=2$. The steps apply directly without any changes to any $k < \frac{n+1}{2}$. When $k = \frac{n+1}{2}$, three out of these profiles contribute non-zero voters for each ranking at a time. Two of the profiles contribute (1) voter each for the first and last places. The profile contributes (-2) each for the $\frac{n+1}{2}$-th place. Hence the sum is 0

*Part 2*: Consider the inner product of $(R_i^n)^T$ and $B_i^n$, for any given $k$. This has non-zero components for all rankings in which (1) candidate $I$ is in the 1-st place and (2) candidate $I$ is in the $n$-th or last place. As there are $(n-1)!$ rankings in which candidate $I$ is 1-st ranked and another $(n-1)!$ rankings in which he/she is last ranked, the non-zero components equal $(n-1)!.(1).(1) - (n-1)!.(1).(-1) = 0$. Hence this pair is orthogonal to each other.

Next assume that $k=2$ and consider the inner product of $(R_i^n)^T$ and $B_j^n$, where $i \neq j$. This has non-zero components for all rankings in which (1) candidate $J$ is in the 1-st place and $I$ is in the 2-nd place (2) candidate $J$ is in the 1-st place and $I$ is in the $(n-1)$-th place (3) candidate $J$ is in the $n$-th place and $I$ is in the 2-nd place and (4) candidate $J$ is in the $n$-th place and $I$ is in the $(n-1)$-th place. The non-zero components equal $-(n-2)! - (n-2)! + (n-2)! + (n-2)! = 0$. Hence these two vectors are orthogonal and the claim is true.

Again, the arguments extend directly without any changes for any $k < \frac{n+1}{2}$. When $k = \frac{n+1}{2}$, the inner product has non-zero components for all rankings in which (1) candidate $J$ is in the 1-st place and $I$ is in the $\frac{n+1}{2}$-th place (2) candidate $J$ is in the $n$-th place and $I$ is in the $\frac{n+1}{2}$-th place. The non-zero components equal $-2(n-2)! + 2(n-2)! = 0$. Hence claim is true for any given $k$.

*Part 3*: Under a $B_i^n$ profile, candidate $I$ is ranked first $(n-1)!$ times and hence receives as many points. Candidate $J$ receives non-zero votes only for rankings in which he/she is ranked first and candidate $I$ is ranked last. There are $(n-2)!$ such rankings each with (-1) voter. Thus every other candidate receives $-(n-2)!$ points. Under a $R_i^n$ profile, with $k = 2$, candidate $I$ is ranked first $(n-1)!$ times and receives as many points. Candidate $J$ receives non-zero votes for every ranking in which (1) $J$ is first ranked and $I$ is second ranked (2) $J$ is first ranked and $I$ is $(n-1)$-th ranked (3) $J$ is first ranked and $I$ is $n$-th ranked. There are $(n-2)!$ rankings in each category. $J$ receives (-1) for each ranking in the first two categories and (1) for each ranking in the last category. Hence $J$ receives $-(n-2)!$ points.

These tallies remain unchanged for any $k < \frac{n+1}{2}$. For $k = \frac{n+1}{2}$, candidate $J$ receives non-zero votes for every ranking in which (1) $J$ is first ranked and $I$ is $\frac{n+1}{2}$-th ranked (2) $J$ is first ranked and $I$ is $n$-th ranked. There are $(n-2)!$ rankings in each category. $J$ receives (-2) for each ranking in the first category and (1) for each ranking in the last category. Hence $J$ receives $-(n-2)!$ points.

The total number of voters in a $B_i^n + K^n$ profile is $2(n-1)! + (n-2)(n-1)! = n!$. The total number of voters in a $R_i^n + K^n$ profile is $2(n-1)! + 2(n-1)! + (n-4)(n-1)! = n!$ for $n > 3$. The normalized plurality scores can be derived using the previous steps. Under a $R_i^n$ profile, each ranking and its reversal has the same number of voters. Hence pairwise scores are a complete tie for each candidate pair.

