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An Achievable Region for the Double Unicast Problem Based on a Minimum Cut Analysis

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***Abstract*—We consider the multiple unicast problem under network coding over directed acyclic networks when there are two source-terminal pairs, $s_1 - t_1$ and $s_2 - t_2$. Current characterizations of the multiple unicast capacity region in this setting have a large number of inequalities, which makes them hard to explicitly evaluate. In this work we consider a slightly different problem. We assume that we only know certain minimum cut values for the network, e.g., mincut(S_i, T_j), where $S_i \subseteq \{s_1, s_2\}$ and $T_j \subseteq \{t_1, t_2\}$ for different subsets S_i and T_j. Based on these values, we propose an achievable rate region for this problem based on linear codes. Towards this end, we begin by defining a base region where both sources are multicast to both the terminals. Following this we enlarge the region by appropriately encoding the information at the source nodes, such that terminal t_i is only guaranteed to decode information from the intended source s_i, while decoding a linear function of the other source. The rate region takes different forms depending upon the relationship of the different cut values in the network.**

I. Introduction

The problem of characterizing the utility of network coding for multiple unicasts is an intriguing one. In the multiple unicast problem there is a set of source-terminal pairs in a network that wish to communicate messages. This is in contrast to the multicast problem where each terminal requests exactly the same set of messages from the source nodes. The multicast problem under network coding is very well understood. In particular, several papers [1][2][3] discuss the exact capacity region and network code construction algorithms for this problem.

However, the multiple unicast problem is not that well understood. A significant amount of previous work has attempted to find inner and outer bounds on the capacity region for a given instance of a network. In [4], an information theoretic characterization for directed acyclic networks is provided. However, explicit evaluation of the region is computationally intractable for even small networks due to the large number of constraints. The authors in [5] propose an outer bound on the capacity region. Price et al. [6] provide an outer bound on the capacity region in a two unicast session network, and provided a network structure in which their outer bound is the exact capacity region. The work of [7] forms a linear optimization to characterize an achievable rate region by packing butterfly structures in the original graph. This approach is limited since only the XOR operation is allowed in each butterfly structure.

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In this work we propose an achievable region for the two-unicast problem using linear network codes. Our setup is somewhat different from the above-mentioned works in that we consider directed acyclic networks with unit capacity edges and assume that we only know certain minimum cut values for the network, e.g., mincut(S_i, T_j), where $S_i \subseteq \{s_1, s_2\}$ and $T_j \subseteq \{t_1, t_2\}$ for different subsets S_i and T_j. This is related to the work of Wang and Shroff [8] (see also [9]) for two-unicast that presented a necessary and sufficient condition on the network structure for the existence of a network coding solution that supports unit rate transmission for each $s_i - t_i$ pair. In this work we consider general rates. Reference [10] is related in the sense that they give an achievable rate region for this problem based on the number of edge disjoint paths for $s_i - t_i$ pair. In our work we propose a new achievable rate region given additional information about the network resources. The work of [11] considered the three unicast session problem in which each source is transmitting at unit rate. Finally, reference [12] applies the technique of interference alignment in the case of three unicast sessions and shows that communication at half the mincut of each source-terminal pair is possible.

This paper is organized as follows. Section II introduces the system model under consideration. Section III contains the precise problem formulation and the derivations of our proposed achievable rate region. Section IV compares our achievable region to existing literature. Due to space limitations, some of the lemma proofs are not given and can be found in [13].

II. System Model

We consider a network represented by a directed acyclic graph $G = (V, E)$. There is a source set $S = \{s_1, s_2\} \in V$ in which each source observes a random process with a discrete integer entropy, and there is a terminal set $T = \{t_1, t_2\} \in V$ in which t_i needs to uniquely recover the information transmitted from s_i at rate R_i. Each edge $e \in E$ has unit capacity and can transmit one symbol from a finite field of size q. If a given edge has a higher capacity, it can be divided into multiple parallel edges with unit capacity. Without loss of generality (W.l.o.g.), we assume that there is no incoming edge into source s_i, and no outgoing edge from terminal t_i. By Menger's theorem, the minimum cut between sets $S_{N_1} \subseteq S$ and $T_{N_2} \subseteq T$ is the number of edge disjoint paths from S_{N_1} to T_{N_2}, and will be denoted by $k_{N_1-N_2}$ where $N_1, N_2 \subseteq \mathcal{N} = \{1, 2\}$. For two unicast sessions, we define the *cut vector* as the vector of the cut values $k_{1-1}, k_{2-2}, k_{1-2}, k_{2-1}, k_{12-1}, k_{12-2}, k_{1-12}, k_{2-12}$ and k_{12-12}.

