1981

The Quenching of Solutions of Semilinear Hyperbolic Equations

Peter H. Chang
University of Nebraska at Omaha

Howard A. Levine
Iowa State University, halevine@iastate.edu

SIAM J. MATH. ANAL.
Vol. 12, No. 6, November 1981

0036-1410/81/1206-0010 $01.00/0

THE QUENCHING OF SOLUTIONS OF SEMILINEAR HYPERBOLIC EQUATIONS*

PETER H. CHANG† AND HOWARD A. LEVINE‡

Abstract. We consider the problem $u_{tt} = u_{xx} + \phi(u(x, t))$, $0 < x < L$, $t > 0$; $u(0, t) = u(L, t) = 0$; $u(x, 0) = u_t(x, 0) = 0$. Assume that $\phi : (-\infty, A) \to (0, \infty)$ is continuously differentiable, monotone increasing, convex, and satisfies $\lim_{u \to A^-} \phi(u) = +\infty$. We prove that there exist numbers L_1 and L_2, $0 < L_1 \leqq L_2$ such that if $L > L_2$, then a weak solution u (to be defined) quenches in the sense that u reaches A in finite time; if $L < L_1$, then u does not quench. We also investigate the behavior of the weak solution for small L and establish the local (in time) existence of u.

1. Introduction. In [3], Kawarada investigated the following nonlinear initial boundary value problem:

$$\text{(P)}\qquad \begin{aligned} &u_t = u_{xx} + \frac{1}{1-u}, && 0 < x < L, \quad t > 0, \\ &u(0, t) = u(L, t) = 0, && t > 0, \\ &u(x, 0) = 0, && 0 \leqq x \leqq L. \end{aligned}$$

There, he established the following interesting results:

(A) If $L > 2\sqrt{2}$, then $u(L/2, t)$ reaches one in finite time.

(B) If $u(L/2, t)$ reaches one in finite time, then $u_t(L/2, t)$ is unbounded in finite time.

Whenever (B) occurs, Kawarada says that *u quenches in finite time.* We shall say that u quenches if (A) occurs. This is a weaker definition than Kawarada's.

In [1] and independently in [6] it was established that there is a number L_0 such that if $L > L_0$, u quenches in finite time while if $L < L_0$, u tends monotonically to the smaller of the two solutions of the stationary problem

$$f''(x) + \frac{1}{1-f(x)} = 0, \qquad 0 < x < L,$$
$$f(0) = f(L) = 0.$$

In [6] it was also shown that this latter situation also obtains at $L = L_0$, where the two stationary solutions coalesce into a single stationary solution. The number L_0 can be found exactly, in fact $L_0 \cong 1.5307 \cdots$. These papers also included extensions to more general nonlinear parabolic problems where the nonlinear term has the same qualitative properties as $1/(1-u)$ (convex, positive, monotone increasing and singular at the right endpoint of $(-\infty, a)$.) The principal tools employed there were the maximum principle and the various comparison theorems derived from it.

* Received by the editors April 2, 1980; and in final form December 12, 1980. Funds for numerical computation provided by the Science and Humanities Research Institute of Iowa State University.

† Department of Mathematics, University of Nebraska at Omaha, Omaha, Nebraska 68182.

‡ Department of Mathematics, Iowa State University, Ames, Iowa 50011. The work of this author was supported in part by the National Science Foundation under Grant MCS 78-02729.

Motivated by the preceding remarks, we were led to examine the analogous problem for the wave equation. That is, we studied the problem

$$\text{(W)}\qquad \begin{aligned} &u_{tt} = u_{xx} + \frac{1}{1-u}, && 0<x<L, \quad t>0, \\ &u(0, t) = u(1, t) = 0, && t>0, \\ &u(x, 0) = u_t(x, 0) = 0, && 0 \leqq x \leqq L. \end{aligned}$$

Although we do not have any physical application in mind, we believe the study of problem (W) to be of theoretical interest. Since parabolic equations are in some sense on the borderline between elliptic and hyperbolic equations, it is of interest to know which of the properties of their solutions are possessed by solutions of the other two types of equations and what form the properties take in these cases. For example, the maximum principle for parabolic equations has a stronger version for elliptic equations and a much weaker version for hyperbolic equations. See [9] and references therein.

The first result we obtained on this problem is contained in Theorem 3.2. For this problem it says that if $L > L_1 \cong 1.418 \cdots$, then u quenches (reaches one) in finite time. Since $L_1 < L_0$, we conjectured that for any $L>0$, u must quench in finite time. However, when (W) was solved numerically for small L, the results obtained seemed to contradict this conjecture.

Guided by the numerical results, we were able to show that if $L < 1.238$, then $u \leqq 0.7732$ for all time. That is, if L is small, u cannot quench, even in infinite time. This result is contained in Theorem 4.1.