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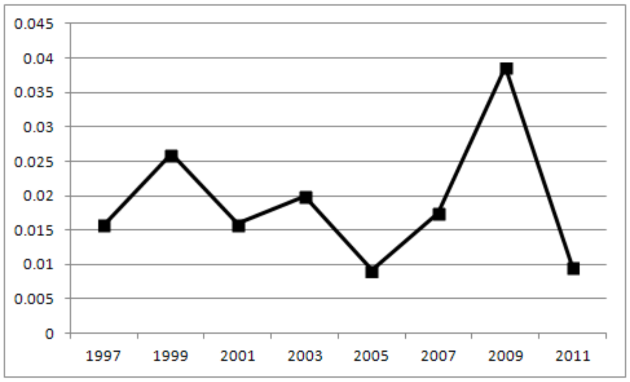
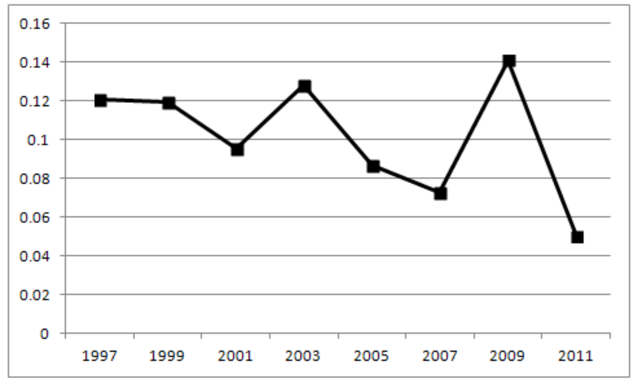
Table 7: Heterogeneity, polarization measures by years

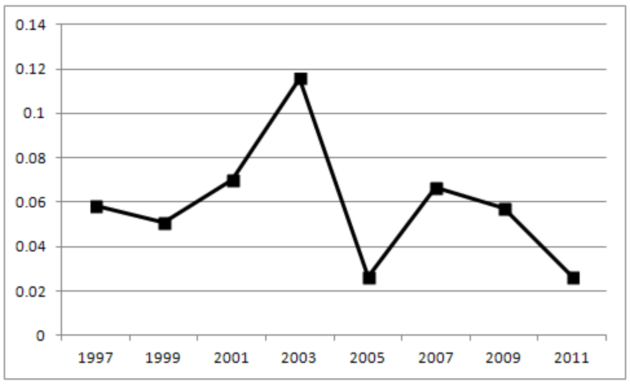
| Years | $\eta_1$ | $\eta_2$ | $\Gamma$ | $\bar{r}$ | Std$(r_i)$ | $\frac{\bar{r}}{\bar{a}}$ | $\bar{r_w}$ | $\frac{\bar{r_w}}{\bar{a_w}}$ | $\Psi$ |
|---|---|---|---|---|---|---|---|---|---|
| 1997 | 0.016 | 0.120 | 0.06 | 0.203 | 0.153 | 0.94 | 0.070 | 0.20 | 0.40 |
| 1999 | 0.026 | 0.119 | 0.05 | 0.173 | 0.110 | 0.88 | 0.071 | 0.24 | 0.61 |
| 2001 | 0.016 | 0.095 | 0.07 | 0.193 | 0.146 | 0.77 | 0.061 | 0.16 | 0.23 |
| 2003 | 0.020 | 0.128 | 0.11 | 0.153 | 0.125 | 0.78 | 0.029 | 0.09 | 0.34 |
| 2005 | 0.009 | 0.087 | 0.03 | 0.224 | 0.171 | 0.85 | 0.089 | 0.22 | 0.34 |
| 2007 | 0.018 | 0.073 | 0.07 | 0.215 | 0.145 | 0.89 | 0.111 | 0.32 | 0.38 |
| 2009 | 0.039 | 0.141 | 0.06 | 0.240 | 0.146 | 0.92 | 0.150 | 0.43 | 0.32 |
| 2011 | 0.010 | 0.050 | 0.03 | 0.240 | 0.151 | 1.06 | 0.126 | 0.38 | 0.26 |



