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Keywords

continuous-time Markov chain, maintenance, probability modeling, proportional hazards model

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Value of condition monitoring for optimal replacement in the proportional hazards model with continuous degradation

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Abstract: This article investigates the value of perfect monitoring information for optimal replacement of deteriorating systems in the Proportional Hazards Model (PHM). A continuous-time Markov chain describes the condition of the system. Although the form of an optimal replacement policy for system under periodic monitoring in the PHM was developed previously, an approximation of the Markov process as constant within inspection intervals led to a counter intuitive result that less frequent monitoring could yield a replacement policy with lower average cost. This article explicitly accounts for possible state transitions between inspection epochs to remove the approximation and eliminate the cost anomaly. However, the mathematical evaluation becomes significantly more complicated. To overcome this difficulty, a new recursive procedure to obtain the parameters of the optimal replacement policy and the optimal average cost is presented. A numerical example is provided to illustrate the computational procedure and the value of condition monitoring. By taking the monitoring cost into consideration, the relationships between the unit cost of periodic monitoring and the upfront cost of continuous monitoring under which the continuous, periodic, or no monitoring scheme is optimal are obtained.

KEYWORDS: CONTINUOUS-TIME MARKOV CHAIN, PROPORTIONAL HAZARDS MODEL, MAINTENANCE, PROBABILITY MODELING

1 INTRODUCTION

Critical infrastructures depend on equipment and systems that deteriorate with age and are subject to failure. Because abrupt failures of assets such as high-voltage power transformers and heavy mining equipment may cause immense economic loss, preventive maintenance is essential. Some of these assets or their electronic components are difficult and/or exorbitantly expensive to repair, and the need for continuous service precludes shutting down the dependent systems while on-site maintenance or repairs are done. In this paper, we consider replacement as the only maintenance option.

Optimal replacement policies for deteriorating systems have been studied for decades (Aven and Bergman, 1986; Lam and Yeh, 1994b), and the recent research effort has been focused on the problem of optimal replacement when some concomitant (condition) information about the system, such as temperature, humidity, vibration levels, or the amount of metal particles in the lubricant, is available. Remote monitoring of condition information is appealing particularly when distance or environmental conditions make regular inspections difficult. Condition monitoring sensors along with information and communication technology increase the visibility of the system's condition and environment while in use. Condition-based maintenance policies, such as those in Banjevic *et al*. (2001), Makis and Jiang (2003), Dieulle *et al*. (2003) and Ghasemi *et al*. (2007), exploit such information to determine when to preventively replace the system. Presumably, policies derived from more frequent observations of condition information have lower cost than those based on less frequent or no observations. The reduction in expected cost provided by frequent monitoring can be used to assess the value of the technology that enables the monitoring.

Condition monitoring may require substantial initial investment in hardware and software installation, in contrast to traditional monitoring which typically incurs a cost associated with each observation. Taking this latter cost into consideration for systems under sequential or periodic monitoring, the optimal monitoring interval is usually determined by searching the possible parameter space within each step of a policy iteration algorithm, such as those in Yeh (1997) and Chiang and Yuan (2001). Continuous monitoring has been studied more recently (Liao *et al.*, 2006). Comparison of periodic and continuous monitoring for a two-state system has been considered by Rosenblatt and Lee (1986). A more general comparative study of sequential and continuous monitoring strategies for a multistate model was presented by Lam and Yeh (1994a) for a deteriorating Markovian system; however, they did not include any cost for continuous monitoring.

The concomitant information may be described by a stochastic process, which most frequently appears in the literature as a semi-Markov or Markov process. Models of the system's failure probability differ according to their approaches for utilizing the condition information. Many researchers assume that the failure process of the system can be described adequately by a multi-state deteriorating Markov or semi-Markov process that leads to failure, and extensive research has been done with such models. For example, Chiang and Yuan (2001) proposed a state-dependent maintenance policy for a Markovian deteriorating system and they showed that many policies presented earlier were special cases of their proposed policy. Bloch-Mercier (2002) studied the preventive maintenance policy for a Markovian deteriorating system when a sequential checking procedure is applied. A dynamic preventive maintenance policy for a multi-state deteriorating system was developed by Chen *et al*. (2003).

For many applications, it is most natural to model failures as dependent on system age in addition to some deterioration process. One way to account for these combined effects is to use the proportional hazards model (PHM), which explicitly includes both the age and the condition information in the hazard function (Makis and Jardine, 1992; Banjevic *et al*., 2001). Makis and Jardine (1992) derived an optimal replacement policy for systems in the PHM with a continuous time Markov chain and periodic monitoring, and presented recursive methods to compute the optimal policy parameters. Banjevic *et al*. (2001) extended Makis and Jardine's model by relaxing the monotonicity assumption of the hazard function and they developed methods for estimating model parameters as well. However, the computations in both papers relied on approximating the concomitant Markov chain as unchanging between inspection epochs. Ghasemi *et al*. (2007) also used the PHM to characterize the system failure process and, under the same discrete time approximation, derived an optimal replacement policy when the condition information of the system is only partially observed.

In this paper, we compare the average cost per unit time of monitoring, replacement and failure under three monitoring schemes: no monitoring which corresponds to age-based replacement, periodic monitoring at various intervals, and continuous monitoring. For periodic monitoring, we follow the model of Makis and Jardine but remove their discrete-time approximation by explicitly accounting for the possibility that the concomitant Markov chain may make transitions among its states between observation epochs. This allows an accurate comparison of monitoring at discrete intervals of different lengths against continuous monitoring (approximated as the interval vanishes) or no monitoring. Accounting for state transitions between observations introduces significant intricacies in the computation of policy parameters.

These are addressed in Sections 3-5. We use conditioning to develop a new recursive procedure to obtain the parameters of the optimal replacement policy and its long-run average cost. We focus on systems with an underlying pure-birth process having an arbitrary number of states and illustrate the reasoning and computations for a three-state deterioration process in detail. In Section 6 we review the optimal replacement age for the no-monitoring scheme. Section 7 illustrates the computation of replacement policy parameters under periodic monitoring and the overall cost comparison of the three monitoring schemes in numerical examples. Based on the numerical results, we illustrate relationships between the costs of periodic or continuous monitoring under which the different monitoring schemes minimize the overall cost. Section 8 concludes.

2 MODEL DESCRIPTION

We assume that the deterioration of the system follows a continuous time process and the system can fail at any time instant. The hazard rate of the system depends both on its age and on the values of concomitant variables that reflect the current system state or the operating environment.

We use average cost per unit time to compare three schemes for monitoring and replacement decision-making. The simplest is to choose a replacement time based only on the age of the system. In this case the cost is due only to replacements and failures. The second scheme is to inspect the condition at discrete time intervals of length Δ. We assume each inspection costs a fixed amount γ. The third is to pay an amount Γ upfront to install equipment and software that will enable continuous monitoring with no additional cost per observation. To evaluate continuous monitoring, we approximate the replacement and failure cost using periodic monitoring with $\Delta \to 0$. The goal is to determine relationships between γ and Γ under which each of these schemes minimizes the total average cost of inspection, failure and replacement per unit time, where Δ is optimized in the second approach.