Because for problem (W) we do not have as useful a maximum principle available, the arguments we use are much different than those used for the parabolic problem.

Rather than studying problem (W), we treat the somewhat more general problem (W′).

$$\text{(W}'\text{)}\qquad \begin{aligned} &u_{tt} = u_{xx} + \varepsilon\varphi(u(x, t)), && 0<x<1, \quad t>0, \quad \varepsilon>0. \\ &u(0, t) = u(1, t) = 0, && t>0, \\ &u(x, 0) = u_t(x, 0) = 0, && 0 \leqq x \leqq 1, \end{aligned}$$

which reduces to (W) when $\varepsilon = L^2$ and $\varphi(u) = 1/(1-u)$, after a change of variables. Here $\varphi: (-\infty, A) \to (0, \infty)$ is continuously differentiable, monotone increasing, convex and satisfies

$$\lim_{u \to A^-} \varphi(u) = +\infty.$$

The solution $u(x, t; \varepsilon)$ for fixed $\varepsilon > 0$, is shown to exist in the weak sense (defined precisely later) on the largest set $[0, 1] \times [0, T)$, where $|u| < A$. If $T = +\infty$, we say that u is a global solution. If $T < \infty$, then $\sup\{u(x, t): (x, t) \in [0, 1] \times [0, T)\} = A$ and we say u quenches (reaches A) in finite time. If $T = +\infty$, and this supremum is A, u quenches in infinite time. Thus, if u does not quench at all, $u \leqq A(1-\delta)$ for some $\delta \in (0, 1)$, on the half strip.

We then summarize our results for (W′) as follows: There exist two numbers ε_1, ε_2, $0 < \varepsilon_1 \leqq \varepsilon_2 < +\infty$ such that if $\varepsilon < \varepsilon_1$, $u(x, t; \varepsilon)$ (the solution of (W′)) cannot quench. If $\varepsilon > \varepsilon_2$ then $u(x, t; \varepsilon)$ quenches in finite time. We do *not* prove $\varepsilon_1 = \varepsilon_2$, although we believe this to be the case. The numerical results indicate that this is so for (W) and that $L_1 = \sqrt{\varepsilon_1} = \sqrt{\varepsilon_2} \cong 1.365 \cdots$. Also, we believe that if $\varepsilon = \varepsilon_1 = \varepsilon_2$, then u quenches in infinite time.

The plan of the paper is as follows: In § 2 we define the notion of a weak solution which we shall use in the sequel. We establish local existence there also. In § 3 we show that if ε is "large" u quenches in finite time whereas in § 4 we show that if ε is "small", u cannot quench at all, even in infinite time. In § 5 we discuss the behavior of u as $\varepsilon \to 0^+$. We conclude with some remarks in the final section.

2. The definition of a weak solution. We say u is a weak solution of (W′) on $D_T \equiv (0,1) \times (0, T)$ if:

(i) u is continuous in $\bar{D}_T$ and satisfies the initial and boundary conditions there.

(ii) $|u| \leqq A(1-\delta)$ on $\bar{D}_T$.

(iii) u has weak derivatives u_x, u_t on D_T and for all $t \in (0, T)$, $u_x, u_t \in L^2(0,1)$.

(iv) For any function $\psi(x,t) \in C^2(\bar{D}_T)$ satisfying the boundary conditions and $0 \leqq t \leqq T$,

$$\begin{aligned}\int_0^1 \psi(x,t)u_t(x,t)\,dx &= \int_0^t \int_0^1 [\psi_\tau(x,\tau)u_\tau(x,\tau) - \psi_x(x,\tau)u_x(x,\tau)]\,dx\,d\tau \\ &\quad + \varepsilon \int_0^t \int_0^1 \psi(x,\tau)\varphi(u(x,\tau))\,dx\,d\tau. \end{aligned} \tag{2.1}$$

(v) The total energy associated with (W′) is conserved, i.e.,

$$E_T(t) = \frac{1}{2}\int_0^1 (u_x^2 + u_t^2)\,dx - \varepsilon \int_0^1 \int_0^{u(x,t)} \varphi(\eta)\,d\eta\,dx = E_T(0) = 0. \tag{2.2}$$

We next examine the question of the local (in time) existence of the weak solution defined above. The singular value of the nonlinearity and the consequent restriction $|u| \leqq A(1-\delta)$ on D_T prevent straightforward application of Reed [10, Thm. 1, p. 5], because the nonlinearity is now not defined on the domain of d^2/dx^2. Nevertheless, a local existence theorem of the desired kind can be obtained from the contraction mapping principle used for hyperbolic systems, to be found in Garabedian [11, p. 110]. One still has to deal with the strange nonlinearity and the boundary conditions however.