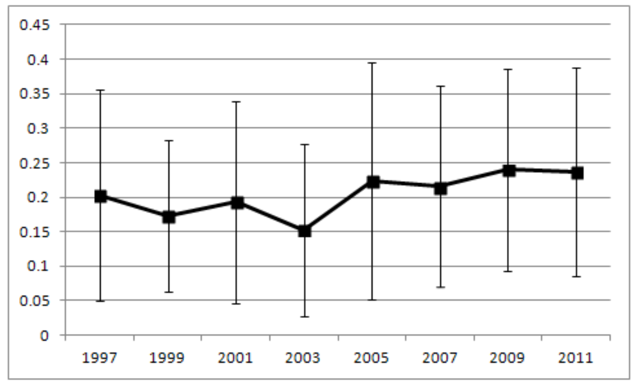

$\eta_1$ vs Election Year



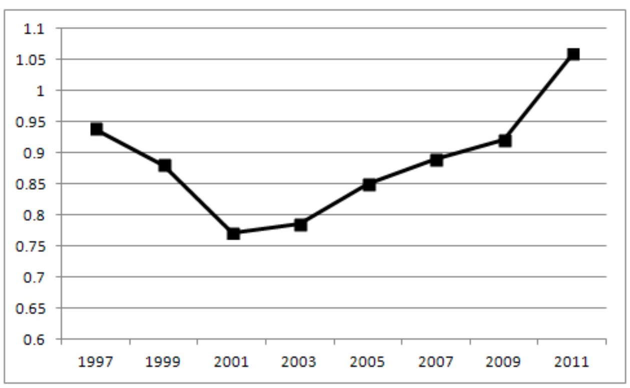

$\eta_2$ vs Election Year



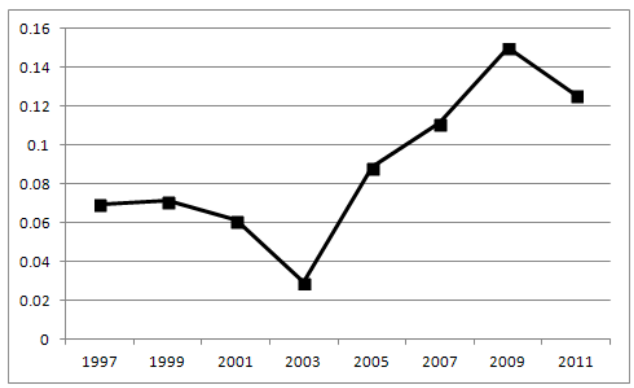

$\Gamma$ vs Election Year



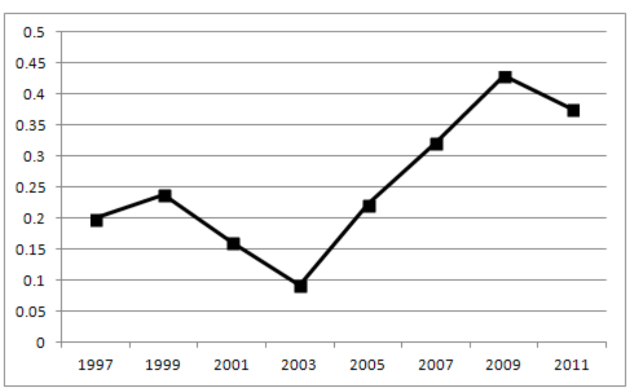

$\bar{r}$ with Std($r_i$) vs Election Year



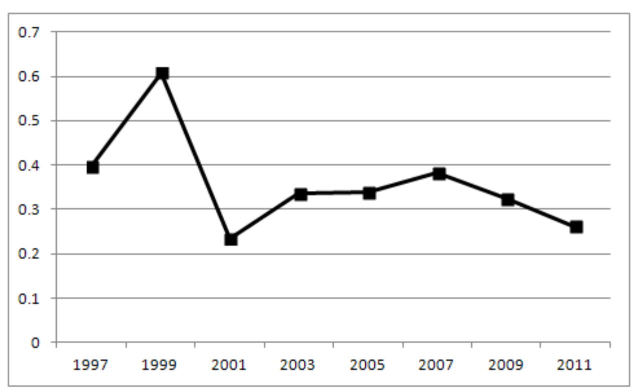

$\frac{\bar{r}}{\bar{a}}$ vs Election Year




$\bar{r_w}$ vs Election Year




$\frac{\bar{r_w}}{\bar{a_w}}$ vs Election Year




$\Psi$ vs Election Year

Table 8: 1997 elections: $a_2 = 0$, $r_4 = 0$

| Status | Candidates | Plurality rank | Borda rank | $a_i$ | Rank by $a_i$ | $r_i$ | Rank by $r_i$ |
|---|---|---|---|---|---|---|---|
| E, 3rd | $C01$ | 6 | 5 | 0.378680243 | 5 | 0.040926482 | 15 |
| D | $C02$ | 10 | 19 | 0 | 19 | 0.419606726 | 1 |
| E, 14th | $C03$ | 9 | 8 | 0.33295654 | 8 | 0.086650186 | 12 |
| E, 1st | $C04$ | 2 | 1 | 0.419606726 | 1 | 0 | 19 |
| D | $C05$ | 15 | 18 | 0.001354005 | 18 | 0.418252721 | 2 |
| E, 1st | $C06$ | 1 | 7 | 0.3532176 | 7 | 0.066389126 | 13 |