Let G_1, G_2, G_3 be the average costs per unit time of the three schemes, respectively, and let g_Δ be the minimum replacement and failure cost per unit time for a periodic monitoring scheme with a fixed interval Δ. Assume r is the interest rate for continuous discounting. Then $G_1 = G_1(\tau)$ where τ is the replacement age, $G_2 = G_2(\Delta) = g_\Delta + \frac{\gamma}{\Delta}$, and $G_3 = \Gamma' + g_0$, where $g_0 = \lim_{\Delta \to 0} g_\Delta$ and $\Gamma' \equiv r\Gamma$ is found from $\Gamma = \int_0^\infty e^{-rt}\Gamma' dt$ as the equivalent average cost per unit time of Γ.

For simplicity, we consider only one concomitant variable (covariate) in this paper. We assume that the operating condition of the system, which is described by the concomitant variable, may be classified into a finite set of states, $S = \{0,1,...,n-1\}$. State 0 is the initial state of a new system. States $1,2,...,n-1$ reflect the increasingly deteriorating working condition of the system. Upon replacement, the system returns to state 0. The transition course among the states is formulated as a diagnostic stochastic process $Z = \{Z_t, t \geq 0\}$ which is a continuous time homogeneous Markov chain on state space S.

A convenient method to include both the age effect and the condition information in the hazard rate function is to employ the proportional hazards model (PHM), which has been applied successfully to engineering reliability problems in recent years (Cox and Oakes, 1984). In the

PHM, the hazard rate of a system is assumed to be the product of a baseline hazard rate $h_0(t)$ dependent only on the age of the system and a positive function $\psi(\cdot)$ that depends only on the values of concomitant variables (in our case, the states of the Z process). Thus, the hazard rate of the system at time t can be expressed as

$$h(t, Z_t) \equiv h_0(t)\psi(Z_t), t \geq 0 .$$

From the above analysis, it is obvious that the key to comparing among different monitoring schemes is to obtain the optimal replacement policy and optimal replacement cost for periodic monitoring. Thus, first we assume that the Z process is under periodic monitoring with a constant cost γ per period. In other words, the states of the Z process are available only at time instants 0, Δ, 2Δ, …, where $\Delta > 0$, in a replacement cycle.

We adopt the following notation in this paper:

t: The age of the system from time of replacement.

T: The time to failure of the system.

$Z = \{Z_t, t \geq 0\}$: A continuous time Markov chain that reflects the condition of the system at age t with $Z_0 = 0$; in general, the effect of the operating environment on the system.

X_k: The sojourn time of the Z process in state k, $k = 0,1,...,n-2$, assumed exponentially distributed.

v_k: The hazard rate of X_k.

$h_0(t)$: The baseline hazard rate, which depends only on the age of the system.

$\psi(Z_t)$: A link function that depends on the state of the stochastic process Z.

Δ: The length of the monitoring interval.

C : The replacement cost without failure, $C > 0$.

K : The additional cost for a failure replacement, $K > 0$.

γ : The monitoring cost per period for periodic monitoring.

Γ : The one-time initial cost for continuous monitoring.

r: Interest rate for continuous discounting.

g_Δ : Minimum replacement and failure cost per unit time for monitoring interval Δ.

In addition, we introduce the following basic assumptions:

1. The system must be kept in working order at all times. Replacement is instantaneous.
2. The continuous time Markov chain Z is a pure birth process, i.e., whenever a transition occurs the state of the system always increases by one. Replacement restarts the process at $Z_0 = 0$ and state n-1 is absorbing. Note that the Markov chain governs how the condition variable evolves without intervention. If maintenance actions other than replacement were considered in the model, this monotonicity assumption would be violated.
3. The baseline hazard rate, $h_0(t)$, is a non-decreasing function of the system age, that is, the system deteriorates with time.
4. The link function, $\psi(Z_t)$, is a non-decreasing function with $\psi(0) = 0$.
5. The practice of periodic monitoring influences neither the diagnostic Z process nor the system failure process.
6. Failure of the system can occur at any time. Upon failure, system replacement is executed immediately.

7. The pair $\left(I_t, Z_t\right)$, where $I_t = 1$ if $T > t$ and 0 otherwise, is a Markov process in the following sense: For any times $0 \le s_0 < s_1 < \ldots < s_{k-1} < s < t$ and states $i_0, i_1, \ldots, i_{k-1}, i, j$,

$$P\left(T > t, Z_t = j \middle| T > s, Z_s = i, Z_{s_{k-1}} = i_{k-1}, \ldots, Z_{s_0} = i_0\right) = P\left(T > t, Z_t = j \middle| T > s, Z_s = i\right).$$

As discussed by Banjevic et al. (2001), Z_t could represent either an "external" covariate such as environmental condition or an "internal" diagnostic variable.

Under periodic monitoring, let $Z_{k\Delta}$ be the condition at time point $k\Delta$ after the most recent replacement. Although condition information is available only at integer multiples of Δ, the continuous time Markov chain Z_t may shift among its discrete values at any time. Then for $t \in [0, \Delta]$, define the expected conditional reliability function

$$\bar{R}(k, Z_{k\Delta}, t) \equiv E\left[P(T > k\Delta + t \mid T > k\Delta, Z_\Delta, ..., Z_{k\Delta})\right] = E\left[\exp\left(-\int_{k\Delta}^{k\Delta+t} h_0(s)\psi(Z_s)ds\right) \middle| Z_{k\Delta}\right] \quad (1)$$

This expression for the reliability function differs from the one in Makis and Jardine (1992). In the previous work, the diagnostic process was approximated as not only unobserved but also unchanging between observation epochs. Approximating $\left\{Z_t, k\Delta < t \le (k+1)\Delta\right\}$ with the single value $Z_{k\Delta}$ allowed a deterministic evaluation of

$$R(k, Z_{k\Delta}, t) \equiv P\left(T > k\Delta + t \mid T > k\Delta, Z_{k\Delta}\right) = \exp\left(-\psi(Z_{k\Delta})\int_{k\Delta}^{k\Delta+t} h_0(s)ds\right).$$

An attempt to apply that formula and others based on the same approximation resulted in the average replacement cost of the "optimal" replacement policy decreasing with Δ, suggesting that less frequent observations would enable better replacement decisions. This counter-intuitive result motivated the more detailed analysis in the next three sections of this paper.

3 OPTIMAL REPLACEMENT POLICY FOR PERIODIC MONITORING.

The form of an optimal replacement policy, which minimizes the long-run expected average replacement cost per unit time for systems in the PHM with fixed Δ, was derived by Makis and Jardine (1992) while the computation of the optimal policy parameters was simplified by the discrete-time approximation of *Z*. To compare costs under different values of Δ while considering the fact that the *Z* process may change state at any time, we find the parameters of an optimal replacement policy and its cost without the discrete-time approximation, given that the form of the replacement policy follows variant 2 of the policy in Makis and Jardine (1992); that is, the system may be replaced preventively either at an observation epoch or immediately if it fails between observation epochs.

