APPLICATION OF FOURIER ELASTODYNAMICS TO DIRECT AND INVERSE PROBLEMS FOR THE SCATTERING OF ELASTIC WAVES FROM FLAWS NEAR SURFACES

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ABSTRACT

In ultrasonic inspection one frequently encounters situations in which flaws lie too close to surfaces (or interfaces) for the applications of the conventional theory of scattering in an infinite host medium. One approach, pursued by Varadan, Pillai, and Varadan, is to rework scattering theory with a Green function suitably modified for the presence of surfaces. Our approach uses a Fourier elastodynamics formalism in which the scattering process is represented by the generalized transfer function for a slab of material containing the flaw. A sufficiently simple elastodynamic system can be built up of separately characterizable subsystems defined by intervals on the z-axis. Fortunately, it turns out that, in the case of propagating waves the generalized transfer function can be simply related to the scattering amplitude, the representation of scattering in the conventional theory. In the case where evanescent waves are present, the generalized transfer function can also be simply related to the scattering amplitude in which the incident and scattered direction $\vec{e}_i$ and $\vec{e}_s$ have been analytically continued from real to complex forms. In some cases it may be a good approximation to ignore completely the effects of evanescent waves. An integral equation has been derived whose solution gives the scattering from a system composed of a flaw and a nearby surface or interface. The computational results pertain to the scattering from spherical voids and inclusions in plastic at various distances from the plastic-water interface with the incident wave propagating through the water.
INTRODUCTION

The conventional theory of the scattering of elastic waves from flaws (or, in general, localized inhomogeneities) assumes that a plane wave of given polarization is incident and that the scattered wave is observed in the far field. This implies that the physical system is composed of an isolated flaw surrounded by an infinite extension of host medium. To deal with the problem of flaws near surfaces or interfaces in the framework of the conventional theory one must revamp the above theory in a fundamental way with a new Green function incorporating a different global model of the medium with the surface or interface appropriately represented.

A different, and in our opinion more convenient, approach to this problem is provided by Fourier elastodynamics, a formalism that enables one to decompose an elastodynamic system into separately characterizable parts. This decomposition can be rigorously achieved if the system is stratified in a simple manner, e.g., by planes perpendicular to the z-axis.

Of particular significance is the fact that the scattering process can be represented by a generalized transfer function relating the near-field scattered waves to the waves incident on a slab of material containing the flaw. It should be noted, however, that both propagating and evanescent waves must be considered in the incident and scattered categories. If only propagating waves are considered then the generalized transfer functions can be simply related to the scattering amplitude characterizing the scattering process in the conventional theory of scattering. On the other hand in the case when evanescent waves are present this simple correspondence cannot be made except by performing a suitable analytical continuation of the scattering amplitude into a domain in which $e^i k_1$ and $e^i k_2$, giving the incident and scattered directions, are complex. Unfortunately it is often not feasible to carry out this process, especially if the conventional scattering amplitude is derived by computational means.

In any case, the scattering from a complex system, composed of a flaw in proximity to an interface separating two different media, can be determined in a straightforward manner by synthesizing such a system from separately characterized parts. It should be re-emphasized that the process of scattering from the flaw can be characterized in a manner that is completely independent of the remaining parts of the system.

In the remaining sections of this report we apply Fourier elastodynamics to the characterization of the total scattering process involving a flaw at various distances from a plastic-water interface with the incident wave propagating through the water.
The transducer is operating in a pulse-echo mode and thus the observed part of the scattered wave is also propagating through water but in a direction opposite to the incident wave. The explicit, detailed form of the theory is limited to spherical voids and inclusions, the symmetry of which enables one to ignore one of the two transverse polarizations. We present results from only normal incidence for a wide range of temporal frequencies and a diversity of incident directions for the case of low temporal frequencies (i.e., the wavelengths are large compared with the flaw size—but not large compared with the distance from the flaw center to the interface).

In the next section we present a rather abbreviated discussion of Fourier elastodynamics. In this discussion we make repeated references to Fourier acoustics, a simpler formalism in which the waves are represented by a scalar field (e.g., the velocity potential) in contrast with the vector field involved in the more complex elastodynamics formalism. To present the theory for both cases in a simple form we have devised an abstract notation that enables one to represent the principal results in a compact manner. The next section after that is devoted to the specialization of the previous results to the case of spherical voids and inclusions near an interface. A later section is devoted to the presentation and discussion of computational results.

FOURIER ELASTODYNAMICS

For treating the problem of scattering from a flaw near a surface or interface, it appears that Fourier elastodynamics (FE) is the appropriate formulation since it provides the means for partitioning the total system into separately characterizable parts. This can be done conveniently and rigorously if the system can be partitioned with respect to a particular axis, which is fortunately true in the case of concern to us here. The main value of FE in this context, aside from its greater convenience, is that it provides a means for utilizing the results of the conventional theory of scattering, i.e., a maximal exploitation of previous investments in scattering theory.

The FE approach to the analysis of elastodynamic systems is a straightforward, although complicated, extension of Fourier optics, a subject discussed in detail by Goodman (1968). The propagation of sound in a gas and/or a liquid can be treated by Fourier acoustics (FA), which is much simpler than FE and is mathematically identical to Fourier optics when suitable substitutions are made. In this section we will attempt to provide a partial understanding of FE by describing it at a rather abstract level with specific realizations in both the acoustic and elastodynamic cases.
In setting up the FE formation we will employ a standard cartesian coordinate system in which the coordinates are x, y, and z and in which the corresponding unit vectors are \( \hat{e}_x, \hat{e}_y, \) and \( \hat{e}_z \) as shown in Fig. 2.1. The z-axis is chosen to be the longitudinal axis of the system. A general position vector in the full three-dimensional space is denoted by

\[ \hat{r} = \hat{e}_x x + \hat{e}_y y + \hat{e}_z z \]  

(2.1)

On the other hand a position vector in the two-dimensional transverse space (i.e., the xy-plane) is denoted by

\[ \hat{r} = \hat{e}_x x + \hat{e}_y y \]  

(2.2)

Finally, a position in the one-dimensional longitudinal space is denoted simply by \( z \). Clearly, the three-dimensional vector \( \hat{r} \) can be written as a sum of its longitudinal and transverse parts, i.e.,

\[ \hat{r} = \hat{r} + \hat{e}_z z \]  

(2.3)

Now let us consider a general vector \( \hat{v} \) written as a sum of longitudinal and transverse parts, i.e.

\[ \hat{v} = \hat{v} + \hat{e}_z z \]  

(2.4)

in which the parts can be obtained by the following relations

\[ v_z = \hat{e}_z \cdot \hat{v} \]  

(2.5a)

\[ v = \hat{l} \cdot \hat{v} \]  

(2.5b)

The transverse unit tensor \( \hat{l} \) is defined by

\[ \hat{l} = \hat{e}_x x + \hat{e}_y y \]  

(2.6)

The magnitude of a transverse vector will be denoted by the same symbol with the underline absent, e.g., \( \hat{r} = |\hat{r}| \) and \( \hat{v} = |\hat{v}| \). The magnitude of a three-dimensional vector will be denoted by the original symbol with the magnitude sign added, e.g., \( |\hat{r}| \) or \( |\hat{v}| \), or sometimes a special symbol will be employed, e.g., \( R = |\hat{r}| \).

