INTRODUCTION

In the last ten years several numerical methods have been developed for the solution of elastic wave scattering problems that have found application in quantitative flaw definition. Before the development of these methods, due to the complexity of Navier's equation which governs wave motion in an elastic continuum, numerical results were available only for circular cylinders and spheres. The elastic wave equation is separable only in polar and spherical coordinates. For other geometries, three types of numerical methods have been developed. They were all originally developed for acoustic and electromagnetic problems governed by the scalar and vector wave equations respectively.

The first type starts with the integral representation of the scattered field for harmonic waves. In the T-matrix method the extinction theorem, namely the nulling of the incident field in the region occupied by the scatterer is used as an extended boundary condition to eliminate the unknown surface or interior fields. In the boundary integral method, by letting the field point approach the surface of the scatterer an integral equation is derived for the surface field. This equation is solved and the computed value of the surface field is substituted in the integral representation for the scattered field which can then be evaluated numerically. The drawback of the boundary integral method is that it is plagued by the occurrence of spurious resonances in the scattered field spectrum at frequencies that correspond to interior resonances. This is not so with the T-matrix method. Interestingly enough, the use of the extinction theorem at selected interior points and solution of the
system in a least squares sense removes the spurious resonances in the boundary integral method.

The second type of method does not start with an integral representation. However, the incident and scattered fields are expanded in spherical functions and forced to satisfy the appropriate boundary conditions on the surface of the scatterer in a mean square sense. The error is minimized with respect to variations in the expansion coefficients to arrive at a set of matrix equations for the scattered field coefficients which are then solved. In this method also one can define a T-matrix that relates the expansion coefficients of the incident field to those of the scattered field. The structure of this T-matrix is however different. This method has been called the method of optimal truncation (MOOT).

The third method is a finite element - eigenfunction method. In this method a finite element approximation is used to describe the fields within the scatterer and its vicinity. The field outside the sphere completely enclosing the scatterer is expanded in vector spherical functions. The unknown coefficients are evaluated by matching the displacement vector and its partial derivatives across the sphere. The advantage of this method is that it provides values of the field everywhere within, in the vicinity and far from the scatterer. Moreover, the scatterer and its vicinity can be inhomogeneous and the method is not as sensitive to scatterer geometry as the matrix methods. The disadvantage is that it is somewhat more expensive in terms of computer time.

A good reference to the T-matrix method is a conference proceedings as well as a survey article of the method. References to MOOT are in the papers of Visscher and Opsal. The finite element method for SH-waves (2-D problems) was discussed by Su and Varadan as well as Su. The plan of this article is to briefly describe the T-matrix method and MOOT as they apply to 2-D, SH-wave problems as well as 3-D elastic wave problems. The two methods are compared for scattering by elliptical cylinders as well as oblate spheroids of various eccentricity as a function of frequency. Convergence, symmetry of the T-matrix as well as the unitarity of $S = 1-2T$ are discussed critically.

**T-MATRIX USING THE EXTENDED BOUNDARY CONDITION**

This method was first developed by P.C. Waterman for acoustic and electromagnetic waves and then simultaneously by Waterman and Varadan and Pao for elastic waves. More recently it has found wide application in a variety of problems involving voids, inclusions, layered scatterers, multiple scatterers, subsurface flaws and surface breaking cracks often in good agreement with corresponding experiments.
2-D SH-Wave Problems

The starting point of the method is the Helmholtz integral representation for the scattered field and the extinction theorem. Consider an infinitely long cylinder of arbitrary cross section with its axis parallel to the z-axis. Plane harmonic shear waves of frequency \( \omega \), polarized in the z-direction and propagating in the x-y plane are incident on the cylinder. Due to the symmetry of the problem, the scattered field is also made up of SH-shear waves only and all fields are independent of the z-coordinate. This is referred to as the problem of anti-plane strain in elasticity (see Fig. 1).