As in Makis and Jardine (1992), let decision 0 represent immediate replacement upon observation of the system state, and decision $+\infty$ correspond to non-replacement (i.e., wait and see). They showed that an optimal replacement policy δ for variant 2 exists and has the following form

$$\delta(k,z)=\begin{cases}+\infty \text{ if } K\left[1-R(k,z,\Delta)\right]<g\int_0^{\Delta}R(k,z,t)dt \\ 0 \text{ otherwise,}\end{cases}$$

where g is the optimal average replacement cost per unit time, k is the number of monitoring intervals since the last replacement and $z=Z_{k\Delta}$ is the condition of the system at age $k\Delta$. This conclusion still holds upon substitution of $R(k,z,t)$ by $\overline{R}(k,z,t)$ in the analysis.

The optimal replacement policy δ is monotonic in the system age and state. It specifies that if the value of g were known and no failure would occur, then the optimal replacement time for a specific condition z would be $k_z\Delta$, where k_z is the minimum integer that satisfies the inequality:

$$K\left[1-\bar{R}(k_z, z, \Delta)\right] < g\int_0^\Delta \bar{R}(k_z, z, t)dt \,. \tag{2}$$

On the other hand, if the system fails before $k_z\Delta$, then it is replaced immediately upon failure.

According to Makis and Jardine (1992), the following algorithm may be employed to find g. Define

$$\phi(d) = \left[C + KP\left(T_d \geq T\right)\right] \big/ E\left[\min\left\{T, T_d\right\}\right] \tag{3}$$

where T_d is the planned replacement time associated with the expected average cost d. Here, under a given replacement policy δ_d, $P\left(T_d \geq T\right)$ is the probability of failure replacement and $E\left[\min\left\{T, T_d\right\}\right]$ is the mean replacement time considering failure. Thus, according to the theory of renewal reward processes (Ross, 2003), $\phi(d)$ is the long-run expected average cost per unit time for policy δ_d.

The algorithm is based on a fixed point result that for any $d_0 \geq 0$, if $d_m = \phi(d_{m-1})$, $m = 1, 2, \ldots$, then $\lim_{m\to+\infty} d_m = g$. It may be described as the following procedure:

Algorithm I

1. Initialize the iteration counter $m = 0$, choose an arbitrary replacement policy, and set d_0 equal to the cost of the chosen policy.
2. For d_m, use (2) to find the planned replacement time $k_i\Delta$ associated with current system condition i, i.e.,

$$k_i = \min\left\{k \geq 0 : K\left[1 - E\left(\bar{R}(k, i, \Delta)\right)\right] \geq d_m \int_0^\Delta E\left(\bar{R}(k, i, t)\right)dt\right\}, i \in S \,. \tag{4}$$

3. Use the replacement policy obtained in step 2 and equation (3) with $d = d_m$ to calculate $d_{m+1} = \phi(d_m)$.

4. If $d_{m+1} = d_m$, stop with $g = d_m$; otherwise, set $m \leftarrow m+1$ and go to step 2.

Actually, Algorithm I is an example of the policy iteration algorithm as discussed by Tijms (1986), who proved that the sequence of d values obtained from a policy improvement algorithm is monotonically decreasing and therefore the algorithm will converge in a finite number of iterations.

For step 1, a good initial choice is $d_0 = (C+K)/E(T)$, which is the long-run average cost of the policy that replaces only at failure. The crucial steps of this iteration procedure are steps 2 and 3; that is, to use equation (4) to identify current parameters of the replacement policy and then use equation (3) to update $d_{m+1} = \phi(d_m)$. The difficulties arise from the calculation of $\bar{R}(k,i,t)$ and the computation of $E(\min\{T, T_d\})$ and $P(T_d \geq T)$ under a given replacement police δ_d. In the next two sections, we will derive formulas for computing $\bar{R}(k,i,t)$, $E(\min\{T, T_d\})$ and $P(T_d \geq T)$ by conditioning.

4 Analysis of the Expected Conditional Reliability Function

4.1 *Definitions*

Here we introduce some new definitions to facilitate the presentation of our method. Based on the assumption and notation in section 2, the sojourn time X_k is exponentially distributed with rate v_k and the X_k's are mutually independent. For convenience, define $X_{n-1} \equiv +\infty$ associated with the absorbing state *n*-1.

For $j \geq 0$ and $i \in S$, given that the age of the system is $j\Delta$ and $Z_{j\Delta} = i$, define

$$S_{ir} = \sum_{k=i}^{r} X_k \,, \quad r \in S \quad \text{and} \quad r \geq i \,.$$

Then $j\Delta + S_{ir}$ is the time point that the *Z* process makes a transition from state i to state $r+1$. Therefore, if $t \in [S_{i,r-1}, S_{ir})$, then $Z_{j\Delta+t} = r$. For convenience, we also define $S_{i,i-1} \equiv 0$, $S_{i,n-1} \equiv +\infty$.

Define $T_R = T - j\Delta$, which is the residual time to failure if no preventive replacement is made. (Note that, for simplicity, dependence on *j* is suppressed in the notation for S_{ir} and T_R).

Then from the expected conditional reliability function (1), it follows that:

$$\overline{R}(j, Z_{j\Delta}, t) = E\left[P(T_R > t \mid j\Delta, Z_{j\Delta})\right] = E\left[\exp\left(-\int_{j\Delta}^{j\Delta+t} h_0(s)\psi(Z_s)ds\right) \mid Z_{j\Delta}\right]. \qquad (5)$$

Next, we evaluate $\overline{R}(j,i,t)$ by conditioning on $S_{ii}, S_{i,i+1}, ..., S_{i,n-2}$. To better illustrate this procedure, first we examine a simple situation where the *Z* process has only three states $\{0,1,2\}$. Then we generalize the formulation of the three-state *Z* process to that of an *n*-state pure birth process.

4.2 Derivation of $\overline{R}(j,i,t)$ for Three-State Z process

As mentioned by Makis *et al*. (2003), a diagnostic process with three working states often is practical; e.g., one can view state 0 as a new system, state 1 as having some deterioration and state 2 as a warning state. Thus, it is helpful to detail the analysis for a three-state *Z* process for both illustrative and practical purposes.

Here, we analyze $\overline{R}(j,0,t)$ only**.** The formulas for $\overline{R}(j,i,t), i = 1,2$, may be deduced similarly and we relegate them to Appendix 1.

For a three-state Z process, we can evaluate $\bar{R}(j,0,t)$ by conditioning on S_{00} and S_{01}. Using the law of total expectation, we have

$$\begin{aligned}\bar{R}(j,0,t) &= E\left[P(T > j\Delta + t \mid T > j\Delta, Z_{j\Delta} = 0)\right] \\ &= E\left[E\left[P(T > j\Delta + t \mid T > j\Delta, Z_{j\Delta} = 0, S_{00}, S_{01})\right]\right].\end{aligned} \tag{6}$$

Given $Z_{j\Delta} = 0$ and for a given $t > 0$, the feasible region of the two-dimensional (S_{00}, S_{01}) space could be divided into 3 sub-regions (cases), as shown in Figure 1; that is, Case 0: $S_{00} \geq t$, Case 1: $S_{00} < t \leq S_{01}$ and Case 2: $S_{01} < t$.