| E, 11th | $C07$ | 11 | 11 | 0.136258229 | 11 | 0.283348497 | 9 |
| D | $C08$ | 18* | 17 | 0.041575915 | 17 | 0.378030811 | 3 |
| D | $C09$ | 14 | 14 | 0.094198304 | 14 | 0.325408422 | 6 |
| D | $C10$ | 16 | 16 | 0.066836377 | 16 | 0.352770349 | 4 |
| D | $C11$ | 8 | 9 | 0.265489111 | 9 | 0.154117615 | 11 |
| E, 2nd | $C12$ | 7 | 2 | 0.401894381 | 2 | 0.017712344 | 18 |
| E, 1st | $C13$ | 5 | 3 | 0.387539479 | 3 | 0.032067247 | 17 |
| E, 1s | $C14$ | 4 | 6 | 0.354014073 | 6 | 0.065592653 | 14 |
| E, 1st | $C15$ | 3 | 4 | 0.384610907 | 4 | 0.034995819 | 16 |
| D | $C16$ | 12 | 13 | 0.108161096 | 13 | 0.31144563 | 7 |
| D | $C17$ | 17* | 12 | 0.07116184 | 12 | 0.348444886 | 5 |
| D | $C18$ | 19* | 15 | 0.126075622 | 15 | 0.293531104 | 8 |
| D | $C19$ | 13 | 10 | 0.188120905 | 10 | 0.231485821 | 10 |
| | Average | | | 0.216407966 | | 0.20319876 | |

E, . = elected, count; D = defeated
$C08$ is ranked 17th, $C18$ is ranked 18th and $C17$ is ranked19th officially, after the first count.

