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Normal Approximations for Computing Confidence Intervals for Log-Location-Scale Distribution Probabilities

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Normal Approximations for Computing Confidence Intervals for Log-Location-Scale Distribution Probabilities

Abstract

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Keywords

Censored data, Maximum likelihood, Quantile

Disciplines

Statistics and Probability

Normal Approximations for Computing Confidence Intervals for Log-Location-Scale Distribution Probabilities

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Abstract

Normal approximation confidence intervals are used in most commercial statistical package because they are easy to compute. However, the performance of such procedures could be poor when the sample size is not large or when there is heavy censoring. A transformation can be applied to avoid having confidence interval endpoints fall outside the parameter space and otherwise improves performance, but the degree of improvement (if any) depends on the chosen function. Some seemingly useful transformation functions will cause the estimated variance blow-up in extrapolation, which makes the performance poor. This article reviews statistical methods to construct confidence intervals for distribution probabilities based on a normal distribution approximation and studies the properties of these confidence interval procedures. Our results suggest that a normal approximation confidence interval procedure based on a studentized statistic, which we call the \hat{z} procedure, has desirable properties. We also illustrate how to apply the \hat{z} procedure to other functions of the parameters and in more general situations.

KEY WORDS: Censored data; Maximum likelihood; Quantile

1 INTRODUCTION

1.1 The Problem

Normal approximation confidence intervals are easy to compute and thus they are used to obtain confidence intervals for functions of parameters (e.g., quantiles and probabilities) in most commercial statistical packages. Although improvements in computer technology have made it possible to use likelihood-based confidence intervals, which are considered to be more accurate and reliable, doing so is not practicable for interactive computing if a large set of confidence intervals need to be computed for plotting purposes.

Although easy to compute, normal approximation confidence interval procedures can have poor performance. For example, a confidence interval for a distribution probability $p = F(t_e)$ for some specified t_e , directly based on a normal approximation for the studentized $\hat{F}(t_e)$, might contain points outside the $[0, 1]$ parameter space when the sample size is small.

A transformation could be used to make an improvement, but the choice of the transformation function is important. Some seemingly useful transformations will cause the estimated variance to blow-up in extrapolation, making the performance of the confidence interval procedure poor.

1.2 Ball Bearing Life Test Data Example

When estimating a probability distribution using a parametric model, it is standard practice, especially in the data analysis, to compute a set of pointwise confidence intervals for quantiles or $F(t)$ and plot them all on one graph (e.g., Minitab 2003, PROC RELIABILITY in SAS 2000, S-PLUS/SPLIDA in Meeker and Escobar 2003, etc. provide such graphics). We will refer to these pointwise sets as “confidence bands.”

To see the transformation effect on the confidence interval procedures, we consider a well-known subset of the Lieblein and Zelen (1956) ball bearing life test data as an example. As described in Lawless (2003, p. 98), this data set has 23 exact observations which are the number of million cycles before failure for each ball bearing. Figure 1 shows the ML estimate and pointwise confidence bands for the cdf of the uncensored

ball bearing life test data on the Weibull probability plot. These confidence bands are computed based on four different confidence interval procedures which will be described in Section 3. Figure 2 is similar, showing the ML estimate and the pointwise confidence bands based on the censored data by assuming the life test ended after the first 10 bearing failures.

One would expect the confidence interval endpoints to converge to 1 for large t_e . However, some transformation confidence intervals, computed both from the uncensored and the censored data, become wider for large t_e . This anomaly is called “bend-back behavior” and is the result of a poor asymptotic approximation. The transformation used to construct the confidence interval has a strong effect on bend-back behavior. The reasons for this behavior will be studied in detail in Section 4.

One might think that when plotting confidence intervals over a wide range of time, simultaneous confidence intervals (e.g., Cheng and Iles 1983) should be used. The common practice is, however, to plot a set of pointwise intervals because most applications call for inferences at a single point in time or for a single quantile. Plotting the entire set of confidence intervals relieves the user from having to specify the particular of the application, making the software easier to use.

1.3 Related Literature and Relationship to Current Work

Statistical methods for log-location-scale distributions, especially with application to lifetime studies are given, for example, in Nelson (1982), Meeker and Escobar (1998), and Lawless (2003). Billman, Antle, and Bain (1972) provided theory and gave limited tables for confidence limits on the parameters and survival probabilities with failure-censored samples from the Weibull distribution. Nelson and Schmee (1979) presented parallel results for the (log) normal distribution. Meeker and Escobar (1995) compared normal approximation methods with likelihood based methods for computing confidence intervals.

This article reviews and extends normal approximation procedures to construct confidence intervals for distribution probabilities. We investigate the properties of a procedure based on a studentized statistic, which we call the \hat{z} procedure. Comparisons

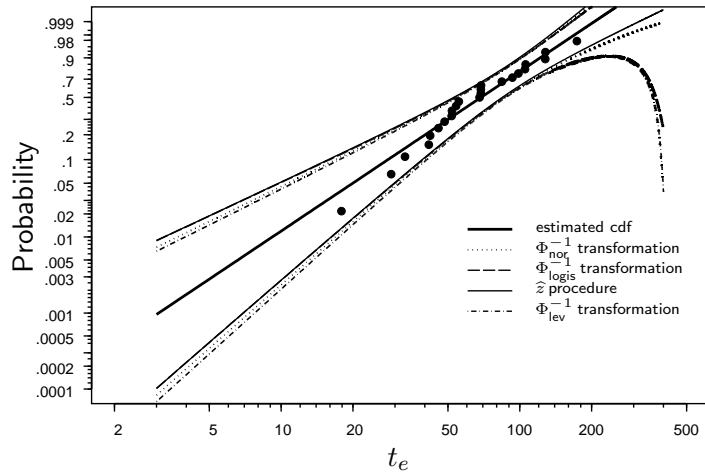


Figure 1: ML Estimate and Pointwise Confidence Bands for cdf from the Uncensored Ball Bearing Life Test Data On Weibull Probability Paper

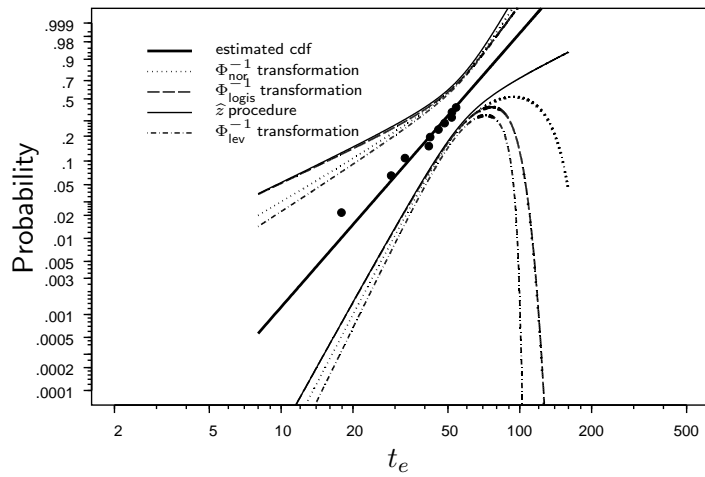


Figure 2: ML Estimate and Pointwise Confidence Bands for cdf from the Censored Ball Bearing Life Test Data On Weibull Probability Paper

between the \hat{z} procedure and other normal approximation procedures show that the \hat{z} procedure has desirable statistical properties. Also we provide examples to illustrate how to use the \hat{z} procedure for other functions of the parameters and in more general situations.

1.4 Overview

The remainder of this paper is organized as follows. Section 2 describes the model and maximum likelihood (ML) estimation. Section 3 describes procedures that have been used to construct confidence intervals for distribution probabilities. Section 4 presents analytical results and simulation results. Section 5 provides some extensions. Section 6 contains concluding remarks and possible areas for future research. Some technical details are given in the appendix.

2 MODEL AND ML ESTIMATION

2.1 Model

The results of this paper apply to location-scale and log-location-scale distributions. A random variable Y belongs to the location-scale family, with location μ and scale σ , if its distribution can be written as

$$F_Y(y; \mu, \sigma) = \Phi\left(\frac{y - \mu}{\sigma}\right), \quad -\infty < y < \infty,$$

where $-\infty < \mu < \infty, \sigma > 0$, and $\Phi(z)$ is the parameter free cdf of $(Y - \mu)/\sigma$. The normal distribution (NOR), the smallest extreme value distribution (SEV), the largest extreme value distribution (LEV), and the logistic distribution (LOGIS) are commonly used location-scale distributions.