We proceed as follows: Let $\delta \in (0,1)$ be fixed. Consider the problem (W′) with nonzero initial data

$$\text{(W'')}\qquad \begin{aligned} &u_{tt} = u_{xx} + \varepsilon\varphi(u(x,t)), && 0<x<1, \quad T \geqq t > 0, \quad \varepsilon > 0, \\ &u(0,t) = u(1,t) = 0, && T \geqq t > 0, \\ &u(x,0) = u_0(x), \\ &u_t(x,0) = v_0(x), \end{aligned}$$

where $u_0, v_0 \in C^1(0,1)$ and $u_0(0) = u_0(1) = 0$. Letting $\|\cdot\|_\infty$ denote the sup norm of a function of x, we assume that

$$(*)\qquad \|u_0\|_\infty + T\|v_0\|_\infty < A(1-2\delta).$$

Define u, u_0, v_0 by odd periodic (with period two) reflection (in x) on $R^1 \times [0, T]$. Define the following function

$$F: R^1 \times [0, \infty) \times (-A, A) \to R^1,$$

by

$$F(x,t,u) = \begin{cases} \varphi(u), & x \in [2n, 2n+1), \\ \varphi(-u), & x \in [2n-1, 2n), \end{cases} \quad n = 0, \pm 1, \pm 2, \cdots.$$

Then by standard arguments, u solves (W'') if and only if u solves, on $R^1\times[0, T)$, the integral equation

$$u(x,t)=u_1(x,t)+\frac{\varepsilon}{2}\int_0^t\int_{x-t+\eta}^{x+t-\eta}F(\xi,\eta,u(\xi,\eta))\,d\xi\,d\eta, \tag{2.3}$$

where

$$u_1(x,t)=\frac{1}{2}(u_0(x+t)+u_0(x-t))+\frac{1}{2}\int_{x-t}^{x+t}v_0(\sigma)\,d\sigma.$$

Clearly, if (*) holds,

$$|||u_1|||_\infty\equiv\sup_{0\leqq t\leqq T}\|u_1(t)\|_\infty<A(1-2\delta).$$

Let B_τ be the Banach space of odd (in x) continuous functions on $R^1\times[0,\tau]$, which vanish on the lines $x=n$, n an integer, and are of period two in x. Let $\bar{B}(u_1, A\delta)$ denote the closed ball of radius $A\delta$ in this Banach space. (Note that $u_1\in B_\tau$.) Define

$$\tilde{T}:\bar{B}(u_1,A\delta)\to B_\tau,$$

by

$$(\tilde{T}u)(x,t)=u_1(x,t)+\frac{\varepsilon}{2}\int_0^t\int_{x-t+\eta}^{x+t-\eta}F(\xi,\eta,u(\xi,\eta))\,d\xi\,d\eta.$$

In view of the definition of F, this map is well defined. It is then easy to check that

$$|||\tilde{T}u-u_1|||_\infty<A\delta, \tag{2.4}$$

$$|||\tilde{T}u-\tilde{T}v|||_\infty<\lambda\,|||u-v|||_\infty,\qquad 0<\lambda<1 \tag{2.5}$$

for $u, v\in B(u_1, A\delta)$ provided

$$\tau<\min\left\{T,\left(\frac{2}{\varepsilon}\right)^{1/2}[\delta A/\varphi((1-\delta)A)]^{1/2},\left(\frac{2}{\varepsilon}\right)^{1/2}[\varphi'((1-\delta)A)]^{-1/2}\right\},$$

so that $\tilde{T}:B(u_1,A\delta)\to B(u_1,A\delta)$ and is a contraction. Thus $\tilde{T}$ has a unique fixed point. This establishes the following theorem.

THEOREM 2.1. *A weak* (C^1) *solution of* (W') *exists on* D_T *if* T *is sufficiently small, for any* $\varepsilon>0$. *The solution is piecewise* C^2 *in* D_T *and* (2.1) *and* (2.2) *hold there. Furthermore, if* u *exists on* D_T *and* $|u|\leqq A(1-\delta)$ *on* $\bar{D}_T$, *then* u *may be continued to* $D_{T+\tau}$ *for* τ *sufficiently small* (*and positive*).

It can be shown that the solution of (2.3) is regular enough that (2.1) and (2.2) hold when $u_0\equiv v_0\equiv 0$. In this case, from (2.3), one easily calculates

$$u_x(x,t)=\frac{\varepsilon}{2}\left(\int_x^{x+t}F(\sigma,x+t-\sigma,u(\sigma,x+t-\sigma))\,d\sigma\right.$$
$$\left.-\int_{x-t}^{x}F(\sigma,\sigma-x+t,u(\sigma,\sigma-x+t))\,d\sigma\right),$$

$$u_t(x,\sigma)=\frac{\varepsilon}{2}\left(\int_x^{x+t}F(\sigma,x+t-\sigma,u(\sigma,x+t-\sigma))\,d\sigma\right.$$
$$\left.+\int_{x-t}^{x}F(\sigma,\sigma-x+t,u(\sigma,\sigma-x+t))\,d\sigma\right),$$