Table 9: 1999 elections: $a_3 = 0$, $r_{18} = 0$

| Status | Candidates | Plurality rank | Borda rank | $a_i$ | Rank by $a_i$ | $r_i$ | Rank by $r_i$ |
|---|---|---|---|---|---|---|---|
| E, 11th | *C*01 | 2 | 3 | 0.344874753 | 3 | 0.02540773 | 22 |
| E, 14th | *C*02 | 4* | 10 | 0.24075288 | 10 | 0.129529603 | 15 |
| D | *C*03 | 19 | 24 | 0 | 24 | 0.370282483 | 1 |
| D | *C*04 | 24 | 23 | 0.05359921 | 23 | 0.316683273 | 2 |
| E, 14th | *C*05 | 5* | 4 | 0.336589054 | 4 | 0.033693429 | 21 |
| E, 13th | *C*06 | 3 | 8 | 0.266562658 | 8 | 0.103719824 | 17 |
| D | *C*07 | 20 | 18 | 0.098987012 | 18 | 0.271295471 | 7 |
| E, 1st | *C*08 | 1 | 6 | 0.302314402 | 6 | 0.067968081 | 19 |
| D | *C*09 | 18 | 19 | 0.097422518 | 19 | 0.272859965 | 6 |
| D | *C*10 | 11 | 16 | 0.134397888 | 16 | 0.235884595 | 9 |
| D | *C*11 | 14 | 20 | 0.091334976 | 20 | 0.278947506 | 5 |
| D | *C*12 | 22 | 22 | 0.064651179 | 22 | 0.305631304 | 3 |
| E, 14th | *C*13 | 10 | 7 | 0.279454437 | 7 | 0.090828046 | 18 |
| D | *C*14 | 21 | 21 | 0.087428112 | 21 | 0.282854371 | 4 |
| D | *C*15 | 16* | 15 | 0.151139721 | 15 | 0.219142762 | 10 |
| E, 14th | *C*16 | 7 | 11 | 0.217337913 | 11 | 0.15294457 | 14 |
| D | *C*17 | 13 | 12 | 0.189360568 | 12 | 0.180921915 | 13 |
| E, 13th | *C*18 | 8 | 1 | 0.370282483 | 1 | 0 | 24 |
| E, 13th | *C*19 | 6* | 5 | 0.332297621 | 5 | 0.037984862 | 20 |
| D | *C*20 | 9 | 2 | 0.349856661 | 2 | 0.020425822 | 23 |
| D | *C*21 | 12 | 14 | 0.165788278 | 14 | 0.204494205 | 11 |
| D | *C*22 | 17 | 13 | 0.187533868 | 13 | 0.182748615 | 12 |
| D | *C*23 | 15* | 9 | 0.255204783 | 9 | 0.1150777 | 16 |
| D | *C*24 | 23 | 17 | 0.108824095 | 17 | 0.261458388 | 8 |
| | Average | | | 0.196916461 | | 0.173366022 | |

E, . = elected, count; D = defeated
*C*02 is ranked 5th, *C*05 is ranked 6th and *C*19 is ranked 4th, officially, after the first count. *C*15 is ranked 15th and *C*23 is ranked 16th, officially, after the first count.

Table 10: 2001 elections: $a_1 = 0$, $r_5 = 0$

| Status | Candidates | Plurality rank | Borda rank | $a_i$ | Rank by $a_i$ | $r_i$ | Rank by $r_i$ |
|---|---|---|---|---|---|---|---|
| D | *C*01 | 18 | 19 | 0 | 19 | 0.443075497 | 1 |
| E, 1st | *C*02 | 3 | 3 | 0.414859265 | 3 | 0.028216232 | 17 |
| E, 9th | *C*03 | 4 | 5 | 0.381725656 | 5 | 0.061349841 | 15 |
| D | *C*04 | 16 | 18 | 0.028673663 | 18 | 0.414401835 | 2 |
| E, 1st | *C*05 | 1 | 1 | 0.443075497 | 1 | 0 | 19 |
| D | *C*06 | 15* | 17 | 0.093947775 | 17 | 0.349127722 | 3 |
| D | *C*07 | 14* | 12 | 0.202280531 | 12 | 0.240794966 | 8 |
| D | *C*08 | 12 | 13 | 0.127292042 | 13 | 0.315783455 | 7 |
| D | *C*09 | 13 | 15 | 0.105527986 | 15 | 0.337547511 | 5 |
| D | *C*010 | 11 | 11 | 0.268463485 | 11 | 0.174612012 | 9 |
| E, 15th | *C*011 | 10 | 8 | 0.3472017 | 8 | 0.095873797 | 12 |
| E, 1st | *C*012 | 2 | 7 | 0.350806973 | 7 | 0.092268524 | 13 |
| D | *C*013 | 17 | 16 | 0.095946023 | 16 | 0.347129474 | 4 |
| D | *C*014 | 9 | 10 | 0.275782371 | 10 | 0.167293126 | 10 |
| E, 15th | *C*015 | 8 | 9 | 0.319238258 | 9 | 0.123837239 | 11 |
| E, 13th | *C*016 | 6 | 2 | 0.417958958 | 2 | 0.025116539 | 18 |
| E, 7th | *C*017 | 7 | 4 | 0.388129682 | 4 | 0.054945816 | 16 |
| E, 14th | *C*018 | 5 | 6 | 0.373479872 | 6 | 0.069595626 | 14 |
| D | *C*019 | 19 | 14 | 0.120370398 | 14 | 0.322705099 | 6 |
| | Average | | | 0.250250533 | | 0.192824964 | |

E, . = elected, count; D = defeated
*C*06 is ranked 14th, *C*07 is ranked 15th, officially, after the first count.