A positive random variable T is a member of the log-location-scale family if $Y = \log(T)$ is a member of the location-scale family. Then the distribution of T is $F(t; \mu, \sigma) = \Phi\{[\log(t) - \mu]/\sigma\}$. The lognormal, the Weibull, the Fréchet, and the loglogistic are among the important distributions of this family. For example, the cdf and pdf of the

Weibull random variable T are

$$F(t; \mu, \sigma) = \Phi_{\text{sev}} \left[\frac{\log(t) - \mu}{\sigma} \right] \quad \text{and} \quad f(t; \mu, \sigma) = \frac{1}{\sigma t} \phi_{\text{sev}} \left[\frac{\log(t) - \mu}{\sigma} \right],$$

where $\Phi_{\text{sev}}(z) = 1 - \exp[-\exp(z)]$ and $\phi_{\text{sev}}(z) = \exp[z - \exp(z)]$ are the standard (i.e., $\mu = 0, \sigma = 1$) smallest extreme value cdf and pdf, respectively. For the lognormal distribution, replace Φ_{sev} and ϕ_{sev} above with Φ_{nor} and ϕ_{nor} , the standard normal cdf and pdf, respectively.

Suppose that T is a lifetime that has a distribution in the log-location-scale family. Frequently, interest is on quantities like the failure probability $F(t_e) = F(t_e; \mu, \sigma)$ at t_e or the p quantile $t_p = \exp[\mu + \Phi^{-1}(p)\sigma]$ of the distribution, where $\Phi^{-1}(p)$ is the p quantile of $\Phi(z)$.

Life tests often result in censored data. Type I (time) censored data result when unfailed units are removed from test at a prespecified time, perhaps due to limited time for study completion. Type II (failure) censored data result when a test is terminated after a specified number of failures, say $2 \leq r \leq n$. If all units fail, the data are called “complete” or “uncensored” data.

2.2 Maximum Likelihood Estimation

For a censored sample with “exact” and “right” censored observations in n independent observations from a log-location-scale random variable T , the likelihood of the data at $\boldsymbol{\theta} = (\mu, \sigma)'$ is

$$L(\boldsymbol{\theta}) = \mathcal{C} \prod_{i=1}^n \left\{ \frac{1}{\sigma t_i} \phi \left[\frac{\log(t_i) - \mu}{\sigma} \right] \right\}^{\delta_i} \left\{ 1 - \Phi \left[\frac{\log(t_i) - \mu}{\sigma} \right] \right\}^{1-\delta_i},$$

where $\delta_i = 1$ if t_i is an “exact” observation and $\delta_i = 0$ if t_i is a right censored observation, ϕ and Φ are, respectively, the location-scale standard pdf and cdf, and \mathcal{C} is a constant that does not depend on the unknown parameters. Standard computer software (e.g., JMP, MINITAB, SAS, S-PLUS/SPLIDA) provide maximum likelihood (ML) estimates of $\boldsymbol{\theta}$ and functions of $\boldsymbol{\theta}$ such as quantiles and probabilities. We denote the ML estimator of $\boldsymbol{\theta}$ by $\hat{\boldsymbol{\theta}} = (\hat{\mu}, \hat{\sigma})$. From the invariance property of ML estimators, the ML estimator

of t_p is $\hat{t}_p = \exp[\hat{\mu} + \Phi^{-1}(p)\hat{\sigma}]$. Similarly, the ML estimator of $F(t)$ at t_e is

$$\hat{F}(t_e) = \Phi \left[\frac{\log(t_e) - \hat{\mu}}{\hat{\sigma}} \right]. \quad (1)$$

See, for example, Chapter 8 in Meeker and Escobar (1998) for more details.

In large samples, the ML estimator $\hat{\boldsymbol{\theta}}$ has a distribution that can be approximated by a bivariate normal distribution $\text{MVN}(\boldsymbol{\theta}, \Sigma)$, where $\Sigma = I_{\boldsymbol{\theta}}^{-1}$. For the location-scale family or log-location-scale family with Type I or Type II censored data, the Fisher information matrix is

$$I_{\boldsymbol{\theta}} = \text{E} \left[-\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right] = \left(\frac{n}{\sigma^2} \right) M. \quad (2)$$

Here $\mathcal{L}(\boldsymbol{\theta}) = \log[L(\boldsymbol{\theta})]$ is the log likelihood of the data, and M is the scaled information matrix denoted by

$$M = \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix},$$

the elements of which do not depend on n and can be computed by the algorithm in Escobar and Meeker (1994), as a function of the proportion failing r/n (if r is fixed) or the expected proportion failing $\Phi[(\log(t_c) - \mu)/\sigma]$ (if r is random and t_c is the censoring time) in a censored sample. Also, we define the scaled covariance matrix Λ by

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix}^{-1}. \quad (3)$$

$I_{\boldsymbol{\theta}}$ can be estimated by the expected Fisher information matrix $I_{\boldsymbol{\theta}}$ evaluated at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$, or the observed local information matrix $\hat{I}_{\boldsymbol{\theta}}$, in which the Hessian matrix is simply evaluated at $\hat{\boldsymbol{\theta}}$, without taking expectations. In our computations we use the observed local information matrix $\hat{I}_{\boldsymbol{\theta}}$. In the following sections, we also use the estimate

$$\hat{\Lambda} = \begin{bmatrix} \hat{\lambda}_{11} & \hat{\lambda}_{12} \\ \hat{\lambda}_{12} & \hat{\lambda}_{22} \end{bmatrix} = \left(\frac{n}{\hat{\sigma}^2} \right) \hat{I}_{\boldsymbol{\theta}}^{-1}.$$

3 NORMAL APPROXIMATION CONFIDENCE INTERVAL PROCEDURES FOR PROBABILITIES

If t_e is the specified time at which an estimate of $F(t)$ is desired, (1) gives the ML estimator of $F(t_e)$. That is $\hat{F}(t_e) = \Phi(\hat{z}_e)$ where $\hat{z}_e = [\log(t_e) - \hat{\mu}]/\hat{\sigma}$. In this section, we outline four different normal approximation procedures for computing a confidence interval for $F(t_e)$. Each procedure is based on an assumption that a particular studentized statistic can be approximated by a normal distribution.

3.1 The \hat{F} Procedure

An approximate $100(1 - \alpha)\%$ confidence interval for $p = F(t_e)$ can be obtained from

$$[\underset{\sim}{p}, \underset{\sim}{\hat{p}}] = \hat{F}(t_e) \mp z_{1-\alpha/2} \hat{\text{se}}_{\hat{F}(t_e)}, \quad (4)$$

where $z_{1-\alpha/2} = \Phi_{\text{nor}}^{-1}(1 - \alpha/2)$ is the $(1 - \alpha/2)$ quantile of the standard normal distribution and the estimated standard error is obtained using the delta method. That is,

$$\hat{\text{se}}_{\hat{F}(t_e)} = \phi(\hat{z}_e) \sqrt{\frac{1}{n} \left(\hat{\lambda}_{11} + 2\hat{\lambda}_{12}\hat{z}_e + \hat{\lambda}_{22}\hat{z}_e^2 \right)}.$$

The interval (4) is based on the NOR(0, 1) approximation for $Z_{\hat{F}(t_e)} = [\hat{F}(t_e) - F(t_e)]/\hat{\text{se}}_{\hat{F}(t_e)}$. With a small to moderate number of failures, however, the approximation could be poor and the interval might contain points outside the $[0, 1]$ interval.

3.2 The Transformation Procedure

A confidence interval procedure based on a transformation $g = g[F(t_e)]$ would have a coverage probability closer to the nominal $100(1 - \alpha)\%$ if $Z_g^* = (\hat{g} - g)/\hat{\text{se}}_g$ has a distribution that is closer than $Z_{\hat{F}(t_e)}$ to a NOR(0, 1).