The relation between the magnitude of a three-dimensional vector and the magnitudes of its longitudinal and transverse components, i.e., \( |\hat{r}| = \sqrt{r^2 + z^2} \) and \( |\hat{v}| = \sqrt{v^2 + z^2} \), is obvious.

Let us now consider a scalar field \( u(\hat{r}) \) \( \exp(-i\omega t) \) in which \( \omega \) is the temporal frequency and \( t \) is the time. In the context of acoustics (i.e., wave motion in a fluid) the above field should be interpreted as the velocity or displacement potential. In the context of FA we should consider the explicit emphasis on the
transverse and longitudinal parts of $\mathbf{r}$, namely $\mathbf{r}$ and $z$, and consequently write

$$u(\mathbf{r}) = u(\mathbf{r}, z) \quad (2.7)$$

We can also represent the field in the domain of the transverse spatial frequency $\mathbf{k}$ according to the well-known relation

$$u(\mathbf{r}, z) = (2\pi)^{-2} \int d\mathbf{k} \ u(\mathbf{k}, z) \exp(\mathbf{i} \mathbf{k} \cdot \mathbf{r}) \quad (2.8)$$

where

$$u(\mathbf{k}, z) = \int d\mathbf{r} \ u(\mathbf{r}, z) \exp(-\mathbf{i} \mathbf{k} \cdot \mathbf{r}) \quad (2.9)$$

in which $d\mathbf{r} = dx dy$ and $d\mathbf{k} = dk dk$ are the differential areas in $\mathbf{r}$ and $\mathbf{k}$ spaces, respectively. Unless otherwise stated the integrations on $\mathbf{r}$ or $\mathbf{k}$ will span all of each of these transverse spaces. Throughout this discussion we will define a function by its arguments, i.e. $u(\mathbf{r}, z)$ and $u(\mathbf{k}, z)$ are different functions of their arguments in accordance with the above integral relation. In FA a strong emphasis is placed upon the representation $u(\mathbf{k}, z)$ involving the transverse spatial frequency $\mathbf{k}$ (hence the presence of the name "Fourier" in FA and in FE).

The generalization of the results of the last paragraph to the $\mathbf{r}$ case is obvious. It should be remarked that the vector field $u(\mathbf{r}, z)$ involved in FE should be interpreted as the displacement field.

As a first step toward putting physical content into FA let us consider the scalar Helmholtz equation

$$(\nabla^2 + K^2) \ u(\mathbf{r}, z) = 0 \quad (2.10)$$
where \( K = \omega/c \) is the total spatial frequency or wave number and where, in turn, \( c \) is the propagation velocity. In the \((k,z)\) domain, the above equation is transformed into the ordinary differential equation (o.d.e.)

\[
\left( \frac{d^2}{dz^2} - k^2 + K^2 \right) u(k,z) = 0
\]  

(2.11)

The solution can be written as a sum of + and - waves, namely

\[
u(k,z) = u_+(k,z) + u_-(k,z)
\]  

(2.12)

These waves at two different values of \( z \) are related according to the expression

\[
u_\pm(k,z) = \exp(\pm i\alpha(z - z_0)) v_\pm(k,z_0)
\]  

(2.13)

where the quantity \( \alpha \) is defined by

\[
\alpha = \sqrt{K^2 - k^2}, \quad 0 \leq k < K
\]

(2.14)

\[= 0, \quad k = K
\]

\[= i\sqrt{K^2 - k^2}, \quad k > K
\]

The above expressions are written under the implicit assumption that \( \omega > 0 \).

When \( k < K \), the quantity \( \alpha \) is of course real and thus in a + wave, for example, the factor \( \exp(i\alpha z) \) corresponds to a propagating wave with the velocity of the wave projected on the \( z \)-axis having a positive sign. However, when \( k > K \), quantity \( \alpha \) is pure imaginary (with a positive sign) and thus in a + wave the factor \( \exp(-|\alpha||z) \) corresponds to a so-called evanescent wave that has pure damping in the positive \( z \)-direction. The revision of the above statements for - waves is obvious. Figure 2.2 illustrates the nature of propagating and evanescent waves for the case of a single transverse spatial frequency \( k = e^{i k} \). This figure shows the dependence of

\[\text{Re} u_+(k,z) = \cos(\alpha z), \quad (2.15)\]

where \( \alpha = \alpha(e^{i k}) \), upon \( k \) and \( z \). It is clear that when \( 0 < k < K \) the field has an oscillatory dependence upon \( z \) and when \( k > K \) the field has an exponentially damped dependence upon \( z \). Furthermore, as \( k_x \) increases from 0 to \( K \) the wavelength in the \( z \)-direction increases from \( 2\pi/K \) to infinity and as \( k_x \) increases further from \( K \) to infinity the attenuation length decreases from infinity to zero.
Fig. 2.2. Nature of propagating and evanescent waves.

It is instructive to consider the Helmholtz Green function defined by the relation

\[(\nabla^2 + K^2) G(\mathbf{r}) = \delta(\mathbf{r}) \quad .\]  

(2.16)

It is of course well known that the Green function is given by

\[G(\mathbf{r}) = -\frac{1}{4\pi} \frac{\exp(iK|\mathbf{r}|)}{|\mathbf{r}|} = G(\mathbf{r}, z) \quad .\]  

(2.17)

In the \((\mathbf{k}, z)\)-domain the Green function reduces to the very simple form

\[G(\mathbf{k}, z) = \frac{1}{2i\alpha} \exp(i\alpha|z|) \quad .\]  

(2.18)

From this result we readily obtain an alternative form of the Green function in the \((\mathbf{r}, z)\)-domain in terms of the Fourier integral

\[G(\mathbf{r}, z) = \frac{1}{(2\pi)^2} \int \frac{dk}{\alpha} \frac{1}{2i\alpha} \exp(ik \cdot \mathbf{r} + i\alpha|z|)\]

\[= \frac{1}{4\pi^2} \int_0^\infty k d\alpha \frac{1}{\alpha} \exp(i\alpha|z|) J_0(kr) \quad .\]  

(2.19)
We can readily identify the propagating and evanescent wave parts of \( G(r, z) \) in terms of the ranges of integration \( 0 < k < K \) and \( K < k < \infty \) with the results denoted by \( G_p \) and \( G_e \). The nature of these functions is clearly illustrated by restricting \( r \) to the xy-plane (i.e., \( z = 0 \)), whereupon

\[
G_p(r, 0) = -\frac{1}{4\pi r} \sin kr = \text{Im}G(r, 0) \quad (2.20a)
\]

\[
G_e(r, 0) = -\frac{1}{4\pi r} \cos kr = \text{Re}G(r, 0) \quad (2.20b)
\]

It is clear that the propagating part of the Green function is nonsingular and that the \(|r|^{-1}\) singularity is contained entirely within the evanescent part.