Table 11: 2003 elections: $a_5 = 0$, $r_6 = 0$

| Status | Candidates | Plurality rank | Borda rank | $a_i$ | Rank by $a_i$ | $r_i$ | Rank by $r_i$ |
|---|---|---|---|---|---|---|---|
| D | $C$01 | 13 | 13 | 0.142250572 | 13 | 0.204323903 | 8 |
| E, 9th | $C$02 | 2 | 5 | 0.322525198 | 5 | 0.024049277 | 16 |
| D | $C$03 | 8 | 16 | 0.074460729 | 16 | 0.272113746 | 5 |
| E, 12th | $C$04 | 7* | 7 | 0.30908604 | 7 | 0.037488436 | 14 |
| D | $C$05 | 19 | 20 | 0 | 2 | 0.346574475 | 1 |
| E, 1st | $C$06 | 1 | 1 | 0.346574475 | 1 | 0 | 20 |
| D | $C$07 | 20 | 18 | 0.02376686 | 18 | 0.322807616 | 3 |
| D | $C$08 | 18 | 17 | 0.056103618 | 17 | 0.290470857 | 4 |
| D | $C$09 | 12 | 14 | 0.125982373 | 14 | 0.220592102 | 7 |
| D | $C$10 | 15 | 12 | 0.161708137 | 12 | 0.184866339 | 9 |
| D | $C$11 | 17 | 19 | 0.018420412 | 19 | 0.328154063 | 2 |
| E, 13th | $C$12 | 9 | 8 | 0.299381604 | 12 | 0.047192871 | 13 |
| E, 12th | $C$13 | 6* | 4 | 0.325076691 | 4 | 0.021497784 | 17 |
| D | $C$14 | 11 | 11 | 0.165160442 | 11 | 0.181414033 | 10 |
| E, 13th | $C$15 | 5 | 9 | 0.254813264 | 9 | 0.091761211 | 12 |
| E, 13th | $C$16 | 10 | 3 | 0.337468958 | 3 | 0.009105517 | 18 |
| D | $C$17 | 14 | 15 | 0.074738277 | 15 | 271836198 | 6 |
| E, 10th | $C$18 | 3 | 2 | 0.344553732 | 2 | 0.002020743 | 19 |
| D | $C$19 | 16 | 10 | 0.181930175 | 10 | 0.164644301 | 11 |
| E, 12th | $C$20 | 4 | 6 | 0.317334567 | 6 | 0.029239908 | 15 |
| | Average | | | 0.194066806 | | 0.152507669 | |

E, . = elected, count; D = defeated
$C$06 is ranked 6th and $C$13 is ranked 7th, officially, after the first count.

Table 12: 2005 elections: $a_2 = 0$, $r_{16} = 0$

| Status | Candidates | Plurality rank | Borda rank | $a_i$ | Rank by $a_i$ | $r_i$ | Rank by $r_i$ |
|---|---|---|---|---|---|---|---|
| D | $C01$ | 13 | 17 | 0.027062433 | 17 | 0.459551143 | 2 |
| D | $C02$ | 18 | 18 | 0 | 18 | 0.486613576 | 1 |
| E, 9th | $C03$ | 3 | 3* | 0.434377909 | 3 | 0.052235667 | 16 |
| E, 9th | $C04$ | 2 | 5 | 0.409046092 | 5 | 0.077567484 | 14 |
| E, 1st | $C05$ | 1 | 6 | 0.405791705 | 6 | 0.080821871 | 13 |
| D | $C06$ | 12 | 12 | 0.150867032 | 12 | 0.335746544 | 7 |
| D | $C07$ | 15 | 16 | 0.066018547 | 16 | 0.420595029 | 3 |
| D | $C08$ | 17 | 15 | 0.071293136 | 15 | 0.41532044 | 4 |
| D | $C09$ | 14 | 13 | 0.094715069 | 13 | 0.391898507 | 6 |
| E, 11th | $C10$ | 9 | 11 | 0.230661564 | 11 | 0.255952012 | 8 |
| D | $C11$ | 16 | 14 | 0.09189506 | 14 | 0.394718516 | 5 |
| D | $C12$ | 11 | 7 | 0.393973868 | 7 | 0.092639708 | 12 |
| E, 10th | $C13$ | 7 | 4 | 0.424111421 | 4 | 0.062502155 | 15 |
| E, 11th | $C14$ | 8 | 8 | 0.37030372 | 8 | 0.116309856 | 11 |
| D | $C15$ | 10 | 10 | 0.257144827 | 10 | 0.229468749 | 9 |
| E, 11th | $C16$ | 6 | 1 | 0.486613576 | 1 | 0 | 18 |
| E, 5th | $C17$ | 4* | 2 | 0.467411315 | 2 | 0.019202262 | 17 |
| E, 10th | $C18$ | 5 | 9 | 0.35467301 | 9 | 0.131940566 | 10 |
| | Average | | | 0.263108905 | | 0.223504671 | |