Usually, g is a monotone increasing and differentiable function of F chosen such that g ranges from $(-\infty, \infty)$, the same range as the normal distribution. Then if Ψ is a differentiable and monotone increasing cdf, $g = \Psi^{-1}$ is a plausible choice for the transformation. For example, $g = \Psi^{-1}$ could be the quantile function corresponding to

any probability distribution with support on the entire real line, such as the location-scale distributions mentioned in Section 2.1. In Meeker and Escobar (1998, p. 190), and Bagdonavičius and Nikulin (2001, p. 88), g was chosen to be the quantile function of the standard logistic distribution.

Let ψ be the corresponding pdf (first derivative of Ψ). A normal approximation $100(1 - \alpha)\%$ confidence interval for $z_e = [\log(t_e) - \mu]/\sigma$ is

$$[z, \tilde{z}] = \Psi^{-1}[\Phi(\hat{z}_e)] \mp z_{1-\alpha/2} \widehat{\text{se}}_{\Psi^{-1}[\Phi(\hat{z}_e)]}.$$

Then a general form for the transformation procedure confidence interval is

$$[p, \tilde{p}] = [\Psi(z), \Psi(\tilde{z})]. \quad (5)$$

By the delta method, and using (2) and (3), an estimator for the standard error of $\Psi^{-1}[\Phi(\hat{z}_e)]$ is

$$\widehat{\text{se}}_{\Psi^{-1}[\Phi(\hat{z}_e)]} = \dot{\Psi}^{-1}[\dot{\Phi}(\hat{z}_e)] \sqrt{\frac{1}{n} (\hat{\lambda}_{11} + 2\hat{\lambda}_{12}\hat{z}_e + \hat{\lambda}_{22}\hat{z}_e^2)},$$

where $\dot{\Psi}^{-1}[\dot{\Phi}(\hat{z}_e)] = \partial\Psi^{-1}[\Phi(\hat{z}_e)]/\partial\hat{z}_e = \phi(\hat{z}_e)/\psi\{\Psi^{-1}[\Phi(\hat{z}_e)]\}$. Therefore

$$[z, \tilde{z}] = \Psi^{-1}[\Phi(\hat{z}_e)] \mp \dot{\Psi}^{-1}[\dot{\Phi}(\hat{z}_e)] \sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{11} + 2\hat{\lambda}_{12}\hat{z}_e + \hat{\lambda}_{22}\hat{z}_e^2)}, \quad (6)$$

where $\gamma_{\alpha,n} = z_{1-\alpha/2}^2/n$.

The choice of the transformation function g is important because it affects the shape of the distribution of Z_g^* and how closely it agrees with a $\text{NOR}(0, 1)$ cdf. For one parameter problems, Spratt (1973) suggested that the shape of the likelihood function could be used to examine how the transformations improve the accuracy of normal approximation. In some cases, his approach could be extended to more than one parameter by considering the profile likelihood function for the quantity of interest.

3.3 The \hat{z} Procedure

Nelson (1982, p. 332) implicitly suggests using the transformation function $g = \Phi^{-1}$ (i.e., the quantile function of the distribution of T .) In this case, $g[\hat{F}(t_e)] = g[\Phi(\hat{z}_e)] = \Phi^{-1}[\Phi(\hat{z}_e)] = \hat{z}_e$. We call this Nelson's \hat{z} -method. Formally, the \hat{z} procedure is based

on the approximation $(\widehat{z}_e - z_e)/\widehat{\text{se}}_{\widehat{z}_e} \sim \text{NOR}(0, 1)$. For the \widehat{z} procedure, the confidence interval in (5) reduces to

$$[\underset{\sim}{p}, \underset{\sim}{\widehat{p}}] = [\Phi(\underset{\sim}{z}), \Phi(\underset{\sim}{\widehat{z}})], \quad (7)$$

where

$$[\underset{\sim}{z}, \underset{\sim}{\widehat{z}}] = \widehat{z}_e \mp \sqrt{\gamma_{\alpha,n}(\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12}\widehat{z}_e + \widehat{\lambda}_{22}\widehat{z}_e^2)}.$$

3.4 The \widehat{t}_p Procedure

Another confidence interval procedure for $F(t_e)$ is related to the confidence bands based on the normal approximation confidence interval procedure for the quantile t_p . This procedure is based on inverting the confidence intervals for the quantiles as illustrated in Figure 3. In particular,

- Compute the confidence intervals $[\underset{\sim}{t}_p, \underset{\sim}{\widehat{t}}_p]$ for the quantiles of the cdf. In Figure 3 the lower endpoints, $\underset{\sim}{t}_p$, and the upper endpoints, $\underset{\sim}{\widehat{t}}_p$, of the confidence intervals are indicated by \leftarrow and \rightarrow , respectively.
- The confidence bands for the cdf $F(t)$, $0 < t < \infty$ are defined as follows. The upper boundary of the confidence band for $F(t)$ is obtained by joining the lower endpoints of the quantile confidence intervals, $\underset{\sim}{t}_p$ and the lower boundary of the confidence bands is obtained by joining the upper endpoints, $\underset{\sim}{\widehat{t}}_p$.
- A pointwise confidence interval for $F(t_e)$ is obtained from the intersections of a vertical line through t_e with the boundaries of the confidence bands for $F(t)$.

In Figure 3, this confidence interval for $F(t_e)$ is indicated with the \updownarrow symbol. Computationally, the confidence interval for $F(t_e)$ is obtained as follows. A normal approximation confidence interval for $\log(t_p)$ is

$$\left[\log(\underset{\sim}{t}_p), \log(\underset{\sim}{\widehat{t}}_p) \right] = \log(\widehat{t}_p) \mp z_{1-\alpha/2} \widehat{\text{se}}_{\log(\widehat{t}_p)}, \quad (8)$$

where, using (2) and (3), the estimated standard error of $\log(\widehat{t}_p)$ is

$$\widehat{\text{se}}_{\log(\widehat{t}_p)} = \widehat{\sigma} \sqrt{\frac{1}{n} \left(\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12}z_p + \widehat{\lambda}_{22}z_p^2 \right)}$$

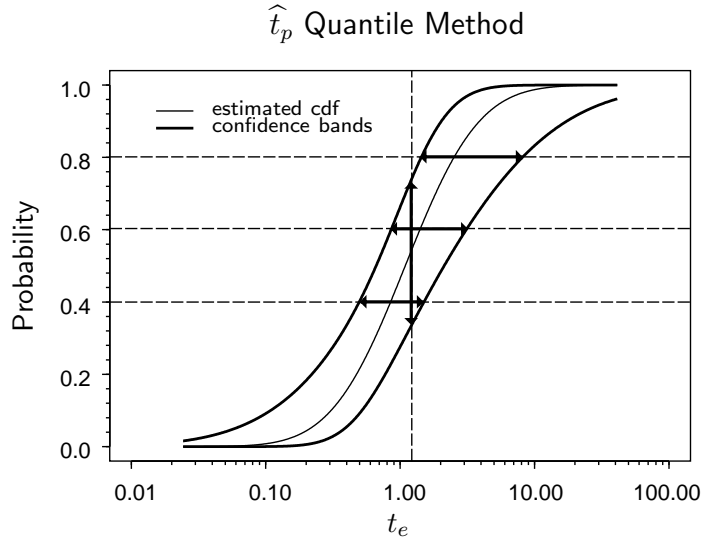


Figure 3: Illustration of the \hat{t}_p Quantile Procedure

and $z_p = \Phi^{-1}(p)$. Thus, the normal approximation confidence interval for t_p is

$$\left[\underline{t}_p, \tilde{t}_p \right] = \exp \left[\log(\hat{t}_p) \mp \hat{\sigma} \sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{11} + 2\hat{\lambda}_{12}z_p + \hat{\lambda}_{22}z_p^2)} \right]. \quad (9)$$

A confidence interval for $F(t_e)$ is given by the solutions \underline{p} and \tilde{p} for the equations

$$\begin{aligned} \log(t_e) &= \log(\hat{t}_p) + \hat{\sigma} \sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{11} + 2\hat{\lambda}_{12}z_{\underline{p}} + \hat{\lambda}_{22}z_{\underline{p}}^2)}, \\ \log(t_e) &= \log(\hat{t}_p) - \hat{\sigma} \sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{11} + 2\hat{\lambda}_{12}z_{\tilde{p}} + \hat{\lambda}_{22}z_{\tilde{p}}^2)}. \end{aligned}$$

Thus

$$[\underline{p}, \tilde{p}] = [\Phi(z), \Phi(\tilde{z})], \quad (10)$$

where

$$[\underline{z}, \tilde{z}] = \hat{z}_e + \frac{\gamma_{\alpha,n}(\hat{\lambda}_{12} + \hat{z}_e \hat{\lambda}_{22})}{1 - \gamma_{\alpha,n} \hat{\lambda}_{22}} \mp \frac{\sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{11} + 2\hat{z}_e \hat{\lambda}_{12} + \hat{z}_e^2 \hat{\lambda}_{22}) - \gamma_{\alpha,n}^2(\hat{\lambda}_{11} \hat{\lambda}_{22} - \hat{\lambda}_{12}^2)}}{1 - \gamma_{\alpha,n} \hat{\lambda}_{22}}$$

The procedure leading to the interval (10) requires that $\gamma_{\alpha,n} \hat{\lambda}_{22} < 1$ for the solution to be unique. If $\gamma_{\alpha,n} \hat{\lambda}_{22} \geq 1$, one or both roots might be complex, infinite, or non-unique.