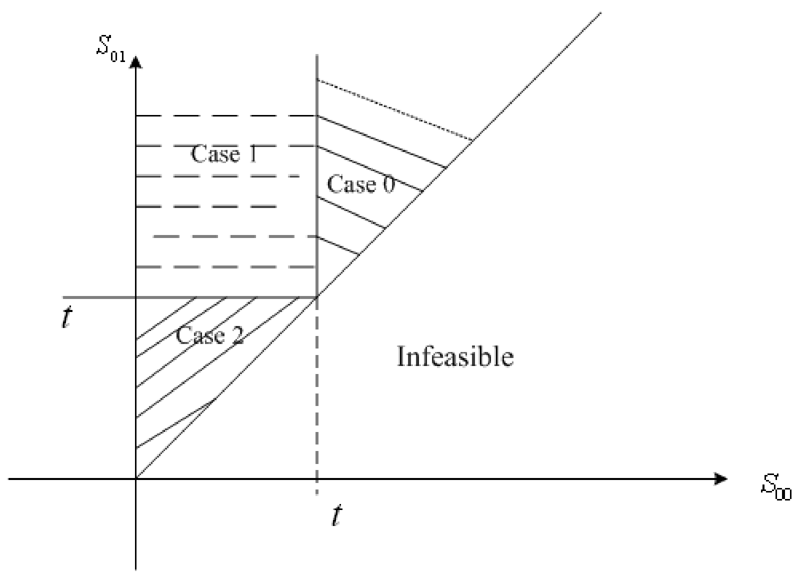

Figure 1 S_{ir} Space partition

Let s_{ir} represent a value (realization) of S_{ir}. Then define conditional cumulative distribution functions (CDF's) of T_R corresponding to the three cases above when $Z_{j\Delta} = 0$.

For $t \leq s_{00}$,

$$F_0^0(j,t) = P(T_R \leq t \mid S_{00} = s_{00}, S_{01} = s_{01}, j\Delta, Z_{j\Delta} = 0) = 1 - \exp\left(-\psi(0)\int_{j\Delta}^{j\Delta+t} h_0(u)\,du\right). \tag{7}$$

For $s_{00} < t \leq s_{01}$,

$$F_1^0(j,t,s_{00}) = P(T_R \le t \mid S_{00}=s_{00}, S_{01}=s_{01}, j\Delta, Z_{j\Delta}=0)$$
$$= 1-\exp\left(-\psi(0)\int_{j\Delta}^{j\Delta+s_{00}} h_0(u)\,du - \psi(1)\int_{j\Delta+s_{00}}^{j\Delta+t} h_0(u)\,du\right). \quad (8)$$

And for $t > s_{01}$,

$$F_2^0(j,t,s_{00},s_{01}) = P(T_R \le t \mid S_{00}=s_{00}, S_{01}=s_{01}, j\Delta, Z_{j\Delta}=0)$$
$$= 1-\exp\left(-\psi(0)\int_{j\Delta}^{j\Delta+s_{00}} h_0(u)\,du - \psi(1)\int_{j\Delta+s_{00}}^{j\Delta+s_{01}} h_0(u)\,du - \psi(2)\int_{j\Delta+s_{01}}^{j\Delta+t} h_0(u)\,du\right). \quad (9)$$

We know that X_0 and X_1 are exponentially distributed and they are independent of each other. In addition, the event $S_{00}=s_{00}, S_{01}=s_{01}$ is equivalent to the event $X_0 = s_{00}, X_1 = s_{01}-s_{00}$. Hence, the joint density function of S_{00}, S_{01} is:

$$f(s_{00},s_{01}) = v_0 e^{-v_0 s_{00}} v_1 e^{-v_1(s_{01}-s_{00})}. \quad (10)$$

Therefore, using equation (6) and setting the relevant integral domains according to the three sub-regions, we get

$$\bar{R}(j,0,t) = \int_t^\infty v_0 e^{-v_0 s_{00}}\left[1-F_0^0(j,t)\right]ds_{00} + \int_t^\infty\int_0^t f(s_{00},s_{01})\left[1-F_1^0(j,t,s_{00})\right]ds_{00}ds_{01}$$
$$+\int_0^t\int_0^{s_{01}} f(s_{00},s_{01})\left[1-F_2^0(j,t,s_{00},s_{01})\right]ds_{00}ds_{01}$$
$$= e^{-v_0 t}\left[1-F_0^0(j,t)\right] + \int_0^t v_0 e^{-v_0 s_{00}} e^{-v_1(t-s_{00})}\left[1-F_1^0(j,t,s_{00})\right]ds_{00} \quad (11)$$
$$+\int_0^t\int_0^{s_{01}} v_0 e^{-v_0 s_{00}} v_1 e^{-v_1(s_{01}-s_{00})}\left[1-F_2^0(j,t,s_{00},s_{01})\right]ds_{00}ds_{01}.$$

4.3 Derivation of $\bar{R}(j,i,t)$ for an n-State Z process

In the situation where the *Z* process has *n* states $\{0,1,...,n-1\}$, the formulas for $\bar{R}(j,i,t)$ may be derived in the same manner as in the three-state situation. Thus, in the following, we will present the formulas for $\bar{R}(j,i,t), i=0,1,...,n-1$, directly.

Let s_{ir} represent a value (realization) of S_{ir}. Define conditional CDF's of T_R when

$Z_{j\Delta} = i$. For $s_{i,i+m-1} < t \le s_{i,i+m}$,

$$F_m^i(j,t,s_{ii},...,s_{i,i+m-1}) = P(T_R \le t \mid S_{ii} = s_{ii}, \cdots, S_{i,i+m} = s_{i,i+m}, j\Delta, Z_{j\Delta} = i)$$
$$= 1 - \exp\left(-\sum_{k=i}^{i+m-1} \psi(k) \int_{j\Delta + s_{i,k-1}}^{j\Delta + s_{i,k}} h_0(u)du - \psi(i+m) \int_{j\Delta + s_{i,i+m-1}}^{j\Delta + t} h_0(u)du \right), m = 0,1,...,n-i-1. \quad (12)$$

The joint density function of $S_{ii}, S_{i,i+1}, ..., S_{i,i+m}$ is

$$f(s_{ii}, s_{i,i+1}, ..., s_{i,i+m}) = v_i e^{-v_i s_{ii}} v_{i+1} e^{-v_{i+1}(s_{i,i+1} - s_{ii})} ... v_{i+m} e^{-v_{i+m}(s_{i,i+m} - s_{i,i+m-1})} \quad (13)$$

for all $m = 0,1,...,n-i-2$.

Thus,

$$\bar{R}(j,i,t)$$
$$= \sum_{m=0}^{n-i-2} \int_t^\infty \int_0^t \int_0^{s_{i,i+m-1}} ... \int_0^{s_{i,i+1}} f(s_{ii},...,s_{i,i+m}) \left(1 - F_m^i(j,t,s_{ii},...,s_{i,i+m-1})\right) ds_{ii} ... ds_{i,i+m-2} ds_{i,i+m-1} ds_{i,i+m} \quad (14)$$
$$+ \int_0^t \int_0^{s_{i,n-2}} ... \int_0^{s_{i,i+1}} f(s_{ii},...,s_{i,n-2}) \left(1 - F_{n-i-1}^i(j,t,s_{ii},...,s_{i,n-2})\right) ds_{ii} ... ds_{i,n-3} ds_{i,n-2}$$

for all $i \in S$, where $f(s_{ii},...,s_{i,i+m})$ is given by (13).