In the case of the vector field \( \mathbf{u}(r) = \mathbf{u}(r, z) \) involved in FE, the governing equation is more complicated than the Helmholtz equation. In a uniform isotropic medium \( \mathbf{u} \) satisfies the well known elastodynamic equation

\[
\rho \omega^2 \mathbf{u} + (\lambda + \mu) \nabla \times \mathbf{u} + \mu \omega^2 \mathbf{u} = 0 \quad (2.21)
\]

where \( \rho \) is the density and where \( \lambda \) and \( \mu \) are the two Lame constants. In the \((k, z)\)-domain the solution can, as before, be written as a sum of \( + \) and \(- \) waves, i.e.,

\[
\mathbf{u}(k, z) = \mathbf{u}^+ (k, z) + \mathbf{u}^- (k, z) \quad (2.22)
\]

The general solution for the \( + \) wave is

\[
\mathbf{u}^+_4(k, z) = \mathbf{e}^+_{\omega L} a^+ \exp(i\alpha_L z) + \mathbf{e}^+_{\omega T} \times b^+ \exp(i\alpha_T z) \quad (2.23)
\]

and correspondingly the solution for the \( - \) wave is

\[
\mathbf{u}^- (k, z) = \mathbf{e}^-_{\omega L} a^- \exp(-i\alpha_L z) + \mathbf{e}^-_{\omega T} \times b^- \exp(-i\alpha_T z) \quad (2.24)
\]

In the above equation \( a^+ \) and \( a^- \) are arbitrary scalar constants and \( b^+ \) and \( b^- \) are arbitrary vector constants. The quantities \( \alpha_L \) and \( \alpha_T \) are the versions of \( \alpha \), defined by \((2.14)\), pertaining to longitudinal (L) and transverse (T) waves, respectively. They are obtained by replacing \( K \) in \((2.14)\) successively by \( K_L \) and \( K_T \) where

\[
K_L = \frac{\omega}{c_L}, \quad (2.25a)
\]

\[
K_T = \frac{\omega}{c_T}, \quad (2.25b)
\]

in which \( c_L \) and \( c_T \) are the velocities of L and T waves given by
\[ c_L = \left( \frac{\lambda + 2\mu}{\rho} \right)^{1/2}, \]  \hspace{1cm} (2.26a)

\[ c_T = \left( \frac{\mu}{\rho} \right)^{1/2}, \]  \hspace{1cm} (2.26b)

respectively. Returning to (2.23) and (2.24) the unit vectors \( \hat{e}_{+L} \) and \( \hat{e}_{-T} \) are the propagation directions for \(+\) and \(-\) L waves and similarly \( \hat{e}_{+T} \) and \( \hat{e}_{-T} \) are the propagation directions for \(+\) and \(-\) T waves. These vectors are parallel to the corresponding total spatial frequencies (or wave numbers)

\[ \hat{k}_{+L} = k + \hat{e}_z \alpha_L, \]  \hspace{1cm} (2.27a)

\[ \hat{k}_{-L} = k - \hat{e}_z \alpha_L, \]  \hspace{1cm} (2.27b)

\[ \hat{k}_{+T} = k + \hat{e}_z \alpha_T, \]  \hspace{1cm} (2.27c)

\[ \hat{k}_{-T} = k - \hat{e}_z \alpha_T. \]  \hspace{1cm} (2.27d)

It should be noted that \( \hat{k}_{+L} \) and \( \hat{k}_{-L} \) (and the corresponding unit vectors) are real when \( k < K_L \) and complex when \( k > K_L \). Similarly \( \hat{k}_{+T} \) and \( \hat{k}_{-T} \) are real when \( k < K_T \) and complex when \( k > K_T \). It is easy to show that

\[ |\hat{k}_{+L}| = |\hat{k}_{-L}| = K_L \]  \hspace{1cm} (2.28a)

\[ |\hat{k}_{+T}| = |\hat{k}_{-T}| = K_T \]  \hspace{1cm} (2.28b)

and thus the magnitudes of the above vectors are always real.

With a planar interface between two media or with a planar free surface other solutions can exist (e.g., Stonely waves or Rayleigh waves). For the sake of brevity we will not discuss these matters here.

To describe the further development of FE and its applications with a minimum of notational complexity, it is expedient to introduce an abstract notation. Although the reduction of complexity is not great in the case of FA we will make the abstract notation sufficiently general to include this case as a special realization. In the abstract notation we will eliminate the explicit reference to all parameters, coordinates, and/or indices except for the longitudinal coordinate \( z \) and the \(+\) or \(-\) subscript indicating whether the field is composed of \(+\) or \(-\) waves. Thus in the case of FA, we have the correspondence.
and in the case of FE

\[ u_+(z) \leftrightarrow u_+^+(k,z) \text{ or } u_+^+(r,z) \]  

(2.30)

The wave case is obvious. Instead of the vector form in FE one could assume that the field is decomposed according to mode types, i.e.

\[ \begin{bmatrix} u_{+L}(k,z) \\ u_{+T\parallel}(k,z) \\ u_{+T\perp}(k,z) \end{bmatrix} \]

(2.31)

or with similar expressions in which \( r \) replaces \( k \). The subscripts \( L, T\parallel, \) and \( T\perp \) refer to the longitudinal and the two transverse modes. The meanings of \( T\parallel \) and \( T\perp \) depend upon the conventions to be used in particular contexts. For example, in the case of scattering it is usually assumed that \( T\parallel \) and \( T\perp \) denote transverse polarizations parallel and perpendicular, respectively, to the scattering plane (the senses are also specified on a conventional basis, but will not be discussed here).

We turn now to the consideration of various transfer functions, a term we will interpret somewhat more loosely than is done in the electrical engineering literature. Scattering from a localized inhomogeneity can be represented by a generalized transfer function representing the input-output properties of a slab of material containing the inhomogeneity. Such a transfer function is defined by the equations

\[ u_+(z_2) = H_{++}(z_2;z_1) u_+(z_1) + H_{+-}(z_2;z_2) u_-(z_2) \]  

(2.32)

\[ u_-(z_1) = H_{-+}(z_1;z_1) u_+(z_1) + H_{--}(z_1;z_2) u_-(z_2) \]

where \( H_{++}, H_{+-}, H_{-+}, \) and \( H_{--} \) are operators and where \( z_1 \) and \( z_2 \) are the longitudinal positions of the imaginary planes bounding the inhomogeneity. It is more convenient to work with a representation of scattering in which the fields are extrapolated to the central plane at \( z = 0 \) the nominal center of the inhomogeneity. The extrapolation is carried out as though only unperturbed host material were present. The resultant "central" form of the transfer functions can be written in the form

\[ u_+(+0) = H_{++} u_+(-0) + H_{+-} u_-(+0) \]
Actually, $H_{++}^f$ should be written $H_{++}^f(+0; -0)$ etc.; however, these arguments are implied by the + and - subscripts if it is understood that the central form is being used with the central plane at $z=0$. More explicitly the leading term $H_{++}^f(-0)$ has the following meanings in the FA and FE cases

$$H_{++}^f u_+ (-0) \Leftrightarrow \int dk' K(k, +0'; k', -0) u_+ (k', -0) \quad \text{(FA)} \tag{2.34a}$$

$$\Leftrightarrow \int dk' \quad \overset{\rightarrow}{H_{++}^f} (k, +0', k', -0) \cdot u_+ (k', -0) \quad \text{(FE)} \tag{2.34b}$$

The generalized transfer function representation of scattering can be simply related to the scattering amplitude pertaining to the conventional theory of scattering. This relation will be discussed more fully and explicitly in the next section.