Appendix A.1 shows that the confidence interval in (10) is asymptotically equivalent to the confidence interval procedure in (7).

4 PROPERTIES OF CONFIDENCE INTERVAL PROCEDURES

This section outlines various statistical properties of the confidence interval procedures described in Section 3.

4.1 Conditions for No Bend-Back of the \hat{z} and \hat{t}_p Procedures

Here, we give conditions under which bend-back will not occur in the \hat{z} and \hat{t}_p procedures. The bend-back behavior arises because the variance of certain functions of the parameters tends to blow-up in extrapolation. We show that the bend-back behavior has a lower probability of occurrence with the \hat{z} and the \hat{t}_p procedure, when compared to the other transformation procedures.

Result 1 *The condition*

$$\gamma_{\alpha,n}\hat{\lambda}_{22} < 1 \tag{11}$$

is necessary and sufficient to assure that the \hat{z} procedure in (7) and the \hat{t}_p procedure in (10) yield confidence intervals for $F(t_e)$ that do not bend-back and their width converges to 0 as t_e increases to ∞ .

Appendix A.2 gives a proof of this result. Section 4.3 shows that for Type II censored data, the condition (11) is satisfied with high probability.

The condition in **Result 1** has an interesting interpretation in the context of a Wald joint confidence region for $\boldsymbol{\theta}$ used to construct a confidence interval for a scalar function of $\boldsymbol{\theta}$. Consider the region $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})' \hat{I}_{\boldsymbol{\theta}} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \leq \chi_{1;1-\alpha}^2 = z_{1-\alpha/2}^2$, that is, $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})' \hat{\Lambda}^{-1} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \leq \gamma_{\alpha,n} \hat{\sigma}^2$, where the χ^2 quantile has one degree of freedom because the confidence interval of interest here is for a scalar function. The minimum value of σ in the joint confidence region is $\sigma_{\min} = \hat{\sigma} (1 - \sqrt{\gamma_{\alpha,n} \hat{\lambda}_{22}})$. Thus condition (11) ensures that the Wald joint confidence region does not include negative values of σ .

4.2 Transformation Effects on Bend-Back

This section investigates the proneness of bend-back behavior when the transformation procedure of Section 3.2 is used to compute confidence intervals for $F(t_e)$. Consider the following reexpression of \underline{z} defined in (6) (we only consider \underline{z} because \tilde{z} is similar)

$$\begin{aligned}\underline{z} &= \Psi^{-1}[\Phi(\hat{z}_e)] - \sqrt{n\gamma_{\alpha,n}} \widehat{\text{se}}_{\Psi^{-1}[\Phi(\hat{z}_e)]} \\ &= \Psi^{-1}[\Phi(\hat{z}_e)] - \frac{\phi(\hat{z}_e)}{\psi\{\Psi^{-1}[\Phi(\hat{z}_e)]\}} \sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{11} + 2\hat{\lambda}_{12}\hat{z}_e + \hat{\lambda}_{22}\hat{z}_e^2)}.\end{aligned}$$

If $\sqrt{n\gamma_{\alpha,n}} \widehat{\text{se}}_{\Psi^{-1}[\Phi(\hat{z}_e)]}$ becomes large in comparison to $\Psi^{-1}[\Phi(\hat{z}_e)]$ for large t_e (i.e., the estimated variance blows-up in extrapolation), \underline{z} will go to $-\infty$. Thus $\Psi(\underline{z})$ will approach 0 for large t_e causing the bend-back anomaly. Formally, consider the ratio

$$\nu = \frac{\sqrt{n\gamma_{\alpha,n}} \widehat{\text{se}}_{\Psi^{-1}[\Phi(\hat{z}_e)]}}{\Psi^{-1}[\Phi(\hat{z}_e)]} = \frac{\hat{z}_e \phi(\hat{z}_e)}{\Psi^{-1}[\Phi(\hat{z}_e)] \psi\{\Psi^{-1}[\Phi(\hat{z}_e)]\}} \frac{\sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{11} + 2\hat{\lambda}_{12}\hat{z}_e + \hat{\lambda}_{22}\hat{z}_e^2)}}{\hat{z}_e}$$

and the limit

$$\lim_{t_e \rightarrow \infty} \nu = \nu_1 \sqrt{\gamma_{\alpha,n} \hat{\lambda}_{22}} \quad \text{where} \quad \nu_1 = \lim_{t_e \rightarrow \infty} \frac{\hat{z}_e \phi(\hat{z}_e)}{\Psi^{-1}[\Phi(\hat{z}_e)] \psi\{\Psi^{-1}[\Phi(\hat{z}_e)]\}}.$$

Because both $\Psi^{-1}[\Phi(\hat{z}_e)]$ and $\sqrt{n\gamma_{\alpha,n}} \widehat{\text{se}}_{\Psi^{-1}[\Phi(\hat{z}_e)]}$ are positive, if $\nu_1 \sqrt{\gamma_{\alpha,n} \hat{\lambda}_{22}} < 1$, then \underline{z} does not approach $-\infty$ as $t_e \rightarrow +\infty$, which ensures that the bend-back anomaly will not occur. Here, we can consider ν_1 as a “transformation effect” because it is determined by the transformation function. Similarly, $\gamma_{\alpha,n} \hat{\lambda}_{22}$ can be considered as a “data effect” because it is determined only by the data. For the \hat{z} procedure $\nu_1 = 1$, which shows that the \hat{z} procedure only has a “data effect.” For the transformation procedures of Section 3.3, if the distribution assumed for the data is Weibull and the transformation function used to construct the intervals is logit, then $\nu_1 = \infty$. This implies that even for a large sample size n , $\Psi(\underline{z}) \rightarrow 0$, as $t_e \rightarrow \infty$. That is, the bend-back behavior will happen, at some point, no matter how large the sample is, as illustrated in Figure 1 and 2.

In general, if ν_1 is larger than 1, $\Psi(\underline{z})$, is more prone to bend-back than the \hat{z} procedure; if ν_1 is less than 1, $\Psi(\underline{z})$ is less prone to bend-back than the \hat{z} procedure. The following result compares the proneness to bend-back of the transformation procedure when compared with the \hat{z} procedure.

Result 2 *In contrast to the \hat{z} procedure, the bend-back proneness of the transformation procedure is determined by the relative monotonicity of $\log \phi[\Phi^{-1}(p)]$ and $\log \psi[\Psi^{-1}(p)]$.*

If

$$\frac{\partial}{\partial p} \log \phi[\Phi^{-1}(p)] - \frac{\partial}{\partial p} \log \psi[\Psi^{-1}(p)] > 0 \text{ as } p \rightarrow 1,$$

the lower band curve will be more prone to bend-back than the \hat{z} procedure. If

$$\frac{\partial}{\partial p} \log \phi[\Phi^{-1}(p)] - \frac{\partial}{\partial p} \log \psi[\Psi^{-1}(p)] < 0 \text{ as } p \rightarrow 1$$

the lower band curve will be less prone to bend-back than the \hat{z} procedure. The result for the upper band is similar.