Let \( W_0 \) and \( W_s \) be the incident and scattered fields and let C be a cross section of the cylinder in the x-y plane. The free space Green's function for 2-D wave propagation is well known to be

\[
g(k|r-r'|) = i\pi H_0^{(1)}(k|r-r'|)
\]

where \( k = \omega/c \) is the wave number for SH-waves. The integral representation for the fields then takes the form

\[
\int [\hat{n}' \cdot \nabla' g(\vec{r},\vec{r}') W_+(\vec{r}) - g(\vec{r},\vec{r}') \hat{n}' \cdot \nabla' + W(\vec{r}')] dC' = \begin{cases} \hat{W}_s(\vec{r}) ; \vec{r} \text{ outside C} \\ \hat{W}_0(\vec{r}) ; \vec{r} \text{ inside C} \end{cases}
\]

In Eq. (2), the + subscript in the terms refers to the value of the total field and its normal derivative on C approached from the outside.

We now expand \( W_0, W_s \) and \( g \) in cylindrical basis functions using polar coordinates \( \vec{r} = (r, \phi) \) as

\[
W_0(\vec{r}) = e^{i \frac{\omega}{c} k_0 \cdot \vec{r}} = \sum_{n \sigma} a_{n \sigma} \text{Re} \phi_{n \sigma}(\vec{r}) \quad (3)
\]

\[
W_s(\vec{r}) = \sum_{n \sigma} f_{n \sigma} \text{Im} \phi_{n \sigma}(\vec{r}) \quad (4)
\]

\[
g(\vec{r},\vec{r}') = i\pi \sum_{n \sigma} \text{Im} \phi_{n \sigma}(\vec{r}) \text{Re} \phi_{n \sigma}(\vec{r}) \quad (5)
\]

and
In Eq. (5), $r_>$ and $r_<$ refer to the greater and lesser of $|\hat{\mathbf{r}}|$ and $|\hat{\mathbf{r}}'|$ respectively and $H_n^{(1)}$ and $J_n$ are cylindrical Hankel and Bessel functions.

If Neumann boundary conditions are satisfied on $C$, this would refer to the case of a cylindrical void with traction free boundary conditions

$$\hat{n} \cdot \nabla_+ W_+(\hat{r}) = 0 \quad \hat{\mathbf{r}} \text{ on } C. \quad (7)$$

Although several choices are possible for representing the surface field $W_+$, we use the following

$$W_+(\hat{r}) = \sum_{n\sigma} \alpha_{n\sigma} \text{Re } \phi_{n\sigma}(\hat{r}) \quad \hat{\mathbf{r}} \text{ on } C. \quad (8)$$

Using Eqs. (4) - (8) in the two forms of Eq. (2), using the orthogonality of the angular parts of the wave functions we obtain
The two matrix equations yield

$$ f = - Q(\text{Re},\text{Re}) \left[ Q(\text{Ou}, \text{Re}) \right]^{-1} a \equiv Ta $$

(10)

where

$$ Q(\text{Ou},\text{Re}) = \pi \int C \hat{n} \cdot \nabla \begin{vmatrix} \frac{\partial}{\partial n} \phi_{\text{mv}} \\ \text{Re} \phi_{\text{mv}} \end{vmatrix} \text{Re} \frac{\partial}{\partial \sigma} dC $$

(11)

Equation (9) is the definition of the T-matrix. The exact T-matrix is an infinite matrix but must be symmetric due to reciprocity and the matrix $S = 1 - 2T$ must be unitary if the medium outside C is non-dissipative.

