ERRATUM: SINGULARITY FORMATION IN CHEMOTAXIS—A CONJECTURE OF NAGAI

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Abstract. In [H. A. Levine and J. Rencławowicz, SIAM J. Appl. Math., 65 (2004), pp. 336–360] we considered the problem $u_t = u_{xx} - (uv_x)_x, v_t = u - av$ on the interval $I = [0, 1]$, where $u_x, v_x = 0$ at the end points, $u(x, 0), v(x, 0)$ are prescribed, and $a > 0$. (It was claimed in that article that there were solutions that blow up in finite time in every neighborhood of the spatially homogeneous steady state $(u, v) = (\mu, \mu/a)$ if $\mu > a$.) Here we correct an estimate and reduce Nagai’s conjecture to the following statement. Let $\sigma = a/(\mu - a), \rho_1 = 1$. If $\lim_{n \to +\infty} \rho_n$ exists, where for $n \geq 2$, $\rho_n^2 \equiv 1/(n - 1)\sum_{j=1}^{n-1} (1 + \sigma/j)\rho_j n_{n-j}$, then the blow up assertion holds.

Key words. chemotaxis, finite time singularity formation, Keller–Segel model

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1. Introduction. In [1] we studied the system $u_t = u_{xx} - (uv_x)_x, v_t = u - av$ on the interval $I = [0, 1]$, where $u_x, v_x = 0$ at the end points, $u(x, 0), v(x, 0)$, are prescribed, and $a > 0$. Nagai and Nakaki [2] showed that there are solutions that are unbounded in finite or in infinite time. We claimed that there were initial conditions for which solutions failed to exist for all time. In our proof we used a differential inequality, the derivation of which was unfortunately flawed. We correct this and make more precise the statement proved in [1].

2. Approximate solution. The notation of [1] is in force here. Because system $u_t = u_{xx} - (uv_x)_x, v_t = u - av$ is autonomous, we can assume the initial values are prescribed at $t = 0$, and that the blow up time, when it exists, is positive. As in [1], define, for any sequence $z(t) = \{z_n(t)\}_{n=1}^{\infty}, G_n(z, z') = (1/2)C^2n((Mz, z')_n + n^{\sigma/2}(z \ast z)_n)$ and $H_n(z, z') = (1/2)C^2n[(T_nMz, z') - (Mz, T_nz')] + an(z, T_nz')$, where $Mz(t) = \{nz_n(t)\}_{n=1}^{\infty}$ and $T_nz(t) = \{z_n+h(t)\}_{n=1}^{\infty}$. Here $|z| = \{z_n\}_{n=1}^{\infty}$ and $(z \ast w)_n = \sum_{k=1}^{n-1} z_kw_{n-k}$. (The sum is zero if $n = 1$.)

The infinite system of ordinary differential equations for the cosine coefficients $h(t) = \{h_n(t)\}_{n=1}^{\infty}$ is

$$\mathcal{L}_nh_n = h''_n + (C^2n^2 + a)h'_n - (\mu - a)C^2n^2h_n = G_n(h, h') + H_n(h, h').$$

The infinite system of ordinary differential equations satisfied by the cosine coefficients for the approximate problem, $g(t) = \{g_n(t)\}_{n=1}^{\infty}$, satisfies $\mathcal{L}_ng_n = G_n(g, g')$. The

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The Nagai conjecture states that if $\mu > a$, there are spatially nonhomogeneous solutions beginning in every small neighborhood of $(\mu, \mu/a)$ which cannot exist for all time.

The spatially homogeneous solution is given by $V(t) = \mu/a + (v_0 - \mu/a)\exp(-at), U(t) = \mu$. One sets $\psi(x, t) = v(x, t) - V(t), u(x, t) = \mu + \psi + a\psi$. Then $h(t)$ is the sequence of cosine coefficients for $\psi(x, t)$. 361
particular sequence \( g(t) \equiv \{a_n(t) = a_n e^{n\lambda t}\}_{n=1}^\infty \) satisfies this system for \( a_1 > 0 \), and for \( n \geq 2 \) and any integer \( M > 0 \) with \( C = 2\pi M, \mu > a \) if

\[
(2.1) \quad 2\lambda|n - a/(4\pi^2M^2)|a_n = \frac{1}{n-1} \sum_{k=1}^{n-1} [\lambda(n-k)k + ak]a_{n-k},
\]