Because of (5), one is interested in characterizing the monotonicity of $\Psi(z)$ as $\hat{z}_e \rightarrow \pm\infty$. But $\Psi(\cdot)$ is monotone because it is a cdf. Thus it suffices to study the monotonicity of z as a function of \hat{z}_e . The derivative of z with respect to \hat{z}_e is

$$\frac{\partial z}{\partial \hat{z}_e} = \left[1 - \left(\frac{\partial L(\hat{z}_e)}{\partial \hat{z}_e} \right) \sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{11} + 2\hat{\lambda}_{12}\hat{z}_e + \hat{\lambda}_{22}\hat{z}_e^2)} + \frac{\sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{12} + \hat{\lambda}_{22}\hat{z}_e)}}{\sqrt{(\hat{\lambda}_{11} + 2\hat{\lambda}_{12}\hat{z}_e + \hat{\lambda}_{22}\hat{z}_e^2)}} \right] \exp[L(\hat{z}_e)],$$

where $L(\hat{z}_e) = \log[\Psi^{-1}\Phi(\hat{z}_e)] = \log[\phi(\hat{z}_e)/\psi\{\Psi^{-1}[\Phi(\hat{z}_e)]\}]$. For the \hat{z} procedure (i.e., $\Psi = \Phi$), we have $L(\hat{z}_e) = 0$ thus $\partial L(\hat{z}_e)/\partial \hat{z}_e = 0$. For other transformations, i.e., $\Psi \neq \Phi$, the monotonicity of $L(\hat{z}_e)$ affects the bend-back proneness. If $L(\hat{z}_e)$ is monotone increasing (i.e., $\partial L(\hat{z}_e)/\partial \hat{z}_e > 0$, as $\hat{z}_e \rightarrow +\infty$), $\partial z/\partial \hat{z}_e$ is more likely to take negative values than the \hat{z} procedure. In this case $\Psi(z)$ is more prone to bend-back than the \hat{z} procedure. Similarly, if $L(\hat{z}_e)$ is monotone decreasing as $\hat{z}_e \rightarrow +\infty$, then $\Psi(z)$ is less prone to bend-back than the \hat{z} procedure. For the case of $\Psi(\tilde{z})$, the argument is similar but we need to consider $\hat{z}_e \rightarrow -\infty$.

Substituting $\Phi^{-1}(p)$ for \hat{z}_e , which will not affect the monotonicity of $L(\hat{z}_e)$ because $\Phi(\cdot)$ is increasing, we get

$$L(p) = \log \frac{\phi[\Phi^{-1}(p)]}{\psi[\Psi^{-1}(p)]} = \log \phi[\Phi^{-1}(p)] - \log \psi[\Psi^{-1}(p)]. \quad (12)$$

Based on the above discussion, the bend-back proneness is determined by the relative monotonicity of $\log \phi[\Phi^{-1}(p)]$ and $\log \psi[\Psi^{-1}(p)]$. This property of $\log \phi[\Phi^{-1}(p)]$ is determined only by the form of the function ϕ and this property of $\log \psi[\Psi^{-1}(p)]$ is determined only by the form of the function ψ .

Figure 4 shows the shape of $L(p)$ for different combinations of Φ (the underlying standardized location-scale distribution assumed for the logarithms of the data) and Ψ^{-1} (the transformation quantile function used in obtaining the confidence intervals). The rows in the figure correspond to Ψ^{-1} . The columns correspond to Φ and they are ordered from the heaviest left tail (SEV) to the lightest left tail (LEV). If the shape of $L(p)$ looks like a \cap (i.e., $L(p)$ is monotone decreasing as $p \rightarrow 1$ and is monotone increasing as $p \rightarrow 0$), both the upper and lower confidence band curves will be relatively less prone to bend-back than the \hat{z} procedure. If the shape of $L(p)$ looks like a \cup , both the upper and lower confidence band curves will be relatively more prone to bend-back than the \hat{z} procedure.

Figure 4 explains the occurrence of bend-back in terms of the tail behavior of the ϕ and ψ . If the tail of ϕ is lighter than that of ψ , the bend-back anomaly is relatively more prone to occur. If the tail of ϕ is heavier than that of ψ , the bend-back anomaly is relatively less prone to occur.

4.3 Probability of Bend-Back for \hat{z} Procedure

In this section, we used a simulation to study the probability of bend-back for the \hat{z} procedure in the case of Type II censoring. We used observed information to estimate standard errors. The condition that we need to check is (11). Table 1 shows the proportion of samples with bend-back behavior when $r = 3$ for a confidence level of 0.95. When the confidence level is 0.95, the bend-back behavior occurred only when $r = 3$. For $r > 3$ no bend-back was observed in 10,000 simulations. Figure 5 shows the proportion of trials in which bend-back was observed for \hat{z} procedure with the lognormal distribution, as a function of different confidence level for several combinations of sample size n and number failing r .

For each distribution in Table 1, as n increases the observed proportion of bend-back occurrences increases. This is related to the fact that, for fixed r and finite n , the variance $\text{Var}(\hat{\sigma})$ is monotone increasing in n . Although we can only compute $\text{Var}(\hat{\sigma})$ through simulation, the increase in the variance can be explained as follows. When n increases the spacing, among the smallest r order statistics decreases. Then, in a

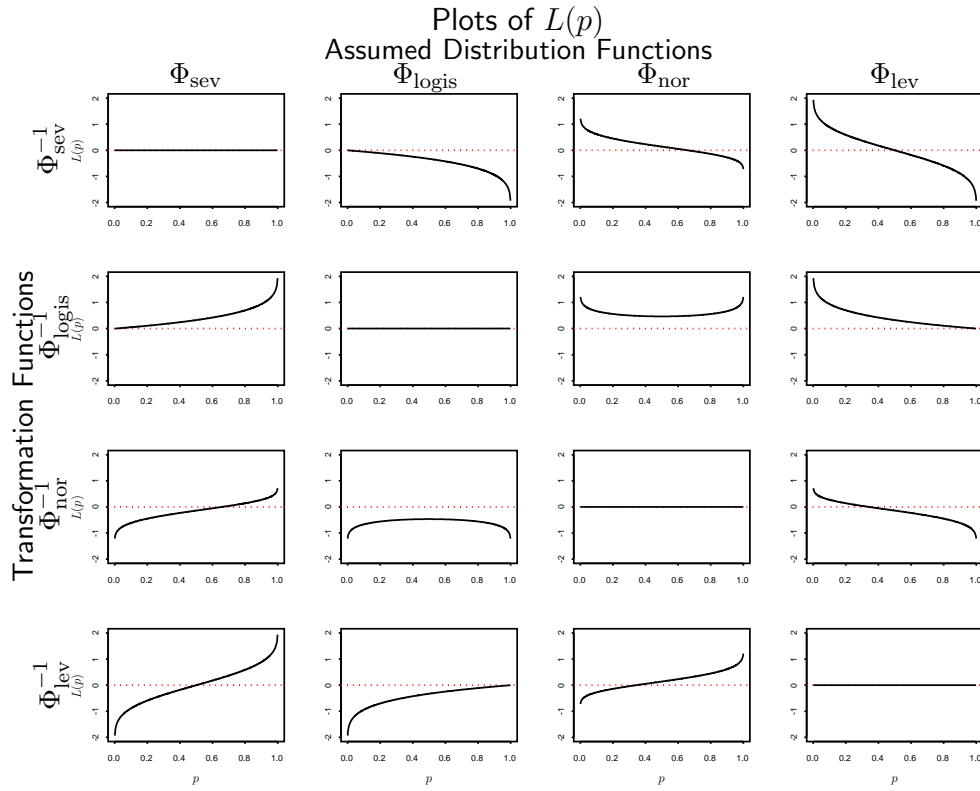


Figure 4: Plots of the $L(p)$ Function in (12). The Columns Correspond to the Assumed Distribution and the Rows Correspond to the Transformation Function

probability plot, the straight line corresponding to the ML estimate of the cdf becomes less stable as n increases because it is largely anchored on r observations that are closer to each other. This implies a larger variance for $1/\hat{\sigma}$, the slope of the line, and thus for $\hat{\sigma}$.

Figure 6 shows the values of $\gamma_{\alpha,n}\lambda_{22}$ obtained using the expected Fisher information in (2). As in Table 1, $r = 3$ and $\gamma_{\alpha,n} = z_{.975}^2/n$. The computation of the expected information is based in the algorithm of Escobar and Meeker (1994) as implemented in Meeker and Escobar (2003). The values of λ_{22} are obtained from Λ in (3).