E, . = elected, count; D = defeated
$C03$ is ranked 4th and $C17$ is ranked 3rd, officially, after the first count.

Table 13: 2007 elections: $a_8 = 0$, $r_{13} = 0$

| Status | Candidates | Plurality rank | Borda rank | $a_i$ | Rank by $a_i$ | $r_i$ | Rank by $r_i$ |
|---|---|---|---|---|---|---|---|
| E, 1st | $C01$ | 1 | 2 | 0.423140965 | 2 | 0.033877992 | 15 |
| E, 9th | $C02$ | 7 | 4 | 0.364579521 | 4 | 0.092439435 | 13 |
| D | $C03$ | 12 | 14 | 0.051708995 | 14 | 0.405309961 | 3 |
| D | $C04$ | 13 | 13 | 0.07408695 | 13 | 0.382932006 | 4 |
| E, 10th | $C05$ | 6 | 7 | 0.268489717 | 7 | 0.18852924 | 10 |
| E, 7th | $C06$ | 3 | 6 | 0.339173315 | 6 | 0.117845642 | 11 |
| D | $C07$ | 14 | 12 | 0.123069604 | 12 | 0.333949352 | 5 |
| D | $C08$ | 15 | 16 | 0 | 16 | 0.457018956 | 1 |
| E, 9th | $C09$ | 5 | 3 | 0.39536522 | 3 | 0.061653736 | 14 |
| D | $C10$ | 16 | 15 | 0.051553466 | 15 | 0.40546549 | 2 |
| E, 8th | $C11$ | 4 | 10 | 0.242232672 | 10 | 0.214786284 | 7 |
| E, 10th | $C12$ | 8 | 8 | 0.265616995 | 8 | 0.191401962 | 9 |
| E, 9th | $C13$ | 9 | 1 | 0.457018956 | 1 | 0 | 16 |
| D | $C14$ | 10 | 11 | 0.19453085 | 11 | 0.262488106 | 6 |
| E, 6th | $C15$ | 2 | 5 | 0.355915612 | 5 | 0.101103345 | 12 |
| D | $C16$ | 11 | 9 | 0.262945546 | 9 | 0.19407341 | 8 |
| | Average | | | 0.241839274 | | 0.215179682 | |