It is to be noted that the above transfer function does not in general conserve the transverse spatial frequency $k$. An exception is the case in which the material properties in the inhomogeneity depend only upon the coordinate $z$; however, such an inhomogeneity is not localized in all directions.

All of the other transfer functions that will concern us do conserve $k$. The first such transfer function is that representing propagation from $z = 0$ to $z = z_1 (z_1 > 0)$, namely

$$u_+(z_1) = H_{10}^f u_+ (0) \quad \text{(2.35)}$$

and in the case of the reverse process

$$u_-(0) = H_{01}^f u_-(z_1) \quad \text{(2.36)}$$

In the case of FA in the $(k,z)$-domain the operators $H_{10}^f$ and $H_{01}^f$ are identical and correspond simply to the multiplication by $\exp(i\alpha z_1)$. In FE the operators $H_{10}^f$ and $H_{01}^f$ are not equal and are more complicated, at least in the vector representation of $u_+(z)$. In the mode representation (see Eq. 2.31) in the $(k,z)$-domain, the operator $H_{10}^f$ is defined by the correspondence

$$H_{10}^f \Leftrightarrow \begin{pmatrix} \exp(i\alpha_L z_1) & 0 & 0 \\ 0 & \exp(i\alpha_T z_1) & 0 \\ 0 & 0 & \exp(i\alpha_T z_1) \end{pmatrix}. \quad \text{(2.37)}$$
a relatively simple form, but it requires the use of consistent polarization conventions throughout the system.

Another $k$-preserving process is transmission and reflection at an interface at $z = z_1$ between two different media. Such a process can be described by the equations

$$u_+ (z_1 + 0) = H_{++}^\text{int} u_+ (z_1 - 0) + H_{+-}^\text{int} u_- (z_1 + 0)$$

$$u_- (z_1 - 0) = H_{++}^\text{int} u_+ (z_1 - 0) + H_{--}^\text{int} u_- (z_1 + 0)$$  \hspace{1cm} (2.38)

in which $H_{++}^\text{int}$ and $H_{--}^\text{int}$ represent the transmission and reflection coefficients for an incident $+$ wave, etc. For the case of FE in the $(k,z)$-domain and in the mode representation we obtain, for example,

$$H_{++}^\text{int} u_+ (z_1 - 0) \leftrightarrow \begin{pmatrix} H_{++LL} (k) & H_{++LT\parallel} (k) & H_{++LT\bot} (k) \\ H_{++T\parallel L} (k) & H_{++T\parallel T\parallel} (k) & H_{++T\parallel T\bot} (k) \\ H_{++T\bot L} (k) & H_{++T\bot T\parallel} (k) & H_{++T\bot T\bot} (k) \end{pmatrix} \begin{pmatrix} u_+ (k, z_1 - 0) \\ u_+ (k, z_1 - 0) \\ u_+ (k, z_1 - 0) \end{pmatrix}$$

and so forth. The elements of the above matrix are given by the existing theory of transmission and reflection; however, this theory is expressed in terms of the vectors $e_+^L$, $e_+^T$, $e_-^L$, and $e_-^T$ given by Eq. (2.27a) to (2.27d) in terms of $k$ and the local material properties. The conservation of $k$ in the present problem is a simple way of expressing Snell's law.

SCATTERING FROM A FLAW NEAR AN INTERFACE

The formalism of FE will now be applied to the direct problem of scattering of elastic waves from a flaw near a surface. The inverse problem will be discussed in a later communication. Here we will consider the case of a spherical void or inclusion in a plastic near the planar plastic-water interface. The scattering experiment is conducted in the pulse-echo manner with the incident wave propagating first through the water and then into the plastic.
We will discuss most of this problem in abstract notation with specialization to more explicit representations near the end of the treatment.

In Fig. 3.1 we present a schematic representation of the scattering experiment. As is shown, the flaw is a spherical void or inclusion of radius, a, placed with its center at a distance, d, from the interface. The incident L-wave in the water is assumed to be propagating in the incident direction \( e^+ \) and the scattered wave is observed in the direction \( -e^+ \). It is assumed that the single transducer involved here is placed in the far field of the flaw.

![Fig. 3.1. Schematic representation of scattering experiment.](image)

In Fig. 3.2 we show the explicit geometrical framework to be used in the FE approach to the problem at hand. The nominal center of the flaw is placed on the z-axis at the origin \( z = 0 \). The planar plastic-water interface is perpendicular to the z-axis and intersects it at the point \( z = z_1 \). The analysis of the multiple interactions between the flaw and the interface begins with the consideration of the + and - waves between the flaw and the interface. The process of scattering from the flaw is described by the relations

\[
\begin{align*}
    u_+ (+0) &= H_{++} u_+ (-0) + H_{+-} u_- (+0) \\
    u_- (-0) &= H_{-+} u_+ (-0) + H_{--} u_- (+0)
\end{align*}
\]
However, since there are no $+\,$ waves to the left of the flaw we must write

\[ u_+ (-0) = 0 \quad (3.3) \]

It is necessary at this point to distinguish between incident and scattered waves. We will define the incident wave as the one that would exist in the absence of the flaw and the scattered wave as the remainder. We therefore decompose the actual wave field according to the relations

\[ u_+ (+0) = u^s_+ (+0) \quad (3.4) \]
\[ u_- (-0) = u^i_- (-0) + u^s_- (-0) \quad (3.5) \]
\[ u_- (+0) = u^i_- (+0) + u^s_- (+0) \quad (3.6) \]

where obviously the super $i$ denotes "incident" and the super $s$ denotes "scattered." It is appropriate to make a similar decomposition of the generalized transfer function for scattering, namely

\[ H_++ = I + H^s_++ \quad (3.7) \]
\[ H_+- = H^s_+- \quad (3.8) \]
\[ H_-+ = H^s_-+ \quad (3.9) \]
\[ H_{--} = I + H^s_{--} \quad (3.10) \]

It is clear that with the operators $H^s_+\,, \, H^s_-, \, H^s_+, \, \text{and } H^s_-$ all set equal to zero the remaining terms on the r.h. sides of (3.7) to (3.10), i.e., $I$ in (3.7) and (3.10), and $0$ in (3.8) and (3.9) represent the transmission of $+$ and $-$ waves through the plane $z = 0$ without scattering.
APPLICATION OF FOURIER ELASTODYNAMICS

For a complete description of all of the relevant processes involved in the scattering from a flaw near an interface, we must include a description of the propagation from the flaw to the interface and back (including of course the reflection at the interface). Such a process is described by the equation

$$u^s_+ (-0) = H_{010} u^s_+ (+0) \quad (3.11)$$

where the operator $H_{010}$ is defined by

$$H_{010} = H_{01} H_{\text{int}} H_{10} \quad (3.12)$$

In the last equation, the operator $H_{10}$ represents the propagation of $+$ waves from $z = +0$ to $z = z_1 - 0$ and $H_{01}$ represents the reverse process, i.e., the propagation of $-$ waves from $z = z_1 - 0$ to $z = +0$. In contrast with the situation in FA, the operators $H_{01}$ and $H_{10}$ are not equal in FE. Finally, the operator $H_{\text{int}}$ represents the process of reflection from the plastic-water interface at $z = z_1$. It is to be noted that in the overall process represented by $H_{010}$, there is no conversion of incident to scattered waves and vice versa.