The network coding model in this work is based on [2]. Assume source s_i needs to transmit at rate R_i. Then the random variable observed at s_i is denoted as $X_i = (X_{i1}, X_{i2}, \cdots, X_{iR_i})$, where each X_{ij} is an element of $GF(q)$; the X_is are assumed to be independent. For linear network codes, the signal on an edge (i,j) is a linear combination of the signals on the incoming edges on i or a linear combination of the source signals at i. Let Y_{e_n} ($tail(e_n) = k$ and $head(e_n) = l$) denote the signal on edge $e_n \in E$. Then,

$$Y_{e_n} = \sum_{\{e_m | head(e_m)=k\}} f_{m,n} Y_{e_m} \text{ if } k \in V \setminus \{s_1, s_2\}, \text{ and}$$

$$Y_{e_n} = \sum_{j=1}^{R_i} a_{ij,n} X_{ij} \text{ if } X_i \text{ is observed at } k.$$

The *local coding vectors* $a_{ij,n}$ and $f_{m,n}$ are also chosen from $GF(q)$. We can also express Y_{e_n} as , $Y_{e_n} = \sum_{j=1}^{R_1} \alpha_{j,n} X_{1j} + \sum_{j=1}^{R_2} \beta_{j,n} X_{2j}$. Then the global coding vector of Y_{e_n} is $[\alpha_n, \beta_n] = [\alpha_{1,n}, \alpha_{2,n}, \cdots, \alpha_{R_1,n}, \beta_{1,n}, \beta_{2,n}, \cdots, \beta_{R_2,n}]$. We are free to choose an appropriate value of the field size q.

In this work, we present an achievable rate region given a subset of the cut values in the cut vector; namely, k_{1-1}, k_{2-2}, k_{1-2}, k_{2-1}, k_{12-1}, k_{12-2}. W.l.o.g, we assume there are k_{i-ij} outgoing edges from s_i and k_{ij-i} incoming edges to t_i. If this is not the case one can always introduce an artificial source (terminal) node connected to the original source (terminal) node by k_{i-ij} (k_{ij-i}) edges. It can be seen that the new network has the same cut vector as the original network.

III. ACHIEVABLE RATE REGION FOR A GIVEN CUT VECTOR

First, suppose that only t_1 is interested in recovering the random variables X_1 and X_2 which are observed at s_1 and s_2 respectively. Denote the rate from s_1 to t_1 and s_2 to t_1 as R_{11} and R_{12}. Then the capacity region C_{t_1}, that is achieved by routing will be

$$\begin{aligned} R_{11} &\leq k_{1-1}, \\ R_{12} &\leq k_{2-1}, \\ R_{11} + R_{12} &\leq k_{12-1}. \end{aligned}$$

The capacity region C_{t_2} for t_2 can be drawn in a similar manner. This is shown in Fig. 1(a). We also find the boundary points a, b, c, d such that their coordinates are $a = (k_{12-1} - k_{2-1}, k_{2-1}), b = (k_{1-2}, k_{12-2} - k_{1-2}), c = (k_{1-1}, k_{12-1} - k_{1-1}), d = (k_{12-2} - k_{2-2}, k_{2-2})$. A simple achievable rate region for our problem can be arrived at by multicasting both sources X_1 and X_2 to both the terminals t_1 and t_2.