5 RECURSIVE FORMULAS FOR MEAN REPLACEMENT TIME AND FAILURE PROBABILITY

5.1 Derivation of $E(\min\{T, T_d\})$ for an n-State Z process

Like $\bar{R}(j,i,t)$, the mean replacement time $E(\min\{T, T_d\})$ and failure probability $P(T_d \ge T)$ may be computed by conditioning on the variables S_{ir}. What's more, they may be calculated efficiently using recursion. Next, we derive a recursive computational procedure for $E(\min\{T, T_d\})$. The failure probability $P(T_d \ge T)$ may be treated similarly and its derivation will be presented directly in Section 5.2.

For a given value $d > 0$, the replacement policy δ_d may be found using (4). Then $k_i\Delta$ is the planned replacement time associated with the current observed system condition, i.

Let random variable

$$T(j,i)=\min\{T,T_d\}-j\Delta$$

be the residual time to replacement given that the age of the system is $j\Delta$, $Z_{j\Delta}=i$ and the replacement policy is δ_d. Define

$$W(j,i)=E\left[T(j,i)\right],$$

so that $W(0,0)=E\left(\min\{T,T_d\}\right)$. From the definitions above, it follows that

$$W(j,i)=0\text{ , for } j\ge k_i,$$

and for $j<k_i$, we will evaluate $W(j,i)$ by conditioning on $S_{ii},S_{i,i+1},...,S_{i,n-2}$. It is natural to assume that $j<k_i$ for the remainder of this section.

Again, using the law of total expectation,

$$W(j,i)=E\left[T(j,i)\right]=E\left[E\left[\left(T(j,i)\,|\,S_{ii},S_{i,i+1},...,S_{i,n-2}\right)\right]\right].$$

According to the state of the *Z* process at time point $(j+1)\Delta$, there are $(n-i)$ cases:

Case *m*: $Z_{(j+1)\Delta}=i+m$, that is $S_{i,i+m-1}<\Delta\le S_{i,i+m}$

$$T(j,i)=\begin{cases}T_R \text{ if } T_R\le\Delta\\ \Delta+W\left(j+1,i+m\right) \text{ if } T_R>\Delta\end{cases}$$

where $m=0,1,...,n-i-1$.

Then for $s_{i,i+m-1}<\Delta\le s_{i,i+m}$, define:

$$\begin{aligned}&W_m^i(j,s_{ii},s_{i,i+1},...,s_{i,i+m-1})=E\left(T(j,i)\,|\,S_{ii}=s_{ii},S_{i,i+1}=s_{i,i+1},...,S_{i,i+m}=s_{i,i+m}\right)\\&=\sum_{k=i}^{i+m-1}\int_{s_{i,k-1}}^{s_{ik}}tdF_k^i(j,t,s_{ii},..,s_{i,k-1})+\int_{s_{i,i+m-1}}^{\Delta}tdF_m^i(j,t,s_{ii},..,s_{i,i+m-1})\\&+(\Delta+W(j+1,i+m))(1-F_m^i(j,t,s_{ii},..,s_{i,i+m-1})),\ \ m=0,1,...,n-i-1.\end{aligned}\tag{15}$$

Note from (15) that the conditional value of $T(j,i)$ is obtained in terms of

$W\left(j+1,i+m\right),m=0,1,...,n-i-1$. Thus this is a recursive expression.

To sum up above, we have

$$W(j,i)=\sum_{m=0}^{n-i-2}\int_{\Delta}^{\infty}\int_{0}^{\Delta}\int_{0}^{s_{i,i+m-1}}..\int_{0}^{s_{i,i+1}}f(s_{ii},...,s_{i,i+m})W_m^i ds_{ii}...ds_{i,i+m-2}ds_{i,i+m-1}ds_{i,i+m}$$
$$+\int_{0}^{\Delta}\int_{0}^{s_{i,n-2}}...\int_{0}^{s_{i,i+1}}f(s_{ii},...,s_{i,n-2})W_{n-1}^i ds_{ii}...ds_{i,n-3}ds_{i,n-2} \quad (16)$$

where the density function $f(s_{ii},...,s_{i,i+m})$ is from (13) and the arguments of $W_m^i(j,s_{ii},s_{i,i+1},...,s_{i,i+m-1})$ as shown in (15) have been dropped for succinctness.

5.2 Derivation of $P(T_d \geq T)$ for an n-State Z process

Define $Q(j,i)=P(T_d \geq T\mid(j,i))$. Then $Q(0,0)=P(T_d \geq T)$ and $Q(j,i)=0$, for $j \geq k_i$. For $j<k_i$ and $s_{i,i+m-1}<\Delta \leq s_{i,i+m}$, define

$$\begin{aligned}&Q_m^i(j,s_{ii},s_{i,i+1},...,s_{i,i+m-1})\\&=E\left[P\left(T \leq T_d \mid S_{ii}=s_{ii},S_{i,i+1}=s_{i,i+1},...,S_{i,i+m}=s_{i,i+m}\right)\right]\\&=F_m^i(j,\Delta,s_{ii},..,s_{i,i+m-1})\\&+Q(j+1,i+m)\left(1-F_m^i(j,t,s_{ii},..,s_{i,i+m-1})\right),\ m=0,1,...,n-i-1.\end{aligned} \quad (17)$$

Then we have

$$Q(j,i)=\sum_{m=0}^{n-i-2}\int_{\Delta}^{\infty}\int_{0}^{\Delta}\int_{0}^{s_{i,i+m-1}}..\int_{0}^{s_{i,i+1}}f(s_{ii},...,s_{i,i+m})Q_m^i ds_{ii}...ds_{i,i+m-2}ds_{i,i+m-1}ds_{i,i+m}$$
$$+\int_{0}^{\Delta}\int_{0}^{s_{i,n-2}}...\int_{0}^{s_{i,i+1}}f(s_{ii},...,s_{i,n-2})Q_{n-i-1}^i ds_{ii}...ds_{i,n-3}ds_{i,n-2} \quad (18)$$

for all $i \in S$ where $f(s_{ii},...,s_{i,i+m})$ is from (13) and Q_m^i is from (17) with arguments suppressed.

6 OPTIMAL AGE-BASED REPLACEMENT

To investigate the value of condition monitoring, we also studied the optimal age-based

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replacement policy as a baseline for comparison.

Without any condition monitoring, preventive replacement would be based only on the age of the system. If $F(t)$ is the distribution function of the failure time and the system is replaced whenever it fails or reaches age τ, then one can find the average replacement rate,

$$\lambda_r(\tau) = \left[\int_0^\tau sf(s)\,ds + \tau\left(1 - F(\tau)\right)\right]^{-1} = \left[\int_0^\tau \left(1 - F(s)\right)ds\right]^{-1},$$

and the corresponding failure rate,

$$\lambda_d(\tau) = F(\tau)\lambda_r(\tau)$$

(see (Ross, 2003), p.461). The optimal replacement age, τ^*, is found by minimizing the total average cost per unit time, which is given by:

$$w(\tau) = C\lambda_r(\tau) + K\lambda_d(\tau) = \left(C + KF(\tau)\right)\left[\int_0^\tau \left(1 - F(s)\right)ds\right]^{-1}. \qquad (19)$$

In the notation of this paper, we have

$$F(t) = 1 - \bar{R}(0,0,t)$$

where $\bar{R}(0,0,t)$ is obtained from equation (14).