### 3-D Elastic Wave Scattering

The integral representation in this case governs the incident and scattered displacement fields and $\vec{U}^O$ and $\vec{U}^S$ outside the surface $S$ of a bounded obstacle. It takes the form

$$ \int_S \left[ \hat{n}' \cdot \hat{\nabla}(\hat{\tau}, \hat{\tau}') - \hat{\nabla}(\hat{\tau}, \hat{\tau}') \cdot (\hat{n}' \cdot \hat{\nabla}(\hat{\tau}, \hat{\tau}')) \right] dS' $$

\[ \begin{cases} \hat{U}^S(\hat{r}) ; \hat{r} \text{ outside } S \\ -\hat{U}^O(\hat{r}) ; \hat{r} \text{ inside } S \end{cases} \]

(12)

where $\hat{G}$ and $\hat{\Sigma}$ are the Green's displacement and stress tensors respectively and $\hat{\tau}$ is the stress tensor which in a linear elastic medium is related to $\hat{\nabla}$ by generalized Hooke's law.

The procedure is the same as for 2-D problems, except that we use vector spherical functions to expand the fields using spherical coordinates $\hat{r}(r, \theta, \phi)$ in the usual manner. The polarization index $\tau = 1, 2, 3$ refers to longitudinal or $P$-waves and the two components of transverse, $S$-waves. The speed of $P$-waves is denoted by $c_p$ and that of $S$-waves by $c_s$. In metals the ratio $c_p/c_s \approx 2$. We define the vector spherical functions as follows.
\[
\begin{align*}
\left\{ \begin{array}{c}
\text{Ou} \\
\text{Re}
\end{array} \right\} \frac{1}{\sqrt{k/k_s}} \nabla \left\{ \begin{array}{c}
\text{Ou} \\
\text{Re}
\end{array} \right\} f_{\ell \ell' m' \sigma'}(r) = 0,
\end{align*}
\]

where
\[
\begin{align*}
\left\{ \begin{array}{c}
\text{Ou} \\
\text{Re}
\end{array} \right\} \phi_{\ell \ell' m' \sigma'}(k r) &= \frac{1}{\sqrt{\lambda(\lambda + 1)}} \\
\left\{ \begin{array}{c}
\text{Ou} \\
\text{Re}
\end{array} \right\} \psi_{\ell \ell' m' \sigma'}(k r) \\
\phi_{\ell \ell' m' \sigma'}(k r) &= \frac{h_{\ell}^{(1)}}{Y_{\ell m}(\theta, \phi)} \\
\psi_{\ell \ell' m' \sigma'}(k r) &= j_{\ell}(k r)
\end{align*}
\]

and
\[
Y_{\ell m}(\theta, \phi) = \sqrt{\frac{\epsilon_0}{\pi} \frac{\lambda - m!}{\lambda + m!} \frac{1}{(2\ell + 1)}} P_{\ell}^{m}(\cos \phi) \sin m \phi, \sigma = \pm 1
\]

In Eqs. (13), \( h_{\ell} \) and \( j_{\ell} \) are the spherical Hankel and Bessel functions, \( P_{\ell}^{m} \) is the associated Legendre polynomial, \( \ell \in [0, \infty] \) and \( m \in [0, \ell] \). We particularly note the normalization of the longitudinal wave function in Eq. (12). This is required to ensure that both P- and S-waves carry the same energy flux outwards through any closed surface circumscribing the origin.

For a void, the boundary condition is that
\[
\hat{n} \cdot \mathbf{\nabla}(r) = 0 ; \quad r \text{ on S.}
\]

Expanding the incident and scattered fields as well as the Green's function in vector spherical functions as in the corresponding 2-D problem, we obtain again
\[
f_{\tau \ell m \sigma} = \sum_{\tau ' \ell ' m' \sigma'} T_{\tau \ell m \sigma, \tau ' \ell ' m' \sigma'} a_{\tau ' \ell ' m' \sigma'}
\]

where
\[
T = -Q(\text{Re}, \text{Re}) \left[ Q(\text{Ou, Re}) \right]^{-1}
\]

