

THE TRANSMISSION/SCATTERING OF ELASTIC WAVES BY  
A SIMPLE INHOMOGENEITY - A COMPARISON OF THEORIES

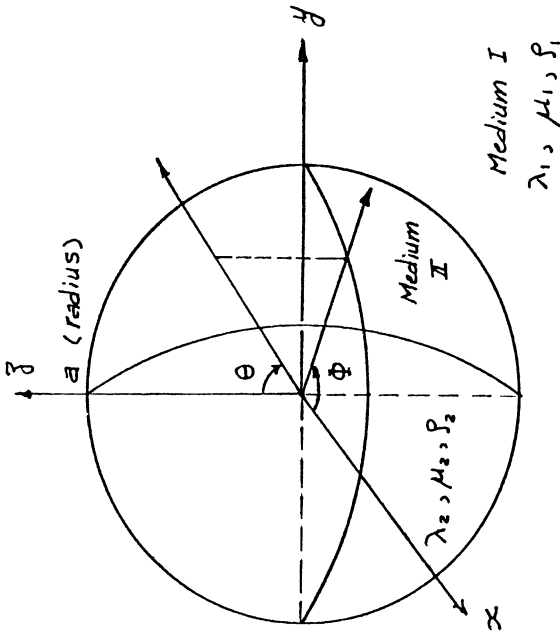
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INTRODUCTION

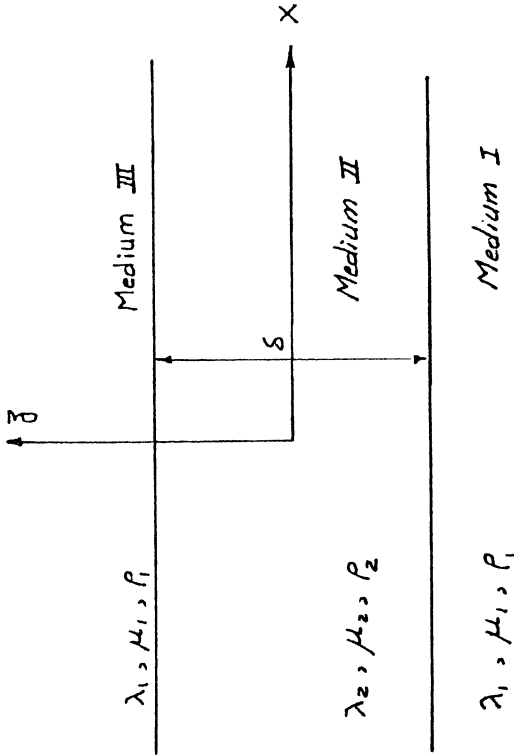
The method of equivalent inclusion, originally developed by Eshelby<sup>1,2</sup> was first applied to study dynamic behavior of composite laminates by Wheeler and Mura<sup>3</sup> where they neglected the presence of the mass density mismatch. Fu<sup>4,5</sup> gave a similar formulation for the problem of a pair of ellipsoidal inhomogeneities embedded in an infinite elastic medium subjected to time-harmonic waves. A complete formulation and presentation of analytic and numerical results for a single ellipsoidal inhomogeneity are subsequently given by Fu<sup>6</sup> following his work on the ultrasonic determination of fracture toughness, References (7) and (8). The eigenstrain approach is employed and the eigenstrains are expanded as a geometric series in position vector.

The extended method of equivalent inclusion developed in References (5) and (6) is applied to study two specific wave problems (i) the transmission of elastic waves in an infinite medium containing a layer of inhomogeneity, and (ii) the scattering of elastic waves in an infinite medium containing a sphere of inhomogeneity. The purpose of the study is to compare results, obtained by using limited number of terms in the eigenstrain expansion, with exact solutions for the layer problem and that for a perfect sphere, References (9-11). Two parameters are singled out for this comparison: the ratio of elastic moduli  $f = (\lambda_2 + 2\mu_2) / (\lambda_1 + 2\mu_1)$  and the ratio of the mass densities  $h = (\rho_2 / \rho_1)$ , Figs. 1 and 2. Results related to material systems same as those given in Truell<sup>12</sup> are presented.



$$u_0 \exp(i\alpha_1 r - i\omega t)$$

Fig. 2. Geometry and materials properties of an elastic spherical inhomogeneity in an infinite elastic medium.



$$u_0 \exp(i\alpha_1 z - \omega t)$$

Fig. 1. Geometry and material properties of the three-layered medium.