For all of the distributions in Figure 6, $\gamma_{\alpha,n}\lambda_{22}$ increases with n . This is a consequence of the increase in the large sample approximation of $\text{Var}(\hat{\sigma})$ as n increases. Using **Result 1** and observing that the left hand side in (11) is an estimator of $\gamma_{\alpha,n}\lambda_{22}$, values in Figure 6 approaching or exceeding 1 identify situations with high probability of bend-back. All of the values for the SEV distribution in Figure 6 exceed 1, indicating high probability of bend-back which is consistent with the large observed proportion of bend-back occurrences for the SEV distribution reported in Table 1. The values of $\gamma_{\alpha,n}\lambda_{22}$ for the LOGIS distribution are similar to the ones for the SEV distribution. The values for the NOR distribution are smaller than the corresponding to the LOGIS distribution, indicating smaller probabilities of bend-back for the NOR distribution. In particular, for $n = 10$, the value $\gamma_{\alpha,n}\lambda_{22} = 0.863$ suggests a low probability of bend-back which corroborates the results in Table 1. For the LEV distribution, even for large n , the values of $\gamma_{\alpha,n}\lambda_{22}$ are smaller than the corresponding values for the other distributions, which is consistent with the bend-back observed proportion reported in Table 1.

4.4 Coverage Probability

In the case of Type II censoring, the confidence interval procedures defined by (4), (5), (7), and (8) can all be calibrated to give coverage probability that is exactly equal to the nominal confidence level. This is because the distributions of the endpoints of the confidence interval procedures do not depend on the true parameters μ and σ . In other words, $Z_{\hat{g}}^* = (\hat{g} - g)/\hat{\text{se}}_{\hat{g}}$ is pivotal because its distribution does not depend on parameters μ and σ . We only need to replace $\mp z_{1-\alpha/2}$ with the $\alpha/2$ and $1 - \alpha/2$ quantiles of $Z_{\hat{g}}^*$.

n	SEV	LOGIS	NOR	LEV
10	1	1	0	0
50	1	1	0.4104	0
100	1	1	0.7341	0
1000	1	1	1	0.4612
10000	1	1	1	0.8558

Table 1: Proportion of Bend-back for the \hat{z} Procedure When $r = 3$, for 95% Confidence Level, Repeated in 10,000 Times

Frequency of Bend-Back: \hat{z} Procedure with Lognormal Distribution

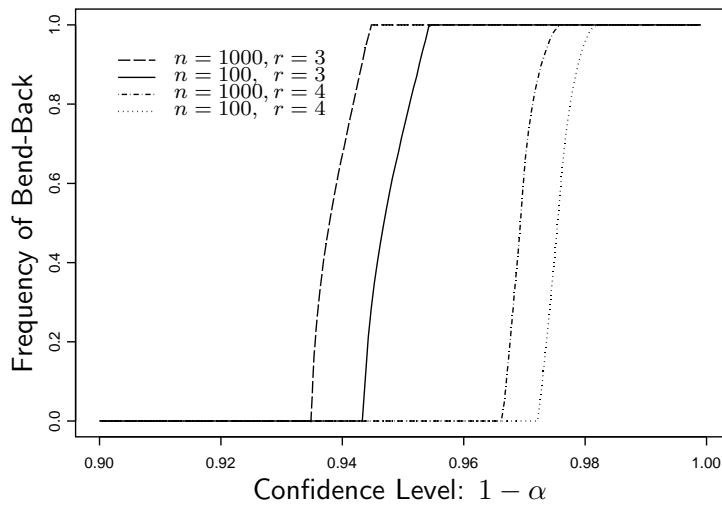


Figure 5: Proportion of Bend-back for the \hat{z} Procedure With Lognormal Distribution in Respect to Different Confidence Level in 10,000 Simulations

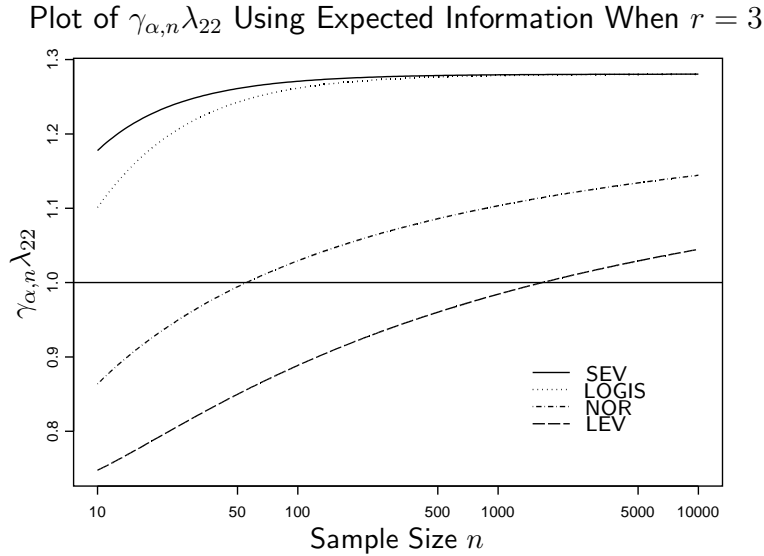


Figure 6: Values of $\gamma_{\alpha,n}\lambda_{22}$ for the Assumed Distribution Using Expected Information When $r = 3$ and $\gamma_{\alpha,n} = z_{.975}^2/n$

These quantiles depend only on n, r, Φ and $p = F(t_e)$, and they can be obtained by Monte Carlo simulation. See Appendix A.3 for details.

For Type I censoring, calibration using a similar simulation based procedure will provide an excellent approximation to the quantiles of $Z_{\hat{g}}^*$, as long as the expected number of failures is not too small.

Because the quantiles of $Z_{\hat{g}}^*$ are not readily available, the distribution of $Z_{\hat{g}}^*$ is commonly approximated by a NOR(0,1) distribution. Then the actual coverage probability of the normal approximation confidence interval procedure is $\Pr(-z_{1-\alpha/2} \leq Z_{\hat{g}}^* \leq z_{1-\alpha/2})$, which can be computed exactly because $Z_{\hat{g}}^*$ does not depend on the true parameters. We used simulation to obtain the quantiles of $Z_{\hat{g}}^*$. Figures 7 and 8 show, respectively, the simulated coverage probabilities for the standard Weibull and lognormal distributions (i.e., $\mu = 0$ and $\sigma = 1$) for t_e ranging from before the .01 quantile to beyond the .99 quantile of the distributions for some combinations of n and r . These figures (and others that we have produced in a more extensive simulation study) show that the coverage probability of the approximate \hat{z} procedure is good when $r > 30$.