Separating out the processes involving only incident waves, we obtain from the above analysis the following pair of equations

$$u^s_+ (+0) = H_{++}^s u^i_+ (+0) + H_{--}^s H_{010} u^s_+ (+0) \quad (3.13)$$

$$u^s_- (-0) = H_{++}^s u^i_- (+0) + H_{010} u^s_+ (+0) + H_{--}^s H_{010} u^s_+ (+0) \quad (3.14)$$

The solution of (3.13) for $u^s_+ (+0)$ then provides the means for the direct computation (using (3.14)) of $u^s_- (-0)$ the forward-scattered $-$ waves propagating to the left of the flaw. The back-scattered $+$ waves $u^s_+ (z_1 + 0)$ propagating into the water to right of the interface are given in terms of $u^s_+ (+0)$ by the relation

$$u^s_+ (z_1 + 0) = H_{++}^\text{int} H_{10} u^s_+ (+0) \quad (3.15)$$

To complete the description, we must relate the incident waves $u^i_- (+0)$ at the flaw to the incident wave $u^i_- (z_1 + 0)$ in the water just to the right of the interface, i.e.

$$u^i_- (+0) = H_{01} H_{\text{int}} u^i_- (z_1 + 0) \quad (3.16)$$

Our interest is limited to the back-scattered waves and hence we can ignore (3.14) for the present time.
The core of our problem is the solution of (3.13), which is actually an integral equation with \( u^S_+ (+0) \) playing the role of the dependent function and with \( H^S_+ H_{010} \) representing the integral operator. The general formal solution (3.13) can be written in the simple form

\[
    u^S_+ (+0) = (I - H^S_+ H_{010})^{-1} H^S_+ u^i_- (+0) \quad (3.17)
\]

In the later stages of the present treatment we will consider a perturbation expansion of the general solution with \( H^S_+ H_{010} \) regarded as the perturbation. The first order solution (the incident wave is regarded here as the zeroth order solution) is then given by

\[
    u^{S(1)}_+ (+0) = H^S_+ u^i_- (+0) \quad (3.18)
\]

and the second order correction is given by the slightly more complicated expression

\[
    u^{S(2)}_+ (+0) = H^S_+ H_{010} H^S_+ u^i_- (+0) \quad . \quad (3.19)
\]

In our later analysis we will not consider higher order corrections.

To form a point of departure for the more explicit analysis immediately following we will rewrite (3.18) and (3.19) in more comprehensive forms including the initial and final processes represented by (3.16) and (3.15), respectively. The first order solution (3.18) can be rewritten in the form

\[
    u^{S(1)}_+ (z_1 +0) = H^{int}_+ H_{10} H^S_+ H_{01} H^{int}_- u^i_- (z_1 +0) \quad . \quad (3.20)
\]

The second order correction can be correspondingly rewritten

\[
    u^{S(2)}_+ (z_1 +0) = H^{int}_+ H_{10} H^{2s}_+ H_{01} H^{int}_- u^i_- (z_1 +0) \quad (3.21)
\]

where \( H^{2s}_+ \) is an effective scattering transfer function given by

\[
    H^{2s}_+ = H^S_+ H_{01} H^{int}_- H_{10} H^S_+ \quad . \quad (3.22)
\]

This process involves a backscatter from the flaw followed by a round-trip to the interface terminating in a second backscatter from the flaw. The fact that (3.21) can be obtained from (3.20) simply by substituting \( H^{2s}_+ \) for \( H^S_+ \) suggests a useful modularization of the computation to be discussed in the next section.
To obtain eventually an effective scattering amplitude for the combined flaw-interface system, i.e., the representation in terms of conventional scattering theory of the process whose input is \( u_+ (z_1 + 0) \) and whose output is \( u_+ (z_1 + 0) \), we must specialize the input to a form representing an incident monochromatic plane wave and must propagate the output into the far field. The input specialization is achieved by requiring the correspondence

\[
\begin{align*}
    u_+^i (z_1 + 0) & \Leftrightarrow u_+^{+i} (k, z_1 + 0) \\
    & = (2\pi)^2 \hat{e}_w \delta(k - k^i)
\end{align*}
\]  

(3.23)

where

\[
    \hat{e}_w = k_w^{-1} k_w^{-1}
\]  

(3.24)

and

\[
    k_w^{-1} = k^i - e_w \alpha^i_w.
\]  

(3.25)

In the above expressions \( k^i \) is the transverse spatial frequency of the incident wave (in both water and plastic) \( k^i_w \) is the total spatial frequency (or propagation vector) in the water, and \( e_w \) is a unit vector defining its propagation direction. The function \( \alpha^i_w \) is a specialization of the function \( \alpha = \alpha(k) \) discussed in the last section to the case of waves in water, i.e.

\[
    \alpha^i_w = \sqrt{K_w^2 - |k^i|^2}
\]  

(3.26)

where

\[
    K_w = \frac{\omega}{c_w}
\]  

(3.27)

in which \( c_w \) is the velocity of propagation of sound in water. In the above expressions we have assumed the inequalities: \(|k^i| < K_w\) and \( \omega > 0 \). The propagation of \( u_+ (z_1 + 0) \) into the far field can be regarded as a final "add-on" and hence it will be discussed later.

We turn now to the task of translating the above results from abstract notation to an explicit notation appropriate for the application of interest. Rather than accomplishing this translation in the general case, followed by a specialization to the application of interest, we have opted for a simplified translation directly to a form suitable for this application by using the symmetry of the flaw. To be more explicit we will use the spherical symmetry of the flaw to deduce that the mode conversions
are impossible. If we define $T_{||}$ to denote the transverse polarization parallel to the plane determined by the vectors $k$ and $\vec{e}_z$ and $T_{\perp}$ to denote the transverse polarization perpendicular to this plane, then neither the process of transmission through the interface nor reflection (on the plastic side) can cause any of the above mode conversions. Thus, with the above definitions, the $T_{\perp}$ mode is never excited and only the $L$ and $T_{||}$ modes need be considered. To simplify writing, we replace the symbol $T_{||}$ by the bare symbol $T$ since there is now no ambiguity about which transverse mode is present.