so that, because $F(x, t, u)$ is piecewise continuous, u_x and u_t are continuous everywhere and differentiable in x and t except on the lines $x = n$, $x + t = n$ or $x - t = n$, where n is an integer. In fact, except on this point set,

$$u_{xx}(x, t) = \varepsilon[F(x+t, 0, 0) + F(x-t, 0, 0)] - \varepsilon F(x, t, u(x, t)) + I,$$

$$u_{tt}(x, t) = [F(x+t, 0, 0) + F(x-t, 0, 0)] + I,$$

$$u_{xt}(x, t) = u_{tx}(x, t) = \frac{\varepsilon}{2}[F(x+t, 0, 0) + F(x-t, 0, 0)] + J$$

and piecewise continuous, where

$$\begin{aligned} I &\equiv \frac{\varepsilon}{2}\int_x^{x+t} F_3(\sigma, x+t-\sigma, u(x+t-\sigma))u_2(\sigma, x+t-\sigma)\, d\sigma \\ &\quad + \frac{\varepsilon}{2}\int_{x-t}^{x} F_3(\sigma, \sigma-x+t, u(\sigma-x+t))u_2(\sigma, \sigma-x+t)\, d\sigma \\ &\equiv I_1 + I_2, \\ J &\equiv I_1 - I_2. \end{aligned}$$

Thus the solution is classical except on

$$\{(x, t) \in R^1 \times [0, \tau] \mid x, x-t, x+t \text{ are integers}\}.$$

It is then an easy matter to show, using care when integrating across jumps in u_{xx}, u_{xt}, u_{tt}, that (2.1) and (2.2) hold.

3. Nonexistence or quenching for large ε. The results here are analogous to those in [5]. We repeat them here for completeness and because the class of nonlinearities here is different from that considered in [5].

Throughout this section, $\varphi : (-\infty, A) \to (0, \infty)$ satisfies the following conditions:

(a) $\varphi > 0$, $\varphi' \geqq 0$, φ is convex;

(b) $\lim_{u \to A^-} \varphi(u) = +\infty$.

Let

$$\Phi(x) = \int_0^x \varphi(s)\, ds \tag{3.1}$$

and

$$H(x) = -\pi^2 \frac{x^2}{2} + \varepsilon \Phi(x), \qquad -\infty < x < A. \tag{3.2}$$

For H, we suppose that $H(x) > 0$ on $(0, A)$ and $\lim_{x \to A^-} H(x) > 0$.

LEMMA 3.1. *Under the above hypothesis*

$$\infty > \int_0^A [H(\sigma)]^{-1/2}\, d\sigma.$$

This is clear since $H(\sigma) = \varepsilon\varphi(0)\sigma + O(\sigma^2)$ for σ small and positive and bounded away from zero near $\sigma = A$.

THEOREM 3.2. *If* $\varepsilon > 0$ *is such that the above holds for* $H(x)$ *and* φ *satisfies* (a), (b) *above, then a weak solution of* (W′) *must quench in finite time.*

Proof. Assume the contrary: that $|u|<A$ for $(x, t)\varepsilon[0, 1]\times[0, \infty)$. Define

$$F(t)\equiv\frac{\pi}{2}\int_0^1 \sin(\pi x)u(x, t)\,dx; \tag{3.3}$$

the choice $\psi(x, t)=t\sin\pi x$ in (2.1) yields

$$\begin{aligned} tF'(t)&=\frac{t\pi}{2}\int_0^{\pi} \sin(\pi x)u_t(x, t)\,dx \\ &=\frac{\pi}{2}\int_0^t\int_0^1 [\sin(\pi x)u_\eta(x, \eta)-\pi\eta\cos(\pi x)u_x(x, \eta)]\,dx\,d\eta \qquad (3.4)\\ &\quad+\frac{\pi\varepsilon}{2}\int_0^t \eta\int_0^1 \sin(\pi x)\varphi(u(x, \eta))\,dx\,d\eta. \end{aligned}$$

Thus $tF'(t)$ is differentiable and hence so is $F'(t)$. Therefore

$$tF''(t)+F'(t)=F'(t)-\frac{t\pi^2}{2}\int_0^1 \cos(\pi x)u_x(x, t)\,dx+t\frac{\pi\varepsilon}{2}\int_0^1 \sin(\pi x)\varphi(u(x, t))\,dx,$$

so that, after integration by parts,

$$F''(t)=-\pi^2F(t)+\frac{\varepsilon\pi}{2}\int_0^1 \sin(\pi x)\varphi(u(x, t))\,dx.$$

The use of Jensen's inequality yields

$$F''(t)\geqq-\pi^2F(t)+\varepsilon\varphi(F(t))\equiv H'(F(t)).$$

Since $F''(0)\geqq\varepsilon\varphi(0)>0$ and $F(0)=F'(0)=0$, we have $F'(t)>0$ and $F(t)>0$ on some interval $(0, \eta)$. Therefore, on this interval

$$\tfrac{1}{2}(F'(t))^2\geqq H(F(t)).$$

From this it follows that F' cannot change sign so that $F(t)\in(0, A)$ for all $t\in[0, \infty)$ and thus

$$\int_0^A [H(\sigma)]^{-1/2}\,d\sigma\geqq\sqrt{2}t$$

for all t which is a contradiction.