where \( \lambda \) is the positive root of \( \lambda^2 + (4\pi^2M^2 + a)\lambda - (\mu - a)4\pi^2M^2 = 0 \). There are positive constants \( a, b, \epsilon, \delta \) with \( ae^n \leq na_n \leq b\epsilon^n \) for all positive integers \( 1 \). From this, it follows that \( \liminf_{n \to \infty}(-\ln na_n)/(\ln \lambda) = T_b \) and \( \limsup_{n \to \infty}(-\ln na_n)/(\ln \lambda) = T_b \). Hence there is a subsequence \( \{a_{n_k}\}_{k=1}^\infty \) such that \( \lim_{k \to \infty}(-\ln n_k a_{n_k})/(n_k \lambda) = T_b \). For this sequence, \( \lim_{k \to \infty} n_k a_{n_k} \exp(n_k \lambda T_b) = 1 \). Set \( a_n = (A_n/n) \exp(-n\lambda T_b) \). On the subsequence, \( A_{n_k} \to 1 \) and

\[
(2.2) \quad \lim_{t \to T_b} \sum_{k=1}^\infty A_{n_k} e^{-n_k \lambda(T_b - t)} = +\infty \quad \text{and} \quad \lim_{t \to T_b} \sum_{k=1}^\infty \frac{A_{n_k} e^{-n_k \lambda(T_b - t)}}{n_k^{1+\delta}} < +\infty
\]

(for any \( \delta > 0 \)).

Now \( T_b \) must be the blow up time for the approximate solution \( g(t) \) in the space \( \ell^1_T(0, T_b) \times \ell^1_T(0, T_b) \). (A sequence \( \{a_n\} \) is in \( \ell^1 \) if \( \{na_n\} \) is in \( \ell^1 \).) To see this, note that as long as \( t \) is in the existence interval,

\[
\|Mg(t)\|_{\ell^1} + \|g'(t)\|_{\ell^1} = \sum_{n=1}^\infty na_n(1 + \lambda)e^{n\lambda t} \geq (1 + \lambda) \sum_{k=1}^\infty n_k a_{n_k} e^{n_k \lambda t}
\]

\[
(2.3) \quad = (1 + \lambda) \sum_{k=1}^\infty A_k e^{-n_k \lambda(T_b - t)}.
\]

Consequently, from the first equation in (2.2), \( g(\cdot) \) must blow up at some time, possibly earlier than \( T_b \). If \( t < T_b \), then \( \liminf_{n \to \infty}(-\ln na_n)/(\ln \lambda) = T_b > T_b - \delta > t \) for some positive \( \delta \). Therefore, for sufficiently large \( N \),

\[
\sum_{n=1}^{\infty} na_n e^{n\lambda t} = \sum_{n=1}^{\infty} a_n e^{n\lambda t} \leq \sum_{n=1}^{\infty} n e^{-n\lambda(T_b - \delta - t)} < \infty.
\]

Set \( s = a/\lambda \). Let \( \{\ln[a_n/2(a_n^2)]/n\}_{n=1}^\infty = \{\ln A_n/n\}_{n=1}^\infty \equiv \{p_n/n\}_{n=1}^\infty \equiv \{p_n\}_{n=1}^\infty \). The \( p_n \) satisfy \( p_1 = -\ln 2 \), and for \( n \geq 2 \),

\[
(1 - a/(4\pi^2M^2n))e^{p_n} = \frac{1}{n-1} \sum_{j=1}^{n-1} (1 + \sigma/j) e^{(p_j + p_{n-j})},
\]

Then we have the following theorem.

**Theorem 1** (Nagai’s conjecture). Let \( \lim_{n \to \infty} \frac{p_n}{n} \) exist. The corresponding solution of the Nagai problem for which \( h_n(0) = g_n(0) \) and \( h_n'(0) = g_n'(0) \) for all \( n \) cannot both exist and be \( \ell^1 \) regular on \( [0, \infty) \). (A solution of the Nagai–Nakaki problem is \( \ell^1 \) regular on an interval \( I = [0, T_b) \) if it exists there and if \( \|Mh(s)\|_{\ell^1} + \|h'(s)\|_{\ell^1} \) is uniformly bounded on compact subsets \( I \).)