Theorem 3.1: Rate pairs (R_1, R_2) belonging to the following set $\mathcal{B}$ can be achieved for two unicast sessions.

$$\begin{aligned} \mathcal{B} = \{ & R_1 \leq \min(k_{1-2}, k_{1-1}), \\ & R_2 \leq \min(k_{2-1}, k_{2-2}), \\ & R_1 + R_2 \leq \min(k_{12-1}, k_{12-2})\}. \end{aligned}$$

Proof: We multicast both the sources to each terminal. This can be done using the multi-source multi-sink multicast result (Thm. 8) in [2]. ■

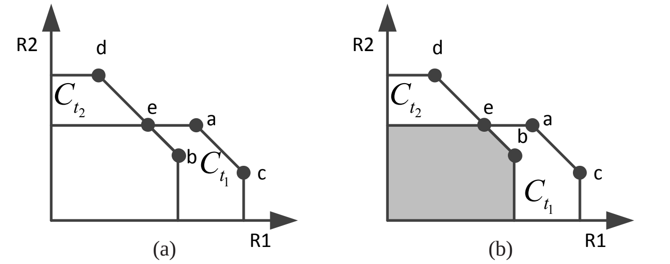

Fig. 1. (a) An example of a capacity region. (b) Base region for the example.

Subsequently we will refer to region $\mathcal{B}$ achieved by multicast as the *base rate region* (the grey region in Fig. 1(b)).

We now move on to precisely formulating the problem. Let Z_i denote the received vector at t_i, X_i denote the transmitted vector at s_i, and H_{ij} denote the transfer function from s_j to t_i. Let M_i denote the encoding matrix at s_i, i.e., M_i is the transformation from X_i to the transmitted symbols on the outgoing edges from s_i. In our formulation, we will let the length of X_i to be k_{i-i} (i.e., the maximum possible). For transmission at rates R_1 and R_2, we introduce precoding matrices $V_i, i = 1, 2$ of dimension $R_i \times k_{i-i}$, so that the overall system of equations is as follows.

$$\begin{aligned} Z_1 &= H_{11} M_1 V_1 X_1 + H_{12} M_2 V_2 X_2, \\ Z_2 &= H_{21} M_1 V_1 X_1 + H_{22} M_2 V_2 X_2. \end{aligned} \quad (1)$$

We say that t_i can receive at rate R_i from s_i if it can decode $V_i X_i$ perfectly. The row dimension of the $V_i's$ can be adjusted to obtain different rate vectors. For $(R_1, R_2) \in \mathcal{B}$, it can be shown that there exist local coding vectors over a large enough field such that the ranks of the different matrices in the first column of Table I are given by the corresponding entries in the third column, which correspond to the maximum possible. Furthermore, by the multi-source multi-sink multicast result, these matrices are such that $[H_{11}M_1 \;\; H_{12}M_2]$ is a full rank matrix of dimension $k_{12-1} \times (R_1 + R_2)$, and $[H_{21}M_1 \;\; H_{22}M_2]$ is a full rank matrix of dimension $k_{12-2} \times (R_1 + R_2)$. In Table I, for instance since the minimum cut between s_1 and t_1 is k_{1-1}, we know that the maximum rank of H_{11} is k_{1-1}. Using the formalism of [2], we can conclude that there is a square submatrix of H_{11} of dimension $k_{1-1} \times k_{1-1}$ whose determinant is not identically zero. Such appropriate submatrices can be found for each of the matrices in the first column of Table I. This in turn implies that their product is not identically zero and therefore using the Schwartz-Zippel lemma, we can conclude that there exists an assignment of local coding vectors so that the rank of all the matrices is simultaneously the maximum possible. For the rest of the paper, we assume that such a choice of local coding vectors has been made. Our arguments will revolve around appropriately modifying the source encoding matrices M_1 and M_2.

Note that there are two boundary points of the base region (the two boundary points may overlap). At point Q_1, we denote the achievable rate pair by (R_1^*, R_2^*) where

$$\begin{aligned} & R_1^* = \min(k_{1-2}, k_{1-1}), \text{ and} \\ & R_2^* = \min(\min(k_{2-1}, k_{2-2}), \min(k_{12-1}, k_{12-2}) - R_1^*). \end{aligned}$$

TABLE I
DIMENSION OF MATRICES

matrix	*dimension*	*rank*
H_{11}	$k_{12-1} \times k_{1-12}$	k_{1-1}
H_{12}	$k_{12-1} \times k_{2-12}$	k_{2-1}
$[H_{11} \quad H_{12}]$	$k_{12-1} \times (k_{1-12} + k_{2-12})$	k_{12-1}
M_1	$k_{1-12} \times R_1$	R_1
H_{21}	$k_{12-2} \times k_{1-12}$	k_{1-2}
H_{22}	$k_{12-2} \times k_{2-12}$	k_{2-2}
$[H_{21} \quad H_{22}]$	$k_{12-2} \times (k_{1-12} + k_{2-12})$	k_{12-2}
M_2	$k_{2-12} \times R_2$	R_2