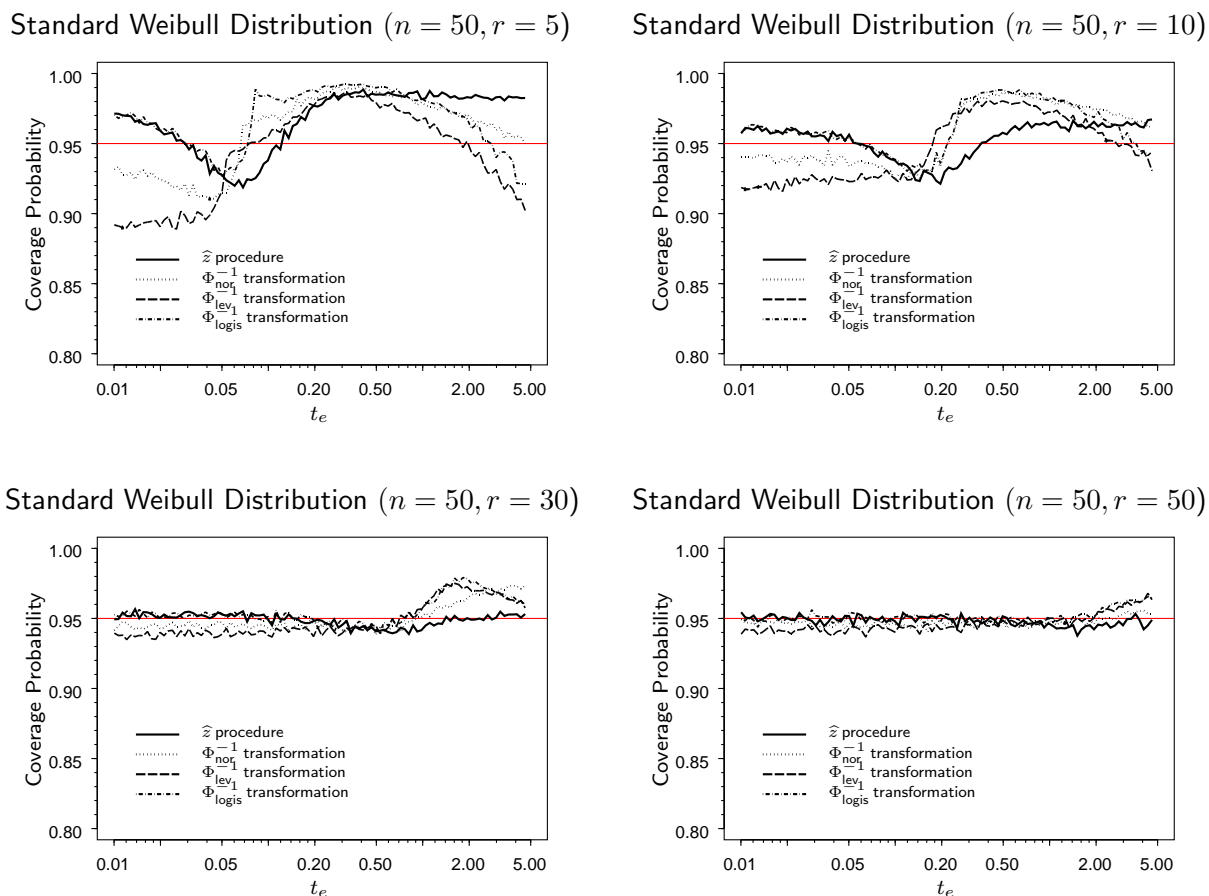


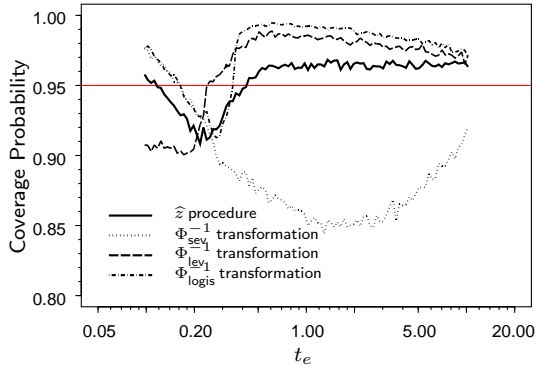
Figure 7: Estimated Coverage Probability for Weibull Distribution Confidence Intervals Based on 10,000 Simulations

Not surprisingly, the transformation methods that are prone to bend-back, however, can have actual coverage probabilities that deviate importantly from nominal, especially in extrapolation outside the range of the data (e.g., for large t_e when r is small and n is large, where bend-back is serious).

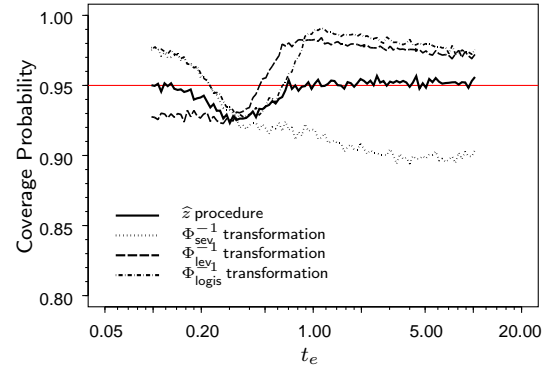
5 EXTENSIONS

The \hat{z} procedure can also be used to construct confidence intervals for the survival function $S(t_e) = 1 - F(t_e)$ and some other functions of the parameters. In general, it

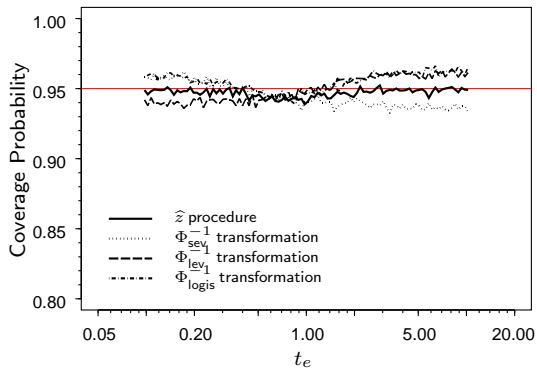
Standard Lognormal Distribution ($n = 50, r = 5$)



Standard Lognormal Distribution ($n = 50, r = 10$)



Standard Lognormal Distribution ($n = 50, r = 30$)



Standard Lognormal Distribution ($n = 50, r = 50$)

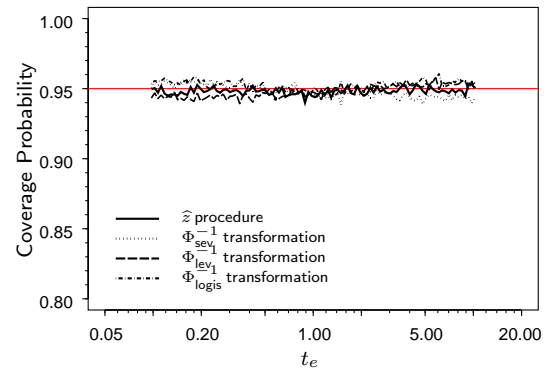


Figure 8: Estimated Coverage Probability for Lognormal Distribution Confidence Intervals Based on 10,000 Simulations

can be used to construct confidence intervals for functions of $z_e = [\log(t_e) - \mu]/\sigma$ that are continuous, strictly monotone, and that have a domain from $-\infty$ to ∞ .

For example, the hazard function for $Y = \log(T)$ which is $h_Y[\log(t_e); \mu, \sigma] = \phi(z_e)/[1 - \Phi(z_e)]$, is often of interest in reliability applications (e.g., Meeker and Escobar 1998, p. 191). The ML estimator of $h_Y[\log(t_e)]$ is $\hat{h}_Y[\log(t_e)] = \phi(\hat{z}_e)/[1 - \Phi(\hat{z}_e)]$. When $h_Y[\log(t_e)]$ is strictly monotone, which is true for most of distributions in the location-scale family (e.g., normal, SEV, and logistic). So the inverse function of $h_Y[\log(t_e)]$ exists. We use the inverse function as the transformation function. The confidence interval for $h_Y[\log(t_e)]$ if $h_Y[\log(t_e)]$ is monotone increasing is

$$[\underset{\sim}{h}, \underset{\sim}{\hat{h}}] = \left[\frac{\phi(\underset{\sim}{z})}{1 - \Phi(\underset{\sim}{z})}, \frac{\phi(\underset{\sim}{\hat{z}})}{1 - \Phi(\underset{\sim}{\hat{z}})} \right].$$

And if $h_Y[\log(t_e)]$ is monotone decreasing, the confidence interval is similar. Here $\underset{\sim}{z}$ and $\underset{\sim}{\hat{z}}$ are given in (7).

For another example, consider a system with s identical, independent components in series with failure time T_1, T_2, \dots, T_s for the components. The cdf of the time to failure of the series system is $F(t_e) = 1 - \prod_{i=1}^s \Pr(t_i > t_e) = 1 - [1 - \Phi(z_e)]^s$. The ML estimator of $F(t_e)$ is $\hat{F}(t_e) = 1 - [1 - \Phi(\hat{z}_e)]^s$. Similarly, the confidence interval for $p = F(t_e)$ is

$$[\underset{\sim}{p}, \underset{\sim}{\hat{p}}] = \left[1 - [1 - \Phi(\underset{\sim}{z})]^s, 1 - [1 - \Phi(\underset{\sim}{\hat{z}})]^s \right],$$

where $\underset{\sim}{z}$ and $\underset{\sim}{\hat{z}}$ are given in (7).