## COMPARISON OF COMPUTATIONAL RESULTS

Details of formulation, governing equations and evaluation of integrals are given in References (6,13,14). Numerical results are presented and compared for the three-layered and spherical inhomogeneity problems. Numerical results were first made known by Truell and his co-workers for a perfect spherical inhomogeneity. In order to compare with these results, the same material properties are used and listed in Table 1.

When the extended equivalent inclusion method is used, there are complex matrices, with dimensions depending upon the dimensionless wavenumber and the differences in elastic moduli and mass density between the inhomogeneity and the matrix, need to be solved. In order to get the best results, the IMSL (International Mathematical and Statistical Libraries) subroutine LEQ2C is used to solve matrices with double precision. The routine applies iterative improvement until the solution is accurate to machine precision. If the matrix is too ill-conditioned to get effective iterative improvement, a terminal error is produced.

Three-Layered Medium Problem

The input parameters for both methods are the dimensionless wavenumber  $\alpha_1 \delta$ , the relative ratio of elastic constants  $f$  and mass density  $h$ , where  $f$  is  $(\lambda_2 + 2\mu_2)/(\lambda_1 + 2\mu_1)$  and  $h$  is  $\rho_2/\rho_1$ . The displacement amplitude  $U_0$  and the stress amplitude  $(\lambda_1 + 2\mu_1)\alpha_1 U_0$  given in the preceding figures are nondimensionalized.

The calculation of the exact solutions is simple and the dimension of the matrix is just  $4 \times 4$ . The calculation of the equivalent inclusion method is relatively complicated. The solution is an infinite series summation, and  $N_f$  (the accepted number of terms to get convergent results) depends on  $\alpha_1 \delta$ ,  $f$  and  $h$ . In the

Table 1. Material Properties

Material	Compressional shear wave		Mass density (g/cm <sup>3</sup> )
	wave velocity (m/s)	velocity (m/s)	
Stainless steel	5790	3100	7.90
Mg	5770	3050	1.74
Al	6568	3149	2.70
Ge	5285	3376	5.36
Polyethylene	1950	540	0.90
Be	12890	8880	1.87

calculation of the displacements and stresses, the summation is considered to be acceptable until the ratio of the current term to the current partial term is less than 0.5%.

In order to make detailed comparison, five cases are studied and selective results are shown here. Figs. 3-5 and Figs. 6,7 display the displacement amplitude and stress amplitude vs  $\alpha_1\delta$ , respectively. From these results, it is shown that the extended method of equivalent inclusion gives excellent results which can be treated as the exact solutions. The values of  $N_f$  in getting the convergent displacement and stress amplitude in the  $f$  third case are listed in Table 2.

### Spherical Inhomogeneity Problem

The input parameters are the dimensionless wave number  $\alpha_1 a$ , the compressional and shear wave velocity and the mass density of the inhomogeneity and the matrix.

For the method of separation of variables, the results of the scattering cross section were plotted in Truell's paper but the specific values for different  $\alpha_1 a$  are not listed.†

For the equivalent inclusion method, there are two independent series summation. The first one is to get the inside function values which are related to  $\alpha_1 a$  and  $\beta_1/\alpha_1$  only. The second one is to get the outside function value which also depends on  $\alpha_1 a$  and  $\beta_1/\alpha_1$ .

From the three-layered problem, it is known that the value of  $N_f$  depends on  $\alpha_1\delta$ ,  $f$  and  $h$ . This is also true for the current problem. In addition, the value of  $N_f$  also depends on  $\beta_1/\alpha_1$ . It should be noted that the dimension of the matrix is proportional to the value of  $N_f$ . Therefore, for large  $N_f$ , the dimension of the matrix will be very large and the derivation and numerical computation to get the scattering cross section is very lengthy and time consuming. Instead of finding the scattering cross section for given  $\alpha_1 a$ , it is set out to find the upper validity of  $\alpha_1 a$  when  $N_f=1$ . The result is called one-term solution when  $N_f$  is equal to 1.  $f$ The equivalence conditions are reduced to, Reference (13):

$$\Delta\rho\omega^2 [f_{sj} [0]A_j + F_{skj} [0]B_{kj}] + A_s = -\Delta\rho\omega^2 H_s \quad (1)$$

$$\begin{aligned} \Delta\lambda\delta_{st} [d_{mmj} [0]A_j + D_{mmjk} [0]B_{jk}] + 2\Delta\mu [d_{stj} [0]A_j + D_{stjk} [0]B_{jk}] \\ + (\lambda_1\delta_{st} B_{mm} + 2\mu_1 B_{st}) = -(\Delta\lambda\delta_{st} E_{mm} + 2\Delta\mu E_{st}) \quad . \quad (2) \end{aligned}$$

†The data shown in this section are calculated from the computer program kindly supplied by Dr. J. Gubernatis.