At point Q_2, we denote the achievable rate pair by (R_1^{**}, R_2^{**}) where

$$R_2^{**} = \min(k_{2-1}, k_{2-2}), \text{ and}$$
$$R_1^{**} = \min(\min(k_{1-2}, k_{1-1}), \min(k_{12-1}, k_{12-2}) - R_2^{**})$$

In Fig. 1(a), these boundary points are $Q_1 = b$ and $Q_2 = e$.

In what follows, we will present our arguments towards increasing the value of R_1 to be larger than R_1^* (these arguments can be symmetrically applied for increasing R_2 as well). For this purpose, we will start with the point Q_1 and attempt to achieve points that are near it but do not belong to $\mathcal{B}$. At Q_1, if $R_1^* = k_{1-1}$, then we cannot increase R_1 due to the cut constraints. Hence, we assume $R_1^* = k_{1-2}$. Furthermore, since $k_{2-2} \geq k_{12-2} - k_{1-2} \geq \min(k_{12-1}, k_{12-2}) - k_{1-2}$, $R_2^* = \min(\min(k_{2-1}, k_{2-2}), \min(k_{12-1}, k_{12-2}) - R_1^*) = \min(k_{2-1}, \min(k_{12-1}, k_{12-2}) - k_{1-2})$.

In this paper we refer to $k_{1-2} + k_{2-1}$ as a measure of the interference in the network and in the subsequent discussion present achievable regions based on its value. We emphasize though that this is nomenclature used for ease of presentation. Indeed a high value of k_{1-2} does not necessarily imply that there is a lot of interference at t_2, since the network code itself dictates the amount of interference seen by t_2. The following lemma will be used extensively.

Lemma 3.2: Consider a system of equations $Z = H_1 X_1 + H_2 X_2$, where X_1 is a vector of length l_1 and X_2 is a vector of length l_2 and $Z \in span([H_1 \quad H_2])$[1]. The matrix H_1 has dimension $z_t \times l_1$, and rank $l_1 - \sigma$, where $0 \leq \sigma \leq l_1$. The matrix H_2 is full rank and has dimension $z_t \times l_2$ where $z_t \geq (l_1 + l_2 - \sigma)$. Furthermore, the column spans of H_1 and H_2 intersect only in the all-zeros vectors, i.e. $span(H_1) \cap span(H_2) = \{0\}$. Then there exists a unique solution for X_2.

A. Low Interference Case

This is the case when $k_{1-2} + k_{2-1} \leq \min(k_{12-1}, k_{12-2})$. At Q_1, from the assumption, it follows that $R_1^* = k_{1-2}, R_2^* = \min(k_{2-1}, \min(k_{12-1}, k_{12-2}) - k_{1-2}) = k_{2-1}$. An example is shown in Fig. 2(a). Furthermore, $Q_1 = Q_2 = e$.

[1]Throughout the paper, $span(A)$ refers to the column span of A.

Our solution strategy is to consider the encoding matrices M_1 and M_2 at the point Q_1, and to introduce a new encoding matrix at s_1, denoted M_1' (with $R_1^* + \delta$ columns) such that $span(H_{11}M_1') \cap span(H_{12}) = \{0\}$. As shown below, this will allow t_1 to decode from s_1 at rate $R_1^* + \delta$ and t_2 to decode from s_2 at rate R_2^*. After the modification, each t_i is guaranteed to decode at the appropriate rate from s_i. A similar argument can then be applied for R_2^* to arrive at the achievable rate region in this case.