Notice that we are invoking part of Theorem 2.1 here, to the effect that if $|u|\leqq A(1-\delta)$ on $[0, 1]\times[0, T]$, then u can be continued as a weak solution on $[0, 1]\times[0, T+t']$ with $|u|\leqq A(1-\delta')(\delta'<\delta)$ and $t'>0$.

Example. For the above problem, we take $\varphi(u)=(1-u)^{-\beta}$, $\beta>0$, $|u|<1$. It is easy to verify that $H(x)>0$ on $(0, 1)$ if either

(i) $\varepsilon\geqq\dfrac{\pi\beta^\beta}{(\beta+1)^{\beta+1}}$

or

(ii) if $\varepsilon<\pi^2\beta^\beta/(\beta+1)^{\beta+1}$ and $(1-\beta)^{-1}\{1-(\pi^2/\varepsilon)x_0+(\pi^2/\varepsilon)((1+\beta)/2)x_0^2\}>0$, where x_0 is the larger root of $x_0(1-x_0)^\beta=\varepsilon/\pi^2$ if $\beta\neq1$ or $\ln(1/(1-x_0))-\pi^2x_0^2/2\varepsilon>0$, where x_0 satisfies $x_0/1-x_0)=\varepsilon/\pi^2$ and is the larger root if $\beta=1$. (This amounts to showing H is positive at a local minimum.)

Since, for $\beta=1$, $x_0(1-x_0)^\beta=\varepsilon/\pi^2$ is readily solved for x_0, we have the following corollary.

COROLLARY 3.3. *If* $\varphi(u)=(1-u)^{-1}$, *then u reaches one in finite time if*

$$\varepsilon > \frac{2\pi^2\theta_0}{(1+2\theta_0)^2} = \varepsilon_2,$$

where $e^{\theta_0}=1+2\theta_0$ *and* $x_0 = 1-e^{-\theta_0}(\theta_0 \cong 1.25643$ *and* $L_2=\sqrt{\varepsilon_2}\cong 1.41766)$.

It is of interest to note that the larger β is, the wider is the range of ε's for which quenching in finite time must occur.

4. Global existence. The energy equation (2.2) can be written as

$$\frac{1}{2}\int_0^1 u_x^2\,dx + \frac{1}{2}\int_0^1 u_t^2\,dx - \varepsilon\int_0^1 \Phi(u(x,t))\,dx = 0, \tag{4.1}$$

where Φ is given by (3.1). We write, for φ as in § 3,

$$\Phi(u) = \int_0^u \varphi(\eta)\,d\eta = \varphi(0)u + \frac{u^2}{2}\psi(u). \tag{4.2}$$

Here ψ will be singular at $u=A$ if and only if $\varphi \notin L^1(0,A)$.

THEOREM 4.1. *For a weak solution u of* (W′) *over* D_T, *if there is* $\delta\in(0,1]$ *such that*

$$\varepsilon < A(1-\delta)\frac{\pi^2}{\pi\varphi(0)+A(1-\delta)M_\delta} \equiv \mathcal{F}(\delta), \tag{4.3}$$

where $M_\delta = \sup_{|u|\leqq A(1-\delta)} \psi(u)$, *where* $0<\delta\leqq 1$, *then*

$$|u(x,t;\varepsilon)| < A(1-\delta) \tag{4.4}$$

for all $(x,t)\in[0,1]\times[0,\infty)$.

Proof. Assume that T is the first number such that

$$\operatorname{Max}\{u(x,t;\varepsilon)\,|\,(x,t)\in[0,1]\times[0,T]\} = A(1-\delta). \tag{4.5}$$

From (4.1), (4.2), Schwarz's and Poincaré's inequalities for $t\in[0,T]$,

$$\begin{aligned}\pi^2\int_0^1 u^2\,dx &\leqq \int_0^1 u_x^2\,dx\\ &\leqq 2\varepsilon\left(\int_0^1 u^2\,dx\right)^{1/2}\left\{\varphi(0)+M_\delta\left(\int_0^1 u^2\,dx\right)^{1/2}\right\}.\end{aligned}$$

From this we obtain the bound

$$\left(\int_0^1 u^2\,dx\right)^{1/2} \leqq 2\varepsilon\varphi(0)(\pi^2-\varepsilon M_\delta)^{-1},$$

the implied denominator on the right-hand side being positive in view of (4.3). If this bound is used in the right hand side of the preceding inequality and if the (sharp) inequality

$$4u^2(x,t) \leqq \int_0^1 u_x^2(x,t)\,dx,$$

is also employed, we obtain the pointwise bound

$$\begin{aligned}u^2(x,t) &\leqq \varepsilon^2\pi^2\varphi^2(0)[\pi^2-\varepsilon M_\delta]^{-2}\\ &\leqq A^2(1-\delta)^2-\delta'\end{aligned} \tag{4.5a}$$

for some $\delta'>0$ (by (4.3)) since the latter inequality is equivalent to (4.3) this contradicts the choice of T in (4.5).