and
\[
Q_{\tau \ell m \sigma, \tau ' \ell ' m' \sigma'}(\text{Ou, Re}) = \frac{k_s}{\rho \omega} \int_{S} \mathbf{\nabla} \left\{ \begin{array}{c}
\text{Ou} \\
\text{Re}
\end{array} \right\} \cdot \mathbf{\nabla} \left\{ \begin{array}{c}
\text{Ou} \\
\text{Re}
\end{array} \right\}, \text{Re} \mathbf{\nabla} \left\{ \begin{array}{c}
\text{Ou} \\
\text{Re}
\end{array} \right\} dS
\]
In Eq. (19), \( \rho \) is the mass density, \( k_s = \omega / c_s \), and \( k_p = \omega / c_p \) are the S- and P-wave numbers respectively. Further \( \mathbf{t} \) is the traction operator.

The T-matrix for elastic waves is also symmetric due to the Betti-Rayleigh reciprocal identity and \( S = 1 - 2T \) is unitary for elastic media. More detailed derivations may be found in Ref. 2.

MOOT – METHOD OF OPTIMAL TRUNCATION

This method was developed by W.M. Visscher for scalar\(^3\) and elastic\(^4\) wave problems. More recently an improved form of MOOT was introduced by Vischer\(^5\) and Opsal\(^6\). The incident and scattered fields outside the obstacle for both 2-D and 3-D problems are expanded as in Eqs. (3) and (4) using cylindrical and spherical basis functions respectively. The details are given only for the case of elastic waves since the scalar wave case is quite similar. In Refs. 3-6, the same definition of the spherical basis functions is used as given in Eq. (12), except that the P-wave function is normalized with \( 1 / k_p \) rather than \( \sqrt{k_p / k_s} \) and the S-wave functions are normalized with a factor of \( 1 / k_s \). It is important to note this since the normalization has an effect on the symmetry properties of the resulting T-matrix.
The boundary conditions on the surface of a void embedded in an elastic solid is that
\[ \hat{t}(\hat{r}) = \hat{n} \cdot \hat{t}'(\hat{r}) ; \hat{r} \text{ on } S = 0 \] (20)

Thus forcing the expansion for the scattered field in outgoing vector spherical functions results in
\[ \sum_{\ell \neq 0} a_{\ell \ell m} \hat{t}(\text{Re} \psi_{\ell \ell m}(\hat{r}')) + f_{\ell \ell m} \hat{t}(\text{Out} \psi_{\ell \ell m}(\hat{r}')) = 0 ; \hat{r} \text{ on } S. \] (21)

We observe that the Rayleigh hypothesis has been explicitly invoked in using this particular expansion of the scattered field directly on \( S \). Thus for scatterer shapes that deviate much from a sphere, this is expected to lead to poor convergence. Equation (21) can be solved by point matching, weighted residual or by Galerkin's method.

In the MOOT, the boundary condition is not satisfied as in Eq. (21), but in a mean square sense as follows
\[ I = \int_S |\hat{t}(\hat{r})|^2 \, dS = 0 \] (22)

However, using the expansion for the scattered field as in Eq. (21), \( I \neq 0 \). The first variation of \( I \) is set equal to zero in order to minimize the error, i.e.
\[ \frac{\partial I}{\partial f_{\ell \ell m}} = 0 \] (23)

This leads to the following set of matrix equations
\[ \sum_{\ell \neq 0} \overline{Q(\text{In,Re})_{\ell \ell m}, \ell' \ell' m' a_{\ell' \ell' m',} a_{\ell \ell m}} + \overline{Q(\text{In,Out})_{\ell \ell m}, \ell' \ell' m',} f_{\ell' \ell' m',} \right) = 0 \] (24)

which leads to
\[ f = - \left[ \overline{Q(\text{In,Out})} \right]^{-1} \overline{Q(\text{In,Re})} a \equiv Ta \] (25)

where 'In' signifies incoming wave functions which are the complex conjugates of the outgoing functions and hence contain spherical Hankel functions of the second kind. Equation (25) is similar in
in structure to Eqs. (17) and (18) but the Q-matrices have a totally
different structure in that the row index is comprised of incoming
functions rather than 'Re' or 'Ou' and takes the form

$$\bar{Q}(\text{In}, \frac{\text{Ou}}{\text{Re}}) = \int_S \tilde{t}(\text{In} \psi_{\tau \ell m \sigma}) \cdot \tilde{t}(\frac{\text{Ou}}{\text{Re}} \psi_{
} \psi_{\tau' \ell' m' \sigma}) \, dS. \tag{26}$$