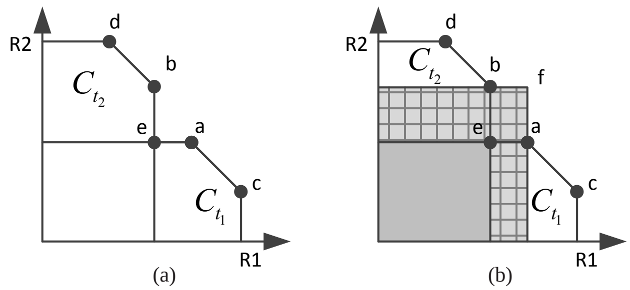

Fig. 2. (a) The capacity regions C_{t_1} and C_{t_2} for an example of low interference case. (b) The achievable rate region for low interference case. For each point in the shaded grey area, both terminals can recover both the sources. In the hatched grey area, for a given rate point, its x-coordinate is the rate for $s_1 - t_1$ and its y-coordinate is the rate for $s_2 - t_2$; the terminals are not guaranteed to decode both sources in this region.

At the point Q_1, the rates are $R_1^* = k_{1-2}, R_2^* = k_{2-1}$. Since both terminals can decode both sources, it holds that

$$rank(H_{i1}M_1) = k_{1-2}, rank(H_{i2}M_2) = k_{2-1}, \text{ and}$$
$$span(H_{i1}M_1) \cap span(H_{i2}M_2) = \{0\} \text{ for } i = 1, 2.$$

By analyzing the properties of the above matrices, we have Theorem 3.4. Before we state the theorem, we first give the following lemma which will be used in proving Theorem 3.4.

Lemma 3.3: Rate Increase Lemma. In the base region, denote the achievable rates at Q_1 as R_1^* and R_2^*, and the corresponding encoding matrices as M_1 and M_2. Let $rank([H_{11} \quad H_{12}M_2]) = r \geq R_1^* + R_2^*$. There exist a series of full rank matrices $\bar{M}_1^{(n)} = [\tilde{M}_1^{(n)} \quad M_1]$ of dimension $k_{1-12} \times (n + R_1^*)$ such that $rank([H_{11}\bar{M}_1^{(n)} \quad H_{12}M_2]) = R_1^* + R_2^* + n$, $0 \leq n \leq (r - R_1^* - R_2^*)$.

Theorem 3.4: Given a cut vector, if $k_{1-2} + k_{2-1} \leq \min(k_{12-1}, k_{12-2})$, then the rate pair in the following region can be achieved.

Region 1:

$$R_1 \leq k_{12-1} - k_{2-1},$$
$$R_2 \leq k_{12-2} - k_{1-2},$$

which is shown in Fig. 2(b).

Proof: In this case, $R_1^* = k_{1-2}$ and $R_2^* = k_{2-1}$ is the boundary point $Q_1 = Q_2$. We will try to find full rank matrix M_1' of dimension $k_{1-12} \times (k_{12-1} - k_{2-1})$ and full rank matrix M_2' of dimension $k_{2-12} \times (k_{12-2} - k_{1-2})$ such that the system of equations can be written as

$$Z_1 = H_{11}M_1'V_1'X_1 + H_{12}M_2'V_2'X_2,$$
$$Z_2 = H_{21}M_1'V_1'X_1 + H_{22}M_2'V_2'X_2,$$

and $V_1'X_1$ can be decoded at t_1, $V_2'X_2$ can be decoded at t_2.

First, note that $rank(H_{12}M_2) = rank(H_{12})$, which implies that $span(H_{12}) = span(H_{12}M_2)$. Therefore $rank([H_{11} \quad H_{12}]) = rank([H_{11} \quad H_{12} \quad H_{12}M_2] = rank([H_{11} \quad H_{12}M_2])$. Together, this implies that $rank([H_{11} \quad H_{12}M_2]) = k_{12-1}$. Using the Rate Increase Lemma, we can find the matrix M_1' such that the following conditions are satisfied: (i) M_1' is a full rank matrix of dimension $k_{1-12} \times (k_{12-1} - k_{2-1})$, (ii) $rank(H_{11}M_1') = k_{12-1} - k_{2-1}$ and (iii) $span(H_{11}M_1') \cap span(H_{12}) = \{0\}$. (i) is from the Rate Increase Lemma. (ii) and (iii) hold because of the following argument. From Rate Increase Lemma and the fact that $rank(H_{12}M_2) = rank(H_{12}) = k_{2-1}$, we will have

$$\begin{aligned}
k_{12-1} &= rank(H_{11}M_1' \quad H_{12}M_2) \\
&= rank(H_{11}M_1') + rank(H_{12}M_2) \\
&\quad - rank(span(H_{11}M_1') \cap span(H_{12}M_2)) \\
&\leq rank(H_{11}M_1') + rank(H_{12}M_2) \\
&\leq rank(M_1') + rank(H_{12}) \\
&= k_{12-1} - k_{2-1} + k_{2-1} = k_{12-1}.
\end{aligned}$$

Then all the inequalities become equalities. (ii) and (iii) are satisfied. Likewise, M_2' can be found with similar conditions.