Actually, we have proved a little more. Since ψ is continuous on $(-\infty, A)$, M_δ is monotone decreasing in δ and continuous on $[0, 1]$ so that $\lim_{\delta\to 0^+} \mathcal{F}(\delta)$ exists. Defining $\mathcal{F}(0)$ to be this limit, we see that $\mathcal{F}$ has a (unique) maximum in $[0, 1)(\mathcal{F}(1)=0)$, and that if the maximum occurs at $\delta_0 \in (0, 1)$, then $\varepsilon < \mathcal{F}(\delta_0)$ implies $|u(x, t; \varepsilon)| < A(1-\delta_0)$ on the half strip while if $\delta_0 = 0$ and $\varepsilon < \mathcal{F}(0)$, then $\varepsilon \leqq \mathcal{F}(\delta_1)$ for all δ_1 sufficiently close to zero so that $|u(x, t; \varepsilon)| < A(1-\delta_1)$, again on the half strip. It is also clear that if $\varphi \in L^1(0, A)$, the maximum must occur in $(0, 1)$.

COROLLARY 4.2. *If* $\varepsilon < \max\{\mathcal{F}(\delta): 0 \leqq \delta \leqq 1\}$, *then* $u(x, t; \varepsilon)$ *can never quench.*

Example. If $\varphi(u) = (1-u)^{-\beta}$, $\beta > 0$, then

$$\psi(u) = 2u^{-2}(1-\beta)^{-1}[1-(1-\beta)u-(1-u)^{1-\beta}].$$

Since, in $(-\infty, 0)$, $u^2\psi(u)$ is concave and vanishes with its first derivative at $u=0$, we have $\psi(u) \leqq 0$ in $(-\infty, 0]$. Furthermore, expanding $\psi(u)$ in a Taylor series about $u=0$, we see that on $[0, 1)$,

$$\psi(u) = 2\sum_{i=0}^{\infty} \frac{1}{(i+2)!}(\pi_{j=0}^{i+1}(\beta+j))u^i$$

and therefore $M_\delta = \psi(1-\delta)$.

For the case $\beta = 1$, $\psi(u) = 2u^{-2}[\ln(1/(1-u)) - u]$. We find, by direct computation, that $\delta_0 = 0.22684$ and that $u \leqq 1-\delta_0$ if $\varepsilon < \mathcal{F}(\delta_0) = (1.2379)^2$. Thus, combining this with the results of the example from the last section, we see that, with reference to (W): If $L > 1.41766 \equiv L_2$, then u quenches in finite time, while if $L < 1.2379 \equiv L_1$, then $u \leqq 0.7732$ for all time.

It is fair to ask whether one could improve the arguments involved in Theorem 4.1 by writing

$$\Phi(u) = \sum_{0}^{N-1} \varphi^{(k-1)}(0)\frac{u^k}{k!} + \left(\frac{1}{N!}\right)u^N\psi_N(u),$$

and employing Holder's inequality on the first $N-1$ terms and the (Sobolev) inequality

$$C^2(N)\left(\int_0^1 |u|^N\,dx\right)^{2/N} \leqq \int_0^1 u_x^2\,dx, \tag{4.6}$$

in place of Poincaré's inequality to obtain a bound on the L^N-norm of u. The constant $C(N)$ is known[1] and the inequality (4.6) is best possible. We did this for $\varphi(u) = 1/(1-u)$ and for various N. However the case $N = 2$ seems to yield the best value of L_2.

The numerical results indicate that if $L > 1.365 \cdots$, then u quenches in finite time while if $L < 1.365 \cdots$, u does not quench, even in infinite time. More precisely, what is observed numerically is the following. The solution, for small ε, has a discrete sequence of local maxima located along the line $x = \frac{1}{2}$, $t > 0$. The first of these local maxima appears to be an absolute maximum, which, as L increases to $1.365 \cdots$, from below, approaches one from below. However, the time to reach this maximum value increases without bound as L increases to $1.365 \cdots$. Moreover, if $L > 1.365 \cdots$, this maximum value is one and as L decreases to $1.365 \cdots$, the time taken to reach one increases without bound. This, if $L = 1.365 \cdots$, then u quenches in infinite time.