The first order result, given by (3.20) as abstract notation, is translated for the present application into the following form using explicit notation

$$u^{s(0)}_{+L} (-k^1_z, z_1 + 0) = \left[ h^{\text{int}}_{+LL} (-k^1_z) \exp(i\alpha^1_z) h^{s}_{+-LL} (-k^1_z, k^1) \right. \times \exp \left( i\alpha^1_z \right) h^{\text{int}}_{--LL} (k^1) \right. \nonumber$$

$$+ h^{\text{int}}_{+LT} (-k^1_z) \exp (i\alpha^1_z) h^{s}_{+-TL} (-k^1_z, k^1) \times \exp (i\alpha^1_z) h^{\text{int}}_{--TL} (k^1) \right. \nonumber$$

$$+ h^{\text{int}}_{+LL} (-k^1_z) \exp (i\alpha^1_z) h^{s}_{++LT} (-k^1_z, k^1) \times \exp (i\alpha^1_z) h^{\text{int}}_{--TL} (k^1) \right] (2\pi)^2 . \quad (3.28)$$

In deriving the above result we have used the special form of the incident wave given by (3.23). The details of the various processes in (3.28) are tabulated in Table I. Some additional explanatory remarks are in order. In the sixth and seventh columns (and only in these) the subscript $w$ refers to water and the subscripts $L$ or $T$ refer to longitudinal or transverse elastic waves in plastic. In the case of the quantity $a$, defined generally by (2.14) as a function of $K$ and $k$, the superscript $i$ means that $k$ is replaced by $k^1$ and the subscript $w$, L, or T means that $K$ is replaced by $K_w$, $K_L$, or $K_T$, respectively. $K_L$ and $K_T$ are defined by Eqs. (2.25a) - (2.26b) and $K_w$ is defined by (3.27).
Table I

<table>
<thead>
<tr>
<th>Process, Abstract Symbol</th>
<th>Medium</th>
<th>+ or - Mode</th>
<th>Propagation Vector</th>
<th>Propagation Direction</th>
<th>Explicit Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transmission through water-plastic interfaces at ( z = z_1 ), ( H_{\text{int}}^{\text{pl}} )</td>
<td>( w ) ( P )</td>
<td>( L ) ( L ) ( L ) ( T )</td>
<td>( k_1 ) ( k_1 ) ( k_1 ) ( k_1 )</td>
<td>( k_{\text{down}} ) ( k_{\text{up}} )</td>
<td>( H_{\text{int}}^{\text{pl}} ) ( (\lambda, \mu) )</td>
</tr>
<tr>
<td>Propagation from ( z = z_1 ) to ( z = 0 ), ( H_{\text{in}} )</td>
<td>( P ) ( P )</td>
<td>( L ) ( L ) ( L ) ( L )</td>
<td>( k_1 ) ( k_1 ) ( \bar{k}_1 ) ( \bar{k}_1 )</td>
<td>( k_{\text{down}} ) ( k_{\text{up}} )</td>
<td>( H_{\text{in}} ) ( (\lambda, \mu) )</td>
</tr>
<tr>
<td>Backscatter, ( H_{\text{out}}^{\text{pl}} )</td>
<td>( P ) ( P ) ( P ) ( P ) ( P ) ( P )</td>
<td>( L ) ( L ) ( L ) ( L ) ( T ) ( T ) ( T )</td>
<td>( \bar{k}_1 ) ( \bar{k}_1 ) ( \bar{k}_1 ) ( \bar{k}_1 ) ( \bar{k}_1 ) ( \bar{k}_1 )</td>
<td>( k_{\text{down}} ) ( k_{\text{up}} )</td>
<td>( H_{\text{out}}^{\text{pl}} ) ( (\lambda, \mu) )</td>
</tr>
<tr>
<td>Propagation from ( z = 0 ) to ( z = z_1 ), ( H_{\text{in}} )</td>
<td>( P ) ( P ) ( P ) ( P ) ( P ) ( P )</td>
<td>( L ) ( L ) ( L ) ( L ) ( T ) ( T ) ( T )</td>
<td>( \bar{k}_1 ) ( \bar{k}_1 ) ( \bar{k}_1 ) ( \bar{k}_1 ) ( \bar{k}_1 ) ( \bar{k}_1 )</td>
<td>( k_{\text{down}} ) ( k_{\text{up}} )</td>
<td>( H_{\text{in}} ) ( (\lambda, \mu) )</td>
</tr>
<tr>
<td>Transmission, ( H_{\text{out}}^{\text{pl}} )</td>
<td>( w ) ( w ) ( P ) ( P ) ( P ) ( P ) ( P )</td>
<td>( L ) ( L ) ( L ) ( L ) ( L ) ( L ) ( L )</td>
<td>( \bar{k}_1 ) ( \bar{k}_1 ) ( \bar{k}_1 ) ( \bar{k}_1 ) ( \bar{k}_1 ) ( \bar{k}_1 )</td>
<td>( k_{\text{down}} ) ( k_{\text{up}} )</td>
<td>( H_{\text{out}}^{\text{pl}} ) ( (\lambda, \mu) )</td>
</tr>
</tbody>
</table>
The second order correction is given in abstract notation by (3.21) with the effective scattering transfer function $H_{+}^{s}$ defined by (3.22). In explicit notation the latter quantity is given by

$$H_{+}^{s,-LL}(-k^i, k^i) = \int dk H_{+}^{s,-LL}(-k^i, k) \exp(i\alpha L z_1) H_{-}^{int}(k) \exp(i\alpha L z_1) H_{+}^{s,-LL}(k, k^i) + \int dk H_{+}^{s,-LT}(-k^i, k) \exp(i\alpha T z_1) H_{+}^{int}(k) \exp(i\alpha T z_1) H_{+}^{s,-TL}(k, k^i)$$

plus terms involving mode conversion on reflection from plastic-wave interface.

Analogous expressions are obtained for the $L \to T$, $T \to L$, and $T \to T$ cases. If the round-trip distance from the flaw to the interface, $2z_1$, is sufficiently large, the factor $\exp(2i\alpha L z)$ for example, will act like a $\delta$-function in $k$ due to oscillating side lobes if the remaining factors in the integrand are sufficiently slowly varying. More explicitly, we make the replacement

$$\exp(2i\alpha L z_1) = B \delta(k)$$

when

$$B = \int dk \exp(2i\alpha L z_1) = -2\pi i K_L \frac{\exp(2iK_L z_1)}{2z_1}$$

in which the last line was obtained with the use of the Fresnel approximation. The result for $T$ waves is an obvious analog of the above expression. Equation (3.29) now reduces to

$$H_{+}^{2s,-LL}(-k^i, k^i) \approx$$

$$H_{+}^{s,-LL}(-k^i, 0) H_{+}^{int}(0) H_{+}^{s,-LL}(0, k^i) \left[-2\pi i K_L \frac{\exp(2iK_L z_1)}{2z_1}\right]$$

$$+ H_{+}^{s,-LT}(-k^i, 0) H_{+}^{int}(0) H_{+}^{s,-TL}(0, k^i) \left[-2\pi i K_T \frac{\exp(2iK_T z)}{2z_1}\right]$$

(3.32)
The contributions due to mode conversion at the plastic-water interface vanish in this approximation. The analogs of (3.32) for the L + T, T + L, and T + T case are obvious. For example, in the L + T case we simply replace the first subscript L in \( H_{+LL}^S(-k^z,0) \) and \( H_{+LT}^S(-k^z,0) \) by T.