Next, since $span(H_{11}M_1') \cap span(H_{12}) = \{0\}$ and $span(H_{12}M_2') \subseteq span(H_{12})$, we will have $span(H_{11}M_1') \cap span(H_{12}M_2') = \{0\}$. By Lemma 3.2 and the above three conditions, t_1 can decode $V_1'X_1$ at rate $k_{12-1} - k_{2-1}$, but cannot decode $V_2'X_2$. By a similar argument, t_2 can decode $V_2'X_2$ at rate $k_{12-2} - k_{1-2}$, but cannot decode $V_1'X_1$. ■

B. High Interference Case

This is the case when $k_{1-2} + k_{2-1} \geq \min(k_{12-1}, k_{12-2})$. Recall that we also assume that $k_{1-2} \leq k_{1-1}$. At Q_1, $R_1^* = k_{1-2}$, $R_2^* = \min(k_{2-1}, \min(k_{12-1}, k_{12-2}) - k_{1-2}) = \min(k_{12-1}, k_{12-2}) - k_{1-2}$. This means that Q_1 and Q_2 are two separated points. An example is shown in Fig. 1(a). In particular, when C_{t_1} is contained in C_{t_2} or vice versa, the achievable region is described by this case.

Our strategy is similar to the one for the previous case, but with important differences. We begin with the rate vector at point Q_1 and then attempt to increase R_1. However, in this particular case we will not be able to increase R_2 and in fact may need to reduce it. This is because at point Q_1, we have $R_2^* = rank(H_{12}M_2) < k_{2-1} = rank(H_{12})$, i.e., the encoding matrix M_2 is such that $rank(H_{12}M_2)$ is strictly less than the maximum possible. Therefore, if we augment M_2 with additional columns to arrive at M_2', it is not possible to assert as before that the $span(H_{11}M_1') \cap span(H_{12}M_2') = \{0\}$. Hence, it may be possible that s_1 cannot be decoded at t_1, (after augmenting M_2 to M_2'). In this situation, we have the following result.

Theorem 3.5: Given a cut vector, if $k_{1-2} + k_{2-1} \geq \min(k_{12-1}, k_{12-2})$ and $k_{1-2} \leq k_{1-1}$, then the rate pair in the following region can be achieved.

Region 2:

$$\begin{aligned}
R_1 &\leq k_{1-1}, \\
R_2 &\leq \min(k_{12-1}, k_{12-2}) - k_{1-2}, \\
R_1 + R_2 &\leq rank([H_{11} \quad H_{12}M_2]).
\end{aligned}$$

Note that in the above characterization, the sum rate constraint depends on $rank([H_{11} \ H_{12}M_2])$; we show a lower bound on $rank([H_{11} \ H_{12}M_2])$ in III-B1. The following lemma that discusses situations in which rates can be traded off between the two unicast sessions is needed for the proof of Thm. 3.5.

Lemma 3.6: Rate Exchange Lemma. Given that $rank([H_{11}M_1 \quad H_{12}M_2]) = rank([H_{11} \ H_{12}M_2]) = r$, where M_1 is a full rank matrix of dimension $k_{1-12} \times (r - R_2)$, M_2 is a full rank matrix of dimension $k_{2-12} \times R_2$. If $M_1' = [\ \vec{\alpha} \quad M_1\ ]$ where $\vec{\alpha}$ is a vector of length k_{1-12} and $rank(H_{11}M_1') = r - R_2 + 1$, then there exists an M_2' such that $span(H_{11}M_1') \cap span(H_{12}M_2') = \{0\}$ where M_2' is a full rank submatrix of M_2 of dimension $k_{2-12} \times (R_2 - 1)$.