[1] $C(p) = (2\pi p)^{1/2}(2/(2+p))^{(p-2)/2p}\Gamma(1+1/p)/\Gamma(2+1/p)$, $1 \leqq p < \infty$. ($C(2) = \pi$, $C(\infty) = 2$.)

5. Perturbation analysis of (W′). In this section we investigate the behavior of solutions of (W′) as $\varepsilon \to 0$. The arguments involved are standard and will only be sketched. We write

$$\varphi(u) = \varphi(0) + \varphi'(0)u + u^2\psi_1(u)$$

and assume that $u \leqq 1-\delta$ implies $|\psi_1(u)| \leqq M(\delta)$. We take $A = 1$ here for convenience. The linear problem

$$\text{(L)} \qquad \begin{aligned} &v_{tt} - v_{xx} = \varepsilon\varphi(0) + \varepsilon\varphi'(0)v,\\ &v(0, t) = v(1, t) = 0,\\ &v(x, 0) = v_t(x, 0) = 0,\end{aligned}$$

is easily solved by elementary methods and found to have the following properties if $\varepsilon < \pi^2/\varphi'(0) \equiv \varepsilon_1$,

(L-1) $\qquad v(x, t; \varepsilon) = \varepsilon v_1(x, t; \varepsilon);$

(L-2) $\qquad |v_1(x, t; \varepsilon)| \leqq M_1$ where M_1 is an absolute constant independent of x, t, ε for $\varepsilon \leqq \varepsilon_1(1-\sigma)$, $\sigma \in [0, 1]$;

(L-3) $\qquad \lim_{\varepsilon\to 0^+}(x, t; \varepsilon) = v_1(x, t; 0)(\not\equiv 0)$, which solves

$$v_{1tt} - v_{1xx} = \varphi(0).$$

The following result then holds.

THEOREM 5.1. *Let u solve* (W″) *on* $[0, 1]\times[0, \infty)$ *and suppose* $|u(x, t; \varepsilon)| \leqq 1-\delta$ *for all* $\varepsilon < \varepsilon'$ *say, on the half strip. Then*

$$u(x, t; \varepsilon) = v(x, t; \varepsilon) + w(x, t; \varepsilon),$$

where

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-\sigma} w(x, t; \varepsilon) = 0$$

for every σ, $0 \leqq \sigma < 2$, *convergence being uniform on compact subsets of* $[0, 1]\times[0, \infty)$.

Proof. The difference $w = u - v$ satisfies (weakly) the equation

$$w_{tt} - w_{xx} = \varepsilon\varphi'(0)w + \varepsilon u^2\psi_1(u),$$

with the same initial and boundary data as u and v. The following energy principle then holds for w

$$\begin{aligned} E(t) &\equiv \frac{1}{2}\int_0^1 w_t^2\,dx + \frac{1}{2}\int_0^1 w_x^2\,dx\\ &= \tfrac{1}{2}\varepsilon\varphi'(0)\int_0^1 w^2\,dx + \varepsilon\int_0^t\int_0^1 u^2\psi_1(u)w_\eta\,dx\,d\eta.\end{aligned}$$

If we write

$$u^2\psi_1(u)w_\eta = \varepsilon v_1 u\psi(u)w_\eta + u\psi_1(u)ww_\eta,$$

choosing ε so small that $\frac{1}{2}\varepsilon\varphi'(0) < \pi^{-2}/2$, we find that

$$E(t) \leqq \varepsilon^2 A\int_0^t\int_0^1 |w_\eta|\,dx\,d\eta + B\varepsilon\int_0^t\int_0^1 |ww_\eta|\,dx\,d\eta, \tag{5.1}$$

where A, B are computable constants depending only on δ, $M(\delta)$, M_1.[2] For (not necessarily the same) A, B, we have further that

$$E(t) \leqq \varepsilon A \int_0^t E(\eta)\, d\eta + \varepsilon^3 Bt, \tag{5.2}$$

where we have used the assumption that $|\psi(u)| \leqq M(\delta)$ if $u \leqq 1-\delta$ and

$$\varepsilon^3 \int_0^1 |w_\eta|\, dx \leqq \varepsilon^3 \left(\int_0^1 |w_\eta|^2\, dx \right)^{1/2} \leqq \tfrac{1}{2}\varepsilon^3 + \tfrac{1}{2}\varepsilon \int_0^1 |w_\eta|^2\, dx$$

and where the arithmetic-geometric mean inequality and Poincaré's inequality have been used in the second term in (5.1).