At this point we turn to the consideration of the relationship between the scattering transfer function \( H_{+LL}^S(k^s,k^l) \), etc) and the scattering amplitude \( A_{LL}(e^s_{+L}, e^s_{-L}, \omega) \), etc.) representing the results of conventional scattering theory. By considering the scattered wave (i.e., the output) from the transfer function in the far-field region of the \((r,z)\) domain we can make a direct comparison with the scattering amplitude. In the case of propagating waves only, we obtain the results

\[
H_{+LL}^S(k^s, k^l) = -\frac{1}{2\pi i\alpha^s_L} A_{LL}(e^s_{+L}, e^s_{-L}, \omega), \quad (3.33a)
\]
\[
H_{+TL}^S(k^s, k^l) = -\frac{1}{2\pi i\alpha^s_T} A_{TL}(e^s_{+T}, e^s_{-T}, \omega), \quad (3.33b)
\]
\[
H_{+LT}^S(k^s, k^l) = -\frac{1}{2\pi i\alpha^s_L} A_{LT}(e^s_{+L}, e^s_{-T}, \omega), \quad (3.33c)
\]
\[
H_{+TT}^S(k^s, k^l) = -\frac{1}{2\pi i\alpha^s_T} A_{TT}(e^s_{+L}, e^s_{-T}, \omega), \quad (3.33d)
\]

in which the various incident and scattered (observer) directions are given by

\[
\hat{e}^s_{-L} = k^s - k^l z a^s_L \quad (3.34a)
\]
\[
\hat{e}^s_{-T} = k^s - k^l z a^s_T \quad (3.34b)
\]
\[
\hat{e}^s_{+L} = k^s + k^l z a^s_L \quad (3.34c)
\]
\[
\hat{e}^s_{+T} = k^s + k^l z a^s_T \quad (3.34d)
\]

The \( \alpha \) functions are defined for propagating waves by the equations

\[
\alpha^s_L = \sqrt{k^2 - |k^l|^2} \quad (3.35a)
\]
\[
\alpha^s_T = \sqrt{k^2 - |k^l|^2} \quad (3.35b)
\]
\[ \alpha_L^s = \sqrt{k_L^2 - |k^s|^2} \]  
\[ \alpha_T^s = \sqrt{k_T^2 - |k^s|^2} \]  
\[ (3.35c) \]  
\[ (3.35d) \]

where \( k_L \) and \( k_T \) are respectively the magnitudes of the total spatial frequencies (i.e., wave number) of longitudinal and transverse waves defined by (2.25a) - (2.26b).

The transfer functions representing transmission and reflection of various waves at the water-plastic interface \( H_{\text{int}}^{LL}(k) \) etc.) can be related to the standard results discussed by Hulb (1973) using different notation. To make contact with the standard results we need to specify the various propagation directions listed in Table 1.

Finally we must consider the last step in the overall process, namely the propagation of scattered waves through the water to the far field of the flaw. We assume that the observer direction is anti-parallel to the incident direction, namely

\[ \bar{e}_{+w} = k_w^{-1} (-k^i + \frac{e_{+w}^i}{z} \alpha_w^i) \]  
\[ (3.36) \]

Since the incident wave has unit amplitude in the \((r,z)\)-domain it follows that the amplitude in the far field in the above direction is given by

\[ u_w(r,z) = \frac{\exp(iKR)}{R} A_{\text{tot}} (e_{+w}^i, e_{-w}^i, \omega) \]  
\[ (3.37) \]

where \( A_{\text{tot}} \) is the scattering amplitude for the total process and where \( R = \sqrt{r^2 + z^2} \) \((r/z = -k^i/\alpha_w^i)\). By arguments similar to those used in the derivation of \((3.33a) - (3.33d)\), we find that

\[ A_{\text{tot}} = \frac{\alpha_w^i}{2\pi i} u_{+L}^s (-k^i, z_1 + 0) \]  
\[ (3.38) \]

In first order \( u_{+L}^s (-k^i, z_1 + 0) \) is given by \((3.28)\).

COMPUTATIONAL RESULTS

In the present section we discuss the computational results obtained for the theory outlined in the previous sections. As we have already stated, both the explicit theory and the associated computations are limited to spherical voids and inclusions, a limitation producing a considerable simplification in the handling of transverse modes. We give the computational results for several
ranges of temporal frequency and for a sequence of values of the
distance from the flaw center to the interface. All computations
are limited to the pulse-echo case. These matters will be dis-
cussed more fully in the next paragraph. Three important gaps
remain to be filled: (a) the dependence of the overall scattering
process upon the incident angle at intermediate to high frequencies
temporal), (b) the nature of the overall scattering process when
both the flaw size and distance from the surface are small compared
with the relevant wavelengths, and (c) the extension of the expli-
cit formulation to the case of nonspherical flaws.

For a given flaw type there are two independent dimensionless
parameters entering this problem, namely $K_a$ and $K_d$ where $K$
is the
characteristic total spatial frequency (a wave number) which we
will usually assume to be that of longitudinal waves in the plas-
tic. As stated before, $a$ is the radius of the spherical flaw and $d$
is the distance from its center to the plastic-water interface. An
important, but not independent, parameter is the ratio $d/a$. The
various scattering regions can be mapped out in the $(K_a, K_d)$-plane
as is shown in Fig. 4.1. For the sake of simplicity we discuss
this matter for the case of $FA$ with only scalar $L$-waves present.
On the horizontal axis we somewhat arbitrarily define three
regimes: Rayleigh ($K_a < 0.5$) intermediate frequency ($0.5 < K_a <
2.0$) and high frequency ($2.0 < K_a$). These regimes refer to the
isolated flaw. We could define three similar regimes on the
vertical axis with similar inequalities in which $K_d$ replaces $K_a$.
To avoid excessive complexity these regimes are not shown in the
figure. In any case they refer to the nature of the interaction
between the flaw and the interface, e.g., here the Rayleigh regime
($a < K_d < 0.5$) implies that the interaction between the flaw and
the interface can be treated as a quasi-static elastic problem and
at the other extreme the high frequency regime means that the round
trip distance from the flaw to the interface and back is many times
the wavelength. Several contours corresponding to constant values
of $a/d$ are shown. The boundary between near-field and far-field
regimes (i.e., near- and far-field relative to the flaw) is shown
as a curved line (in spite of the fact that this is actually a
fuzzy boundary).

In the present treatment we consider two cases: $K_a$ small ($K_a <
0.5$) and $K_a$ not necessarily small ($0 < K_a < 10$), but with $K_d$ not
small. Here we will interpret $K$ as $K_L$ for plastic. In discussing
inequalities involving $K_a$ and $K_d$ it should be kept in mind when
interpreting the role of transverse waves that $K_T$ is roughly twice
$K_L$ at a given temporal frequency. In the first case we consider a
number of incident directions and also two flaw types (void and
steel inclusion). In the second case we consider only normal inci-
dence and one flaw type (void). The limitation to normal incidence
was due to the fact that the computer program for $T + L$ and $T + T$
scattering in the range ($0 < K_a < 10$) was found to be defective and
thus the mode conversions involved with oblique incidence could not
be handled fully. The regime in which both Ka and Kd are small
requires a special analysis which has not yet been completed.

The relevant material properties are the density ρ, the longi­
tudinal propagation velocity cL, and the transverse propagation
velocity cT. In the present computations attenuation will be neg­
lected. In Table II below we present the values of ρ, cL, and cT
for water, plastic (polymethylmethacrylate), and steel that were
used in the present computations. In the case of a void we assumed
that ρ, cL, and cT are all zero. Throughout these computations we
consistently use the following units; gram (gm), centimeter (cm),
and microsecond (μsec). We use four different values of the ratio
d/a, namely 1.65, 4.57, 8.86, and 13.68. Although a number of
angles of incidence (θ = cos⁻¹ (−ez · e_w)) were used in the
computations, only two will be reported, i.e., θ = 0° and 10°.