Proof of Theorem 3.5. Given that $k_{1-2} + k_{2-1} \geq \min(k_{12-1}, k_{12-2})$ and $k_{1-2} \leq k_{1-1}$, we will extend the rate region from Q_1 where $R_1^* = k_{1-2}$, $R_2^* = \min(k_{12-1}, k_{12-2}) - k_{1-2}$. At Q_1, we need to increase R_1 while keeping R_2 as large as possible. By the Rate Increase Lemma, we can achieve the rate point $R_1' = rank([H_{11} \ H_{12}M_2]) - R_2^*$, $R_2' = R_2^*$. The corresponding encoding matrices are M_1' and M_2. When we want to further increase R_1', we could use the Rate Exchange Lemma repeatedly. Hence, when R_1' is increased by δ, R_2' is decreased by δ where $0 \leq \delta \leq \min(R_2^*, k_{1-1} - R_1')$ ($\delta \leq k_{1-1} - R_1'$ comes from the fact that R_1' can be increased to at most k_{1-1}). Terminal t_1 can decode messages both from s_1 at rate $R_1'' = R_1' + \delta$ and s_2 at rate $R_2'' = R_2' - \delta$. Denote the new set of encoding matrices as M_1'' and M_2''.

At t_2, because M_2'' is a submatrix of M_2, $span(H_{22}M_2'') \subseteq span(H_{22}M_2)$. Furthermore, we have $span(H_{21}M_1'') \subseteq span(H_{21}) = span(H_{21}M_1)$, since $R_1^* = k_{1-2}$. Hence, from the above argument, we will have $span(H_{21}M_1'') \cap span(H_{22}M_2'') = \{0\}$ since $span(H_{21}M_1) \cap span(H_{22}M_2) = \{0\}$. Then by Lemma 3.2, we can decode at $R_2'' = R_2' - \delta$ from s_2, but not decode any messages from s_1. ■

A similar analysis for Q_2 allows us to increase R_2, resulting in the following extended region.

Corollary 3.7: Given a cut vector, if $k_{1-2} + k_{2-1} \geq \min(k_{12-1}, k_{12-2})$ and $k_{2-1} \leq k_{2-2}$, then the rate pair in the following region can be achieved.

Region 3:

$$\begin{aligned}
R_1 &\leq \min(k_{12-1}, k_{12-2}) - k_{2-1}, \\
R_2 &\leq k_{2-2}, \\
R_1 + R_2 &\leq rank([H_{21}M_1 \quad H_{22}]).
\end{aligned}$$

The overall rate region is the convex hull of base region, Region 2 and Region 3 which is shown in Fig. 3(a), where boundary segment $d - f$ is achieved via timesharing.

We note that the idea of increasing one rate while decreasing the other can also be applied to the region obtained in low interference case. Since $rank([H_{11} \ H_{12}M_2]) = k_{12-1}$ and $rank([H_{21}M_1 \ H_{22}]) = k_{12-2}$, we can obtain the following two new regions for low interference case.

Region 2':

$$R_1 \leq k_{1-1}$$
$$R_2 \leq k_{2-1}$$
$$R_1 + R_2 \leq k_{12-1}$$

Region 3':

$$R_1 \leq k_{1-2}$$
$$R_2 \leq k_{2-2}$$
$$R_1 + R_2 \leq k_{12-2}$$

Finally, the achievable rate region for low interference case is the convex hull of the region 1, 2' and 3' shown in Fig. 3(b), where the boundary segment $d-f$ and $f-c$ is achieved via timesharing.

1) Lower bound of $rank([H_{11}\ H_{12}M_2])$*:* Next, we investigate the lower bound of $rank([H_{11}\ \ H_{12}M_2])$. In the following argument, R_1^* and R_2^* denote the rate at boundary point Q_1, and M_1 and M_2 denote the corresponding encoding matrices. First note that $rank([H_{11}\ H_{12}M_2]) \geq rank(H_{11}) = k_{1-1}$ and $rank([H_{11}\ H_{12}M_2]) \geq rank([H_{11}M_1\ H_{12}M_2]) = R_1^* + R_2^*$. Next we will also find another nontrivial lower bound of $rank([H_{11}\ \ H_{12}M_2])$ by the following lemma.