Table 2. The Value of  $N_f$  for the Three-Layered Problem

$\alpha_1 \delta$	Ge in Al f=1.285 h=1.985	Al in Ge f=0.778 h=0.504	Polyethylene f=90.790 h=2.078	in Be	Mg in St f=0.219 h=0.220	St in Mg f=4.572 h=4.540
0.5	6	4	6		6	6
1.0	6	6	6		6	8
2.0	8	6	6		8	8
4.0	12	8	8		10	12
6.0	16	10	8		14	14
8.0	18	14	8		16	18
10.0	22	16	10		20	20

From the formulas in Appendix II Ref. (13), it is found that  $F_{skj}[0]$  and  $d_{stj}[0]$  are equal to zero, which make the equations (1) and (2) become uncoupled, i.e. the difference in mass density and the difference in elastic constants will have no coupled effects. After some manipulation, it is found that the non-vanishing variables are  $A_3$ ,  $B_{11}$ ,  $B_{22}$  and  $B_{33}$  with  $B_{11}$  equal to  $B_{22}$  by symmetry. For low  $\alpha_1 a$ , the closed form solutions of these variables can be obtained and closed form of the scattering cross section can also be obtained.

Fig. 8 displays the scattering cross-section vs  $\alpha_1 a$  from the two methods. It is found if  $\alpha_1 a$  is less than 1, the tendency of the one term solution is good compared to the exact solution. For certain material system the one-term solution represents a good approximation up to medium frequency range.

## DISCUSSION AND CONCLUSION

The accuracy of the solution by taking limited number of terms in the geometric series for eigenstrain decreases as the wavenumber increases, as expected, Figs. 10, 11. What is striking is that, for certain material systems, a single one-term solution compares well with the exact solution for a sphere, say within +5% for the scattering cross section, up to wavenumber in the medium frequency range,  $ka > 1$ , Figs. 8 and 9. The wavenumber at which the total cross section is of peak value is often closely predicted. The programming is simple and computations inexpensive. Solution of better accuracy can be obtained by either taking more terms in the eigenstrain expansion series or by adopting an iteration scheme using the one-term solutions as the first approximation. This work is studied as a special case for the scattering of an ellipsoid<sup>6</sup>.

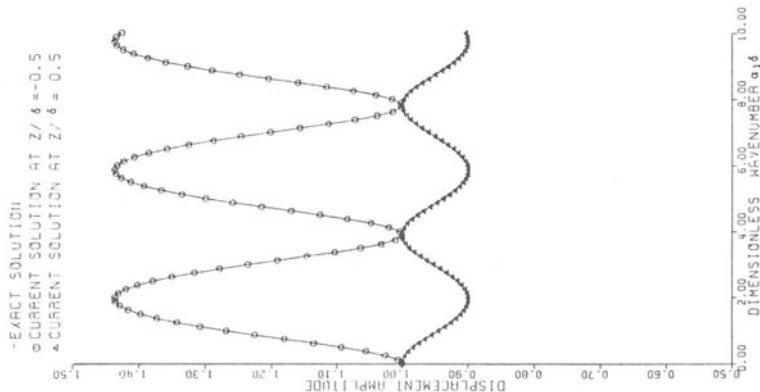


Fig. 3. Displacement amplitude as a function of  $\alpha\delta$  for the three-layered problem, Al in Ge.

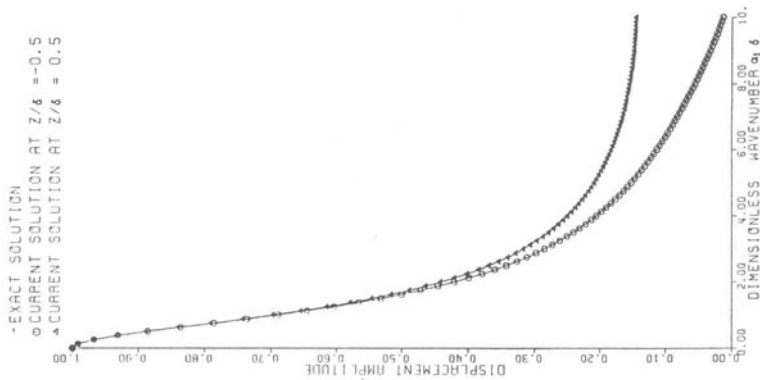


Fig. 4. Displacement amplitude as a function of  $\alpha\delta$  for the three-layered problem, Be in Polyethylene.