Table II Material Properties

<table>
<thead>
<tr>
<th>Substance</th>
<th>ρ (gm cm⁻³)</th>
<th>cL (cm μsec⁻¹)</th>
<th>cT (cm μsec⁻¹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>1.000</td>
<td>0.150</td>
<td>-</td>
</tr>
<tr>
<td>Plastic</td>
<td>1.180</td>
<td>0.272</td>
<td>0.134</td>
</tr>
<tr>
<td>Steel</td>
<td>8.000</td>
<td>0.582</td>
<td>0.310</td>
</tr>
</tbody>
</table>

We first consider a spherical void with normal incidence
(θ = 0), with various values of d/a, and with K_{ja} in the range from
0 to 10. The results of the first order computations, in which the
interaction between the void and the interface is completely ig­
nored, are presented in Fig. 4.2. These results are independent of
d/a. The first plot gives the absolute value of the scattering
amplitude, i.e., ||A||, (henceforth called "amplitude") as a
Fig. 4.2. First order scattering for a spherical void (d/a = any value, θ = 0°).

Fig. 4.3. Second order correction (d/a = 1.65, θ = 0°).

function of $K_la$ and the second plot gives the phase shift, i.e., ArgA expressed in degrees, also as a function of $K_la$. It should be emphasized that in all of the phase-shift calculations that we have subtracted a constant phase shift corresponding to the first order process in which the actual scatterer is replaced by a fictitious point scatter. The reader will note that these results look almost identical to those for an isolated spherical void in an infinite host medium. The only difference is that the vertical scale of the amplitude here is about 60% greater than that of the isolated void. This is due to the narrowing effect on the scattered wave due to the propagation from plastic into water, whereupon the far-field amplitude is increased by the longitudinal velocity ratio $c_L$(plastic)/$c_L$(water). The transmission losses in propagating both ways across the plastic-water interface decrease the actual far-field amplitude by a small amount (~13%) from that given by the velocity ratio.

In Fig. 4.3 we present for d/a = 1.65 the second order correction due to the interaction between the flaw and the interface being treated in first order (only one round trip propagation between the flaw and the interface is considered). To obtain the total corrected scattering amplitude one must add the complex amplitudes represented in the present and previous figures. The
plots give the absolute value (i.e. amplitude) and the argument (i.e. phase shift) of the correction as functions of $K_L a$. The correction amplitude is smaller than the first order amplitude by a factor of approximately 15, which is not surprising. However, the oscillatory character of the curve is enhanced due to the fact the correction involves the square of the scattering amplitude of an isolated flaw. The downward slope of the correction phase shift curve has been reduced markedly compared with the first order phase shift. In the latter, the downward slope is associated with the fact that the front face reflection arrives early by two units of time relative to the arrival of a pulse scattered from the center. A unit of time is the time required for a longitudinal wave to travel through the plastic over a distance equal to the sphere radius $a$. In the case of the correction process the two units of lead time are cut down by the delay in traveling from the front face to the interface and back. This delay is equal to $2(1.65 - 1) = 1.30$ units and thus the net lead time is $2.00 - 1.30 = 0.70$ units and thus the downward slope in the phase shift is roughly 1/3 as much as before.

We consider the computational results for $d/a = 4.57$ and $8.86$. The first order results given above are valid for other values of $d/a$ as stated before. The second order corrections are presented in Figs. 4.4 and 4.5 for $d/a = 4.57$ and $8.86$, respectively. The plots of amplitude vs $K_L a$ are similar to that in the previous case of $d/a = 1.65$ except that the vertical scales decrease in an expected manner as $d/a$ increases. The phase shift plots show positive slopes that increase as $d/a$ increases due to the increasing delay associated with the longer round trips between the flaw and the interface.

It is desirable to devote more attention to the dependence of the correction amplitude (which we will denote by $||A(2)||$) upon $d/a$.

In Table III we list estimates of the asymptotic values of $||A(2)||$ along with the corresponding values of $d/a$. The near-constancy of the product of these two quantities supports the hypothesis that the correction amplitude is inversely proportional to the distance $d$.

| $||A(2)||_{asymp}$ | $d/a$ | $||A(2)||_{asymp}$ | $X(d/a)$ |
|--------------------|------|--------------------|----------|
| 0.045              | 1.65 | 0.074              |
| 0.0165             | 4.57 | 0.075              |
| 0.0085             | 8.86 | 0.075              |
| 0.0055             | 13.68| 0.075              |
We now turn our attention to the case of a spherical void in the Rayleigh regime (i.e., $0 < K_L a < 0.5$) with selected values of $d/a$ and the angle of incidence $\theta$. We first present a set of results for normal incidence ($\theta = 0$). In Fig. 4.6 a plot of the first order amplitude vs $K_L a$ is given. The phase shift in this case is so close to $0^\circ$ for $K_L a < 0.5$ that a meaningful plot was not possible. As would be expected, the amplitude plot is very close to parabolic. In Figs. 4.7 and 4.8 are given the second order corrections for $d/a = 4.57$ and 13.68, respectively. The correction amplitude appears to be closely proportional to $(K_L a)^4$, which is to be expected since the square of the Rayleigh regime scattering amplitude for the isolated flaw enters the second order correction. As before, the vertical scale of the correction amplitude curves appears to be inversely proportional to the distance $d$. The phase-shift curves have positive slopes proportional to the delay associated with the round trip between the flaw center and the interface.
We turn next to a consideration of results obtained in the Rayleigh regime for the case of oblique incidence (i.e., $\theta = 10^\circ$). With oblique incidence mode conversion occurs in transmission through the interface for both the incident and scattered waves. To accentuate the relative phase shifts between the different modes we have chosen the largest value of $d/a$, namely 13.68. The first order results are given in Fig. 4.9. The effect of the relative phase shifts associated with different modes is clearly manifested. The amplitude versus $K_{La}$ curve shows nonparabolic behavior, particularly at the higher values of $K_{La}$. Furthermore, the phase-shift curve deviates significantly from the smooth, almost constant curve obtained in the normal-incidence case in which no mode conversions can occur. The second order corrections are represented in Fig. 4.10. The effects of relative phase shifts pertaining to different modes are clearly visible.

An analogous set of results have been obtained for the case of a spherical steel inclusion in the Rayleigh regime ($K_{La} < 0.5$). The results are very similar to those obtained for the void except that the first order results involve a phase shift of $180^\circ$ (instead of $0^\circ$) in the limit $K_{La} \to 0$. The second order corrections are rather similar to those obtained in the void case. Because of space limitations we will not give a detailed presentation of these results here.
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REFERENCES

DISCUSSION

J.E. Gubernatis (Los Alamos National Laboratory): I'm a little confused about how you actually came up with this transfer function you speak of. You mentioned something about the far field. Just how do you calculate the far field values by the exact scattering from the defect?

J.M. Richardson (Rockwell International Science Center): Are you asking how I relate the transfer function to the conventional scattering amplitude?

J.E. Gubernatis: Yes.

J.M. Richardson: Simply take an incident wave on the flaw represented by the transfer of function. I get the near field scattered wave, and then I go to the far field by using conventional elastodynamics to relate the near field to the far field. The far field scattered wave, of course, is something that allows you to make a direct correspondence. This particular experiment is related to a homogeneous medium everywhere which relates directly to the conventional scattering amplitude.