3. **Estimate.** Inequality (7.5) of [1] is incorrect. The correct form of the upper bound for the norm of \( g - h = w \), \( \|Mw(t)\|_{\ell^1} + \|w'(t)\|_{\ell^1} \), is based on the following (finite) system of ordinary differential equations:

\[
(3.1) \quad \mathcal{L}_n w_n = \mathcal{G}_n(h - g, h') + \mathcal{G}_n(g, h' - g') + \mathcal{H}_n(h, h') = \mathcal{G}_n(w, h') + \mathcal{G}_n(g, w') + \mathcal{H}_n(h, h')
\]

and, for some \( B > 0 \) depending perhaps on \( \tau \) but not on \( w, w', h, h', g, g' \), is given by
(3.2)\[
\|Mw(t)\|_{\ell^1} + \|w'(t)\|_{\ell^1} \leq I(t) + J(t) + B \int_0^t \left( \frac{\|Mh(s)\|_{\ell^1} + \|h'(s)\|_{\ell^1}}{\sqrt{t-s}} \right) ds
\]
\[
+ B \int_0^t \left( \frac{\|Mw(s)\|_{\ell^1} + \|w'(s)\|_{\ell^1}) (\|Mh(s)\|_{\ell^1} + \|h'(s)\|_{\ell^1}) ds, \right.
\]
where
\[
I(t) + J(t) = \int_0^t \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} n(n-k)|g_k||w_{n-k}|e^{-dn^2(t-s)} ds
\]
\[
\leq \int_0^t \sum_{k=1}^{\infty} |g_k|e^{-(d/2)k^2(t-s)} \left( \sum_{n=1}^{\infty} |w_n| e^{-d(n+k)^2(t-s)} \right) ds
\]
\[
\leq c \int_0^t \sum_{k=1}^{\infty} A_k e^{-(d/2)k^2(t-s) - \lambda k(T_b-t)} \|Mw(s)\|_{\ell^1} \frac{ds}{\sqrt{t-s}}
\]
\[
\leq c \int_0^t \left\{ \sum_{k=1}^{\infty} A_k e^{-(d/2)k^2 + k\lambda(t-s)} \right\} \|Mw(s)\|_{\ell^1} \frac{ds}{\sqrt{t-s}} \equiv c \int_0^t W(t-s) \frac{\|Mw(s)\|_{\ell^1}}{\sqrt{t-s}} ds.
\]
In the same manner,
\[
J(t) = \int_0^t \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} n^2 |g_k||w_{n-k}|e^{-dn^2(t-s)} ds
\]
\[
\leq \int_0^t \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (n+k)^2 |g_k||w_n| e^{-d(n+k)^2(t-s)} ds
\]
\[
\leq \int_0^t \sum_{k=1}^{\infty} |k| g_k e^{-(d/2)k^2(t-s)} \left( \sum_{n=1}^{\infty} \frac{(n+k)^2}{kn} e^{-(d/2)(n+k)^2(t-s)} |w_n| \right) ds.
\]
From the inequality \((k+l)/kl \leq 2\),
\[
J(t) \leq c' \int_0^t \left\{ \sum_{k=1}^{\infty} A_k e^{-(d/2)k^2 + k\lambda(t-s)} \right\} \|Mw(s)\|_{\ell^1} \frac{ds}{\sqrt{t-s}} \equiv c' \int_0^t W(t-s) \frac{\|Mw(s)\|_{\ell^1}}{\sqrt{t-s}} ds.
\]
In view of (2.2), \(\lim_{t \to T_b} \sum_{k=1}^{\infty} A_k k^{-2} e^{-k\lambda(T_b-t)} < +\infty\). Thus \(W(t)\) is in every \(L^p[0,T_b]\) space for \(1 \leq p < \infty\). With \(f(t) = \|Mw(t)\|_{\ell^1} + \|w'(t)\|_{\ell^1}\), we see \(f(t) \leq \int_0^t W(t-s) f(s)/\sqrt{t-s} ds + \Phi(h(t))\). From Hölder’s inequality with \(1/p + 1/r + 1/q = 1\).
and $1 < r < 2$, there is a constant $K > 0$ such that $f(t) \leq K \left[ \int_0^t f(s)^q \, ds \right]^{1/q} + \Phi(h(t))$ on $[0, T_b)$. From Gronwall’s inequality, if $h$ is global, $f(t)$ is bounded on $[0, T_b)$. From the first sum in (2.2) and the triangle inequality, this is impossible.

Other minor errors in [1]. Page 345, equation (5.1): Replace $k(\rho_n + h_k w_n + h_k w_n + h_k w_n)$ by $n(\rho_n + h_k w_n + h_k w_n)$. Page 349, equation in line 13: $c\sqrt{t}$ should be replaced by $c\sup[0,T] \sqrt{t}$.

REFERENCES