7 Numerical Illustration

To illustrate our model and its use in assessing the value of monitoring information, we consider the following numerical example. Assume that the baseline distribution is Weibull with hazard rate

$$h_0(t) = \frac{\beta t^{\beta-1}}{\alpha^\beta},$$

where $\alpha = 1, \beta = 2$, and let $\psi(Z_t) = \exp(2Z_t)$, $C = 5$ and $K = 25$. Assume the stochastic process Z has three states $\{0,1,2\}$ with transition rates $v_0 = v_1 = -\ln(0.4), v_2 = 0$. Since the forms of $h_0(t)$ and $\psi(Z_t)$ are predefined, the PHM here is parametric rather than semi-parametric as described in Cox *et al*. (1984).

7.1 Replacement Policy under Periodic Monitoring

With $\Delta = 1$ in Algorithm I, we initialize $d_0 = (C+K)/E(T) = 46.8823$, which is the cost of the policy that replaces only at failure. Then we illustrate how the first iteration for finding g proceeds below. Other iterations are similar.

Iteration 1: $d_0 = 46.8823$. For $Z_t = i = 0$, we get $k_0 = 1$ from (2) and (14). Thus $W(1,0) = 0$, $Q(1,0) = 0$. Similarly, for $i = 1$ and $i = 2$, we get $k_1 = 1$ and $k_2 = 1$. Thus $W(1,1) = 0$, $Q(1,1) = 0$, $W(1,2) = 0$, $Q(1,2) = 0$. Based on these value, we obtain $W(1,0) = 0.5943$ from (16) and $Q(1,0) = 0.8410$ from (18).

The complete results are shown in Table 1. The policy iteration algorithm converges after a single iteration to the optimal average cost $g = 43.7905$. The algorithm was implemented in *Mathematica*® for precise and efficient numerical evaluation of multiple integrals.

Table 1 An Illustration of the Computation Procedure (three states)

d	k_0	k_1	k_2	$W(0,0)$	$Q(0,0)$	$\phi(d)$
46.8823	1	1	1	0.5943	0.8410	43.7905
43.7905	1	1	1	0.5943	0.8410	43.7905

To study the effect of the interval between observations, we varied Δ from 0.001 (to approach the case with continuous monitoring) to 10 (to approximate the situation without monitoring). Table 2 shows the optimal policies and replacement costs for various values of Δ with three-state Z process. Notably, if no preventive replacement is done, the mean time to failure of the system may be obtained from

$$E(T) = \int_0^\infty \bar{R}(0,0,t)dt = 0.6399, \quad (20)$$

which agrees with the value of $W(0,0)$ when $\Delta = 10$. Table 2 indicates that as the inspection interval Δ decreases, the optimal replacement cost also decreases. This result is expected because with smaller Δ values we obtain more timely information about the system, and thus can respond to condition deterioration more promptly.

Table 2 Effect of Changing Δ on the Optimal Policy and Cost with Comparison to Age-Based Replacement

Δ	k_0	k_1	k_2	$W(0,0)$	$Q(0,0)$	g_Δ	m^*	$w(m^*\Delta)$
0.001	487	66	9	0.3690	0.1606	24.4286	285	32.4929
0.01	48	6	1	0.3664	0.1616	24.6698	29	32.4972
0.05	9	1	1	0.3553	0.1658	25.7381	6	32.5318
0.1	4	1	1	0.3329	0.1602	27.0455	3	32.5318
0.2	2	1	1	0.3444	0.2062	29.4829	2	34.0449
1	1	1	1	0.5943	0.8410	43.7905	1	43.7905
10	1	1	1	0.6399	1.0000	46.8844	1	46.8844

However, the opposite behavior occurred when we applied the discrete approximation formulas from Makis and Jardine (1992) directly to acquire the optimal policies. The results are shown in Table 3. To apply their discrete-time formulas, by uniformization we converted the continuous time Markov chain Z discussed above to a discrete-time Markov chain, which makes a transition every Δ units of time and has the transition probability matrix

$$P = \begin{bmatrix} 0.4^{\Delta} & 1-0.4^{\Delta} & 0 \\ 0 & 0.4^{\Delta} & 1-0.4^{\Delta} \\ 0 & 0 & 1 \end{bmatrix},$$

and we assume that all else are held equal.

Since we ignored possible transitions between inspection intervals, there is no wonder that the results in Table 3 are all overoptimistic, that is, for the same Δ, the optimal replacement cost in Table 3 is smaller than that in Table 2. One apparent problem of Table 3 is that as Δ increases from 0.001 to 0.2, the optimal replacement cost unexpectedly decreases. (We expected the optimal replacement cost to increase with Δ because less frequent observations lead to less information available, based on which it is impossible to make better decisions.) Another problem is that the average replacement time $W(0,0)$ when $\Delta=1$ or $\Delta=10$ is larger than the mean time to failure of the system (20). Despite these drawbacks, the results for $\Delta=0.001$ indicate that the discrete-version formulas from Makis and Jardine do provide an accurate approximation for the continuous time model when Δ is sufficiently small.

Table 3 Optimal policies of various Δ according to Makis and Jardine (1992)

Δ	k_0	k_1	k_2	$W(0,0)$	$Q(0,0)$	g
0.001	488	66	9	0.3695	0.1606	24.3967

0.01	49	7	1	0.3720	0.1624	24.3503
0.05	10	1	1	0.3821	0.1692	24.1569
0.1	5	1	1	0.3907	0.1734	23.8946
0.2	2	1	1	0.3491	0.1819	23.6061
1	1	1	1	0.7468	0.6321	27.8553
10	1	1	1	0.8862	1	33.8514

7.2 Comparison with Age-Based Replacement

To weigh the benefit of condition information against its cost, we can compare the optimal replacement cost of the policy based on more or less frequent monitoring to that of the age-based replacement policy. We also compute the optimal age-based replacement policy, shown with its cost in the last two columns of Table 2. The optimal replacement age, τ^*, is found numerically by minimizing (19) using a heuristic search technique. To compare with the condition-based replacement policy, we constrain it to be an integer multiple, m^*, of Δ. The numerical results quantify the savings $w(m^*\Delta) - g$ that are obtained with small values of Δ by having access to more frequent observations of the product's condition. These cost savings could justify the investment in equipment and software required to monitor the condition frequently.

The additional cost of a failure replacement, K, is usually difficult to estimate. But it could be very high for critical systems, often several times bigger than the regular replacement cost. Table 4 shows the impact of this cost on the optimal replacement policy and average cost when $\Delta = 0.01$. As expected, for larger values of K, the cost savings $w(m^*\Delta) - g$ provided by condition monitoring is more substantial, which implies the great importance of the condition information in critical systems.