Gronwall's inequality applied to (5.2) yields

$$E(t) \leqq A_1(t)\varepsilon^2 + A_2(t)\varepsilon^3,$$

where $A_1(t)$ and $A_2(t)$ are uniformly bounded on $[0, T]$ for $\varepsilon \in [0, \varepsilon_0]$ say. Putting this back into (5.2) yields (for different A_1, A_2 with the aforementioned properties)

$$E(t) \leqq A_1(t)\varepsilon^3 + A_2(t)\varepsilon^4 = O(\varepsilon^3). \tag{5.3}$$

This can be improved, at the expense of worse order constants, by using the following consequence of (5.1),

$$E(t) \leqq A\varepsilon^2 \int_0^t \sqrt{E(\eta)}\, d\eta + B\varepsilon \int_0^t E(\eta)\, d\eta. \tag{5.4}$$

Use of (5.3) in (5.4) yields

$$E(t) = O(\varepsilon^{7/2}).$$

Using this in (5.4) once again, we find $E(t) = O(\varepsilon^{15/4})$ etc. Thus, on every compact subset of $[0, 1] \times [0, \infty)$ and for every $\delta' > 0$,

$$E(t) = O(\varepsilon^{4-\delta'}) \quad \text{as } \varepsilon \to 0.$$

From the inequality

$$w^2(x, t) \leqq \frac{1}{4} \int_0^1 w_x^2\, dx \leqq \tfrac{1}{2} E(t),$$

we see that on every compact subset of $[0, 1] \times [0, \infty)$,

$$\lim_{\varepsilon \to 0^+} \varepsilon^{2-\delta'} w(x, t, \varepsilon) = 0,$$

uniformly.

In particular, $w/\varepsilon \to 0$ uniformly on compact subsets of the half strip. Thus, for small ε, $v_1 = v/\varepsilon$ will make the dominant contribution to u/ε. This function is, for $\varphi(u) = 1/(1-u)$,

$$v_1(x, t, \varepsilon) = \frac{4}{\pi} \sum_1^\infty \frac{[1 - \cos((2n+1)^2\pi^2 - \varepsilon)^{1/2} t]}{(2n+1)[(2n+1)^2\pi^2 - \varepsilon]} \sin((2n+1)\pi x).$$

[2] Here $M(\delta) = \sup\{|\psi_1(u)|, -\infty < u \leqq 1-\delta\}$. Actually this supremum need only be taken over $[-(1-\delta), 1-\delta]$ in view of (4.5a).

This fact has been observed numerically. That is, we observed numerically that for small ε, u was not only bounded away from one but also was oscillatory. This seemed surprising in view of our (mistaken) belief that since u quenched in finite time for L's less than L_0, u would quench in finite time for all $L > 0$.

Finally, we make the following observations: We can extend the results of the paper to the case of nonzero, appropriately restricted initial data. Furthermore it is possible to obtain analogous results in higher dimensions, at least in so far as §§ 3 and 4 are concerned since only Poincaré's inequality and the positivity of the first eigenfunction for the membrane problem are used. Preliminary calculations indicate that results along the lines of this paper and those in [1], [6] may be possible if the nonlinearity appears in the boundary condition. The second author is currently investigating this possibility.

Acknowledgment. The authors thank the referees for several helpful comments and suggestions.

REFERENCES

[1] A. ACKER AND W. WALTER, *On the global existence of solutions of parabolic differential equations with a singular nonlinear term*, Nonlinear Anal., 2 (1978), pp. 499–505.

[2] A. DOUGLIS, *Existence theorems for hyperbolic systems*, Comm. Pure Appl. Math., 5 (1952), pp. 119–154.

[3] H. KAWARADA, *On the solutions of initial boundary value problem for* $u_t = u_{xx} + 1/(1-u)$, Pub. RIMS Kyoto Univ., 10 (1975), pp. 729–736.

[4] P. D. LAX, *Nonlinear hyperbolic equations*, Comm. Pure Appl. Math., 6 (1953), pp. 231–258.

[5] H. A. LEVINE, *On the nonexistence of global weak solutions of some properly and improperly posed problems of mathematical physics: the method of unbounded Fourier coefficients*, Math. Ann., 214 (1975), pp. 205–220.

[6] H. A. LEVINE AND J. T. MONTGOMERY, *The quenching of solutions of some nonlinear parabolic equations*, this Journal, 11 (1980), pp. 842–847.

[7] J. RAUCH, *Singularities of solutions to semilinear wave equations*, J. Math. Pures Appl., 58 (1979), pp. 299–308.

[8] L. E. PAYNE, *Improperly Posed Problems in Partial Differential Equations*. CBMS Regional Conference Series in Applied Mathematics 22, Society for Industrial and Applied Mathematics, Philadelphia, 1975.

[9] M. H. PROTTER AND H. F. WEINBERGER, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1967.

[10] M. REED, *Abstract Nonlinear Wave Equations*, Lecture Notes on Mathematics 507, Springer Verlag, New York, 1976.

[11] P. R. GARABEDIAN, *Partial Differential Equations*, John Wiley, New York, 1964.

[12] J. RAUCH AND M. REED, *Propagation of singularities for semilinear hyperbolic equations in one space variable*, Ann. Math., 111 (1980), pp. 531–552.