E, . = elected, count; D = defeated

Table 14: 2009 elections: $a_{21} = 0$, $r_{14} = 0$

| Status | Candidates | Plurality rank | Borda rank | $a_i$ | Rank by $a_i$ | $r_i$ | Rank by $r_i$ |
|---|---|---|---|---|---|---|---|
| D | $C01$ | 18 | 19 | 0.100249059 | 19 | 0.399637763 | 3 |
| E, 17th | $C02$ | 10 | 10 | 0.264162972 | 10 | 0.235723819 | 12 |
| E, 1st | $C03$ | 1 | 2 | 0.470059345 | 2 | 0.029827446 | 20 |
| D | $C04$ | 17 | 18 | 0.109561943 | 18 | 0.390324849 | 4 |
| D | $C05$ | 15 | 15 | 0.159057867 | 15 | 0.340828924 | 7 |
| E, 15th | $C06$ | 6 | 7 | 0.331563707 | 7 | 0.168323085 | 15 |
| D | $C07$ | 16 | 16 | 0.153278278 | 16 | 0.346608513 | 6 |
| E, 16th | $C08$ | 4 | 6 | 0.422708423 | 6 | 0.077178369 | 16 |
| D | $C09$ | 13 | 14 | 0.171308213 | 14 | 0.328578579 | 8 |
| D | $C10$ | 21 | 20 | 0.094439678 | 20 | 0.405447114 | 2 |
| D | $C11$ | 20 | 17 | 0.141528433 | 17 | 0.358358358 | 5 |
| E, 16th | $C12$ | 7 | 8 | 0.284016159 | 8 | 0.215870633 | 14 |
| E, 17th | $C13$ | 8 | 5 | 0.42332809 | 5 | 0.076558702 | 17 |
| E, 1st | $C14$ | 2 | 1 | 0.499886792 | 1 | 0 | 21 |
| D | $C15$ | 14 | 12 | 0.219237094 | 12 | 0.280649697 | 10 |
| D | $C16$ | 9 | 11 | 0.250363459 | 11 | 0.249523333 | 11 |
| E, 1st | $C17$ | 3 | 3 | 0.451838744 | 3 | 0.048048048 | 19 |
| D | $C18$ | 12 | 9 | 0.283706325 | 9 | 0.216180466 | 13 |
| D | $C19$ | 11 | 4 | 0.445796987 | 4 | 0.054089804 | 18 |
| D | $C20$ | 19 | 13 | 0.189624148 | 13 | 0.310262644 | 9 |
| E, 17th | $WI01$ | 5 | 21 | 0 | 21 | 0.499886792 | 1 |
| | Average | | | 0.260272177 | | 0.239614615 | |

E, . = elected, count; D = defeated

Table 15: 2011 elections: $a_8 = 0$, $r_1 = 0$

| Status | Candidates | Plurality rank | Borda rank | $a_i$ | Rank by $a_i$ | $r_i$ | Rank by $r_i$ |
|---|---|---|---|---|---|---|---|
| E, 1st | $C01$ | 1 | 1 | 0.460430347 | 1 | 0 | 18 |
| E, 9th | $C02$ | 4 | 3 | 0.393626618 | 3 | 0.066803729 | 16 |
| E, 14th | $C03$ | 6 | 8 | 0.280711324 | 8 | 0.179719023 | 11 |
| E, 13th | $C04$ | 7 | 11 | 0.230369018 | 11 | 0.230061328 | 8 |
| E, 1st | $C05$ | 3 | 6 | 0.334550108 | 6 | 0.125880239 | 13 |
| D | $C06$ | 13 | 15 | 0.051566073 | 15 | 0.408864274 | 4 |
| D | $C07$ | 16 | 16 | 0.027006804 | 16 | 0.433423542 | 3 |
| D | $C08$ | 18 | 18 | 0 | 18 | 0.460430374 | 1 |
| D | $C09$ | 12 | 12 | 0.115530661 | 12 | 0.344899686 | 7 |
| D | $C10$ | 17 | 17 | 0.02006979 | 17 | 0.440360557 | 2 |
| E, 14th | $C11$ | 9 | 7 | 0.300004895 | 7 | 0.160425452 | 12 |
| D | $C12$ | 11 | 4 | 0.354312208 | 4 | 0.106118139 | 15 |
| E, 13th | $C13$ | 5 | 2 | 0.41154957 | 2 | 0.048880777 | 17 |
| D | $C14$ | 14 | 13 | 0.112446766 | 13 | 0.347983581 | 6 |
| E, 1st | $C15$ | 2 | 5 | 0.342165439 | 5 | 0.118264907 | 14 |
| E, 14th | $C16$ | 8 | 10 | 0.258557632 | 10 | 0.201872714 | 9 |
| D | $C17$ | 10 | 9 | 0.270571534 | 9 | 0.0.189858812 | 10 |
| D | $C18$ | 15 | 14 | 0.061579989 | 14 | 0.0.398850358 | 5 |
| | Average | | | 0.223613821 | | 0.236816526 | |

E, . = elected, count; D = defeated