Table 4 Effect of Increasing K on the Optimal Policy and Cost when $\Delta = 0.01$ with Comparison to Age-Based Replacement

K	k_0	k_1	k_2	$W(0,0)$	$Q(0,0)$	g	m^*	$w(m^*\Delta)$
$K = 5C = 25$	91	12	2	0.5150	0.3637	27.3659	29	32.4972
$K = 10C = 50$	33	4	1	0.2773	0.0879	33.8817	20	43.6787
$K = 20C = 100$	23	3	1	0.2052	0.0465	47.0403	15	58.4512

7.3 Optimal Monitoring Scheme

We compare age-based, periodic monitoring and continuous monitoring based on total average cost per unit time. Without monitoring, the optimal value of G_1 is obtained in section 6 by minimizing (19). We denote it as $G_1^* = G_1(\tau^*)$. The cost of the periodic monitoring scheme, G_2, is a function of the inspection interval, Δ. Its optimal value, denoted as $G_2^* = G_2(\Delta^*)$, is obtained numerically by searching the Δ space. The continuous monitoring cost, G_3, achieves its optimal value, G_3^*, when the system is under the optimal replacement policy of continuous monitoring, which we approximate by letting Δ approach 0. If $G_3^* = \min\left\{G_1^*, G_2^*, G_3^*\right\}$, then a one-time investment in continuous monitoring is worthwhile. Similarly, a smaller value of G_1^* than both G_2^* and G_3^* means that it is not worthwhile to devote any effort to collecting information on the system condition. This case can happen if the covariates we study have an insignificant influence on the system hazard rate or the cost ratio $(C+K)/C$ is small. The optimal monitoring scheme is therefore determined by comparison among the values of G_1^*, G_2^*, G_3^*.

In our numerical example of Table 2, we have $G_1^* = 32.4929$ and $G_3^* = 24.4286 + \Gamma'$ (approximating g_0 as $\hat{g}_0 = g_{0.001}$). For simplicity, we restrict the value of Δ to a finite set $\Lambda = \{0.01, 0.05, 0.1, 0.2, 1, 10\}$. Then

$$G_2^* = \min_{\Delta \in \Lambda} \left(g_\Delta + \frac{\gamma}{\Delta} \right).$$

Figure 2 displays a plot $G_2(\Delta) - G_1^*$ to compare between G_2^* and G_1^*. The contour of G_2^* is highlighted with bold black. It is clear that if γ is smaller than approximately 0.6 (exact value is 0.6020), we can choose a proper Δ to make the periodic monitoring scheme better than no monitoring.

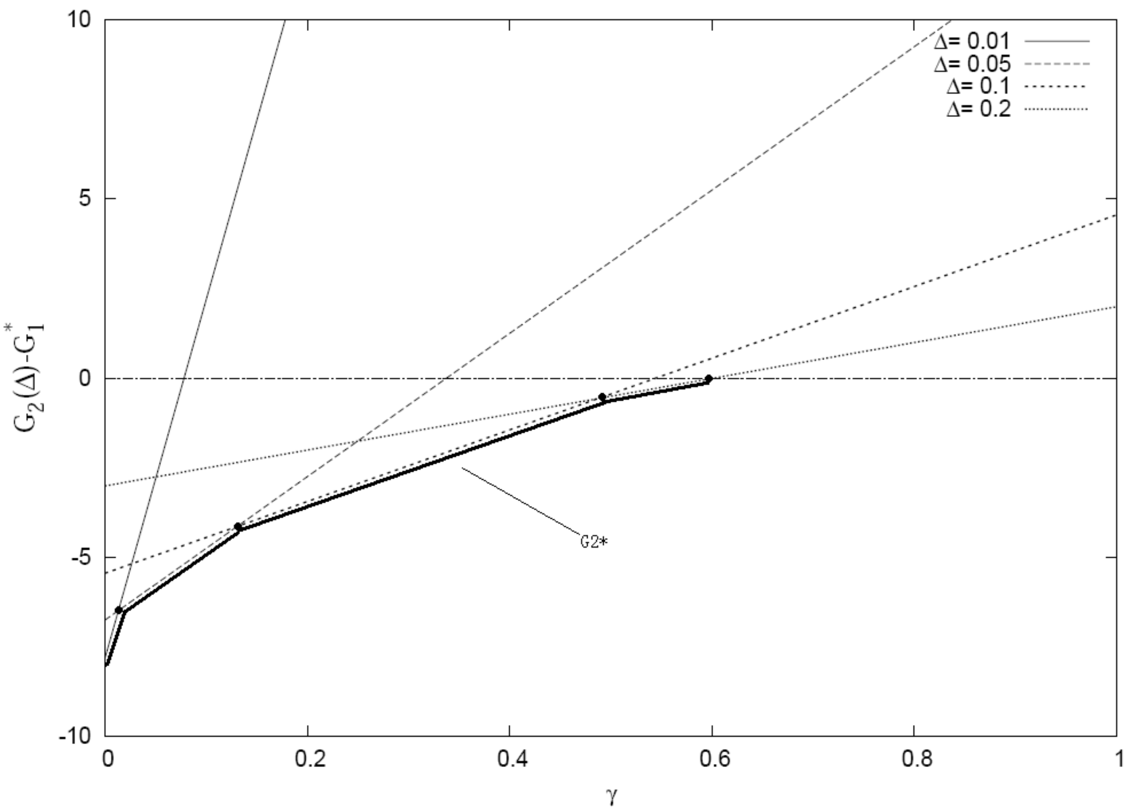

Figure 2 Comparison between G_1^* and G_2^*

We would like to know under what conditions the continuous monitoring scheme would be the best option. Clearly, $\Gamma' \le w\left(m*\left(0.001\right)\right) - g_{0.001} = 8.0643$ is necessary for $G_3^* \le G_1^*$. Besides that, when $\gamma \le 0.6020$, for $G_3^* \le G_2^*$ we must have:

- if $0.4875 < \gamma \le 0.6020$, then $\Gamma' \le \gamma/0.2 + g_{0.2} - \hat{g}_0 = 5\gamma + 5.0543$;
- if $0.1307 < \gamma \le 0.4875$, then $\Gamma' \le \gamma/0.1 + g_{0.1} - \hat{g}_0 = 10\gamma + 2.6169$;
- if $0.0134 < \gamma \le 0.1307$, then $\Gamma' \le \gamma/0.05 + g_{0.05} - \hat{g}_0 = 20\gamma + 1.3095$
- if $\gamma \le 0.0134$, then $\Gamma' \le \gamma/0.01 + g_{0.01} - \hat{g}_0 = 100\gamma + 0.2412$.

This analysis indicates that when it comes to choosing a proper monitoring scheme for a specific system, it is important to weigh the benefit of monitoring against its cost carefully. Although condition-based maintenance often leads to a lower cost than age-based maintenance, this is not always the case. In our numerical example, the combinations of monitoring costs γ and Γ' under which the different monitoring schemes are optimal are shown in Figure 3. Note that the boundary between continuous and periodic monitoring could be described as the critical $r\Gamma$ being a concave piecewise-linear function of γ. This occurred when we restricted the value of Δ to a finite set; we conjecture that if the value of Δ is allowed to vary continuously, the critical $r\Gamma$ would be a smooth increasing concave function of γ. One implication of this concave shape is as follows. Suppose that current costs lie in the region where periodic monitoring is optimal; i.e., the initial cost, Γ, to set up continuous monitoring is prohibitively expensive relative to the periodic monitoring cost, γ. If γ increases, for example due to growth in labor costs, then the drop in Γ required to make continuous monitoring worthwhile becomes disproportionately smaller.