Lemma 3.8: Given $rank([H_{11}\ \ H_{12}]) = k_{12-1}$, $rank(H_{12}) = k_{2-1}$ and $rank([H_{12}M_2]) = l$, we have $rank([H_{11}\ H_{12}M_2]) \geq k_{12-1} - k_{2-1} + l$.

Proof: By the assumed conditions, there are k_{2-1} columns in H_{12} that are linearly independent, and in H_{11}, we can find a subset of $k_{12-1} - k_{2-1}$ columns denoted H'_{11} such that $span(H'_{11}) \cap span(H_{12}) = \{0\}$ and $rank(H'_{11}) = k_{12-1} - k_{2-1}$, which further imply that $rank([H'_{11}\ H_{12}]) = k_{12-1}$.

Since $span(H_{12}M_2) \subseteq span(H_{12})$ this means that $span(H'_{11}) \cap span(H_{12}M_2) = \{0\}$. Then $rank([H'_{11}\ H_{12}M_2]) = rank(H'_{11}) + rank(H_{12}M_2) - 0 = k_{12-1} - k_{2-1} + l$. Hence, $rank([H_{11}\ \ H_{12}M_2]) \geq rank([H'_{11}\ H_{12}M_2]) = k_{12-1} - k_{2-1} + l$. ■

Together with the two lower bounds above, we have $rank([H_{11}\ H_{12}M_2]) \geq \max(k_{1-1}, k_{12-1} - k_{2-1} + R_2^*, R_1^* + R_2^*)$. A case where $\max(k_{1-1}, k_{12-1} - k_{2-1} + R_2^*, R_1^* + R_2^*) = k_{12-1} - k_{2-1} + R_2^*$ is shown in Fig. 3(a).

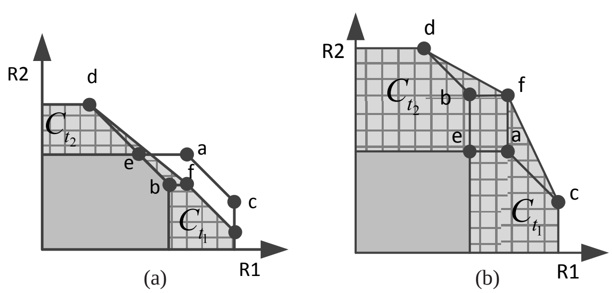

Fig. 3. (a) The extended rate region for high interference case. (b) The final extended rate region for low interference case. For each point in the shaded grey area, both terminals can recover both the sources. In the hatched grey area, for a given rate point, its x-coordinate is the rate for $s_1 - t_1$ and its y-coordinate is the rate for $s_2 - t_2$; the terminals are not guaranteed to decode both sources in this region.

IV. Comparison with existing results

The authors in [8] and [9] explore the case when each source transmits one symbol at a time, or equivalently, $R_1 = R_2 = 1$ in detail, whereas we allow arbitrary rate pairs. Reference [10], also consider the scenario where the rates are arbitrary. Assuming that $k_{2-2} \leq k_{1-1}$, the basic region in [10] is

Region EF09:

$$R_1 + 2R_2 \leq k_{1-1}$$
$$R_2 \leq k_{2-2}$$

They also extend the region using the knowledge of k_{1-2}, k_{2-1} and other cut conditions arising from the network topology (see section IV of [10]). A comparison between our region and theirs indicates that there are example networks where there exist rate points that belong to our region but not to Region EF09. Conversely, there are instances of networks where points that belong to Region EF09, do not fall within our region. The work of [10] can be interpreted in part as an interference nulling scheme, and in future work it may be possible to incorporate this within our approach. The work of [6] considers several different cuts defined in the graph and propose an outer bound for the network capacity. Moreover, they provide certain network structures where the outer bound is tight. Since our work deals with an inner bound, it is qualitatively different. Finally Das et al. [12] have used interference alignment for the case of three unicast sessions, and are able to achieve a rate that is half the mincut for each unicast session. While this is an interesting result for a harder problem, the case of two unicast sessions considered here is different since each connection has only one interferer and the alignment problem does not exist. Moreover, achieving half the mincut for each session can be trivially achieved by timesharing in our problem. In that sense a comparison between our results and theirs is not possible.

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