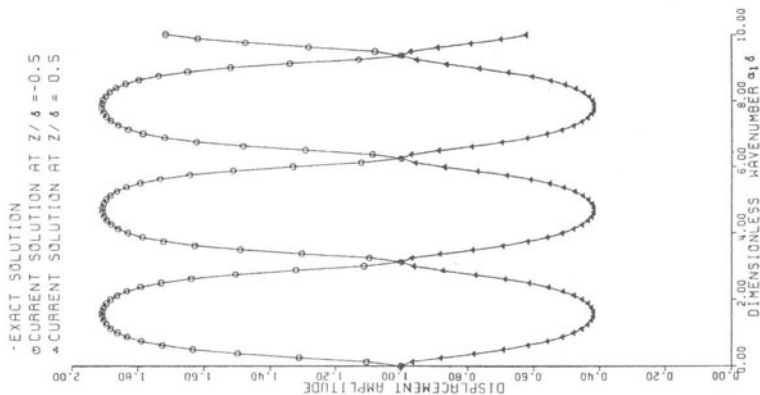


Fig. 5. Displacement amplitude as a function of  $\alpha\delta$  for the three-layered problem, Mg in Stainless Steel.

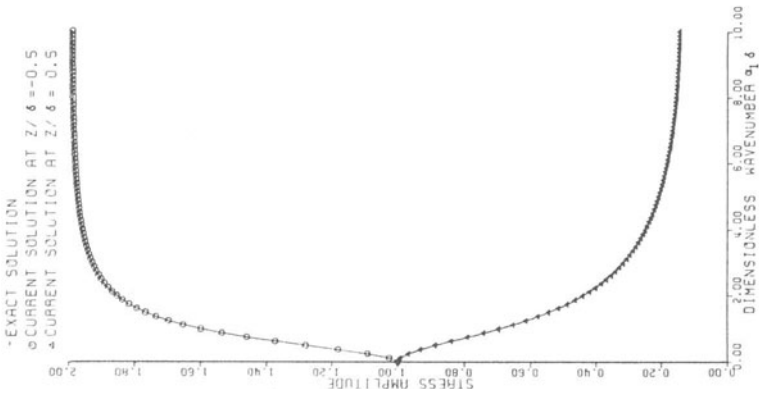


Fig. 6. Stress amplitude as a function of  $\alpha_1 \delta$  for the three-layered problem, Be in Polyethylene.

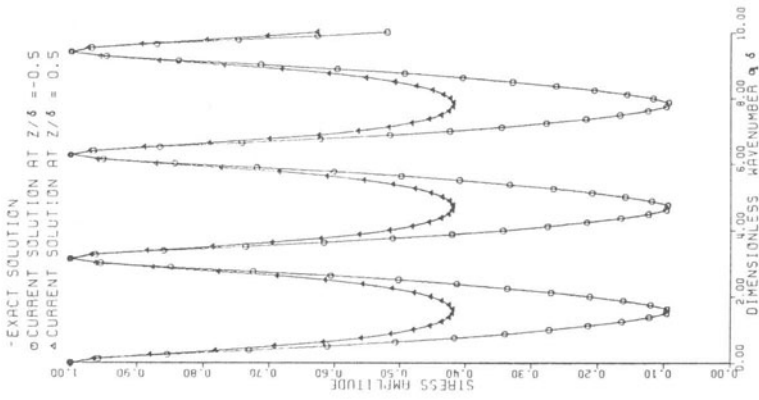


Fig. 7. Stress amplitude as a function of  $\alpha_1 \delta$  for the three-layered problem, Mg in Stainless Steel.

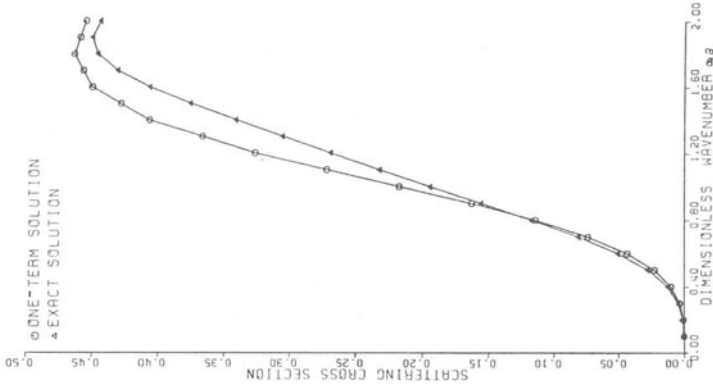


Fig. 8. Scattering cross section as a function of  $\alpha_1 \delta$  for the spherical inhomogeneity problem, Al in Ge.