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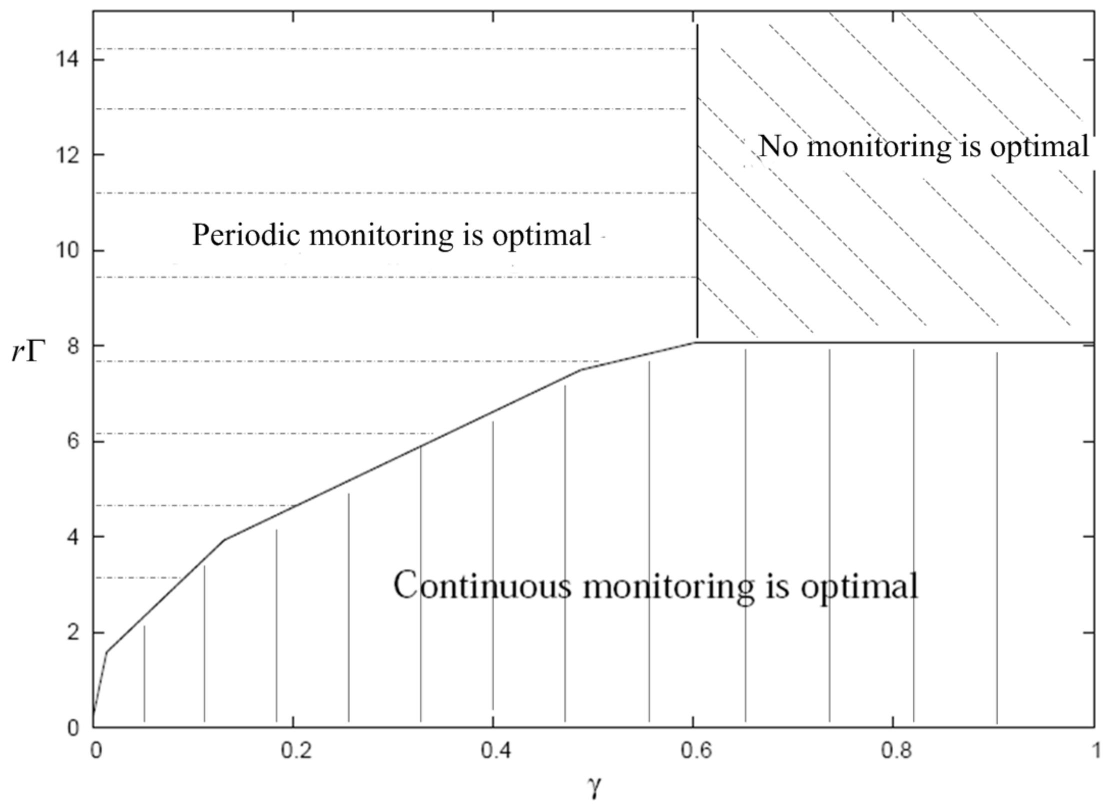

Figure 3 Optimal cost regions for different monitoring schemes

8 CONCLUSION

In this paper, we investigated a condition-based replacement problem under various monitoring schemes for a deteriorating system with concomitant conditions described by a continuous time Markov chain. The proportional hazards model was applied to describe the failure time of this system. For such a model, although the form of the optimal replacement policy under periodic monitoring was given by Makis and Jardine (1992), computing the optimal policy parameters for a system with a continuous time diagnostic process is delicate. First, a recursive procedure was developed to obtain the optimal average cost and the parameters of the optimal policy for system with an n-state pure birth process. Then a numerical example with n=3 illustrated the computational procedure as well as the evaluation of condition information with more or less frequent monitoring. At last by taking the monitoring cost into consideration, we obtained the relationships between the cost γ of each inspection under periodic monitoring and the upfront cost Γ of continuous monitoring, under which the continuous, periodic or no monitoring scheme minimizes the total average cost per unit time. Specifically, in the numerical example, no monitoring (i.e., age-based replacement) is optimal if both γ and Γ exceed certain values; and, for a fixed interest rate, the critical Γ on the boundary between continuous and periodic monitoring optimality is a concave increasing function of γ.

Extensions of this research could include generalizing the one-dimensional covariate vector to multi-dimensional. Then the *Z* process would be a general Markov chain rather than a pure birth process. It could evolve along multiple paths, which would make the calculation of policy parameters by conditioning extremely intricate. In addition, the Markovian assumption of the diagnostic process could be relaxed to a semi-Markovian process, which allows arbitrary sojourn time distributions. Also in this paper, we assumed that the condition of the product is assessed perfectly, but in real situations it is only partially observed. The value of condition monitoring would be estimated more accurately by considering the element of uncertainty added by partial observations. Although Ghasemi *et al*. (2007) solved the partial observation problem on Makis and Jardine's model using dynamic programming, the approximation of the Z process as constant within inspection intervals was left intact. Further extensions could generalize the underlying failure model. Using a different model to relate the concomitant information to system failure time distribution, such as a scale-accelerated failure time (SAFT) model (Meeker and Escobar, 1998), could be of great practical value. In this case, both the optimal policy and its calculation must be reconsidered.

APPENDIX 1 FORMULAS FOR $\overline{R}(j,i,t)$ WITH $i=1,2,$ FOR THREE-STATE Z PROCESS

A. Formulas for $\overline{R}(j,1,t)$

Define conditional CDF's of T_R when $Z_{j\Delta}=1$. For $t \le s_{11}$, we have

$$F_0^1(j,t) = P(T_R \le t \mid S_{11}=s_{11}, j\Delta, Z_j=1) = 1-\exp\left(-\psi(1)\int_{j\Delta}^{j\Delta+t} h_0(u)\,du\right),$$

and for $t > s_{11}$, we have

$$\begin{aligned} F_1^1(j,t,s_{11}) &= P(T_R \le t \mid S_{11}=s_{11}, j\Delta, Z_j=1) \\ &= 1-\exp\left(-\psi(1)\int_{j\Delta}^{j\Delta+s_{11}} h_0(u)\,du - \psi(2)\int_{j\Delta+s_{11}}^{j\Delta+t} h_0(u)\,du\right) \end{aligned}.$$

Then we have

$$\overline{R}(j,1,t) = e^{-v_1 t}(1-F_0^1(j,t)) + \int_0^t v_1 e^{-v_1 s_{11}}(1-F_1^1(j,t,s_{11}))\,ds_{11}.$$

B. Formulas for $\overline{R}(j,2,t)$

Define conditional CDF's of T_R when $Z_{j\Delta}=2$,

$$F_0^2(j,t) = P(T_R \le t \mid j\Delta, Z_j=2) = 1-\exp\left(-\psi(2)\int_{j\Delta}^{j\Delta+t} h_0(u)\,du\right).$$

Then we have

$$\overline{R}(j,2,t) = 1-F_0^2(j,t).$$

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