6 CONCLUDING REMARKS AND AREAS FOR FURTHER RESEARCH

This paper gives a summary of confidence interval procedures for distribution probabilities of log-location-scale distributions. The properties of each procedure are discussed and comparisons are made. We recommend use of the \hat{z} procedure to construct confidence intervals for the distribution probabilities. In the case of Type II censoring, when the number of failures $r \geq 4$ the chance of bend-back behavior is negligible. Furthermore, the \hat{z} procedure can be calibrated by simulation to give exact coverage probabilities for

Type II censoring. Also, this procedure can be used for some other functions of the parameters as shown Section 5.

This work can be extended to the following areas:

- The \hat{z} procedure can be extended directly to regression problems, under the usual log-location-scale model that assumes fixed explanatory variables and independent observations. The only difference is the dimensionality of the matrices and vectors involved in the computations.
- For Type I censoring or other cases, such as multiple censoring, condition (11) can be easily checked by simulation to see how often the bend-back anomaly happens. The actual coverage probability will, however, depend on the model parameters because the number of failures is a discrete random variable in Type I censoring. Also, the actual coverage probability will depend on the distribution of the censoring mechanism and its parameters in the case of multiple censoring.
- For non-log-location-scale distribution situation, the \hat{z} procedure can not been applied directly. A similar method for these distributions could, however, be developed.
- Similar issues arise in the construction of simultaneous confidence bands, such as those described by Cheng and Iles (1983).

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A PROOFS

A.1 Asymptotic Equivalence of the \hat{z} and \hat{t}_p Procedures

Here, we will prove the asymptotic equivalence of confidence bands from the \hat{z} procedure in (7) and the \hat{t}_p procedure in (10). In either case, the confidence band for the cdf can be expressed as $[p, \tilde{p}] = [\Phi(\underline{z}), \Phi(\tilde{z})]$. We only consider the lower band because the upper band is similar. For confidence bands defined by (7), $\underline{z}_1 = \hat{z}_e - \sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{11} + 2\hat{\lambda}_{12}\hat{z}_e + \hat{\lambda}_{22}\hat{z}_e^2)}$. For confidence bands defined by (10),

$$\underline{z}_2 = \hat{z}_e + \frac{\gamma_{\alpha,n}(\hat{\lambda}_{12} + \hat{z}_e\hat{\lambda}_{22})}{1 - \gamma_{\alpha,n}\hat{\lambda}_{22}} - \frac{\sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{11} + 2\hat{z}_e\hat{\lambda}_{12} + \hat{z}_e^2\hat{\lambda}_{22}) - \gamma_{\alpha,n}^2(\hat{\lambda}_{11}\hat{\lambda}_{22} - \hat{\lambda}_{12}^2)}}{1 - \gamma_{\alpha,n}\hat{\lambda}_{22}}.$$

Note that

$$\frac{\underline{z}_2 - \hat{z}_e}{\underline{z}_1 - \hat{z}_e} = \frac{\sqrt{(\hat{\lambda}_{11} + 2\hat{z}_e\hat{\lambda}_{12} + \hat{z}_e^2\hat{\lambda}_{22}) - \gamma_{\alpha,n}(\hat{\lambda}_{11}\hat{\lambda}_{22} - \hat{\lambda}_{12}^2)} - \sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{12} + \hat{z}_e\hat{\lambda}_{22})}}{(1 - \gamma_{\alpha,n}\hat{\lambda}_{22})\sqrt{(\hat{\lambda}_{11} + 2\hat{\lambda}_{12}\hat{z}_e + \hat{\lambda}_{22}\hat{z}_e^2)}} \rightarrow 1$$

as $n \rightarrow \infty$ because $\gamma_{\alpha,n} = z_{1-\alpha/2}^2/n \rightarrow 0$ (holding r/n or expectation of r/n constant). Thus the confidence bands defined by (7) and defined by (10) are asymptotically equivalent.

A.2 Proof of Result 1

To ensure that the bend-back behavior does not happen in the \hat{z} procedure, \underline{z} and \tilde{z} in (7) should be strictly monotone increasing and go to $+\infty$ as $t_e \rightarrow +\infty$ and go to $-\infty$ as $t_e \rightarrow 0$. We need to study the monotonicity of \underline{z} and \tilde{z} . Taking derivatives with respect to \hat{z}_e , the confidence intervals are monotone increasing if and only if

$$\frac{\partial \tilde{z}}{\partial \hat{z}_e} = 1 + \frac{\sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{12} + \hat{\lambda}_{22}\hat{z}_e)}}{\sqrt{\hat{\lambda}_{11} + 2\hat{\lambda}_{12}\hat{z}_e + \hat{\lambda}_{22}\hat{z}_e^2}} > 0 \quad (13)$$

$$\frac{\partial \underline{z}}{\partial \hat{z}_e} = 1 - \frac{\sqrt{\gamma_{\alpha,n}(\hat{\lambda}_{12} + \hat{\lambda}_{22}\hat{z}_e)}}{\sqrt{\hat{\lambda}_{11} + 2\hat{\lambda}_{12}\hat{z}_e + \hat{\lambda}_{22}\hat{z}_e^2}} > 0. \quad (14)$$

Equations (13) and (14) are both satisfied if and only if

$$\frac{\sqrt{\gamma_{\alpha,n}} \left| \widehat{\lambda}_{12} + \widehat{\lambda}_{22} \widehat{z}_e \right|}{\sqrt{\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12} \widehat{z}_e + \widehat{\lambda}_{22} \widehat{z}_e^2}} < 1$$

which is equivalent to

$$(1 - \gamma_{\alpha,n} \widehat{\lambda}_{22}) \left(\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12} \widehat{z}_e + \widehat{\lambda}_{22} \widehat{z}_e^2 \right) + \gamma_{\alpha,n} \left(\widehat{\lambda}_{11} \widehat{\lambda}_{22} - \widehat{\lambda}_{12}^2 \right) > 0. \quad (15)$$

The left hand side of (15) is positive for all \widehat{z}_e if and only if $\gamma_{\alpha,n} \widehat{\lambda}_{22} \leq 1$. This is so because if the matrix $\widehat{\Lambda}$ is positive definite, then $(\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12} \widehat{z}_e + \widehat{\lambda}_{22} \widehat{z}_e^2) > 0$ and $(\widehat{\lambda}_{11} \widehat{\lambda}_{22} - \widehat{\lambda}_{12}^2) > 0$. We exclude the situation that $\gamma_{\alpha,n} \widehat{\lambda}_{22} = 1$ because in this case the width of the confidence intervals does not converge to 0 as t_e goes to ∞ .

For the \widehat{t}_p procedure, a similar argument can be applied with the conclusion that $\gamma_{\alpha,n} \widehat{\lambda}_{22} < 1$ is the necessary and sufficient condition to ensure that the confidence bands are well behaved. In Section 3.4, we have already seen that $\gamma_{\alpha,n} \widehat{\lambda}_{22} < 1$ is the existence condition for (10), proving the result.

A.3 Coverage Probability for the Special Case of Type II Censoring

In Lawless (2003, p. 562), the following results are presented for the location-scale family with Type II censoring. $(\widehat{\mu} - \mu)/\widehat{\sigma}, \sigma/\widehat{\sigma}$, and $(\widehat{\mu} - \mu)/\sigma$ are pivotal quantities. And $\widehat{z}_i = [\log(t_i) - \widehat{\mu}]/\widehat{\sigma}, i = 1, \dots, r$ are ancillary statistics. Now consider

$$Z_{\widehat{g}}^* = \frac{g[\widehat{F}(t_e)] - g[F(t_e)]}{\widehat{se}_{\widehat{g}}} = \frac{\Psi^{-1}[\Phi(\widehat{z}_e)] - \Psi^{-1}[\Phi(z_e)]}{\dot{\Psi}^{-1}\dot{\Phi}(\widehat{z}_e) \sqrt{\frac{1}{n} \left(\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12} \widehat{z}_e + \widehat{\lambda}_{22} \widehat{z}_e^2 \right)}}$$

which is a function of \widehat{z}_e, z_e and $\widehat{z}_i, i = 1, \dots, r$. Because $\widehat{z}_e = [\widehat{\mu} - \mu - \Phi^{-1}(p)\sigma]/\widehat{\sigma}$, where $p = \Phi(z_e)$, is a pivotal quantity, the distribution of $Z_{\widehat{g}}^*$ depends only on n, r, ϕ, g and p , and does not depend on the parameters μ and σ .

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