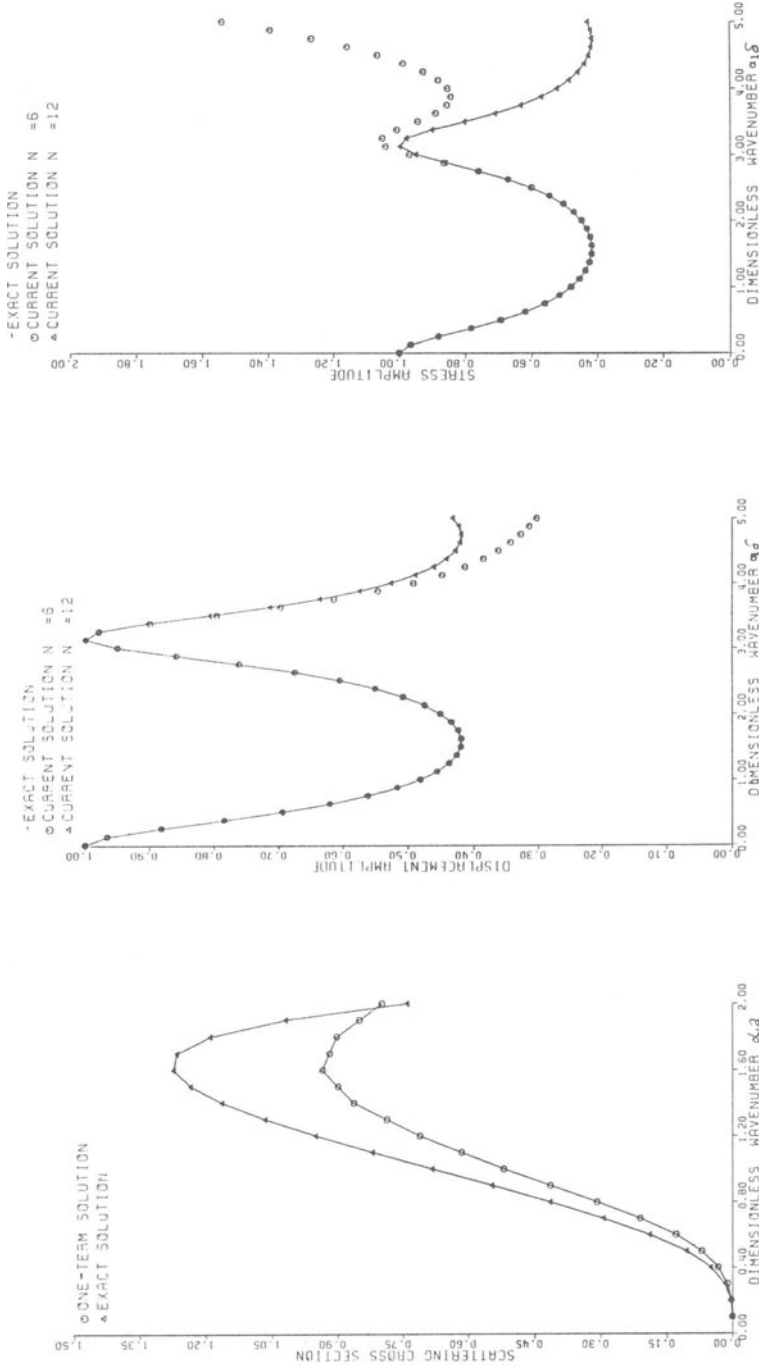


Fig. 9. Scattering cross section as a function of  $\alpha_1 \delta$  for the spherical inhomogeneity problem, Mg in Stainless Steel.

Fig. 10. Displacement amplitude at the point  $z/\delta=0.5$  as a function of  $\alpha_1 \delta$  for the three-layered problem, Stainless Steel in Mg.

Fig. 11. Stress amplitude at the point  $z/\delta=0.5$  as a function of  $\alpha_1 \delta$  for the three-layered problem, Stainless Steel in Mg.



## ACKNOWLEDGEMENT

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