In the MOOT also, the infinite T-matrix is expected to be sym­
metric and the S-matrix unitary for elastic media. But in numerical
calculations, only truncated matrices are used and the symmetry and
unitarity properties are used as a test of convergence.

Recently, due to difficulties caused by forcing the expansion of
the scattered field in outgoing spherical functions on a non-spherical
surface (i.e., the Rayleigh hypothesis), Visscher has tried to
improve the scheme by surrounding the scatterer by a sphere. A new
expansion is used to describe the field between the surface S and the
spherical surfaces (see Fig. 2). Visscher chooses regular functions
for this purpose. Continuity of traction and displacement are
required across S. Again all boundary conditions are satisfied in a
mean square sense and the error is minimized with respect to varia­
tions in the expansion coefficients. This results in the following
complicated relationship between the expansion coefficients of the
scattered and incident fields

$$f = Ta \tag{27}$$

where

$$T = -\left[ X - \n \right]^-1 X^+_n \tag{28}$$

and

$$X_{\tau \ell m \sigma, \tau' \ell' m' \sigma} = Z^2 \int_S \tilde{t}(\text{In} \psi_{\tau \ell m \sigma}) \cdot \tilde{t}(\text{Ou} \psi_{\tau' \ell' m' \sigma}) \, dS$$

$$+ \int_S \tilde{t}(\text{In} \psi_{\tau \ell m \sigma}) \cdot \tilde{t}(\text{Ou} \psi_{\tau' \ell' m' \sigma}) \, dS \tag{29}$$

and

$$X_S = \int_S \tilde{t}(\text{Re} \psi_{\tau \ell m \sigma}) \cdot \tilde{t}(\text{Re} \psi_{\tau' \ell' m' \sigma}) \, dS \tag{30}$$

For any matrix M,
\[ M_n = \frac{1}{2} (M + \Delta M); \quad \Delta = \delta_{TT}, \delta_{LL}, \delta_{nm}, (-1)^{r+\alpha+m} \] (31)

In Eq. (29), \( Z \) is an adjustable parameter. The matrix \( \tilde{X} \) can be obtained from Eq. (29) by replacing the 'In' functions by 'Re' functions and for \( \tilde{X} \) both functions in Eq. (29) are taken as 'Re' functions.

In Refs. 5 and 6, the authors demonstrate that this complicated form of the T-matrix leads to improved convergence. We note that Eq. (28) involves more than 5 distinct matrices. Programming this procedure would require very large computer resources and would demand a very high precision machine. In the comparisons that are made in what follows, the new version of MOOT is not tested directly. Some comments are made with respect to published results using the new version.

NUMERICAL RESULTS AND COMPARISON

Both the T-matrix approach as well as MOOT were programmed on an AMDAHL 470 computer using double precision arithmetic. The scalar 2-D problem was tested for a cylinder of elliptical cross-section with aspect ratio \( b/a = 0.5 \) and \( 0.1 \) respectively. For the 3-D elastic wave case, scattering by oblate spheroidal voids of aspect ratio \( b/a = 0.5 \) and \( 0.2 \) were calculated for a wide range of frequencies and scattering geometries.

General Procedure

For both 2-D and 3-D problems the Q-matrices were first generated. The numerical integration on the boundary of the ellipse or spheroid was performed by using a Gauss-Legendre formula with 31 points on the interval \( 0 - \pi/2 \). For the spheroid the \( \phi \) integration was done analytically resulting in block diagonal matrices in the azimuthal index \( m \). All symmetries in the Q-matrices were first tested and then enforced. The Bessel and Hankel functions were generated by recursion formulas as also the associated Legendre polynomials. The same set of subroutines were used for programming the T-matrix approach as well as MOOT. We remark that for the former only one Q-matrix is sufficient since \( Q(Re,Re) = \text{Real} (Q(Re,Re)) \) but in MOOT, \( Q(Re,Re) \) and \( Q(Re,Re) \) must be separately generated and stored. As remarked in the previous section the more complicated form of MOOT was not programmed since the computations are expected to be too expensive.

Symmetry of the T-matrix

In both cases, the necessary Q-matrices were inverted using a Gaussian elimination procedure with partial pivoting. The T-matrix
was then set up for each case. The unitarity test was not performed for MOOT, since this check is a much weaker one than the symmetry check. The symmetry of the T-matrix generated by both approaches was checked according to the following criterion

\[ \text{Error (\( \varepsilon \))} = \frac{\left| T_{i}^{j} - T_{j}^{i} \right|}{\max \left| T_{\ell \ell'} \right|} \]

If the deviation from perfect symmetry \( \varepsilon \) was > 0.1, the corresponding T-matrix element as well as the \( \varepsilon \) was printed out. In all cases tested, the symmetry check using the extended boundary condition approach \( \varepsilon \) never exceeded 0.1, whereas using MOOT, values of \( \varepsilon \) were printed that ranged from 0.11 - 0.35. This is an indication that MOOT fails to account for mode conversion properly (see also convergence plot for the mode converted amplitude).

We should further point out that if we normalized the spherical function according to Refs. 3-6, i.e. without the factor \( \sqrt{k_p}/k_s \), the symmetry and convergence tests were much poorer. It is clear that improperly normalized wave functions in the elastic wave case will lead to inaccuracies in the mode converted amplitudes. This argument will also apply to the improved form of MOOT.

In Ref. 3 some erroneous statements were made about the symmetry of the T-matrix as derived originally by Waterman. In all tests that were made in this study and in previous studies, the symmetry of the T-matrix was never assumed but always tested. It is true that Waterman introduced a numerical procedure to reduce the number of steps in inverting the Q-matrix and generating the T-matrix. This was achieved by assuming that the truncated T-matrix is symmetric and the S-matrix unitary. It must be emphasized that this was always done after testing the T-matrix for symmetry by straight inversion. In our own experience Waterman's procedure did not result in much saving of computer time and in the last 3 or 4 years we have discontinued the use of this procedure. Thus the T-matrix is always tested for symmetry and if \( \varepsilon > 0.1 \), the corresponding element printed.

**Convergence**

Both methods were tested for convergence by plotting the value of the scattering cross-section for P- incidence as a function of truncation size of the matrix. In Refs. 3-6, it is stated that MOOT results in a convergent sequence for the partial waves. From our experience with MOOT as the figures indicate, this is true. But in all cases tested, the rate of convergence of MOOT is always very much slower than the T-matrix approach. For example at \( k_p a = 4.0 \) for an oblate spheroid of \( b/a = 0.5 \), the T-matrix method has converged at \( \ell = 13 \) whereas MOOT has not converged even at \( \ell = 20 \) although it is converging uniformly (see Figs. 3 and 4). For the mode converted
Fig. 3. Comparison of Convergence of back scattering cross-section from a spheroidal void in Ti for $\theta_{\text{inc}}=0$ versus truncation size $\ell_{\text{max}}$; *---* T-matrix; — MOOT.

Fig. 4. Comparison of Convergence of bistatic cross-section (P+P) from a spheroidal void in Ti for $\theta_{\text{inc}}=30^\circ$, $\theta_{\text{obs}}=135^\circ$ versus truncation size $\ell_{\text{max}}$; *---* T-matrix; — MOOT.
Fig. 5. Comparison of Convergence of bistatic shear scattering cross-section (P→S) from a spheroidal void in Ti for $\theta_{\text{inc}}=90^\circ$, $\theta_{\text{obs}}=135^\circ$ versus truncation size $l_{\text{max}}$; *-*-* T-matrix; ----- MOOT.

Fig. 6. Back scattering cross-section of elliptic cylindrical void for SH-wave incidence versus shear wavenumber; T-matrix; ----- MOOT. ($l_{\text{max}} = 25$ for both methods.)
Fig. 7. Back scattering cross-section (P→P) of a spheroidal void in Ti as a function of P-wavenumber; ——— T-matrix; *-*-* MOOT. (\(l_{\text{max}} = 20\) for both methods.)

Fig. 8. Bistatic shear scattering cross-section (P→S) of spheroidal void in Ti, \(\theta_{\text{obs}} = 90^\circ\) versus P-wavenumber; ——— T-matrix; *-*-* MOOT. (\(l_{\text{max}} = 20\) for both methods.)
Fig. 9. Back scattering cross-section (P→P) of spheroidal void in Ti versus P-wavenumber; T-matrix; *-*-* MOOT. ($k_{max} = 20$ for both methods.)

Fig. 10. Bistatic shear scattering cross-section (P→S) of spheroidal void in Ti, $\theta_{obs}=90^\circ$ versus P-wavenumber; T-matrix; *-*-* MOOT. ($k_{max} = 20$ for both methods.)
cross section, with P-waves incident perpendicular to the symmetry axis, the convergence of MOOT is very poor (see Fig. 5). This of course reflects the inaccuracy resulting from a T-matrix that is not symmetric.

**Frequency Dependence**

In Fig. 6 the frequency dependence of the cross section is plotted for SH-wave incidence on an elliptical cylinder of aspect ratio b/a = 0.5. At the higher values of ka, there is poor agreement between the two methods. In Fig. 7 the scattering cross section for P-wave incidence along the symmetry axis of an oblate spheroid of aspect ratio b/a = 0.5 is plotted using both methods. In Figure 8 the bistatic cross section of the mode converted shear waves is plotted for b/a = 0.5. The agreement is very poor. For a very flat oblate spheroid, b/a = 0.2, the earlier version of MOOT tested here gives erroneous results as shown in Fig. 9. However, the T-matrix results compare very well with integral equation solutions for the penny shaped crack. These results may also be compared with Fig. 13 of Ref. 6 which uses the new version of MOOT as well as the comparison of convergence between the two methods which is shown in Fig. 9 of Ref. 6.

In Fig. 10, the mode converted shear scattering cross section is plotted for an oblate spheroidal void with b/a = 0.2 observed at 90° to the symmetry axis. There is no agreement between MOOT and T-matrix. The spacing of the minima for the T-matrix method however can be explained on the basis of interference between specular reflection at the edge and a creeping wave. The symmetry error is less than 0.1 for the T-matrix whereas it approaches 0.35 with MOOT.

Finally we remark that MOOT becomes quite complicated when applied to penetrable scatterers, more so with the new form of MOOT, whereas the T-matrix approach has already been applied to scattering by elastic obstacles in water with excellent agreement between theory and experiment.

**SUMMARY**

Two matrix methods, namely Waterman's T-matrix approach and MOOT were compared for 2-D and 3-D elastic wave problems. Convergence, symmetry and frequency dependence of the scattering cross section were compared for ellipses and spheroidal cavities of different aspect ratios. Although the T-matrix method and MOOT are not always in agreement, in no way do we imply that any of the published results using MOOT are in error. It must be further emphasized that we have not on our own tested the new version of MOOT as used in Refs. 5 and 6. From the published results, this new version appears to converge more quickly as the aspect ratio decreases but it requires excessive amounts of computer resources.
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